9.1 Solving Equations

Solving equations is one of the most important bits of maths. Solving means finding the value of an unknown letter (usually x) that satisfies the equation, which you do by rearranging it until it's in the form 'x = number'.

Learning Objective — Spec Ref A17:
Be able to solve algebraic equations.

Prior Knowledge Check:
Be able to rearrange algebraic expressions. See Section 6.

To solve an equation, you're going to have to do some rearranging. To rearrange, you always have to **do the same thing to both sides** of the equation — so if you add 7 to one side, you have to add 7 to the other side as well. If you **multiply** or **divide** by a number, you have to multiply or divide **everything** on both sides. Remember, you want to end up with 'x **= number**' — so keep this in mind when rearranging.

Example 1

Solve: a) $15 - 2x = 7$ b) $\frac{x}{3} + 2 = -3$

a)
1. Add $2x$ to both sides (to make the coefficient of x positive).
2. Subtract 7 from both sides.
3. Divide both sides by 2.

$$15 - 2x = 7$$
$$15 = 7 + 2x$$
$$8 = 2x$$
$$\mathbf{x = 4}$$

b)
1. Start by subtracting 2 from both sides. Be careful with the negative numbers here.
2. Multiply both sides by 3 to get rid of the fraction.

$$\frac{x}{3} + 2 = -3$$
$$\frac{x}{3} = -5$$
$$\mathbf{x = -15}$$

Tip: If it helps, write out the working in full:

$$15 - 2x = 7$$
$$15 - 2x + 2x = 7 + 2x$$
$$15 = 7 + 2x$$
$$15 - 7 = 7 + 2x - 7$$
$$8 = 2x$$
$$8 \div 2 = 2x \div 2$$
$$4 = x$$

Exercise 1

Q1 Solve each of the following equations.

a) $x + 9 = 12$ b) $x - 7.3 = 1.6$ c) $12 - x = 9$

d) $9x = 54$ e) $-5x = 50$ f) $40x = -32$

Q2 Find the value of x by solving the following equations.

a) $\frac{x}{3} = 2$ b) $\frac{x}{2} = 3.2$ c) $-\frac{x}{0.2} = 3.2$ d) $\frac{2x}{5} = 6$

Q3 Solve each of the following equations.

a) $8x + 10 = 66$ b) $1.8x - 8 = -62$ c) $8 - 7x = 22$

d) $\frac{x}{2} - 1 = 2$ e) $15x + 12 = 72$ f) $1.5x - 3 = -24$

g) $17 - 10x = 107$ h) $-\frac{2x}{3} - \frac{3}{4} = \frac{1}{4}$ i) $-\frac{3x}{5} + \frac{1}{3} = \frac{2}{3}$

If the equation has **brackets** in it, **expand** the brackets (see p.77-79) before solving the equation.

Example 2

Solve the equation $8(x + 2) = 36$.

1. Expand the brackets by multiplying them out. $8(x + 2) = 36$
   $8x + 16 = 36$
2. Subtract 16 from both sides. $8x = 20$
3. Divide both sides by 8. $\mathbf{x = 2.5}$

Tip: You could also start by dividing both sides by 8 to get $x + 2 = (36 \div 8) = 4.5$.

Exercise 2

Q1 Solve each of the following equations by expanding the brackets.

a) $7(x + 4) = 63$ b) $13(x - 4) = -91$ c) $18(x - 3) = -180$ d) $2.5(x + 4) = 30$

e) $3.5(x + 6) = 63$ f) $4.5(x + 3) = 72$ g) $315 = 21(6 - x)$ h) $171 = 4.5(8 - x)$

Q2 Multiply both sides of the equation $\frac{1}{x-2} = 3$ by $(x - 2)$, and hence solve the equation $\frac{1}{x-2} = 3$.

Q3 Solve each of the following equations.

a) $\frac{1}{x} = 2$ b) $\frac{2}{x} = 5$ c) $\frac{12}{x-2} = 4$ d) $\frac{3}{1-2x} = 2$

If you have x-terms on **both sides** of the equation, rearrange so you have all the **x-terms** on **one side** of the equation, and all the **numbers** on the **other side**. You can then **collect like terms** (see p.75) and solve.

Example 3

Solve the equation $5(x + 2) = 3(x + 6)$.

1. Multiply out the brackets. $5(x + 2) = 3(x + 6)$
   $5x + 10 = 3x + 18$
2. Rearrange so that all the x-terms are on one side and all the numbers are on the other side. $5x - 3x = 18 - 10$
   $2x = 8$
3. Divide by 2 to find the value of x. $\mathbf{x = 4}$

Exercise 3

Q1 Solve each of the following equations.

a) $6x - 4 = 2x + 16$ b) $17x - 2 = 7x + 8$ c) $6x - 12 = 51 - 3x$

d) $5x - 13 = 87 - 5x$ e) $10x - 18 = 11.4 - 4x$ f) $4x + 9 = 6 - x$

Q2 Solve each of the following equations.

a) $3(x + 2) = x + 14$ b) $5(x + 3) = 2x + 57$ c) $7(x - 7) = 2(x - 2)$

d) $5(x - 4) = 3(x + 8)$ e) $4(x - 3) = 3(x - 8)$ f) $11(x - 2) = 3(x + 6)$

Q3 Solve each of the following equations.

a) $7\left(2x + \frac{1}{7}\right) = 8\left(3x - \frac{1}{2}\right)$ b) $7(x - 1) = 4(6.2 - 2x)$ c) $-3(x - 3) = 8(0.7 - x)$

If you have **fractions** on **both sides** of the equation, you can **cross-multiply** to save time. All you do is multiply the **numerator** of each fraction by the **denominator** of the other.

$\frac{a}{b} = \frac{c}{d}$ becomes $a \times d = c \times b$

Example 4

Solve the equation $\frac{x-2}{2} = \frac{6-x}{6}$.

1. Cross-multiply. This is the same as multiplying both sides by 2 and by 6.
2. Expand the brackets and solve for x.

$\frac{x-2}{2} = \frac{6-x}{6}$

$6(x-2) = 2(6-x) \Rightarrow 6x - 12 = 12 - 2x$

$\Rightarrow 8x = 24 \Rightarrow \mathbf{x = 3}$

Exercise 4

Q1 Solve the following equations.

a) $\frac{x}{4} = 1 - x$ b) $\frac{x}{3} = 8 - x$ c) $\frac{x}{5} = 11 - 2x$

d) $\frac{x}{3} = 2(x - 5)$ e) $\frac{x}{2} = 4(x - 7)$ f) $\frac{x}{5} = 2(x + 9)$

Q2 Solve the following equations.

a) $\frac{x+4}{2} = \frac{x+10}{3}$ b) $\frac{x+2}{2} = \frac{x+4}{6}$ c) $\frac{x-2}{3} = \frac{x+4}{5}$

d) $\frac{x-6}{5} = \frac{x+3}{8}$ e) $\frac{x-2}{4} = \frac{15-2x}{3}$ f) $\frac{x-4}{6} = \frac{12-3x}{2}$

If you get an equation with an x^2 in it, you need to rearrange until you have 'x^2 = number', then take **square roots**. Be careful — whenever you take the square root of a number, you get **two solutions** as the answer can be **positive or negative**. See p.91 and p.99-103 for more about square roots and surds.

Example 5

Solve: a) $3x^2 = 75$ b) $4x^2 + 5 = 33$

a) 1. Divide both sides by 3.

2. Take the square root of both sides, giving the two possible answers.

$3x^2 = 75$

$x^2 = 25$

$\mathbf{x = \pm 5}$

b) 1. Rearrange the equation until you end up with 'x^2 = number'.

2. Take the square root. 7 isn't a square number, so the answer is a surd.

$4x^2 + 5 = 33$

$4x^2 = 28$

$x^2 = 7$

$\mathbf{x = \pm\sqrt{7}}$

Tip: The ± symbol means 'plus or minus'. It's just a quick way of writing '$x = 5$ or $x = -5$'.

Exercise 5

Q1 Find the two values of x that satisfy each of the following equations.

a) $x^2 = 16$ b) $x^2 + 10 = 35$ c) $3x^2 = 27$

d) $2x^2 + 1 = 99$ e) $\frac{3x^2}{10} = 1.2$ f) $\frac{x^2+2}{x^2-4} = \frac{11}{10}$

Q2 Solve each of the following equations.

a) $2x^2 = 4$ b) $x^2 + 7 = 13$ c) $3x^2 + 1 = 40$

9.2 Forming Equations from Word Problems

In this section, you'll see how to turn words into maths in order to solve problems with real-life contexts.

Learning Objectives — Spec Ref A21:
- Be able to set up an algebraic equation for a given situation.
- Interpret the solution of an algebraic equation in context.

Prior Knowledge Check: Shape properties, including angles, perimeter and area. See Sections 20 and 27.

If you're given a **wordy problem**, you can use the information given in the question to **write** an equation.

If you're given information about one or more **unknown quantities**, call one quantity x. Then, if someone has "3 more", this would be $(x + 3)$, if someone has "3 times as many", it would be $3x$, etc. If you know the **total**, add up your different expressions, then set this equal to the total. If you're told the **difference**, you should subtract.

Once you've written your equation, you can **solve** it to find the value you need. You might need to **interpret** your answer in the **context** of the question in order to find some **other values**, too.

Example 1

The sum of three consecutive numbers is 63. What are the numbers?

1. Call the first number x, then the other two numbers are $(x + 1)$ and $(x + 2)$.
2. Form an equation in x and solve it.
3. Don't forget to find the other two numbers once you've found x.

$$x + (x + 1) + (x + 2) = 63$$
$$3x + 3 = 63$$
$$3x = 60$$
$$x = 20$$

So the numbers are **20**, **21** and **22**.

Tip: There's more about writing things like consecutive numbers algebraically coming up on p.119.

Exercise 1

Q1 I think of a number. I double it, and then add 3. The result equals 19.

a) Write the above description in the form of an equation.

b) Solve your equation to find the number I was thinking of.

Q2 I think of a number. I divide it by 3, and then subtract 11. The result equals –2. What number was I thinking of?

Q3 The sum of four consecutive numbers is 42. What are the numbers?

Q4 Anna, Bill and Christie are swapping football stickers. Bill has 3 more stickers than Anna. Christie has twice as many stickers as Anna. The three of them have 83 stickers in total. How many stickers does each person have?

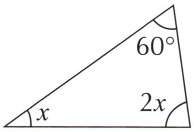
Q5 Deb, Eduardo and Fiz are raising money for charity. Eduardo has raised £6 more than Deb. Fiz has raised three times as much as Eduardo. The three of them have raised £106.50 in total. How much did each of them raise?

Q6 Stacey is three years older than Macy. Tracy is twice as old as Stacey. The three of them have a combined age of 41. How old is each person?

You might have to use **properties of shapes** to form an equation which you can then use to find side lengths, angles, area or perimeter — take a look at Sections 20 and 27 for more.

Example 2

Use the triangle to write an equation involving x.
Solve your equation to find x.



1. The angles in a triangle always add up to 180° — use this to form an equation.
$$x + 2x + 60° = 180°$$
$$3x + 60° = 180°$$
$$3x = 120°$$
2. Solve the equation.
$$\mathbf{x = 40°}$$

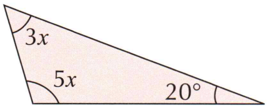
Example 3

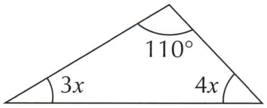
A rectangle has sides of length $5x$ cm and $(2x + 1)$ cm. The perimeter of the rectangle is 44 cm. Find the lengths of the sides of the rectangle.

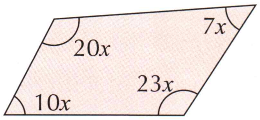
1. Form an equation. Remember to add the length of each side twice to find the perimeter.
$$5x + 5x + (2x + 1) + (2x + 1) = 44$$
$$14x + 2 = 44$$
$$14x = 42$$
2. Solve to find x.
$$x = 3$$
3. Use x to find the side lengths of the rectangle.
So the rectangle has sides of length:
$5 \times 3 =$ **15 cm** and $(2 \times 3) + 1 =$ **7 cm**.

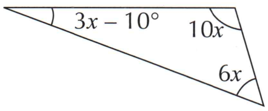
Exercise 2

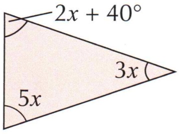
Q1 For each shape below, write an equation and solve it to find the value of x.

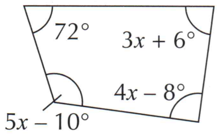
a) 

b) 

c) 

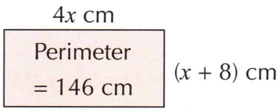
d) 

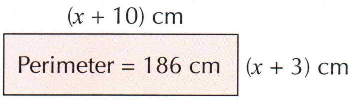
e) 

f) 

Q2 A triangle has angles of size x, $2x$ and $(70° - x)$. Find the value of x.

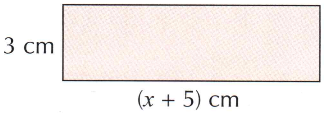
Q3 For each shape below, (i) find the value of x, and (ii) find the area of the shape.

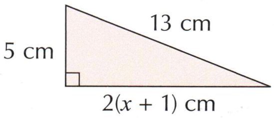
a) 

b) 

Q4 A rectangle has sides of length $(4 - x)$ cm and $(3x - 2)$ cm.
The perimeter of the rectangle is 8.8 cm. Find the area of the rectangle.

Q5 The triangle and rectangle shown below have the same area. Find the perimeter of each shape.





9.3 Identities

Identities look a bit like equations with an extra line, but there's an important difference — equations are only true for a particular value (or values) of x, but identities are always true, no matter what.

Learning Objective — Spec Ref A6:
Understand the difference between equations and identities.

An **equation** is a way of showing that two expressions are equal for some particular values of an unknown.
Identities are like equations, but are **always true**, for **any value** of the unknown.
Identities have the symbol '$\equiv$' instead of '='.

E.g. $x - 1 = 2$ is an equation — it's only true when $x = 3$.
$x + 1 \equiv 1 + x$ is an identity — it's always true, whatever the value of x.

In identity questions, you should **rearrange** the expressions on **either side** to see if they're the **same**.
You **don't** need to take things to the other side, like you would if you were solving an equation.

Example 1

In which of the following equations could you replace the '=' sign with '$\equiv$'?
(i) $6 + 4x = x + 3$ (ii) $x(x - 1) = -(x - x^2)$

1. You can rearrange equation (i) to give $3x = -3$.
   This has only one solution ($x = -1$), so: **$6 + 4x = x + 3$ isn't an identity.**
2. If you expand the brackets in (ii) you get $x^2 - x = -x + x^2$.
   Both sides are the same, so: **$x(x - 1) \equiv -(x - x^2)$ is an identity.**

Example 2

Find the value of k if $(x + 2)(x - 3) \equiv x^2 - x + k$.

1. Expand the brackets on the left hand side.
2. There's an x^2 and a $-x$ on both sides already, so to make both sides identical, k must be -6.

$$(x + 2)(x - 3) \equiv x^2 - x + k$$
$$x^2 + 2x - 3x - 6 \equiv x^2 - x + k$$
$$x^2 - x - 6 \equiv x^2 - x + k$$
$$\mathbf{k = -6}$$

Exercise 1

Q1 For each of the following, state whether or not you could you replace the box with the symbol '$\equiv$'.

a) $x - 1 \ \square\ 0$ b) $x^2 - 3 \ \square\ 3 - x^2$ c) $3(x + 2) - x \ \square\ 2(x + 3)$

d) $x^2 + 2x + 1 \ \square\ (x + 1)^2$ e) $4(2 - x) \ \square\ 2(4 - 2x)$ f) $4x^2 - x \ \square\ 2(x^2 - 2x)$

Q2 Find the value of a if:

a) $2(x + 5) \equiv 2x + 1 + a$ b) $ax + 3 \equiv 5x + 2 - (x - 1)$ c) $(x + 4)(x - 1) \equiv x^2 + ax - 4$

d) $(x + 2)^2 \equiv x^2 + 4x + a$ e) $4 - x^2 \equiv (a + x)(a - x)$ f) $(2x - 1)(3 - x) \equiv ax^2 + 7x - 3$

Q3 Prove that: a) $(x + 5)^2 + 3(x - 1)^2 \equiv 4(x^2 + x + 7)$ b) $3(x + 2)^2 - (x - 4)^2 \equiv 2(x^2 + 10x - 2)$

9.4 Proof

Proof questions might seem a bit confusing — it's not always obvious where to start and where you want to end up. Don't worry though, there are some handy tricks you can use to help you to get started.

Proof

Learning Objective — Spec Ref A6:
Be able to prove that mathematical statements are true.

Proof is all about showing that something is **true**. For example, you can **prove** that two expressions are identical (like on the previous page) by **rearranging** one into the other.
You can use these facts to make proof questions much easier:

- Any **even number** can be written as $\mathbf{2n}$ — i.e. as "2 × an integer".
- Any **odd number** can be written as $\mathbf{2n + 1}$ — i.e. as "(2 × an integer) + 1".
- **Consecutive numbers** can be written as $\mathbf{n}$, $\mathbf{(n + 1)}$, $\mathbf{(n + 2)}$, etc. **Consecutive even** numbers can be written as $2n$, $(2n + 2)$, $(2n + 4)$, etc. and **consecutive odd** numbers as $(2n + 1)$, $(2n + 3)$, $(2n + 5)$, etc.
- The **sum**, **difference** and **product** of integers is **always** an integer.

Proof questions can cover any topic, from algebra to geometry to statistics.

Example 1

Prove that the sum of any three odd numbers is odd.

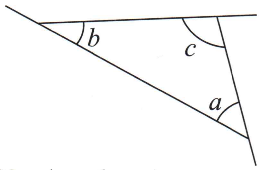
1. Take three odd numbers. $2a + 1$, $2b + 1$ and $2c + 1$, where a, b and c are integers.
2. Add them together and rearrange into the form $2n + 1$ (where n is an integer).

$2a + 1 + 2b + 1 + 2c + 1 = 2a + 2b + 2c + 2 + 1$
$= 2(a + b + c + 1) + 1 = 2n + 1$ where $n = (a + b + c + 1)$ is an integer
So the sum of three odd numbers is **odd**.

Tip: The question doesn't say the numbers are consecutive, so you need to use a different letter for each number.

Example 2

Prove that the sum of the exterior angles of a triangle is 360°.



1. Draw a quick sketch. Call the interior angles a, b and c.

The exterior angles are: $180° - a$, $180° - b$ and $180° - c$
So their sum is: $(180° - a) + (180° - b) + (180° - c) = 540° - (a + b + c)$

2. The angles in a triangle add up to 180°. $= 540° - 180° = \mathbf{360°}$

Exercise 1

PROBLEM SOLVING

Q1 Prove that the sum of three consecutive integers is a multiple of 3.

Q2 Prove that the difference between 8^{12} and 12^7 is a multiple of 4.

Q3 $2x + a = 7(x - 2a) - 5$. Prove that if a is odd then x is even.

Q4 Prove that the sum of two consecutive square numbers is always odd.

Q5 The range of a set of positive numbers is 8. Each number in the set is multiplied by 3. Show that the new range is 24.

Q6 Prove that, for any integer n, n^2 is larger than the product of the two integers either side of n.

Q7 Triangle numbers are formed by the expression $\frac{1}{2}n(n + 1)$.
Prove that the sum of two consecutive triangle numbers is a square number.

Q8 A data set with 5 values has a mean of 12. Each value in the data set is increased by 1. Show that the new mean is 13.

Disproof by Counter Example

Learning Objective — Spec Ref A6:
Find a counter example to show that a statement isn't true.

To show that a statement is **false**, you can just find an **example** where it **doesn't work**. This is called a **counter example**. There are usually many counter examples you could give, but you only need to find one.

Example 3

Disprove the following statement by finding a counter example:
"The difference between two consecutive square numbers is always prime."

Try consecutive square numbers until you find a pair that doesn't work:

1 and 4 — difference = 3 (prime)
4 and 9 — difference = 5 (prime)
9 and 16 — difference = 7 (prime)
16 and 25 — difference = 9 (NOT prime) so **the statement is false**.

Tip: You don't have to go through loads of examples if you can spot one that's wrong straightaway — you could go straight to 16 and 25.

Example 4

Paz says, "If $x > y$, then $x^2 > y^2$". Is she correct? Explain your answer.

Try some values for x and y:
$x = 2, y = 1$: $x > y$ and $x^2 = 4 > 1 = y^2$
$x = 5, y = 2$: $x > y$ and $x^2 = 25 > 4 = y^2$
$x = -1, y = -2$: $\mathbf{x > y}$ **but** $\mathbf{x^2 = 1 < 4 = y^2}$

So **Paz is wrong** as the statement does not hold for all values of x and y.

Tip: When looking for a counter example, try different types of number, e.g. positive and negative.

Exercise 2

Disprove the following statements by finding a suitable counter example.

Q1 "The sum of three consecutive integers is always bigger than each individual number."

Q2 "The difference between any two prime numbers is always an even number."

Q3 "The sum of two square numbers is never a square number."

Q4 "If $x > 0$, then $x^2 \geq x$."

Q5 "If the median of a data set is equal to the mode, then the mean is also the same."

Q6 "If you draw three points on a grid, joining them up with straight lines always forms a triangle."

9.5 Iterative Methods

Iterative methods give you another way to find solutions to equations. They're most useful when an equation is too hard to solve in the normal way (i.e. when the equation involves high powers of x).

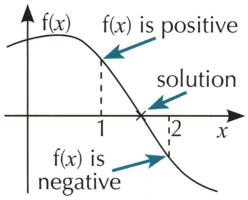
Change of Sign Methods

Learning Objective — Spec Ref A20:
Use change of sign iterative methods to find approximate solutions to equations.

Prior Knowledge Check:
Be able to round to a given degree of accuracy. See Section 2.

Iterative methods can be used to find approximate solutions to equations. The idea is that by repeating steps of the method, you'll get **close enough** to the solution to know it to a **given degree of accuracy**. For example, if you can show that a solution is between 1.62 and 1.63, then you know that the solution will be 1.6 when rounded to 1 d.p. (as all numbers between 1.62 and 1.63 round to 1.6).

Iterative methods for solving equations often involve looking for a **change of sign**. If you have an equation of the form '$f(x) = 0$', $f(x)$ will **change from positive to negative** (or vice versa) as x goes past a solution. So if there's a change of sign between two values, then there must be a **solution** between those values. For example, if f(1) is positive, but f(2) is negative, then there is a solution to $f(x) = 0$ **somewhere between 1 and 2**, as shown in the diagram on the right.



Method 1

To find an approximate solution, **go through values in order** until there's a change of sign. Each time you find a change of sign, **increase** the number of **decimal places** until you know the solution to the required degree of accuracy.

For example, in the diagram above, you know that there's a change of sign between $x = 1$ and $x = 2$, so **add a decimal place** and start testing more values **in order**, i.e. find f(1.1), f(1.2), f(1.3), etc. If you find that $f(x)$ changes sign between $x = 1.6$ and $x = 1.7$, then the solution must be **between 1.6 and 1.7**. Then add another decimal place and start testing f(1.61), f(1.62), f(1.63) etc.

Repeat this method until you know the solution to the level of accuracy you need. You must always test values with **one more decimal place** than the final answer requires — e.g. if you want an answer correct to 2 d.p., you need to test values with 3 d.p. so that you know whether to **round up or down**.

Example 1

The equation $x^3 + 1 = 5x$ has a solution between 2 and 3.
By considering values in this interval, find this solution to 1 decimal place.

1. Rearrange the equation into the form '$f(x) = 0$': $x^3 - 5x + 1 = 0$
2. Try the values of x between 2 and 3 with 1 d.p. (2.1, 2.2, etc.) in order until there's a sign change.

   f(2.1) is negative and f(2.2) is positive.
   So there is a solution between 2.1 and 2.2.
3. Now you need to decide whether the answer should round up to 2.2 or down to 2.1. Try the values of x between 2.1 and 2.2 with 2 d.p. in order until there's a sign change.

   f(2.12) is negative and f(2.13) is positive, so $2.12 < x < 2.13$.
   All the values in this interval round down to 2.1, so the solution is $x = 2.1$ (to 1 d.p.).

x	$x^3 - 5x + 1$	
2.0	−1	Negative
2.1	−0.239	Negative
2.2	0.648	Positive
2.11	−0.156069	Negative
2.12	−0.071872	Negative
2.13	0.013597	Positive

Exercise 1

Use Method 1 to answer all the questions in this Exercise.

Q1 A solution to the equation $x^2 + x - 10 = 0$ lies between 2 and 3.

a) Evaluate $x^2 + x - 10$ for $x = 2$, $x = 2.5$ and $x = 3$.

b) State whether the solution to $x^2 + x - 10 = 0$ is greater than or less than 2.5.

c) Work out whether the solution is greater than or less than the following:

(i) 2.6 (ii) 2.7 (iii) 2.8

d) Find the solution to $x^2 + x - 10 = 0$ correct to 1 d.p.

Q2 A solution to $x^2 + 2x = 30$ lies between 4.5 and 4.6. Copy and complete the table on the right, adding extra rows as required, to find this solution to 1 d.p.

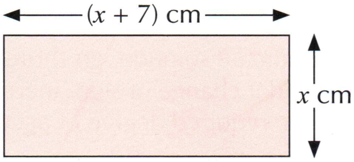
x	$x^2 + 2x - 30$	
4.5	–0.75	Negative
4.6		
4.51		
4.52		

Q3 The following equations have solutions between 5 and 6. Find the solutions correct to 1 d.p.

a) $x^3 + 5x = 170$ b) $x^3 - 3x = 133$

Q4 The solution to the equation $2^x = 20$ lies between 4 and 5. Find x correct to 1 decimal place.

Q5 The rectangle on the right has an area of 100 cm².



a) Write down an equation in x.

b) Given that $7 < x < 8$, find x correct to 1 decimal place.

c) Find the lengths of the sides of the rectangle, correct to 1 decimal place.

Method 2

This method also involves finding a change of sign — but rather than testing the values in order, this time you try a value roughly in the **middle** of the interval where there's a sign change. Depending on whether this value is positive or negative, you can use it as the new upper or lower limit, reducing the size of the interval by about half.

For example, if f(1) is positive and f(2) is negative, start by finding **f(1.5)**. If f(1.5) is **positive**, then you know that there's a change of sign between 1.5 and 2, so you've narrowed the interval down to $1.5 < x < 2$. Next, you can try **either 1.7 or 1.8** (they're both roughly in the middle of the interval). Use your **judgement** here — if f(1.5) is **closer to 0** than f(2) is, it might be better to try 1.7. Keep going until you've narrowed the interval down to two values that are **0.1** apart (e.g. $1.6 < x < 1.7$). At this point, you can start testing values with **2 decimal places.**

Once you've narrowed it down to two possible final answers, you need to test **one more value** to see whether to **round up or down**. E.g. if the question asks for the solution to **2 d.p.** and you know that $1.62 < x < 1.63$, then the final answer will be **either 1.62 or 1.63**. Find **f(1.625)** — all values between 1.62 and 1.625 **round down** to 1.62, and values between 1.625 and 1.63 **round up** to 1.63.

This method is usually **quicker** than Method 1.

Example 2

The equation $x^4 + 4x - 6 = 0$ has two solutions, one positive and one negative.
a) Show that the positive solution lies between 1 and 2.
b) Use an iterative method to find the positive solution correct to 1 d.p.

a) Find the value of $x^4 + 4x - 6$ at $x = 1$ and $x = 2$.

$f(1) = 1 + 4 - 6 = -1$ (negative)
$f(2) = 16 + 8 - 6 = 18$ (positive)
There's a change of sign between 1 and 2, so **there is a solution in this interval.**

b) 1. You know that the solution lies between 1 and 2, so try a value in the middle of the interval. $x = 1.5 \Rightarrow f(x) = 5.0625$

2. f(1.5) is positive and f(1) is negative, from part a): f(1.5) is positive, so $1 < x < 1.5$

3. Repeat this until you've narrowed it down to 2 possible answers.
   $f(1.2) = 0.8736$ is positive, so $1 < x < 1.2$
   $f(1.1) = -0.1359$ is negative, so $1.1 < x < 1.2$

4. So the solution to 1 d.p. is either 1.1 or 1.2. Split the interval in half to see what it rounds to.
   $f(1.15) = 0.3490...$ is positive, so $1.1 < x < 1.15$
   All values in this interval round down, so the solution is $\mathbf{x = 1.1}$ (to 1 d.p.)

Tip: Here, we've tried f(1.2) rather than f(1.3) because f(1) is closer to 0 than f(1.5) is.

Exercise 2

Use Method 2 to answer all the questions in this Exercise.

Q1 A solution to $x^2 + x = 35$ lies between 5.4 and 5.5. Copy and complete the table on the right, adding extra rows as required, to find this solution correct to 2 d.p.

x	$x^2 + x - 35$	
5.4	–0.44	Negative
5.5		
5.45		

Q2 Each of the following equations has a solution between 0 and 10. Find this solution correct to 2 d.p.
a) $x^2 + x = 23$ b) $x^2 + 2x = 17$ c) $x^2 + 5x = 62$

Q3 Find, to 3 d.p., the solution to the equation $x^2 + x = 48$ that lies between 6.4 and 6.5.

Q4 The equation $x^3 + 4x = 21$ has a solution between 2.2 and 2.3. Find this solution, correct to 3 decimal places.

Q5 A cannonball is fired upwards from the ground. After x seconds, its height (h) in metres is given by the formula $h = 100x - 5x^2$. It travels upwards for 10 seconds until it reaches a height of 500 m, then falls back towards the ground. PROBLEM SOLVING
a) How many seconds does it take the ball to reach a height of 200 m as it rises? Give your answer correct to 1 decimal place.
b) After how many seconds does the ball first reach a height of 400 metres? Give your answer correct to 1 decimal place.
c) As it falls back down to the ground, it passes a height of 200 metres for a second time. Find the number of seconds after being fired that it passes this height. Give your answer correct to 1 decimal place.

Recursive Iteration

Learning Objective — Spec Ref A20:
Use recursive iteration to find approximate solutions to equations.

Recursive iteration is when you put a starting value into an **iteration formula**, complete the calculation, then **put the result back into** the iteration formula. Each iteration gets you **closer** to the solution, so you can **keep going** until you've got the answer to the degree of accuracy you need. You can assume that you've reached the final answer when **two consecutive results** are **the same** when rounded to the required number of decimal places.

You'll be **given** the formula to use in the question, sometimes in the form of an '**iteration machine**' — these just explain the method that you should follow, step by step.

Example 3

Use the iteration machine below, with $x_0 = 2$, to find a solution to the equation $x^3 + 2x - 7 = 0$ to 1 d.p.

1. Begin with x_n → 2. Find the value of x_{n+1} using the formula $x_{n+1} = \sqrt[3]{7 - 2x_n}$ → 3. If $x_n = x_{n+1}$ rounded to 1 d.p. then stop. If not, go back to step 1 and repeat using x_{n+1}.

1. Substitute x_0 into the formula to find x_1. $x_1 = \sqrt[3]{7 - 2(2)}$ $x_1 = 1.44224957...$
2. Put x_1 back into the formula to find x_2. $x_2 = \sqrt[3]{7 - 2(1.44224957...)}$ $x_2 = 1.602535155...$
3. Repeat until two consecutive terms round to the same number to 1 d.p. $x_3 = \sqrt[3]{7 - 2(1.602535155...)}$ $x_3 = 1.559796392...$

x_2 and x_3 both round to 1.6 to 1 d.p. so $\boldsymbol{x = 1.6}$ is a solution.

Exercise 3

Q1 Using the following iteration machine, find a solution to the equation $-2x^2 + 3x + 1 = 0$ to 1 d.p. Use the starting value $x_0 = 1$.

1. Begin with x_n → 2. Find the value of x_{n+1} using the formula $x_{n+1} = \sqrt{\frac{3x_n + 1}{2}}$ → 3. If $x_n = x_{n+1}$ rounded to 1 d.p. then stop. If not, go back to step 1 and repeat using x_{n+1}.

Q2 Use the iteration machine below to find a solution to the equation $x^3 + 4x - 2 = 0$ to 2 d.p. Use the starting value $x_0 = 0$.

1. Begin with x_n → 2. Find the value of x_{n+1} using the formula $x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 + 4}$ → 3. If $x_n = x_{n+1}$ rounded to 2 d.p. then stop. If not, go back to step 1 and repeat using x_{n+1}.

Q3 Use the iteration formula $x_{n+1} = \frac{2x_n^2 + 11}{4x_n}$ to find a solution to $2x^2 = 11$ to 3 d.p. Use the starting value $x_0 = 3$.

Q4 Using the iteration $x_{n+1} = \sqrt[3]{\frac{7 - 2x_n}{2}}$, find a solution to $2x^3 + 2x - 7 = 0$ to 3 d.p. Use $x_0 = 1$ as the starting value.

Review Exercise

Q1 Solve the following equations.

a) $\frac{x+8}{3} = 4$ b) $13 - 3.5x = 34$ c) $7(x - 3) = 3(x - 6)$

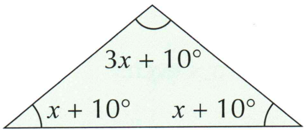
d) $12(x - 3) = 4(6 + 2x)$ e) $\frac{x-2}{5} = \frac{9-x}{3}$ f) $\frac{2x}{5} = 18 - 2x$

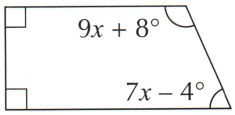
Q2 I think of a number. I subtract 4, then divide by 5. The result equals 15.

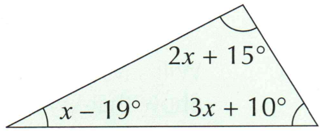
a) Write this information in the form of an equation.

b) Solve the equation to find the number I was thinking of.

Q3 Find the value of x in each of the following.

a)


b)


c)


Q4 For each of the following, state whether you could replace the box with the symbol '≡'.

a) $4x \,\square\, 10$ b) $2(3x + 6) \,\square\, 3(2x + 4)$ c) $3(2 - 3x) + 2 \,\square\, 7x$

Q5 Find the value of a if:

a) $3(x + a) \equiv 12 + 3x$ b) $4(1 - 2x) \equiv 2(a - 4x)$ c) $3(x^2 - 2) \equiv a(6x^2 - 12)$

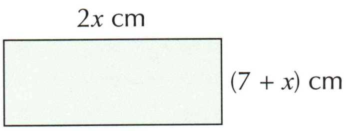
Q6 a) Prove that the sum of 5 consecutive integers is always a multiple of 5.

b) Mark says "If a and b are prime numbers, then $(2ab - 1)$ is always a prime number."
Show that Mark is wrong.

Q7 Each of the following equations has a solution for $0 < x < 10$.
Use an iterative method to find this solution, correct to 1 d.p.

a) $x^2 + 4x = 100$ b) $x^4 = 30 - 10x$ c) $x^4 + 2x^2 + 5x = 20$

Q8 The area of this rectangle is 17 cm². Given that $x < 3$, use an iterative method to find the value of x, correct to 1 d.p.



Q9 Using the following iteration machine, find a solution to the equation $4x^3 + 2x^2 - 5 = 0$ to 2 d.p.
Use the starting value $x_0 = 1$.

1. Begin with x_n →

2. Find the value of x_{n+1} using the formula $x_{n+1} = \sqrt[3]{\frac{5 - 2x_n^2}{4}}$ →

3. If $x_n = x_{n+1}$ rounded to 2 d.p. then stop. If not, go back to step 1 and repeat using x_{n+1}.

Exam-Style Questions

Q1 Solve the following equations:

a) $\frac{2(x+3)}{13} = 1$

[2 marks]

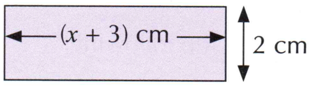
b) $\frac{x-10}{10} = \frac{10-x}{3}$

[2 marks]

c) $3(x^2 + 7) = 4(x^2 - 1)$

[2 marks]

Q2 The area of this rectangle is 12 cm². Find the perimeter of the rectangle.



[3 marks]

Q3 a) Will says "The product of two different square numbers is always even." Show that Will is not correct.

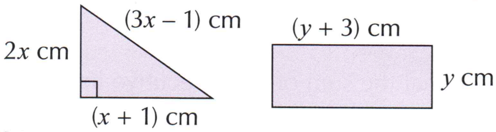
[1 mark]

b) Veronica says "When two prime numbers which are both bigger than 2 are added, the answer is always even." Show that Veronica is correct.

[3 marks]

Q4 The triangle and rectangle shown have the same area. The perimeter of the triangle is 12 cm.

a) Find the area of the triangle.



[3 marks]

b) Use an iterative method to find the value of y, correct to 1 decimal place.

[4 marks]

Q5 The equation $x^2 - 4x + 1 = 0$ has two solutions.

a) Show that one of these solutions lies in the interval $3 < x < 4$.

[2 marks]

b) Use the iteration $x_{n+1} = \sqrt{4x_n - 1}$ to find this solution to 1 decimal place. Use a starting value of $x_0 = 3$.

[3 marks]

Q6 Lol, Maddie and Norm took part in a javelin competition. Lol threw x m. The distance that Maddie threw is the square of the distance that Lol threw. Norm threw 30 m. The total distance of their three throws put together is 128 m.

a) Use the information given to write an equation in x.

[1 mark]

b) The value of x lies in the interval $0 < x < 10$. Use an iterative method to find the distance that Lol threw, correct to the nearest cm.

[4 marks]