7.1 Squares, Cubes and Roots

The square of a number is the product of the number with itself — e.g. the square of 4 is 4 × 4 = 16. Cubes are the product of a number with itself and then itself again — the cube of 4 is 4 × 4 × 4 = 64. Roots are the inverse (or 'opposite') operation — i.e. the square root of x is the number which multiplies with itself to give x and the cube root of y is the number which multiplies with itself and itself again to give y.

Learning Objective — Spec Ref N6:
Evaluate powers and roots.

Prior Knowledge Check:
Know how to multiply and divide by negative numbers — see p.4.

You write squares and cubes using **powers** — '4 squared' would be written $\mathbf{4^2}$ and '4 cubed' would be $\mathbf{4^3}$.

If you square a number, you always get a **positive** answer because you're multiplying the **same signs** together. On the other hand, cubing a number can give a **positive or negative** answer since there are **three multiples** of the same sign — the answer will be the same sign as whatever the **original** number was.

Since squaring only gives positive results, **negative numbers** don't have square roots. However, positive numbers have **two** square roots — for instance, 4 × 4 = 16 and –4 × –4 = 16 so the square roots of 16 are 4 and –4. You can write the positive square root of x as $\sqrt{x}$ and the negative square root as $-\sqrt{x}$.
All numbers, either **positive or negative**, have **exactly one cube root**, which you can write as $\sqrt[3]{x}$.

Example 1

Find: a) 3^2 b) the square roots of 9 c) $\sqrt[3]{-27}$ d) $\sqrt{10^2 - 6^2}$

a) The square of 3 is 3 × 3.

$3^2 = 3 \times 3 = \mathbf{9}$

b) $3^2 = 9$ so a square root of 9 is 3.
But don't forget the second square root:
–3 × –3 = 9 so –3 is also a square root.

$\sqrt{9} = \mathbf{3}$

$-\sqrt{9} = \mathbf{-3}$

c) $(-3)^3 = -27$, so the cube root is –3.

$\sqrt[3]{-27} = \mathbf{-3}$

d) The square root and cube root symbols act like brackets. Evaluate the expression inside before you find the root.

$\sqrt{10^2 - 6^2} = \sqrt{100 - 36}$
$= \sqrt{64} = \mathbf{8}$

Tip: Always be careful with the sign of your answer when squaring and cubing. Anything squared is positive, and a positive number cubed is positive, but a negative number cubed is negative.

Exercise 1

Q1 Find (i) the square and (ii) the cube of the following numbers:
a) 5 b) 10 c) –2 d) 0.1

Q2 Find the square roots of:
a) 36 b) 10 000 c) –16 d) 81

Q3 Find the cube root of:
a) 8 b) –64 c) 1000 d) –1

Q4 Calculate: a) $\sqrt{4 \times 10^2}$ b) $\sqrt[3]{(3+5)^2}$ c) $\sqrt[3]{3^2 \times 2^3 + 12^2}$

7.2 Indices and Index Laws

Powers are a useful shorthand that allow you to write repeated multiplications with just two symbols — a base and an index (plural: indices). Squares and cubes are two simple examples of powers.

Indices

Learning Objective — Spec Ref A4:
Work with numbers in index notation.

Prior Knowledge Check: Be able to multiply variables — see p.76.

Index notation (i.e. powers) can be used to show **repeated multiplication** of a number or letter. For example, $2 \times 2 \times 2 \times 2 = 2^4$. This is read as "2 **to the power** 4".

In index notation, the **base** is the value that you're multiplying (here, 2) and the **index** is the number of instances of that value (here, 4).

base → 2^4 ← index

Example 1

Rewrite the following using index notation: a) $3 \times 3 \times 3 \times 3 \times 3$ b) $b \times b \times b \times b \times c \times c$

a) There are five lots of 3. $3 \times 3 \times 3 \times 3 \times 3 = \mathbf{3^5}$

b) There are four *b*'s and two *c*'s. $b \times b \times b \times b \times c \times c = b^4 \times c^2 = \mathbf{b^4c^2}$

Example 2

Simplify $\left(\frac{2}{5}\right)^2$.

With powers of fractions, the power is applied to both the top and bottom of the fraction.

$$\left(\frac{2}{5}\right)^2 = \frac{2^2}{5^2} = \mathbf{\frac{4}{25}}$$

Exercise 1

Q1 Using index notation, simplify the following.

a) $2 \times 2 \times 2 \times 2 \times 2$ b) $7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7$ c) $3 \times 3 \times x \times x \times x \times y \times y$

Q2 Rewrite the following as powers of 10.

a) $10 \times 10 \times 10$ b) 10 million c) 100 000 000

Q3 Using a calculator, evaluate the following.

a) 3^4 b) 2^8 c) 3^{10} d) 9^5
e) 3×2^8 f) $8 + 2^5$ g) $8^7 \div 4^6$ h) $(9^3 + 4)^2$

Q4 Write the following as fractions without indices.

a) $\left(\frac{1}{2}\right)^2$ b) $\left(\frac{1}{2}\right)^3$ c) $\left(\frac{1}{4}\right)^2$ d) $\left(\frac{2}{3}\right)^2$
e) $\left(\frac{3}{10}\right)^2$ f) $\left(\frac{3}{2}\right)^3$ g) $\left(\frac{5}{3}\right)^4$ h) $\left(\frac{4}{3}\right)^3$

Laws of Indices

Learning Objective — Spec Ref N7/A4:
Use the laws of indices to simplify expressions.

The **laws of indices** let you **simplify** complicated-looking expressions that involve **powers**.

$a^m \times a^n = a^{m+n}$	When **multiplying** powers with the **same base**, you **add** the indices.
$a^m \div a^n = a^{m-n}$	When **dividing** powers with the **same base**, you **subtract** the indices.
$(a^m)^n = a^{m \times n}$	When **raising** one power to another, **multiply** the indices.
$a^1 = a$	Anything to the power **1** is just **itself**.
$a^0 = 1$	Anything to the power **0** is **1**.

Example 3

Simplify the following, leaving the answers in index form.

a) $3^8 \times 3^5$ **b)** $p^8 \div p^5$ **c)** $(17^7)^2$

a) You're multiplying two terms, so add the indices. $3^8 \times 3^5 = 3^{8+5} = \mathbf{3^{13}}$

b) You're dividing two terms, so subtract the indices. $p^8 \div p^5 = p^{8-5} = \mathbf{p^3}$

c) For one power raised to another power, multiply the indices. $(17^7)^2 = 17^{7 \times 2} = \mathbf{17^{14}}$

Tip: The first two rules only work when the base numbers are the same. You can't simplify something like $3^2 \times 2^3$ using these rules.

Exercise 2

Q1 Simplify the following, leaving your answers in index form.

a) $3^2 \times 3^6$ b) $10^7 \div 10^3$ c) $a^6 \times a^4$ d) $(4^3)^3$

e) $8^6 \div 8^1$ f) 7×7^6 g) $(c^5)^4$ h) $\frac{b^8}{b^5}$

i) $f^{75} \div f^0$ j) $\frac{20^{228}}{20^{210}}$ k) $(g^{11})^8$ l) $(14^7)^d$

Q2 For each of the following, find the number that should replace the square.

a) $q^8 \div q^3 = q^{\blacksquare}$ b) $8^{\blacksquare} \times 8^{10} = 8^{12}$ c) $(6^{10})^4 = 6^{\blacksquare}$ d) $(15^6)^{\blacksquare} = 15^{24}$

e) $(9^{\blacksquare})^{10} = 9^{30}$ f) $r^7 \times r^{\blacksquare} = r^{13}$ g) $5^{\blacksquare} \div 5^6 = 5^7$ h) $12^{14} \div 12^{\blacksquare} = 12^7$

Q3 Simplify each expression. Leave your answers in index form.

a) $3^2 \times 3^5 \times 3^7$ b) $5^4 \times 5 \times 5^8$ c) $(p^6)^2 \times p^5$ d) $(9^4 \times 9^3)^5$

e) $7^3 \times 7^5 \div 7^6$ f) $8^3 \div 8^9 \times 8^7$ g) $(12^8 \div 12^4)^3$ h) $(q^3)^6 \div q^4$

Q4 Simplify each expression. Leave your answers in index form.

a) $\frac{3^4 \times 3^5}{3^6}$ b) $\frac{s^8 \times s^4}{s^3 \times s^6}$ c) $\left(\frac{6^3 \times 6^9}{6^7}\right)^3$ d) $\frac{2^5 \times 2^5}{(2^3)^2}$

e) $\frac{5^5 \times 5^5}{5^8 \div 5^3}$ f) $\frac{10^8 \div 10^3}{10^4 \div 10^4}$ g) $\frac{(t^6 \div t^3)^4}{t^9 \div t^4}$ h) $\frac{(8^5)^7 \div 8^{12}}{8^6 \times 8^{10}}$

Q5 a) Write: (i) 4 as a power of 2 (ii) 4^5 as a power of 2 (iii) $2^3 \times 4^5$ as a power of 2

b) Write: (i) 9×3^3 as a power of 3 (ii) $5 \times 25 \times 125$ as a power of 5 (iii) 16×2^6 as a power of 4

Negative Indices

Learning Objective — Spec Ref N7/A4:
Work with negative indices.

Prior Knowledge Check:
Know how to find the reciprocal of a number — see p.35.

You can evaluate powers that have a **negative index** by taking the **reciprocal** of the base (i.e. turning it upside down — the reciprocal of a is $\frac{1}{a}$ and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$) and making the index **positive**.

$$a^{-m} = \frac{1}{a^m}$$

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m = \frac{b^m}{a^m}$$

Example 4

Evaluate 5^{-3}

1. Take the reciprocal of the base and make the index positive.
2. Evaluate the index in the denominator.

$$5^{-3} = \frac{1}{5^3} = \frac{1}{5\times5\times5} = \mathbf{\frac{1}{125}}$$

Exercise 3

Q1 Write the following as fractions.

a) 4^{-1} b) 2^{-2} c) 3^{-3} d) 2×3^{-1}

Q2 Write the following in the form a^{-m}.

a) $\frac{1}{5}$ b) $\frac{1}{11}$ c) $\frac{1}{3^2}$ d) $\frac{1}{2^7}$

Q3 Simplify the following.

a) $\left(\frac{1}{2}\right)^{-1}$ b) $\left(\frac{1}{3}\right)^{-2}$ c) $\left(\frac{5}{2}\right)^{-3}$ d) $\left(\frac{7}{10}\right)^{-2}$

Example 5

Simplify the following: a) $y^4 \div \frac{1}{y^3}$ b) $z^8 \times (z^4)^{-2}$

a) 1. Rewrite $\frac{1}{y^3}$ as a negative index.
2. Subtract the indices.

$$y^4 \div \frac{1}{y^3} = y^4 \div y^{-3}$$
$$= y^{4-(-3)} = \mathbf{y^7}$$

b) 1. Multiply the indices to simplify.
2. Now add the indices.
3. Anything to the power 0 is 1.

$$z^8 \times (z^4)^{-2} = z^8 \times z^{4\times(-2)} = z^8 \times z^{-8}$$
$$= z^{8+(-8)} = z^0$$
$$= \mathbf{1}$$

Tip: You could also do part a) by dividing fractions: $y^4 \div \frac{1}{y^3} = y^4 \times y^3$

Tip: Simplify $(z^4)^{-2}$ before combining the terms.

Example 6

Evaluate $2^4 \times 5^{-3}$. Give the answer as a fraction.

1. Turn the negative index into a fraction.
2. Evaluate the powers.

$$2^4 \times 5^{-3} = 2^4 \times \frac{1}{5^3}$$
$$= \frac{2^4}{5^3} = \mathbf{\frac{16}{125}}$$

Exercise 4

Q1 Simplify the following. Leave your answers in index form.

a) $5^4 \times 5^{-2}$ b) $g^6 \div g^{-6}$ c) $2^{16} \div \frac{1}{2^4}$ d) $k^{10} \times k^{-6} \div k^0$

e) $\left(\frac{1}{p^4}\right)^5$ f) $\left(\frac{l^{-5}}{l^6}\right)^{-3}$ g) $\frac{n^{-4} \times n}{(n^{-3})^6}$ h) $\left(\frac{10^7 \times 10^{-11}}{10^9 \div 10^4}\right)^{-5}$

Q2 a) Write the number 0.01 as:

(i) a fraction of the form $\frac{1}{a}$ (ii) a fraction of the form $\frac{1}{10^m}$ (iii) a power of 10.

b) Rewrite the following as powers of 10:

(i) 0.1 (ii) 0.00000001 (iii) 0.0001 (iv) 1

Q3 Evaluate the following. Write the answers as fractions.

a) $3^2 \times 5^{-2}$ b) $2^{-3} \times 7^1$ c) $\left(\frac{1}{2}\right)^{-2} \times \left(\frac{1}{3}\right)^2$ d) $6^{-4} \div 6^{-2}$

e) $(-9)^2 \times (-5)^{-3}$ f) $8^{-5} \times 8^3 \times 3^3$ g) $10^{-5} \div 10^6 \times 10^4$ h) $\left(\frac{3}{4}\right)^{-1} \div \left(\frac{1}{2}\right)^{-3}$

Fractional Indices

Learning Objective — Spec Ref N7/A4:
Work with fractional indices.

If the index is a **fraction**, you can rewrite the power as a **root**. The **denominator** tells you the root to use.

$a^{\frac{1}{2}} = \sqrt{a}$	If the index is $\frac{1}{2}$ then you replace the power with the **square** root $\sqrt{\ }$.
$a^{\frac{1}{3}} = \sqrt[3]{a}$	If the index is $\frac{1}{3}$ then you replace the power with the **cube** root $\sqrt[3]{\ }$.
$a^{\frac{1}{m}} = \sqrt[m]{a}$	More generally, if the index is $\frac{1}{m}$ then you replace the power with the ***m*th root** $\sqrt[m]{\ }$.
$a^{\frac{n}{m}} = (\sqrt[m]{a})^n$	If the numerator **isn't 1**, i.e. the index is $\frac{n}{m}$, you still use the ***m*th root** $\sqrt[m]{\ }$ but you also need to **raise the root** to the power of n.

Example 7

Evaluate $27^{\frac{2}{3}}$

1. Split up the index using $(a^m)^n = a^{m \times n}$. $27^{\frac{2}{3}} = 27^{\frac{1}{3} \times 2} = (27^{\frac{1}{3}})^2$
2. Write the fractional index as a root. $= (\sqrt[3]{27})^2$
3. Evaluate the root — the cube root of 27 is 3. $= 3^2$
4. Evaluate the remaining power. $= \mathbf{9}$

Tip: You could write $(27^{\frac{1}{3}})^2$ or $(27^2)^{\frac{1}{3}}$. Here, it's much easier to find the cube root of 27 first and then square it.

Exercise 5

Q1 Rewrite the following expressions in the form $\sqrt[m]{a}$ or $(\sqrt[m]{a})^n$.

a) $a^{\frac{1}{5}}$ b) $a^{\frac{3}{5}}$ c) $a^{\frac{2}{5}}$ d) $a^{\frac{5}{2}}$

Evaluate the following expressions.

Q2 a) $64^{\frac{1}{2}}$ b) $64^{\frac{1}{3}}$ c) $16^{\frac{1}{4}}$ d) $1\,000\,000^{\frac{1}{2}}$

Q3 a) $125^{\frac{2}{3}}$ b) $9^{\frac{3}{2}}$ c) $1000^{\frac{5}{3}}$ d) $8000^{\frac{4}{3}}$

7.3 Standard Form

Standard form is useful for writing very big or very small numbers in a more convenient way — e.g. 56 000 000 000 would be 5.6×10^{10} in standard form and 0.000 000 003 45 would be 3.45×10^{-9} in standard form. Any number can be written in standard form and they all follow the same rules.

Standard Form

Learning Objective — Spec Ref N9:
Write and interpret numbers in standard form.

Any number in **standard form** (or standard index form) must be written **exactly** like this:

A can be **any number** between 1 and 10 (but not 10 itself) → $A \times 10^n$ ← n can be **any integer**

There are a few vital things you need to know about numbers in standard form:

- The **front number**, A, must always be **between 1 and 10** (i.e. $1 \le A < 10$).
- The power of 10, n, is how far the **decimal point moves.**
- n is **positive** for **BIG** numbers and n is **negative** for **SMALL** numbers.

Tip: It's handy to think of the decimal point moving, but it's the digits that shift around it.

Example 1

Write these numbers in standard form.

a) 360 000 **b) 0.000036** **c) 146.3 million**

a) 1. Move the decimal point until 360 000 becomes 3.6. The decimal point has moved 5 places.
2. The number is big so n must be +5.

3 6 0 0 0 0 . 0 = $\mathbf{3.6 \times 10^5}$

b) 1. As in part a), the decimal point moves 5 places again to make the number 3.6.
2. The number is small so n must be –5.

0 . 0 0 0 0 3 6 = $\mathbf{3.6 \times 10^{-5}}$

c) 1. Write the number out in full.
2. Convert to standard form using the same method as in parts a) and b).

146.3 million = 146.3 × 1 000 000
= 146 300 000
= $\mathbf{1.463 \times 10^8}$

Tip: Common wrong answers to part c) are 146.3×10^6 (which isn't in standard form as $A > 10$) and 1.463×10^6 (which is too small).

Exercise 1

Write the following numbers in standard form.

Q1 a) 250 b) 1100 c) 48 000 d) 5 900 000
e) 2 750 000 f) 8560 g) 808 080 h) 930 078

Q2 a) 0.0025 b) 0.0067 c) 0.0303 d) 0.000056
e) 0.375 f) 0.07070 g) 0.00000000021 h) 0.0005002

Q3 a) 0.00567×10^9 b) 95.32×10^2 c) 0.034×10^{-4} d) $845\,000 \times 10^{-3}$

Example 2

Write the following standard form numbers as ordinary numbers.

a) $\mathbf{3.5 \times 10^3}$ **b)** $\mathbf{4.67 \times 10^{-5}}$

a) 1. The power is positive so the number will be big.
2. The decimal point moves 3 places.

$3.5 \times 10^3 = \mathbf{3500.0}$

b) 1. The power is negative so the number will be small.
2. The decimal point moves 5 places.

$4.67 \times 10^{-5} = \mathbf{0.0000467}$

Exercise 2

Q1 Write the following as ordinary numbers.

a) 3×10^6 b) 9.4×10^4 c) 8.8×10^5 d) 4.09×10^3
e) 1.989×10^8 f) 6.69×10^1 g) 7.20×10^0 h) 3.56×10^{-6}
i) 8.88×10^{-5} j) 1.9×10^{-8} k) 6.69×10^{-1} l) 7.05×10^{-6}

Multiplying and Dividing in Standard Form

Learning Objective — Spec Ref N9:
Multiply and divide numbers in standard form.

Prior Knowledge Check: Know how to use the laws of indices (p.93).

To **multiply** or **divide** numbers in standard form, first **rearrange** the calculation so that the **front numbers** and the **powers of 10** are **together** — for example, rewrite $(4 \times 10^6) \times (8 \times 10^2)$ as $(4 \times 8) \times (10^6 \times 10^2)$. Then multiply or divide the front numbers and use the **laws of indices** (see p.93) to multiply or divide the powers of 10. Finally, make sure your answer is still in **standard form** — if not, use the method from p.96.

Example 3

Calculate $\mathbf{(2.4 \times 10^7) \times (5.2 \times 10^3)}$**. Give your answer in standard form.**

1. Rearrange to put the front numbers and powers of 10 together.
2. Multiply the front numbers and use the laws of indices.
3. 12.48 isn't between 1 and 10 so this isn't in standard form. Convert 12.48 to standard form.
4. Add the indices again to get the answer in standard form.

$$(2.4 \times 5.2) \times (10^7 \times 10^3)$$
$$= 12.48 \times 10^{7+3}$$
$$= 12.48 \times 10^{10}$$
$$= 1.248 \times 10 \times 10^{10}$$
$$= \mathbf{1.248 \times 10^{11}}$$

Example 4

Calculate $\mathbf{(9.6 \times 10^7) \div (1.2 \times 10^4)}$**. Give your answer in standard form.**

1. Rewrite as a fraction.
2. Separate the front numbers and powers of 10.
3. Simplify the two fractions.

$$\frac{9.6 \times 10^7}{1.2 \times 10^4} = \frac{9.6}{1.2} \times \frac{10^7}{10^4}$$
$$= 8 \times 10^{7-4} = \mathbf{8 \times 10^3}$$

Exercise 3

Q1 Calculate the following. Give your answers in standard form.

a) $(3 \times 10^7) \times (4 \times 10^{-4})$
b) $(7 \times 10^9) \times (9 \times 10^{-4})$
c) $(2 \times 10^5) \times (3.27 \times 10^2)$
d) $(3.4 \times 10^{-4}) \times (3 \times 10^2)$
e) $(2 \times 10^{-5}) \times (8.734 \times 10^5)$
f) $(1.2 \times 10^4) \times (5.3 \times 10^6)$

Q2 Calculate the following. Give your answers in standard form.

a) $(3.6 \times 10^7) \div (1.2 \times 10^4)$
b) $(8.4 \times 10^4) \div (7 \times 10^8)$
c) $(1.8 \times 10^{-4}) \div (1.2 \times 10^8)$
d) $(4.8 \times 10^3) \div (1.2 \times 10^{-2})$
e) $(8.1 \times 10^{-1}) \div (0.9 \times 10^{-2})$
f) $(13.2 \times 10^5) \div (1.2 \times 10^4)$

Adding and Subtracting in Standard Form

Learning Objective — Spec Ref N9:
Add and subtract numbers in standard form.

To **add** or **subtract** numbers in standard form, you first need to make the powers of 10 **the same**.
Do this by **multiplying** the **smaller power** of 10 by an appropriate power of 10 —
but make sure to **balance** it out by **dividing** the **front number** by the **same** power of 10.
Then add or subtract the **front numbers** and make sure your answer is still in **standard form** at the end.

Example 5

Calculate: **a)** $(3.7 \times 10^4) + (2.2 \times 10^3)$ **b)** $(1.1 \times 10^3) - (9.2 \times 10^2)$

Give your answers in standard form.

a) 1. The powers of 10 don't match. To change 10^3 into 10^4, you need to multiply it by 10. But then you need to divide 2.2 by 10 to balance it out.

$(3.7 \times 10^4) + (2.2 \times 10^3)$
$= (3.7 \times 10^4) + (0.22 \times 10^4)$
$= (3.7 + 0.22) \times 10^4$

2. Add the front numbers.

$= \mathbf{3.92 \times 10^4}$

b) 1. Change 9.2×10^2 so that both numbers are multiplied by 10^3.

$(1.1 \times 10^3) - (9.2 \times 10^2)$
$= (1.1 \times 10^3) - (0.92 \times 10^3)$

2. Subtract the front numbers.

$= (1.1 - 0.92) \times 10^3$

3. Convert into standard form.

$= 0.18 \times 10^3 = \mathbf{1.8 \times 10^2}$

Tip: Remember, just because a number ends in $\times 10^n$ doesn't mean it's in standard form. The number in front must be between 1 and 10.

Exercise 4

Calculate the following, giving your answers in standard form.

Q1 a) $(5.0 \times 10^3) + (3.0 \times 10^2)$
b) $(1.8 \times 10^5) + (3.2 \times 10^3)$
c) $(6.2 \times 10^{-2}) + (4.9 \times 10^{-1})$
d) $(6.9 \times 10^{-4}) + (3.8 \times 10^{-5})$
e) $(3.7 \times 10^{-1}) + (1.1 \times 10^0)$
f) $(5.5 \times 10^7) + (5.5 \times 10^8)$

Q2 a) $(5.2 \times 10^4) - (3.3 \times 10^3)$
b) $(7.2 \times 10^{-3}) - (1.5 \times 10^{-4})$
c) $(6.5 \times 10^2) - (3 \times 10^{-1})$
d) $(8.4 \times 10^2) - (6.3 \times 10^0)$
e) $(8.4 \times 10^4) - (8.3 \times 10^2)$
f) $(28.4 \times 10^{-1}) - (9.3 \times 10^{-2})$

7.4 Surds

Surds are expressions with irrational square roots in them (irrational numbers are ones that you can't write as fractions, e.g. π or $\sqrt{2}$). If a question involving surds asks for an exact answer, you have to leave the surds in.

Multiplying and Dividing Surds

Learning Objective — Spec Ref N8/A4:
Use multiplication and division to simplify surds.

Prior Knowledge Check: Be able to find the factors of a number. See p.8.

To **multiply** or **divide** two surds, combine them into a **single surd** using the rules below.

$\sqrt{a}\times\sqrt{b}=\sqrt{a\times b}$ e.g. $\sqrt{2}\times\sqrt{3}=\sqrt{2\times3}=\sqrt{6}$. Also, $(\sqrt{b})^2=\sqrt{b}\times\sqrt{b}=\sqrt{b\times b}=\sqrt{b^2}=b$

$\sqrt{a}\div\sqrt{b}=\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$ e.g. $\sqrt{8}\div\sqrt{2}=\frac{\sqrt{8}}{\sqrt{2}}=\sqrt{\frac{8}{2}}=\sqrt{4}=2$

This is useful for **simplifying** expressions containing surds — the aim is to make the number under the root as **small** as possible or **get rid** of the root completely. To do this, split the number up into two **factors**, one of which should be a **square number**. You can then take the **square root** of this square number to simplify. When you're done, you'll be left with an expression of the form $a\sqrt{b}$ where a and b are **integers** and b is as small as possible.

Example 1

Simplify $\sqrt{72}$.

1. Break 72 down into factors — one of them needs to be a square number. $\sqrt{72}=\sqrt{36\times2}$
2. Write as two roots multiplied together. $=\sqrt{36}\times\sqrt{2}$
3. Evaluate $\sqrt{36}$. $=\mathbf{6\sqrt{2}}$

Tip: You could have done $\sqrt{72}=\sqrt{9}\times\sqrt{8}=3\sqrt{8}$ — but then you'd also have to simplify $\sqrt{8}$ using the same method.

Example 2

Find $\sqrt{5}\times\sqrt{15}$. Simplify your answer.

1. Use the rule $\sqrt{a}\times\sqrt{b}=\sqrt{a\times b}$. $\sqrt{5}\times\sqrt{15}=\sqrt{5\times15}=\sqrt{75}$
2. Now find factors of 75 so you can simplify — remember that one of the factors needs to be a square number. $=\sqrt{25\times3}$ $=\sqrt{25}\times\sqrt{3}$ $=\mathbf{5\sqrt{3}}$

Tip: You could also have done $\sqrt{5}\times\sqrt{15}$
$=\sqrt{5}\times\sqrt{5\times3}$
$=\sqrt{5}\times\sqrt{5}\times\sqrt{3}$
$=5\times\sqrt{3}=5\sqrt{3}$

Exercise 1

Q1 Simplify:

a) $\sqrt{12}$ b) $\sqrt{20}$ c) $\sqrt{50}$ d) $\sqrt{32}$

e) $\sqrt{108}$ f) $\sqrt{300}$ g) $\sqrt{98}$ h) $\sqrt{192}$

Q2 Rewrite the following in the form $a\sqrt{b}$, where a and b are integers. Simplify your answers where possible.

a) $\sqrt{2} \times \sqrt{24}$ b) $\sqrt{3} \times \sqrt{12}$ c) $\sqrt{3} \times \sqrt{24}$

d) $\sqrt{2} \times \sqrt{10}$ e) $\sqrt{40} \times \sqrt{2}$ f) $\sqrt{3} \times \sqrt{60}$

g) $\sqrt{7} \times \sqrt{35}$ h) $\sqrt{50} \times \sqrt{10}$ i) $\sqrt{8} \times \sqrt{24}$

Example 3

Find $\sqrt{40} \div \sqrt{10}$.

1. Use the rule $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$.
2. Do the division inside the square root.
3. Simplify.

$$\sqrt{40} \div \sqrt{10} = \sqrt{\frac{40}{10}} = \sqrt{4} = \mathbf{2}$$

Tip: Don't forget to simplify at the end by evaluating any roots of square numbers.

Example 4

Simplify: **a)** $\sqrt{\frac{1}{4}}$ **b)** $\sqrt{\frac{49}{125}}$

a) Rewrite as two roots, then evaluate $\sqrt{1}$ and $\sqrt{4}$.

$$\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \mathbf{\frac{1}{2}}$$

b) 1. Use the rule $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

2. Simplify the surds in the numerator and denominator separately.

$$\sqrt{\frac{49}{125}} = \frac{\sqrt{49}}{\sqrt{125}} = \frac{7}{\sqrt{25 \times 5}} = \mathbf{\frac{7}{5\sqrt{5}}}$$

Tip: You'll usually have to simplify a surd further so that it doesn't have any roots in the denominator. This is covered on page 102.

Exercise 2

Q1 Calculate the exact values of the following. Simplify your answers where possible.

a) $\sqrt{90} \div \sqrt{10}$ b) $\sqrt{72} \div \sqrt{2}$ c) $\sqrt{200} \div \sqrt{8}$ d) $\sqrt{243} \div \sqrt{3}$

e) $\sqrt{294} \div \sqrt{6}$ f) $\sqrt{80} \div \sqrt{10}$ g) $\sqrt{120} \div \sqrt{10}$ h) $\sqrt{180} \div \sqrt{3}$

i) $\sqrt{180} \div \sqrt{9}$ j) $\sqrt{96} \div \sqrt{6}$ k) $\sqrt{484} \div \sqrt{22}$ l) $\sqrt{210} \div \sqrt{35}$

Q2 Simplify the following as far as possible.

a) $\sqrt{\frac{1}{9}}$ b) $\sqrt{\frac{4}{25}}$ c) $\sqrt{\frac{49}{121}}$

d) $\sqrt{\frac{100}{64}}$ e) $\sqrt{\frac{18}{200}}$ f) $\sqrt{\frac{2}{25}}$

g) $\sqrt{\frac{108}{147}}$ h) $\sqrt{\frac{27}{64}}$ i) $\sqrt{\frac{98}{121}}$

Adding and Subtracting Surds

Learning Objective — Spec Ref N8/A4:
Add and subtract surds.

Prior Knowledge Check: Know how to simplify surds (p.99) and collect like terms (p.75).

You can simplify expressions containing surds by **collecting like terms** — but you can **only** add or subtract terms where the number under the root is **the same**. So you can do $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$ but $\sqrt{2} + \sqrt{3}$ can't be simplified — $\sqrt{a} + \sqrt{b}$ **doesn't equal** $\sqrt{a+b}$.
You'll probably have to **simplify** individual terms first to make the surd parts match.

Example 5

Simplify $\sqrt{12} + 2\sqrt{27}$.

1. Break 12 and 27 down into factors and use the rule $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$.
2. Simplify by taking roots of any square numbers.
3. The surds are the same, so collect like terms.

$$\sqrt{12} + 2\sqrt{27} = \sqrt{4 \times 3} + 2\sqrt{9 \times 3}$$
$$= \sqrt{4} \times \sqrt{3} + 2 \times \sqrt{9} \times \sqrt{3}$$
$$= 2\sqrt{3} + 2 \times 3 \times \sqrt{3} = 2\sqrt{3} + 6\sqrt{3}$$
$$= \mathbf{8\sqrt{3}}$$

Exercise 3

Simplify the following as far as possible.

Q1 a) $2\sqrt{3} + 3\sqrt{3}$ b) $7\sqrt{7} - 3\sqrt{7}$ c) $2\sqrt{3} + 3\sqrt{7}$
d) $2\sqrt{32} + 3\sqrt{2}$ e) $2\sqrt{27} - 3\sqrt{3}$ f) $5\sqrt{7} + 3\sqrt{28}$

Q2 a) $2\sqrt{125} - 3\sqrt{80}$ b) $\sqrt{108} + 2\sqrt{300}$ c) $5\sqrt{294} - 3\sqrt{216}$

Multiplying Brackets Using Surds

Learning Objective — Spec Ref N8/A4:
Expand brackets involving surds.

Prior Knowledge Check: Be able to expand brackets. See p.77-79.

Multiply out brackets with surds in them in the same way as you multiply out brackets with **variables**. After expanding, **simplify** the surds that remain if possible. Here are two **special cases** to keep in mind:

- $(a+\sqrt{b})^2 = a^2 + 2a\sqrt{b} + \sqrt{b}^2 = \mathbf{a^2 + 2a\sqrt{b} + b}$
- $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - a\sqrt{b} + a\sqrt{b} - \sqrt{b}^2 = \mathbf{a^2 - b}$ — this is the **difference of two squares** (page 84).

Example 6

Expand and simplify $(3 - \sqrt{7})^2$.

1. Write as two sets of brackets and expand.
2. Simplify $\sqrt{7} \times \sqrt{7}$ (p.99).
3. Collect like terms.

$$(3 - \sqrt{7})^2 = (3 - \sqrt{7})(3 - \sqrt{7})$$
$$= (3 \times 3) + (3 \times -\sqrt{7})$$
$$+ (-\sqrt{7} \times 3) + (-\sqrt{7} \times -\sqrt{7})$$
$$= 9 - 3\sqrt{7} - 3\sqrt{7} + \sqrt{7 \times 7}$$
$$= 9 - 3\sqrt{7} - 3\sqrt{7} + 7 = \mathbf{16 - 6\sqrt{7}}$$

Tip: You could just use the first of the special cases above to expand these brackets.

Example 7

Expand and simplify $(1+\sqrt{3})(2-\sqrt{8})$.

1. Expand the brackets first.

$$(1+\sqrt{3})(2-\sqrt{8})$$

2. Simplify the surds — here you can simplify $\sqrt{8}$ and $\sqrt{24}$.

$$= (1\times 2)+(1\times -\sqrt{8})+(\sqrt{3}\times 2)+(\sqrt{3}\times -\sqrt{8})$$
$$= 2-\sqrt{8}+2\sqrt{3}-\sqrt{24}$$
$$= 2-\sqrt{4\times 2}+2\sqrt{3}-\sqrt{4\times 6}$$
$$= \mathbf{2-2\sqrt{2}+2\sqrt{3}-2\sqrt{6}}$$

Exercise 4

Expand these brackets and simplify where possible.

Q1 a) $(2+\sqrt{3})^2$ b) $(1+\sqrt{2})(1-\sqrt{2})$ c) $(5-\sqrt{2})^2$
d) $(3-3\sqrt{2})(3-\sqrt{2})$ e) $(5+\sqrt{3})(3+\sqrt{3})$ f) $(7+2\sqrt{2})(7-2\sqrt{2})$

Q2 a) $(2+\sqrt{6})(4+\sqrt{3})$ b) $(4-\sqrt{7})(5-\sqrt{2})$ c) $(1-2\sqrt{10})(6-\sqrt{15})$

Rationalising the Denominator

Learning Objective — Spec Ref N8/A4:
Rationalise the denominator.

'**Rationalising the denominator**' means 'getting rid of surds from the bottom of a fraction'. For the simplest type, you do this by **multiplying** the top and bottom of the fraction by the surd.

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}}\times\frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b}\times\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

Example 8

Rationalise the denominators of: a) $\frac{5}{2\sqrt{15}}$ b) $\frac{2}{\sqrt{8}}$

a) 1. Multiply by $\frac{\sqrt{15}}{\sqrt{15}}$ to eliminate $\sqrt{15}$ from the denominator. Remember that $\sqrt{15}\times\sqrt{15} = 15$.

$$\frac{5}{2\sqrt{15}} = \frac{5}{2\sqrt{15}}\times\frac{\sqrt{15}}{\sqrt{15}}$$
$$= \frac{5\sqrt{15}}{2\sqrt{15}\times\sqrt{15}} = \frac{5\sqrt{15}}{2\times 15}$$

2. Simplify the fraction.

$$= \frac{5\sqrt{15}}{30} = \mathbf{\frac{\sqrt{15}}{6}}$$

b) 1. Multiply by $\frac{\sqrt{8}}{\sqrt{8}}$.

$$\frac{2}{\sqrt{8}} = \frac{2}{\sqrt{8}}\times\frac{\sqrt{8}}{\sqrt{8}} = \frac{2\sqrt{8}}{\sqrt{8}\times\sqrt{8}}$$

2. Simplify the surd.

$$= \frac{2\sqrt{4\times 2}}{8} = \frac{2\times 2\sqrt{2}}{8}$$

3. Simplify the fraction.

$$= \frac{4\sqrt{2}}{8} = \mathbf{\frac{\sqrt{2}}{2}}$$

Tip: You could also simplify the surd while it's still on the denominator and then multiply top and bottom by the surd that remains.

Exercise 5

Q1 Rationalise the denominators of the following fractions. Simplify your answers as far as possible.

a) $\frac{6}{\sqrt{6}}$ b) $\frac{8}{\sqrt{8}}$ c) $\frac{5}{\sqrt{5}}$ d) $\frac{1}{\sqrt{3}}$

e) $\frac{15}{\sqrt{5}}$ f) $\frac{9}{\sqrt{3}}$ g) $\frac{7}{\sqrt{12}}$ h) $\frac{12}{\sqrt{1000}}$

Q2 Rationalise the denominators of the following fractions. Simplify your answers as far as possible.

a) $\frac{1}{5\sqrt{5}}$ b) $\frac{1}{3\sqrt{3}}$ c) $\frac{3}{4\sqrt{8}}$ d) $\frac{3}{2\sqrt{5}}$

e) $\frac{2}{7\sqrt{3}}$ f) $\frac{1}{6\sqrt{12}}$ g) $\frac{10}{7\sqrt{5}}$ h) $\frac{5}{9\sqrt{10}}$

If the denominator is the **sum** or **difference** of an integer and a surd (e.g. $1 + \sqrt{2}$), use the **difference of two squares** (see p.101) to eliminate the surd: $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$. So if the denominator is $\boldsymbol{a + \sqrt{b}}$, multiply by $\boldsymbol{a - \sqrt{b}}$. If it's $\boldsymbol{a - \sqrt{b}}$ then multiply by $\boldsymbol{a + \sqrt{b}}$.

Example 9

Rationalise the denominator of $\frac{2+2\sqrt{2}}{1-\sqrt{2}}$.

1. Multiply top and bottom by $(1 + \sqrt{2})$ to get rid of the surd in the denominator.

$$\frac{2+2\sqrt{2}}{1-\sqrt{2}} = \frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})}$$

2. Expand the brackets in the numerator and denominator

$$= \frac{2+2\sqrt{2}+2\sqrt{2}+4}{1+\sqrt{2}-\sqrt{2}-2}$$

3. Simplify any remaining surds and the fraction.

$$= \frac{6+4\sqrt{2}}{-1} = \mathbf{-6-4\sqrt{2}}$$

Tip: Always multiply by the numbers in the denominator but change the sign of the surd.

Exercise 6

Q1 Rationalise the denominators of the following fractions. Simplify your answers as far as possible.

a) $\frac{1}{2+\sqrt{2}}$ b) $\frac{5}{1-\sqrt{7}}$ c) $\frac{10}{5+\sqrt{11}}$ d) $\frac{9}{12-3\sqrt{17}}$

Q2 Rewrite the following as fractions with rational denominators in their simplest form.

a) $\frac{\sqrt{2}}{2+3\sqrt{2}}$ b) $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ c) $\frac{2+\sqrt{3}}{1-\sqrt{3}}$

d) $\frac{1-\sqrt{5}}{2-\sqrt{5}}$ e) $\frac{1+2\sqrt{2}}{1-2\sqrt{2}}$ f) $\frac{7+8\sqrt{2}}{9+5\sqrt{2}}$

Q3 Show that $\frac{1}{1-\frac{1}{\sqrt{2}}}$ can be written as $2 + \sqrt{2}$.

Q4 Show that $\frac{1}{1+\frac{1}{\sqrt{3}}}$ can be written as $\frac{3-\sqrt{3}}{2}$.

Review Exercise

Q1 Simplify the following. Leave your answers in index form.

a) $7^6 \times 7^9$ b) $d^{-4} \div d^6$ c) $(4^8)^3$

d) $9^{-2} \times \sqrt[4]{9}$ e) $(c^{10} \div c^2)^{\frac{1}{4}}$ f) $2^4 \times \frac{1}{\sqrt[3]{2}} \times 2^{-\frac{1}{2}}$

Q2 Simplify the following.

a) $(27m)^{\frac{1}{3}}$ b) $(y^4z^3)^{-\frac{3}{4}}$ c) $\left(\frac{b^9}{64c^3}\right)^{\frac{2}{3}}$ d) $\sqrt[4]{u^2} \times (2u)^{-2}$

Q3 Write the following as powers of 2.

a) 8 b) $\sqrt{2}$ c) $8\sqrt{2}$ d) $\frac{1}{8\sqrt{2}}$

Q4 Write the following expressions.

a) $3 \times \sqrt[3]{3}$ as a power of 3 b) $16\sqrt{4}$ as a power of 4 c) 5 as a power of 25

d) 2 as a power of 8 e) $\frac{\sqrt{10}}{1000}$ as a power of 10 f) $\frac{81}{\sqrt[3]{9}}$ as a power of 3

Q5 Albert measured the length of his favourite hair each day for three days. On the first day it grew 3.92×10^{-4} m, on the second day it grew 3.77×10^{-4} m, and on the third day it grew 4.09×10^{-4} m. By how much in metres did the hair grow in total over the three days? Give your answer in standard form.

Q6 The Hollywood film 'The Return of Dr Arzt' cost $\$3.45 \times 10^8$ to make. It made a total of $\$8.9 \times 10^7$ at the box office. What was the loss made by the film in standard form?

Q7 To 3 significant figures, the mass of the Earth is 5.97×10^{24} kg, and the mass of the Sun is 3.33×10^5 times the mass of the Earth. What is the mass of the Sun in kg? Give your answer in standard form to 3 significant figures.

Q8 Write the following as simply as possible.

a) $\sqrt{96} + 5\sqrt{18}$ b) $\frac{\sqrt{48} + \sqrt{363}}{5}$ c) $\sqrt{8} - \frac{\sqrt{36}}{2}$

d) $-\sqrt{98} - \sqrt{2} \times \sqrt{162}$ e) $12\sqrt{99} \div \sqrt{176}$ f) $\frac{\sqrt{88}}{\sqrt{32}} + \sqrt{1100}$

Q9 Write the following in the form $a + b\sqrt{c}$ where a, b and c are integers.

a) $\frac{121\sqrt{7}}{-\sqrt{11}}$ b) $\frac{60}{6 + \sqrt{6}}$ c) $\frac{7 + \sqrt{2}}{\sqrt{8} - 3}$

Exam-Style Questions

Q1 Simplify:

a) $2x^3 \times 4x^4$

[1 mark]

b) $(3y^2)^4$

[2 marks]

c) $5z^0$

[1 mark]

Q2 Showing every step of your working, prove that $\left(\frac{4}{9}\right)^{-\frac{3}{2}} = 3\frac{3}{8}$.

[3 marks]

Q3 A square has a side length of $(3 + 2\sqrt{5})$ cm. Work out the area of the square in cm^2, giving your answer in the form $a + b\sqrt{5}$ where a and b are integers.

[2 marks]

Q4 A smartphone has a 2.4 gigahertz (GHz) processor which means it can do 2400 million calculations in a second.

a) Write 2400 million in standard form.

[1 mark]

b) Work out how many calculations the smartphone can do in a minute, giving your answer in standard form.

[2 marks]

c) To download a particular app to this phone will require it to do 1.8×10^{11} calculations. Work out how long in seconds this will take, giving your answer as an ordinary number.

[2 marks]

Q5 A rectangle has a width of $(5 - \sqrt{10})$ cm and an area of $\sqrt{360}$ cm^2. Find the length of the rectangle, giving your answer in the form $a + b\sqrt{10}$ where a and b are integers.

[4 marks]

Q6 Earth is approximately 4.54×10^9 years old. Humans are thought to have evolved around 2.5×10^5 years ago. For what percentage of the age of the Earth have humans been present? Give your answer as an ordinary number to 3 significant figures.

[4 marks]

Q7 $x = y \times 10^z$ where $4 < y < 10$
Find an expression for x^2 in standard form.

[3 marks]