Section 6 Expressions

6.1 Simplifying Expressions

Algebraic expressions involve variables (letters that represent numbers) and don't contain an equals sign — e.g. x, $3a + b$, $y^2 + z^2$. You can simplify them to avoid writing the same thing over and over.

Learning Objective — Spec Ref A3/A4:
Simplify algebraic expressions by collecting like terms.

Terms are the individual parts of an expression separated by plus or minus signs. E.g. in the expression $2x + 6 - 3xy$, the terms are $2x$, 6 and $3xy$.

Expressions can sometimes be **simplified** by **collecting like terms** — these are terms that contain **exactly the same** combination of letters. For example, $4xy + xy - 3xy = 2xy$.

Tip: Remember that in algebraic notation, $3xy$ means $3 \times x \times y$ — the × signs are left out to make it a bit clearer.

Example 1

Simplify the expression $x + x^2 + yx + 7 + 4x + 2xy - 3$

1. There are four sets of like terms:
   (i) terms involving just x
   (ii) terms involving x^2
   (iii) terms involving xy (or yx, which means the same)
   (iv) terms involving just numbers
2. Collect the different sets together separately.

$$
\begin{aligned}
& x + x^2 + yx + 7 + 4x + 2xy - 3 \\
&= (x + 4x) \\
&\qquad + x^2 \\
&\qquad\quad + (xy + 2xy) \\
&\qquad\qquad + (7 - 3) \\
&= \mathbf{5x + x^2 + 3xy + 4}
\end{aligned}
$$

Exercise 1

Q1 Simplify these expressions by collecting like terms.

a) $c + c + c + d + d$
b) $x + y + x + y + x - y$
c) $3m + m + 2n$
d) $3a + 5b + 8a + 2b$
e) $6p + q + p + 3q$
f) $4b + 8c - b - 5c$
g) $x + 7 + 4x + y + 5$
h) $13m + 7 + 2n - 8m - 3$
i) $13a - 5b + 8a + 12b + 7$

Q2 Simplify the following expressions by collecting like terms.

a) $x^2 + 3x + 2 + 2x + 3$
b) $x^2 + 4x + x^2 + 2x + 4$
c) $x^2 + 2x^2 + 4x - 3x$
d) $3p^2 + 6q + p^2 - 4q + 3p^2$
e) $8 + 6p^2 - 5 + pq + p^2$
f) $6b^2 + 7b + 9 - 4b^2 + 5b - 2$

Q3 Simplify the following expressions by collecting like terms.

a) $ab + cd - xy + 3ab - 2cd + 3yx + 2x^2$
b) $pq + 3pq + p^2 - 2qp + q^2$
c) $3ab - 2b + ab + b^2 + 5b$
d) $4abc - 3bc + 2ab + b^2 + 5b + 2abc$

Multiplying and Dividing Variables

Learning Objectives — Spec Ref A1/A4:

- Use and interpret algebraic notation.
- Use the laws of indices when multiplying and dividing variables.

Prior Knowledge Check: Use the laws of indices. See p.92-93.

When you've got numbers and variables **multiplied** or **divided** by each other, you should deal with the numbers and each letter **separately**, using the **laws of indices** to write each term as **simply** as possible.

The most useful laws of indices are: $a^m \times a^n = a^{m+n}$ $(a^m)^n = a^{m \times n}$ $a^m \div a^n = a^{m-n}$

There are some more rules that might help you with these questions:

- $ab^2 = a \times b \times b$ — only the b is squared.
- $(ab)^2 = ab \times ab = a \times a \times b \times b = a^2b^2$ — the whole bracket is squared.
- $2ab \times 3a = 2 \times 3 \times a \times a \times b = 6a^2b$ — multiply the numbers and variables separately.
- $6a^5 \div 3a^3 = (6 \div 3)(a^5 \div a^3) = 2a^2$ — divide the numbers and variables separately.
- $(-a)^2 = (-a) \times (-a) = (-1) \times (-1) \times a \times a = a^2$ — squaring a negative makes it positive.

Example 2

Simplify: a) $b \times b \times b \times b$ b) $4a \times 5b$ c) $12x^3 \div 2x^2$ d) $(ab^2)^2$

a) If a variable is multiplied by itself, write it as a power. $b \times b \times b \times b = \mathbf{b^4}$

b) Multiply the numbers and variables separately. $4a \times 5b = 4 \times 5 \times a \times b = \mathbf{20ab}$

c) Divide the numbers and letters separately. $12x^3 \div 2x^2 = (12 \div 2)(x^3 \div x^2) = \mathbf{6x}$

d) Square the bracket and simplify the a's and the b's. $(ab^2)^2 = ab^2 \times ab^2 = a \times a \times b^2 \times b^2 = \mathbf{a^2b^4}$

Exercise 2

Q1 Simplify the following expressions.

a) $x \times x \times x$ b) $2y \times 3y$ c) $8p \times 2q$ d) $3a \times 7a$

e) $5x \times 3y$ f) $m \times m \times m \times m$ g) $12a \times 4b$ h) $6p \times 8p$

Q2 Simplify the following expressions.

a) $p \times pq$ b) $4a^2 \times 5a$ c) $4ab \times 2ab$ d) $3i^2 \times 8j^3$

e) $9n^2m \div 3n^2$ f) $12a^2 \div 4a$ g) $16p^3q \div 2p^2$ h) $6abc \times 5a^2b^3c^4$

Q3 Expand the brackets in these expressions.

a) $(2y)^2$ b) $(r^2)^2$ c) $(3c)^3$ d) $(2z^2)^3$

e) $a(2b)^2$ f) $5(q^2)^2$ g) $(xy)^2$ h) $i(jk)^2$

6.2 Expanding Brackets

Most of the time, when brackets show up, you want to expand them to get rid of them. Single brackets are pretty straightforward, but when you have two or three sets of brackets there's a lot to keep track of.

a(b + c)

Learning Objective — Spec Ref A4:
Be able to expand a single term multiplied by a bracket.

You can **expand** (or remove) brackets by multiplying everything **inside** the brackets by the letter or number **in front** — remember $a(b + c) = a \times (b + c) = (a \times b) + (a \times c)$.

$$\boldsymbol{a(b + c) = ab + ac}$$

$$\boldsymbol{a(b - c) = ab - ac}$$

Example 1

Expand the brackets in these expressions: a) $8(r - 3s)$ b) $m(n + 7)$ c) $-(q - 4)$

a) Multiply each term in the brackets by 8. $8(r - 3s) = (8 \times r) - (8 \times 3s) = \mathbf{8r - 24s}$

b) Multiply each term in the brackets by m. $m(n + 7) = (m \times n) + (m \times 7) = \boldsymbol{mn + 7m}$

c) You can think of $-(q - 4)$ as $-1 \times (q - 4)$. A minus sign outside the brackets reverses the sign of everything inside the brackets.

$$\begin{aligned} -(q - 4) &= (-1 \times q) - (-1 \times 4) \\ &= (-q) - (-4) \\ &= \boldsymbol{-q + 4} \end{aligned}$$

Example 2

Simplify the expression $3(x + 2) - 5(2x + 1)$.

1. Multiply out both sets of brackets.
2. Collect like terms.

$$\begin{aligned} 3(x + 2) - 5(2x + 1) &= (3x + 6) - (10x + 5) \\ &= 3x + 6 - 10x - 5 \\ &= (3x - 10x) + (6 - 5) = \boldsymbol{-7x + 1} \end{aligned}$$

Exercise 1

Q1 Expand the brackets in these expressions.

a) $2(a + 5)$ b) $p(q + 2)$ c) $6(5 - r)$ d) $t(14 - t)$

e) $3(u + 8v)$ f) $6(4x + 5y)$ g) $p(3q - 8)$ h) $r(5 - s)$

Q2 Expand the brackets in these expressions.

a) $-(x + 2)$ b) $-(n - 11)$ c) $-y(4 + y)$ d) $-v(v - 5)$

e) $-6(5g - 3)$ f) $-8(7 - w)$ g) $-4y(2y + 6)$ h) $-4p(u - 7)$

Q3 Simplify the following expressions.

a) $2(z + 3) + 4(z + 2)$ b) $4(10b - 5) + (b - 2)$ c) $11(5x - 3) - (x + 2)$

d) $5(2q + 5) - 2(q - 2)$ e) $4p(3p + 5) - 3(p + 1)$ f) $4(t + 1) - 7t(8t - 11)$

(a + b)(c + d)

Learning Objective — Spec Ref A4:
Be able to expand two brackets multiplied together.

When expanding **pairs** of brackets, multiply each term in the left bracket by each term in the right bracket. If each bracket contains two terms (brackets like this are called **binomials**), you can use **FOIL** to keep track of which terms you need to multiply:

FIRST — multiply the first term from each bracket
OUTSIDE — multiply the terms on the outside
INSIDE — multiply the terms on the inside
LAST — multiply the last term from each bracket

$$(a + b)(c + d) = ac + ad + bc + bd$$

This method will give you **four terms**, but sometimes you'll be able to simplify by **collecting like terms**.

Example 3

Expand the brackets in the expression: a) $(q + 4)(p + 3)$ b) $3(x - 2)^2$

a) Multiply each term in the left bracket by each term in the right bracket using FOIL.

$$(q + 4)(p + 3) = (q \times p) + (q \times 3) + (4 \times p) + (4 \times 3)$$
$$= \mathbf{pq + 3q + 4p + 12}$$

Tip: Letters are typically written in alphabetical order — so put pq here, not qp.

b) 1. Write out $(x - 2)^2$ as two brackets.
2. Use FOIL to expand the brackets — leave the '3 ×' alone for now.
3. Collect like terms, then multiply each term in the brackets by 3.

$$3(x - 2)^2 = 3 \times (x - 2)(x - 2)$$
$$= 3 \times (x^2 - 2x - 2x + 4)$$
$$= 3 \times (x^2 - 4x + 4)$$
$$= \mathbf{3x^2 - 12x + 12}$$

Tip: Be extra careful with the minus signs here — remember that $(-2) \times (-2) = 4$, not -4.

Exercise 2

Q1 Expand the brackets in the following expressions.

a) $(a + 2)(b + 3)$
b) $(j + 4)(k - 5)$
c) $3(j - 2)(k + 4)$
d) $(x + 6)(y + 2)$
e) $(x - 4)(y - 1)$
f) $8(g + 5)(h + 9)$
g) $2(w - 6)(z - 8)$
h) $7(a - 7)(b - 8)$

Q2 Expand the brackets in the following expressions. Simplify your answer.

a) $(x + 8)(x + 3)$
b) $(b + 2)(b - 4)$
c) $(a - 1)(a + 2)$
d) $(d + 7)(d + 6)$
e) $(c + 5)(3 - c)$
f) $(y - 8)(6 - y)$
g) $2(x + 2)(x + 1)$
h) $(z - 12)(z + 9)$
i) $3(y + 2)(3 - y)$
j) $4(b - 3)(b + 2)$
k) $6(x - 2)(x - 4)$
l) $12(a + 9)(a - 8)$

Q3 Expand and simplify the following expressions.

a) $(x + 1)^2$
b) $(x + 4)^2$
c) $(x - 2)^2$
d) $(x + 5)^2$
e) $(x - 3)^2$
f) $(x - 6)^2$
g) $4(x + 1)^2$
h) $2(x + 5)^2$
i) $3(x - 2)^2$
j) $2(x + 6)^2$
k) $5(x - 3)^2$
l) $2(x - 4)^2$

$(a + b)(c + d)(e + f)$

Learning Objective — Spec Ref A4:
Be able to expand three brackets multiplied together.

When expanding triple brackets, do it in **stages** — first multiply two brackets together, then multiply the result by the remaining bracket (it **doesn't matter** what **order** you multiply the brackets in). You can't use FOIL if you have **more than 2 terms** in a bracket, so in this case just multiply every term in one bracket by every term in the other bracket.

$$\begin{aligned}(a + b)(c + d)(e + f) &= (a + b)(ce + cf + de + df)\\ &= a(ce + cf + de + df) + b(ce + cf + de + df)\\ &= ace + acf + ade + adf + bce + bcf + bde + bdf\end{aligned}$$

Example 4

Expand and simplify the following expressions: a) $(r + 2)(s + 1)(t + 4)$ b) $5(y - 1)^3$

a)

1. Multiply each term in the second bracket by each term in the third bracket.
2. Multiply the result by each term in the first bracket — simplify if possible.

$$\begin{aligned}(r + 2)(s + 1)(t + 4) &= (r + 2)(st + 4s + t + 4)\\ &= r(st + 4s + t + 4) + 2(st + 4s + t + 4)\\ &= \mathbf{rst + 4rs + rt + 4r + 2st + 8s + 2t + 8}\end{aligned}$$

b)

1. Write out $(y - 1)^3$ as $(y - 1)(y - 1)(y - 1)$.
2. Multiply each term in the second bracket by each term in the third bracket.
3. Collect like terms before multiplying the result by each term in the first bracket.
4. Collect like terms before multiplying everything in the bracket by 5.

$$\begin{aligned}5(y - 1)^3 &= 5 \times (y - 1)(y - 1)(y - 1)\\ &= 5 \times (y - 1)(y^2 - y - y + 1)\\ &= 5 \times (y - 1)(y^2 - 2y + 1)\\ &= 5 \times [y(y^2 - 2y + 1) - (y^2 - 2y + 1)]\\ &= 5 \times (y^3 - 2y^2 + y - y^2 + 2y - 1)\\ &= 5 \times (y^3 - 3y^2 + 3y - 1)\\ &= \mathbf{5y^3 - 15y^2 + 15y - 5}\end{aligned}$$

Exercise 3

Q1 Expand the brackets in the following expressions. Simplify where possible.

a) $(a + 1)(b + 1)(c + 2)$
b) $(m - 5)(n - 1)(p - 3)$
c) $(-3 + z)(5 - 2a)(b + 3)$
d) $(x + 3)(y + 1)(y + 2)$
e) $(y - 4)(y - 6)(y - 3)$
f) $(1 - 3z)(z + 5)(z - 3)$
g) $2(z + 3)(z + 2)(z + 1)$
h) $4(w - 4)(w - 5)(w - 2)$
i) $3(3 - 2q)(3q + 3)(q - 2)$

Q2 Expand and simplify the following expressions.

a) $(x + 3)^3$
b) $(x - 2)^3$
c) $(x + 4)^3$
d) $(3 - x)^3$
e) $3(x - 1)^3$
f) $2(x + 5)^3$
g) $-(5 - 2x)^3$
h) $-4(3 - 3x)^3$
i) $\frac{1}{4}(2x + 4)^3$

6.3 Factorising — Common Factors

Factorising an expression means adding brackets in where there weren't any before.
It's called factorising because you need to look for common factors of all the different terms.

Learning Objective — Spec Ref A4:
Factorise expressions by taking out common factors.

Prior Knowledge Check:
Be able to find the HCF of a set of numbers. See p.13.

Factorising is the opposite of expanding brackets. You look for a **common factor** of all the terms in an expression, and 'take it outside' the brackets. These common factors could be **numbers**, **variables**, or **both**. To factorise an expression **fully**, you need to find the **highest common factor** of all the terms.

When factorising **variables**, you'll need to remember how to **divide** two powers: $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n}$

Example 1

Factorise the expression $12x - 18y$.

1. 6 is the highest common factor of $12x$ and $18y$. So 6 goes outside the brackets.

   $12x - 18y = 6(\quad - \quad)$

2. Divide each term by the common factor, and write the results inside the brackets.

   $12x \div 6 = 2x$
   and $18y \div 6 = 3y$
   So $12x - 18y = \mathbf{6(2x - 3y)}$

Tip: If you used 2 or 3 as the factor instead of the HCF of 6, you would get an expression that wasn't fully factorised, e.g. $2(6x - 9y)$.

Example 2

Factorise $3x^2 + 2x$.

1. x is the only common factor of $3x^2$ and $2x$. So x goes outside the brackets.

   $3x^2 + 2x = x(\quad + \quad)$

2. Divide each term by the common factor, and write the results inside the brackets.

   $3x^2 \div x = 3x$ and $2x \div x = 2$
   So $3x^2 + 2x = \mathbf{x(3x + 2)}$

Exercise 1

Q1 Factorise the following expressions.

a) $2a + 10$
b) $3b + 12$
c) $15 + 3y$
d) $28 + 7v$
e) $5a + 15b$
f) $9c - 12d$
g) $3x + 12y$
h) $21u - 7v$
i) $4a^2 - 12b$
j) $3c + 15d^2$
k) $5c^2 - 25f$
l) $6x - 12y^2$

Q2 Factorise the following expressions.

a) $3a^2 + 7a$
b) $4b^2 + 19b$
c) $2x^2 + 9x$
d) $4x^2 - 9x$
e) $21q^2 - 16q$
f) $15y - 7y^2$
g) $7y + 15y^2$
h) $27z^2 + 11z$
i) $10d^3 + 27d$
j) $4y^3 - 13y^2$
k) $11y^3 + 3y^4$
l) $22w - 5w^4$

Sometimes the highest common factor of all the terms might have both **numbers and variables** in it. It's usually best to work out the highest common factor for the numbers and for each letter **separately**.

E.g. if you need to factorise $2a^3b + 4a^2b^2$, find:

- the HCF of 2 and 4 (**2**)
- the HCF of a^3 and a^2 ($\boldsymbol{a^2}$)
- the HCF of b and b^2 ($\boldsymbol{b}$)

So the HCF of $2a^3b$ and $4a^2b^2$ is $\boldsymbol{2a^2b}$, and $2a^3b + 4a^2b^2 = \boldsymbol{2a^2b}(a + 2b)$.

Example 3

Factorise the expression $\boldsymbol{15x^2 - 10xy}$.

1. 5 and x are common factors of $15x^2$ and $10xy$. So $5x$ goes outside the brackets.	$15x^2 - 10xy = 5x(\quad - \quad)$
2. Divide each term by the common factor.	$15x^2 \div 5x = 3x$ and $10xy \div 5x = 2y$
3. Write the results inside the brackets.	$15x^2 - 10xy = \boldsymbol{5x(3x - 2y)}$

Tip: The first term doesn't have a y in it, so the HCF won't either.

Exercise 2

Q1 The expression $4x^3y^2 + 8xy^4$ contains two terms.

a) What is the highest numerical common factor of both terms?

b) What is the highest power of x that is common to both terms?

c) What is the highest power of y that is common to both terms?

d) Factorise the expression.

Q2 Factorise the following expressions.

a) $15a + 10ab$
b) $12b + 9bc$
c) $16xy - 4y$
d) $21x + 3xy$
e) $24uv + 6v$
f) $10p^2 + 15pq$
g) $12q^2 - 18pq$
h) $30ab^2 + 25ab$
i) $14x^2 - 28xy^2$
j) $8ab^2 + 10a^2b$
k) $12pq - 8p^3$
l) $24x^3y - 16x^2$

Q3 Factorise the following expressions.

a) $x^6 + x^4 - x^5$
b) $8a^2 + 17a^6$
c) $12y^6 + 6y^4$
d) $24b^2c^3 - 8c$
e) $25z^2 + 13z^6$
f) $12p^2 + 15p^5q^3$
g) $9a^4b + 27ab^3$
h) $15b^4 - 21a^2 + 18ab$
i) $22pq^2 - 11p^3q^3$
j) $16x^2y - 8xy^2 + 2x^3y^3$
k) $36x^7y^2 + 8x^2y^9$
l) $5x^4 + 3x^3y^4 - 25x^3y$

Q4 Factorise the following expressions.

a) $13x^2y^2 + 22x^6y^3 + 20x^5y^3$
b) $16a^5b^5 - a^4b^5 - ab^3$
c) $21p^6q^2 - 14pq^2 + 7p^3q$
d) $14xy^4 + 13x^5y^3 - 5x^6y^4$
e) $16c^6d^5 - 14c^3 + 8c^3d^6$
f) $18jk + 21j^3k^6 - 15j^2k^3$
g) $36x^2y^4 - 72x^5y^7 + 18xy^3$
h) $20a^4b^5 + 4a^3b^4 - 5a^6b^{15}$
i) $11x^2y^3 + 11x^3y^2 + 66xy^5$

6.4 Factorising — Quadratics

So far, all the factorising you've seen has been the reverse of expanding brackets of the form a(b + c). Doing the reverse of expanding double brackets is up next, and it's going to be pretty useful in later sections.

Quadratic Expressions

Learning Objective — Spec Ref A4:
Be able to factorise a quadratic expression.

A **quadratic expression** is an expression where the highest power of the variable (e.g. x) is **2** — they have the form $\boldsymbol{ax^2 + bx + c}$ where a, b and c are constants ($a \neq 0$).

You can **factorise** some quadratics into the form $\boldsymbol{(dx + e)(fx + g)}$, but it's usually **not clear** what the values of d, e, f, and g are. The method you use **depends** on whether or not there's a number in front of the x^2.

Factorising $x^2 + bx + c$

If there's **no number** in front of the x^2 (i.e. $\boldsymbol{a = 1}$), then you can follow these steps to factorise the expression:

- Write out the brackets as $(x \qquad)(x \qquad)$ — don't put the signs or numbers in yet.
- Find pairs of numbers that **multiply to give** $\boldsymbol{c}$ — this is just like finding **factors** (see p.8). You can **ignore the sign** of c for now.
- Choose the pair of numbers that also **add** or **subtract** to give $\boldsymbol{b}$ (ignoring signs here too).
- Write one number in each bracket, then fill in the + or – signs so that b and c work out with the **correct signs**. Check they're right by expanding the brackets to get back to the original expression.

If c is **positive**, then the two brackets will have the same sign (both + or both –), and if c is **negative** then the signs will be different (one + and one –).

Example 1

Factorise: a) $\boldsymbol{x^2 + 6x + 8}$ b) $\boldsymbol{x^2 + 2x - 15}$

a)	1. Find all the pairs of numbers that multiply to give 8 (i.e. the factor pairs of 8).	1×8 or 2×4
	2. Find the pair that add/subtract to give 6.	$1 + 8 = 9,\ 8 - 1 = 7$ $2 + 4 = 6,\ 4 - 2 = 2$
	3. You need +2 and +4 to give b (+6), so both brackets should have + signs.	So $x^2 + 6x + 8 = \boldsymbol{(x + 4)(x + 2)}$
	4. Check your answer by expanding the brackets.	$(x + 4)(x + 2) = x^2 + 2x + 4x + 8$ $= x^2 + 6x + 8$
b)	1. Find all the pairs of numbers that multiply to give 15.	1×15 or 3×5
	2. Find the pair that add/subtract to give 2.	$1 + 15 = 16,\ 15 - 1 = 14$ $3 + 5 = 8,\ 5 - 3 = 2$
	3. You need +5 and –3 to give +2, so put a + with the 5 and a – with the 3.	So $x^2 + 2x - 15 = \boldsymbol{(x + 5)(x - 3)}$
	4. Check your answer by expanding the brackets.	$(x + 5)(x - 3) = x^2 - 3x + 5x - 15$ $= x^2 + 2x - 15$

Exercise 1

Q1 Factorise each of the following expressions.

a) $x^2 + 7x + 6$

b) $a^2 + 7a + 12$

c) $x^2 + 8x + 7$

d) $z^2 + 8z + 12$

e) $x^2 + 5x + 4$

f) $v^2 + 6v + 9$

Q2 Factorise each of the following expressions.

a) $x^2 + 4x + 3$

b) $x^2 - 6x + 8$

c) $x^2 - 7x + 10$

d) $x^2 - 5x + 4$

e) $y^2 + 3y - 10$

f) $x^2 + 2x - 8$

g) $s^2 + 3s - 18$

h) $x^2 - 2x - 15$

i) $t^2 - 4t - 12$

Factorising ax² + bx + c

When a is not 1, there are a few extra steps you need to do in order to factorise.

- Start by finding all the pairs of numbers that multiply to give a. Write out a separate set of brackets for each pair, writing the two numbers **in front of the *x*'s**. For example, if you had $4x^2$, you would write out $(2x \quad)(2x \quad)$ and $(x \quad)(4x \quad)$.
- Now list all the pairs of numbers that **multiply to give *c*, ignoring the sign** of c for now.
- Here's the tricky bit — try putting **each pair** of numbers in the brackets until you find one that gives you the right value for b. Check them by working out the '**OI**' bits from 'FOIL' — these are the bits that should **give you *bx*** when you add/subtract them. Make sure to try the pairs **both ways round**, e.g. if your factor pair was 2 and 3, try both $(x \quad 3)(4x \quad 2)$ and $(x \quad 2)(4x \quad 3)$.
- Once you've found the right combination, write the + or – signs so that it works out.

Example 2

Factorise: a) $2x^2 + 7x + 3$ b) $6x^2 - 11x - 10$

a)	1. The $2x^2$ has to come from $x \times 2x$.	$(x \quad)(2x \quad)$
	2. $c = 3$ has only got one pair of factors.	1×3
	3. Find the potential values of b — try adding and subtracting.	$(x \quad 1)(2x \quad 3)$: O = $3x$, I = $2x$ $\Rightarrow$ $5x$ or x $(x \quad 3)(2x \quad 1)$: O = x, I = $6x$ $\Rightarrow$ $7x$ or $5x$
	4. You need $+x$ and $+6x$ to make $+7x$, so both brackets should have + signs.	So $2x^2 + 7x + 3 =$ $\mathbf{(x + 3)(2x + 1)}$
	5. Check by expanding the brackets.	$(x + 3)(2x + 1) = 2x^2 + x + 6x + 3 = 2x^2 + 7x + 3$
b)	1. The $6x^2$ could either come from $2x \times 3x$ or from $x \times 6x$.	$(2x \quad)(3x \quad)$ or $(x \quad)(6x \quad)$
	2. $c = 10$ has two pairs of factors.	1×10 or 2×5
	3. Try pairs of numbers in the brackets until you get $11x$.	$(2x \quad 1)(3x \quad 10)$: O = $20x$, I = $3x$ $\Rightarrow$ $23x$ or $17x$ $(2x \quad 10)(3x \quad 1)$: O = $2x$, I = $30x$ $\Rightarrow$ $32x$ or $28x$ $(2x \quad 2)(3x \quad 5)$: O = $10x$, I = $6x$ $\Rightarrow$ $16x$ or $4x$ $(2x \quad 5)(3x \quad 2)$: O = $4x$, I = $15x$ $\Rightarrow$ $19x$ or $11x$
	4. You need $+4x$ and $-15x$ to make $-11x$, so put a + with the 2 ($2x \times +2 = +4x$) and a – with the 5 ($3x \times -5 = -15x$).	So $6x^2 - 11x - 10 =$ $\mathbf{(2x - 5)(3x + 2)}$
	5. Check by expanding the brackets.	$(2x - 5)(3x + 2) = 6x^2 + 4x - 15x - 10 = 6x^2 - 11x - 10$

Exercise 2

Q1 Factorise the following expressions.

a) $2x^2 + 3x + 1$
b) $3x^2 - 16x + 5$
c) $5x^2 - 17x + 6$
d) $2t^2 - 5t - 12$
e) $2x^2 - 13x + 6$
f) $3b^2 - 7b - 6$
g) $5x^2 + 12x - 9$
h) $2x^2 - 3x + 1$
i) $7a^2 + 19a - 6$
j) $11x^2 - 62x - 24$
k) $7z^2 + 38z + 15$
l) $3y^2 - 26y + 16$

Q2 Factorise the following expressions.

a) $6x^2 + 7x + 1$
b) $6x^2 - 13x + 6$
c) $15x^2 - x - 2$
d) $10x^2 - 19x + 6$
e) $12u^2 - 5u - 3$
f) $14w^2 + 25w + 9$

Difference of Two Squares

Learning Objective — Spec Ref A4:
Be able to factorise a difference of two squares.

Prior Knowledge Check:
Recognise square numbers and find square roots. See p.91.

Some quadratic expressions have no middle term, e.g. $x^2 - 49$. When you factorise these, you get two brackets that are **the same**, except that one has a **+ sign** and one has a **– sign**. For example, $x^2 - 49$ factorises to $(x + 7)(x - 7)$.

More generally: $\boldsymbol{a^2 - b^2 = (a + b)(a - b)}$ This is known as the **difference of two squares**.

Example 3

Factorise: a) $16x^2 - 9$ b) $5y^2 - 80z^2$ c) $x^2 - 7$

a) $16x^2 = 4x \times 4x$ — write the expression in the form $a^2 - b^2$, then use the formula.

$$\begin{aligned}16x^2 - 9 &= (4x)^2 - 3^2\\ &= \mathbf{(4x + 3)(4x - 3)}\end{aligned}$$

b) You can take a factor of 5 out of the expression, so do this first. Then write the bit inside the bracket in the form $a^2 - b^2$ and use the formula.

$$\begin{aligned}5y^2 - 20z^2 &= 5(y^2 - 4z^2)\\ &= 5(y^2 - (2z)^2)\\ &= \mathbf{5(y + 2z)(y - 2z)}\end{aligned}$$

c) 7 isn't a square number, so write 7 as $(\sqrt{7})^2$ so that you can use the formula.

$$\begin{aligned}x^2 - 7 &= x^2 - (\sqrt{7})^2\\ &= \boldsymbol{(x + \sqrt{7})(x - \sqrt{7})}\end{aligned}$$

Tip: See p.99-103 for more about surds.

Exercise 3

Q1 Factorise each of the following expressions.

a) $x^2 - 25$
b) $x^2 - 9$
c) $x^2 - 36$
d) $x^2 - 81$
e) $x^2 - 64$
f) $b^2 - 121$
g) $x^2 - 1$
h) $c^2 - 4d^2$

Q2 Factorise each of the following expressions.

a) $4x^2 - 49$
b) $36x^2 - 4$
c) $9x^2 - 100$
d) $25x^2 - 16$
e) $16z^2 - 1$
f) $27t^2 - 12$
g) $98x^2 - 2$
h) $7p^2 - 175$
i) $x^2 - 11$
j) $n^2 - 51$
k) $4x^2 - 3$
l) $3x^2 - 15$

6.5 Algebraic Fractions

Algebraic fractions are just fractions with algebraic expressions in the numerator and/or denominator. You can do all the things you'd do with regular fractions — e.g. simplify, add, subtract, multiply or divide.

Simplifying Algebraic Fractions

Learning Objective — Spec Ref A4:
Be able to simplify algebraic fractions.

Prior Knowledge Check:
Be able to simplify fractions. See p.28.

Algebraic fractions can be **simplified** just like regular fractions. Look for **common factors** between the numerator and denominator (you might have to do some **factorising**) and then **cancel** them. The only difference is that the factors may be **algebraic expressions** — e.g. $4x$, y^2z, $(5a + b)$, etc.

Example 1

Simplify the following algebraic fractions: a) $\boldsymbol{\frac{3x^2y^4}{21xy^5}}$ **b)** $\boldsymbol{\frac{x^2-16}{x^2+8x+16}}$

a)

1. 3 is a common factor, so it can be cancelled.

$$\frac{3x^2y^4}{21xy^5} = \frac{\cancel{3}\times x^2\times y^4}{\cancel{3}\times 7\times x\times y^5}$$

2. x is a common factor, so it can be cancelled.

$$= \frac{x^{\cancel{2}}\times y^4}{7\times \cancel{x}\times y^5}$$

3. y^4 is a common factor, so it can be cancelled.

$$= \frac{x\times \cancel{y^4}}{7\times y^{\cancel{5}}} = \frac{x}{7\times y} = \boldsymbol{\frac{x}{7y}}$$

b) Factorise the numerator and the denominator, then cancel the common factor of $(x + 4)$.

$$\frac{x^2-16}{x^2+8x+16} = \frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(x+4)}$$

$$= \boldsymbol{\frac{x-4}{x+4}}$$

Tip: The numerator here is a difference of two squares.

Exercise 1

Q1 Simplify:

a) $\frac{2x}{x^2}$ b) $\frac{49x^3}{14x^4}$ c) $\frac{25s^3t}{5s}$ d) $\frac{26a^2b^3c^4}{52b^5c}$

Q2 Simplify:

a) $\frac{7x}{5x-x^2}$ b) $\frac{48t-6t^2}{8s^2t}$ c) $\frac{3cd}{8c+6c^2}$ d) $\frac{12}{4a^2b^3+8a^7b^9}$

Q3 Simplify:

a) $\frac{4st+8s^2}{8t^2+16st}$ b) $\frac{15xz+15z}{25xyz-25yz}$ c) $\frac{6xy-6x}{3y-3}$ d) $\frac{3a^2b+5ab^2}{7ab^3}$

Q4 Simplify each of the following fractions as far as possible.

a) $\frac{2x-8}{x^2-5x+4}$ b) $\frac{6a-3}{5-10a}$ c) $\frac{x^2+7x+10}{x^2+2x-15}$

d) $\frac{x^2-7x+12}{x^2-2x-8}$ e) $\frac{x^2+4x}{x^2+7x+12}$ f) $\frac{2t^2+t-45}{4t^2-81}$

Adding and Subtracting Algebraic Fractions

Learning Objective — Spec Ref A4:
Add and subtract algebraic fractions.

Prior Knowledge Check:
Be able to add and subtract fractions. See p.31-32.

Algebraic fractions should be treated in just the same way as numerical fractions. To add or subtract them, they need to have a **common denominator**. You can get them over a common denominator by multiplying the **top and bottom** of each fraction by the **denominators** of the others — this could be a **number**, or an **expression** like $3x + 2$.

Example 2

Express, as a single fraction: a) $\frac{x-2}{3} + \frac{2x+3}{4}$ b) $\frac{2x-1}{x+1} - \frac{3}{x-2}$

a) 1. First, find a common denominator — multiply the top and bottom of the first fraction by 4, and the second fraction by 3.

$$\frac{x-2}{3} + \frac{2x+3}{4} = \frac{4\times(x-2)}{4\times3} + \frac{3\times(2x+3)}{3\times4}$$
$$= \frac{4(x-2)}{12} + \frac{3(2x+3)}{12}$$

2. Expand the brackets in the numerator and simplify as much as possible.

$$= \frac{4(x-2)+3(2x+3)}{12}$$
$$= \frac{4x-8+6x+9}{12} = \mathbf{\frac{10x+1}{12}}$$

b) 1. Find a common denominator by multiplying the top and bottom of the first fraction by $(x - 2)$, and of the second fraction by $(x + 1)$.

$$\frac{2x-1}{x+1} - \frac{3}{x-2} = \frac{(2x-1)(x-2)}{(x+1)(x-2)} - \frac{3(x+1)}{(x+1)(x-2)}$$
$$= \frac{(2x-1)(x-2)-3(x+1)}{(x+1)(x-2)}$$

2. Expand the brackets in the numerator and simplify.

$$= \frac{2x^2-5x+2-3x-3}{(x+1)(x-2)}$$

3. $2x^2 - 8x - 1$ doesn't factorise nicely, so leave it as it is.

$$= \mathbf{\frac{2x^2-8x-1}{(x+1)(x-2)}}$$

Exercise 2

Q1 Express each of these as a single fraction, simplified as far as possible.

a) $\frac{x}{4} + \frac{x}{5}$ b) $\frac{x}{2} - \frac{x}{3}$ c) $\frac{2b}{7} + \frac{b}{6}$ d) $\frac{5z}{6} - \frac{4z}{9}$

Q2 Express each of these as a single fraction, simplified as far as possible.

a) $\frac{x-2}{5} + \frac{x+1}{3}$ b) $\frac{2t+1}{4} + \frac{t-1}{3}$ c) $\frac{3x-1}{4} + \frac{2x+1}{6}$ d) $\frac{c+2}{c} + \frac{c+1}{2c}$

Q3 Express each of these as a single fraction, simplified as far as possible.

a) $\frac{2}{x+1} + \frac{1}{x-3}$ b) $\frac{x-2}{x-1} - \frac{x+1}{x+2}$ c) $\frac{2a-3}{a+2} + \frac{3a+2}{a+3}$

d) $\frac{x+2}{3x-2} + \frac{x-3}{2x+1}$ e) $\frac{s-2}{3s-1} + \frac{s+1}{3s+2}$ f) $\frac{x-2}{5} + \frac{x+1}{3x}$

g) $\frac{y+3}{y+1} - \frac{y+2}{y+3}$ h) $\frac{x-3}{x+2} - \frac{2x+1}{x+1}$ i) $\frac{2x+3}{x-2} - \frac{x-4}{x+3}$

If the denominators of the fractions you're adding or subtracting share a **common factor**, you **don't need to** multiply by that factor to get the fractions over a common denominator (similar to how you can do $\frac{1}{4} + \frac{1}{6} = \frac{3}{12} + \frac{2}{12}$ rather than using a common denominator of $4 \times 6 = 24$).

Example 3

Express $\frac{x}{x^2+3x+2} + \frac{x-1}{x+2}$ as a single fraction.

1. Factorise the terms in each denominator.
2. Here the denominators have a common factor, $(x + 2)$. Multiply the top and bottom of the second fraction by $(x + 1)$ to make the denominators the same.
3. Simplify the numerator by expanding the brackets.
4. $x^2 + x - 1$ doesn't factorise, so leave it as it is.

$$\frac{x}{x^2+3x+2} + \frac{x-1}{x+2}$$
$$= \frac{x}{(x+1)(x+2)} + \frac{x-1}{x+2}$$
$$= \frac{x}{(x+1)(x+2)} + \frac{(x-1)(x+1)}{(x+2)(x+1)}$$
$$= \frac{x+(x^2+x-x-1)}{(x+1)(x+2)}$$
$$= \boldsymbol{\frac{x^2+x-1}{(x+1)(x+2)}}$$

Example 4

Show that $\frac{4}{(3x-5)(x+1)} - \frac{1}{(3x-5)(x-1)} = \frac{1}{x^2-1}$.

1. The denominators of the fractions are already factorised. They have a common factor of $(3x - 5)$, so you just need to multiply the left fraction by $(x - 1)$ and the right fraction by $(x + 1)$.
2. Simplify the numerator.
3. Cancel the factor of $(3x - 5)$ from the top and bottom. Expanding the brackets in the denominator gives you the final answer.

$$\frac{4}{(3x-5)(x+1)} - \frac{1}{(3x-5)(x-1)}$$
$$= \frac{4(x-1)}{(3x-5)(x+1)(x-1)} - \frac{x+1}{(3x-5)(x+1)(x-1)}$$
$$= \frac{4x-4-x-1}{(3x-5)(x+1)(x-1)}$$
$$= \frac{3x-5}{(3x-5)(x+1)(x-1)}$$
$$= \frac{1}{(x+1)(x-1)} = \boldsymbol{\frac{1}{x^2-1}}$$

Exercise 3

Q1 Express each of the following as a single fraction, simplified as far as possible.

a) $\frac{x-2}{(x-3)(x+1)} + \frac{5}{(x-3)}$

b) $\frac{3x}{(x+1)(x+2)} + \frac{1}{x+2}$

c) $\frac{1}{(x+4)} - \frac{(x-2)}{(x+4)(x+3)}$

Q2 Express each of the following as a single fraction, simplified as far as possible.

a) $\frac{z}{z^2+3z+2} + \frac{10}{z+1}$

b) $\frac{x-3}{x^2+x-6} + \frac{2x}{x-2}$

c) $\frac{x-3}{x^2+4x} + \frac{3}{x+4}$

d) $\frac{a+1}{a^2-2a-3} - \frac{a+1}{a^2+4a+3}$

Q3 Express each of the following as a single fraction, simplified as far as possible.

a) $\frac{2t+1}{t^2+3t} - \frac{t+2}{t^2+4t+3}$

b) $\frac{x}{x^2-9} + \frac{x+2}{x^2-5x+6}$

c) $\frac{3x}{x^2-4x+3} + \frac{x}{x^2-5x+4}$

d) $\frac{2y-1}{y^2-3y+2} - \frac{y-1}{y^2-5y+6}$

Multiplying and Dividing Algebraic Fractions

Learning Objective — Spec Ref A4:
Multiply and divide algebraic fractions.

Prior Knowledge Check:
Be able to multiply and divide fractions. See p.33-35.

The method for multiplying and dividing algebraic fractions is exactly the same as with numeric fractions — if you're multiplying, just times the **numerators together** and the **denominators together**.
If you're dividing, **flip the second fraction** and change it to a multiplication, e.g. $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$.

To make the calculations easier, it's usually best to **fully factorise** the numerator and denominator of each fraction, then **cancel any common factors**. Once you've got a multiplication, you can cancel factors that appear on the top and bottom of **either fraction** — so $\frac{(x+1)}{(x+2)}\times\frac{(x+3)}{2(x+1)}$ would cancel to $\frac{1}{(x+2)}\times\frac{(x+3)}{2}$.

Example 5

Express, as a single simplified fraction: a) $\frac{x-2}{x^2+6x+8}\times\frac{2x+4}{x^2+2x-8}$ b) $\frac{3x+6}{x^2-9}\div\frac{x+2}{x^2+4x+3}$

a) 1. Factorise each term as far as possible, then cancel any factor which appears on the top and bottom of either fraction.

$$\frac{x-2}{x^2+6x+8}\times\frac{2x+4}{x^2+2x-8}=\frac{\cancel{x-2}}{\cancel{(x+2)}(x+4)}\times\frac{2\cancel{(x+2)}}{\cancel{(x-2)}(x+4)}$$

2. Multiply the fractions like normal.

$$=\frac{1}{(x+4)}\times\frac{2}{(x+4)}=\frac{\mathbf{2}}{\mathbf{(x+4)^2}}$$

b) 1. This is a division, so turn the second fraction over and change the sign to a multiplication.

$$\frac{3x+6}{x^2-9}\div\frac{x+2}{x^2+4x+3}=\frac{3x+6}{x^2-9}\times\frac{x^2+4x+3}{x+2}$$

2. Factorise each term as far as possible. Cancel any factor which appears on both the top and the bottom of either fraction and multiply.

$$=\frac{3\cancel{(x+2)}}{\cancel{(x+3)}(x-3)}\times\frac{(x+1)\cancel{(x+3)}}{\cancel{(x+2)}}$$

$$=\frac{3}{(x-3)}\times\frac{(x+1)}{1}=\frac{\mathbf{3(x+1)}}{\mathbf{x-3}}$$

Exercise 4

Q1 Express each of the following as a single fraction, simplified as far as possible.

a) $\frac{x}{y}\times\frac{3}{x^2}$ b) $\frac{2a}{4b^2}\times\frac{5b}{a^3}$ c) $\frac{t}{2}\times\frac{24st}{6t}$ d) $\frac{64xy^2}{9y}\times\frac{3x^3}{16x^2y}$

e) $\frac{1}{x+4}\times\frac{3x}{x+2}$ f) $\frac{6a+b}{12}\times\frac{3a}{b+1}$ g) $\frac{1}{2z+5}\times\frac{3z}{z-1}$ h) $\frac{t^2+5}{12}\times\frac{4t^3}{1-t}$

Q2 Express each of the following as a single fraction, simplified as far as possible.

a) $\frac{4x}{3y^2}\div\frac{2x}{12y^4}$ b) $\frac{18a}{6b^2}\div\frac{a}{20b}$ c) $\frac{3x^3y^5}{4x^5y}\div\frac{xy}{28}$ d) $\frac{ab^2}{15}\div\frac{a^2}{5b}$

e) $\frac{y-2}{3y+2}\div\frac{y-2}{4y^4}$ f) $\frac{2c+1}{3d^2}\div\frac{cd}{18}$ g) $\frac{1-x^3y^5}{25x^2}\div\frac{y}{5}$ h) $\frac{12t^5}{6t+3t^3}\div\frac{18t^2}{9t}$

Q3 Express each of the following as a single fraction, simplified as far as possible.

a) $\frac{x^2-16}{x^2+5x+6}\times\frac{x+3}{x+4}$ b) $\frac{x^2+4x+3}{x^2+6x+8}\times\frac{x+4}{2x+6}$ c) $\frac{z^2+3z-10}{z^2+4z+3}\times\frac{z^2+6z+5}{z-2}$

d) $\frac{x^2-4x+3}{x^2+9x+20}\div\frac{x^2-x-6}{x^2+7x+12}$ e) $\frac{y^2-5y+6}{y^2+y-20}\div\frac{y-2}{3y-12}$ f) $\frac{t^2-9}{t^2+3t+2}\div\frac{t^2+6t+9}{t^2+8t+7}$

Review Exercise

Q1 Simplify the following expressions.

a) $3x - 2x + 5x - x$ b) $3x + 2y - 4x + 5y$ c) $3x^2 + 5x - 7 + 2x^2 - 3x + 4$

d) $2a \times 3a$ e) $8p \times 2q$ f) $x^2 \times xy$

g) $22z^4 \div 2z$ h) $27g^3h^3 \div 9gh^2$ i) $x^2(4x)^3$

Q2 Simplify each of the following expressions by expanding the brackets.

a) $6(x + 3) + 3(x - 4)$ b) $4(a + 3) - 2(a + 2)$ c) $5(p + 2) - 3(p - 4)$

d) $2(2x + 3) - 3(2x + 1)$ e) $x(x + 3) + 2(x - 1)$ f) $2x(2x + 3) + 3(x - 4)$

Q3 Expand and simplify each of the following expressions.

a) $(x - 3)(x + 4)$ b) $(3x + 1)(2x + 5)$ c) $(x + 1)(x - 2)(x + 3)$

d) $-(t - 8)(t - 1)(t + 1)$ e) $(3x - 2)^2$ f) $(x + 2)^3$

Q4 Factorise each of the following expressions as far as possible.

a) $4x - 8$ b) $6a + 3$ c) $5t - 10$ d) $3x + 6xy$

e) $8xy - 12x^2$ f) $a^2b - 2ab + ab^2$ g) $16x^2 + 12x^2y - 8xy^2$ h) $14x^3 + 7x^2y - 7xy^4$

Q5 Factorise each quadratic.

a) $a^2 + 6a + 8$ b) $x^2 + 4x + 3$ c) $z^2 - 5z + 6$ d) $x^2 + 3x - 18$

e) $x^2 - 3x - 10$ f) $2x^2 + 5x + 2$ g) $3m^2 - 8m + 4$ h) $3x^2 - 5x - 2$

i) $4g^2 + 4g + 1$ j) $16a^2 - 25$ k) $4c^2 - 196$ l) $81t^2 - 121$

Q6 Express each of the following as a single, simplified, algebraic fraction.

a) $\frac{a}{4} + \frac{a}{8}$ b) $\frac{2x}{3} - \frac{2x}{7}$ c) $\frac{y-1}{2} + \frac{y+1}{3}$

d) $\frac{2}{x+3} + \frac{4}{x-1}$ e) $\frac{5}{z+2} - \frac{3}{z+3}$ f) $\frac{z+2}{z+3} + \frac{z+1}{z-2}$

g) $\frac{3}{x^2+4x+3} + \frac{2}{x^2+x-6}$ h) $\frac{4}{t^2+6t+9} - \frac{3}{t^2+3t}$ i) $\frac{x}{x^2-16} + \frac{x-2}{x^2-5x+4}$

Q7 Simplify the following expressions.

a) $\frac{x^2-4}{3x-3} \times \frac{9}{2x-4}$ b) $\frac{8}{x} \div \frac{6}{x^2}$ c) $\frac{y-1}{2} \div \frac{x+1}{3}$

d) $\frac{x^2-7x+12}{x^2+3x+2} \times \frac{x+1}{x-3}$ e) $\frac{s^2+4s+3}{s^2-16} \div \frac{s+1}{s+4}$ f) $\frac{x^2+x-12}{x^2-4} \times \frac{x^2+2x}{3x+12}$

g) $\frac{a^2-7a+10}{a^2+5a+6} \times \frac{a^2+2a-3}{a^2-3a-10}$ h) $\frac{b^2+5b+6}{b^2+6b+5} \div \frac{2b+6}{3b+3}$ i) $\frac{x^2+8x+15}{x^2+4x-12} \div \frac{x^2+4x+3}{x^2+8x+12}$

Exam-Style Questions

Q1 Expand and simplify:

a) $x(x^2 - 4y) + 9xy$

[2 marks]

b) $(2x - 7)^2$

[2 marks]

Q2 Use an algebraic equivalent to the expression $x^2 - y^2$ to work out the value of $145^2 - 55^2$.

[2 marks]

Q3 Two expressions A and B are defined such that:

$A = 5x^2 + 9xy$ $\qquad\qquad$ $B = 3x(x - y)$

Find an expression for $A - B$ in terms of x and y.
Give your answer in a fully factorised form.

[3 marks]

Q4 a) Factorise $5a^2 - 6a$

[1 mark]

b) Use your answer to part a) to factorise $5(2x + 3)^2 - 6(2x + 3)$.
Give your answer in its simplest form.

[2 marks]

Q5 $6(3x - y) - 4(x + 5y) = a(7x - by)$

Find the values of a and b.

[3 marks]

Q6 Simplify fully $\dfrac{6x^2 + 18x}{2x^2 - 4x - 30}$.

[3 marks]

Q7 Expand $(x + 3)(x - 2)^2$, simplifying your answer as much as possible.

[3 marks]