30.1 Congruence and Similarity

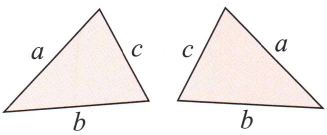
Congruent shapes are exactly the same shape and size. Similar shapes are exactly the same shape but different sizes. Translated, rotated and reflected shapes are congruent to the original shapes, while enlarged shapes are similar to the original shapes.

Congruent Triangles

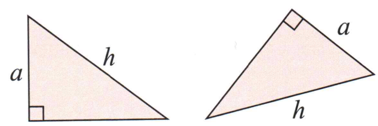
Learning Objectives — Spec Ref G5/G6:
- Know the congruence conditions for triangles.
- Be able to prove that two triangles are congruent.

Prior Knowledge Check: Properties of parallel lines and polygons, circle theorems and Pythagoras' theorem (see Sections 20, 21 and 25).

In congruent shapes, all **side lengths** and **angles** on one shape are **identical** to the side lengths and angles on the other. However, you don't need to know **all** the side lengths and angles to show that two **triangles** are congruent — just show that they satisfy any **one** of the following '**congruence conditions**'.

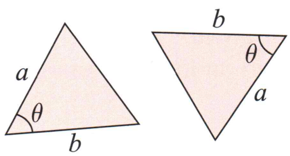


Side, Side, Side:
The three sides on one triangle are the same as the three sides on the other triangle.

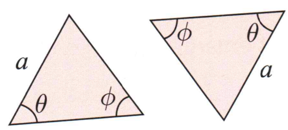


Right angle, Hypotenuse, Side:
Both triangles have a right angle, both triangles have the same hypotenuse and one other side is the same.

Tip: If any of these conditions are true then you can use trigonometry to show that the other sides and angles in the triangles must also be the same.



Side, Angle, Side:
Two sides and the angle between them on one triangle are the same as two sides and the angle between them on the other triangle.

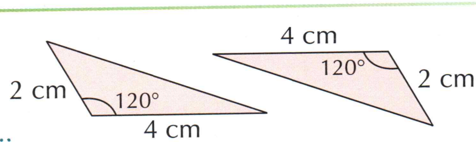


Angle, Angle, Side:
Two angles and an opposite side on one triangle are the same as two angles and the corresponding opposite side on the other triangle.

Be very careful when using 'Side, Angle, Side' and 'Angle, Angle, Side' — you have to have the **correct combination** of sides and angles.

Example 1

Are these two triangles congruent? Give a reason for your answer.



Look at the conditions listed above...

Two of the sides and the angle between them are the same on both triangles.

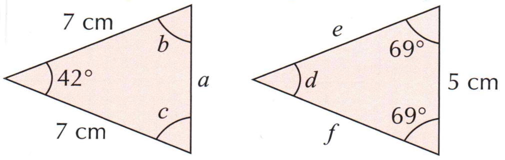
Condition SAS holds, so the triangles **are congruent**.

Tip: The conditions are often abbreviated to SAS, SSS, AAS and RHS.

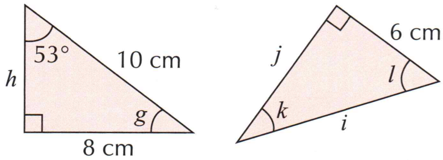
Exercise 1

Q1 The following pairs of triangles are congruent. Find the values marked with letters.

a)

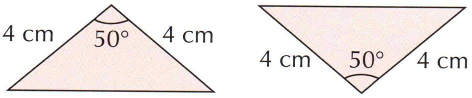


b)

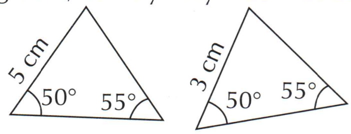


Q2 For each of the following, decide whether the two triangles are congruent.
In each case explain either how you know they are congruent, or why they must not be.

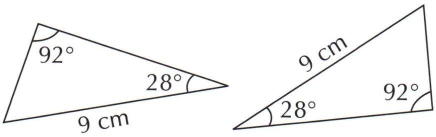
a)



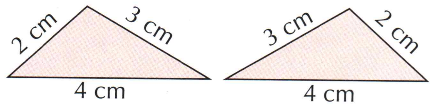
b)



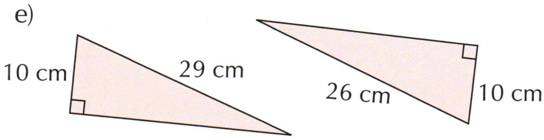
c)



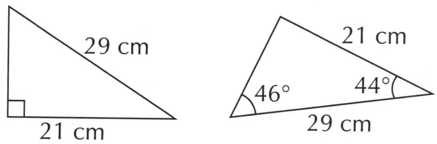
d)



e)



f)



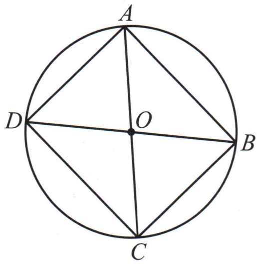
If it's not obvious which of the conditions apply, you might have to figure out some of the side lengths or angles for yourself. Properties of **parallel lines** and **polygons**, **circle theorems** and **Pythagoras' theorem** will all come in handy here — have a look at Sections 20, 21 and 25 if you need a reminder.

Example 2

***ABCD* is a cyclic quadrilateral within the circle with centre *O*.**

PROBLEM SOLVING

a) Prove that triangles *AOD* and *BOC* are congruent.



Use properties of vertically opposite angles and circles to match sides and angles in the two triangles.

Vertically opposite angles are equal, so angle *AOD* = angle *BOC*.

Sides *OA*, *OD*, *OB* and *OC* are all radii, so *OA* = *OC* and *OD* = *OB*.

Two sides and the angle between them are the same, which means condition SAS holds, so the triangles **are congruent**.

b) Hence prove that triangles *ABD* and *BCD* are congruent.

This time, you need to use your result from part a) and circle theorems.

Triangles *ABD* and *BCD* lie in a semicircle so are right-angled, which means angle *BAD* = angle *BCD* = 90°.

Side *BD* is the hypotenuse and is common to both triangles.

From part a), triangles *AOD* and *BOC* are congruent, so *AD* = *BC*.

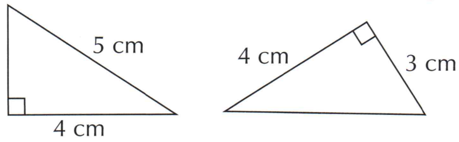
Both triangles have a right angle, the same hypotenuse and one other side the same, which means condition RHS holds, so the triangles **are congruent**.

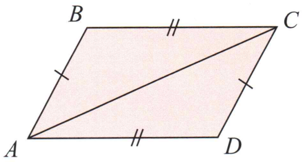
Tip: *BD* is a diameter as it passes through the centre of the circle.

Exercise 2

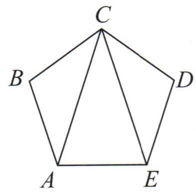
PROBLEM SOLVING

Q1 Show that the two triangles on the right are congruent.

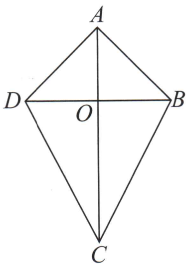




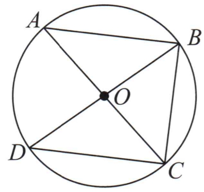
Q2 *ABCD* is a parallelogram. Prove that triangles *ABC* and *ADC* are congruent.



Q3 *ABCDE* is a regular pentagon. Prove that triangles *ABC* and *CDE* are congruent.



Q4 *ABCD* is a kite and *O* is the point where the diagonals of the kite intersect. Prove that *BOC* and *DOC* are congruent triangles.



Q5 *AC* and *BD* are diameters of the circle with centre *O*. Prove that triangles *ABC* and *DCB* are congruent.

Similar Shapes

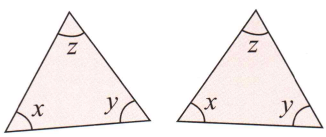
Learning Objectives — Spec Ref G6:

- Know the conditions for similar triangles.
- Be able to prove that two triangles are similar.
- Use scale factors to find missing sides and angles.

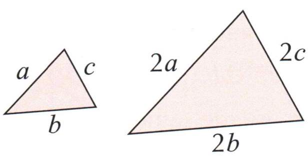
Prior Knowledge Check:
Properties of triangles and other 2D shapes (see Section 20) and Pythagoras' theorem (Section 25).

Similar Triangles

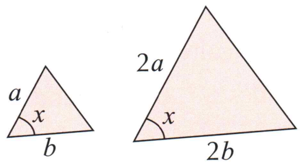
Similar shapes have the **same angles** as each other, but the **side lengths** are **different** — they are enlarged by the same scale factor. Two **triangles** are similar if they satisfy **any one** of the following conditions.



All the **angles** on one triangle are the **same** as the angles on the other triangle.



All the **sides** on one triangle are in the **same ratio** as the corresponding sides on the other triangle.



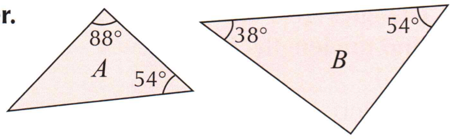
Two sides on one triangle are in the **same ratio** as the corresponding sides on the other triangle, and the **angle between** is the **same** on both triangles.

You might have to work out some **side lengths** or **angles** for yourself before you can decide if two triangles are similar. For example, if you're told **two** angles in one triangle, you can use the fact that the angles in a triangle **add up to 180°** to find the missing angle, then compare it to the other triangle.

To find missing side lengths, you might have to use properties of **isosceles triangles** or other **2D shapes**. You might even have to use **Pythagoras' theorem** or **trigonometry** if it's a right-angled triangle.

Example 3

Are these two triangles similar? Give a reason for your answer.



1. Start by finding the missing angles in each triangle.

   Triangle A: missing angle = 180° – 88° – 54° = 38°
   Triangle B: missing angle = 180° – 38° – 54° = 88°

2. Look at the conditions for similarity and see if any of them are true for these triangles.

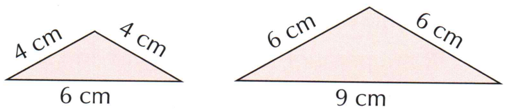
   All the angles in triangle A are the same as the angles in triangle B.

   So the triangles **are similar**.

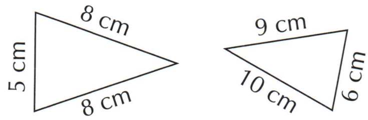
Exercise 3

Q1 For each of the following, decide whether the two triangles are similar.
In each case explain either how you know they are similar, or why they must not be.

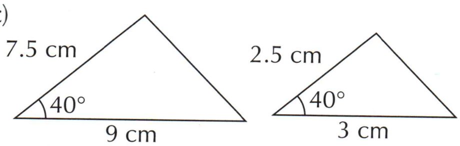
a)



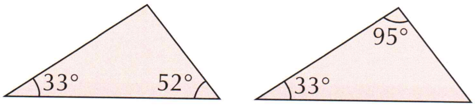
b)



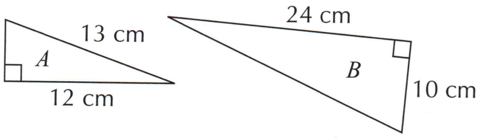
c)

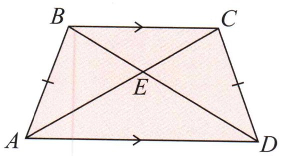


d)



Q2 Show that triangles A and B are similar.





Q3 $ABCD$ is an isosceles trapezium. Prove that triangles BEC and AED are similar.

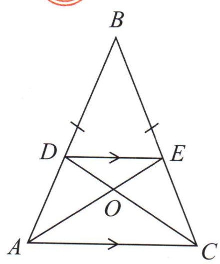
PROBLEM SOLVING

Q4 ABC is an isosceles triangle.

PROBLEM SOLVING

a) Prove that triangles ABC and DBE are similar.

b) Find another pair of triangles that are similar but not congruent.
Explain your answer.



Scale Factors

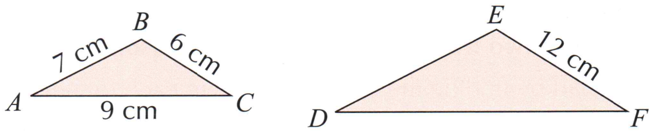
If you **know** that two shapes are **similar**, you can use that fact to find **missing sides** and **angles** — just remember that similar shapes have **all angles** the **same**, and sides in the **same ratio**.

All you have to do to find missing sides is find the **scale factor** — the number you have to multiply all the side lengths in one shape by to get the side lengths in the other shape.
To find the scale factor to get you from shape A to a similar shape B, use this **formula**:

$$\textbf{scale factor} = \frac{\textbf{side length of shape } B}{\textbf{side length of shape } A}$$

Example 4

Triangles *ABC* and *DEF* are similar. Find the lengths *DE* and *DF*.



1. Find the scale factor that gets you from *ABC* to *DEF*.

   scale factor $= \frac{EF}{BC} = \frac{12}{6} = 2$

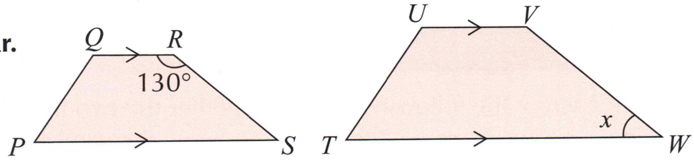
2. Use this scale factor to find the missing side lengths.

   So $DE = 2AB = 2 \times 7 =$ **14 cm**
   and $DF = 2AC = 2 \times 9 =$ **18 cm**

Tip: You could describe the sides as being in the ratio 1 : 2 — see p.400.

Example 5

Trapeziums *PQRS* and *TUVW* are similar. Find angle *x*.



1. The trapeziums are similar, so they will have the same angles.

   x = angle *RSP*, as equivalent angles in similar shapes are equal.

2. Use the properties of trapeziums and parallel lines to find the equivalent angle on *PQRS*.

   Sides *PS* and *QR* are parallel, so angles *QRS* and *RSP* are allied angles, which means they add up to 180°.

3. Calculate the size of this angle.

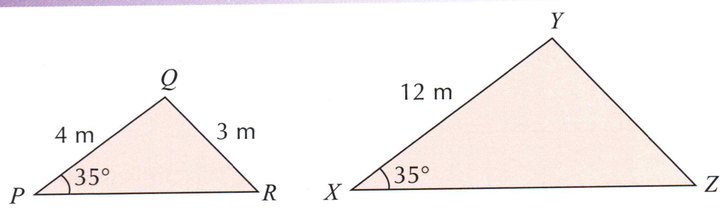
   So angle *RSP* = 180° – 130° = 50°, which means $\boldsymbol{x = 50°}$.

Exercise 4

Q1 Triangles *PQR* and *XYZ* are similar.

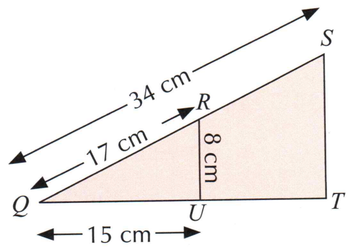
a) Find the scale factor that takes you from *PQR* to *XYZ*.

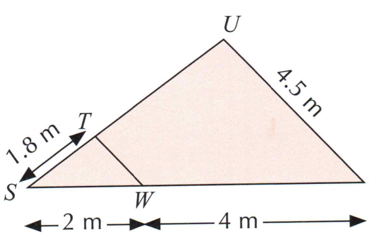
b) Find the length *YZ*.



Q2 The diagram on the right shows two similar triangles, *QRU* and *QST*. Find the following lengths.

a) *ST* b) *QT* c) *UT*



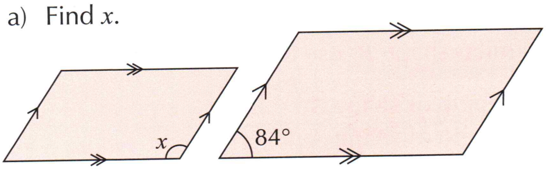


Q3 The diagram on the left shows two similar triangles, *STW* and *SUV*. Find the following lengths.

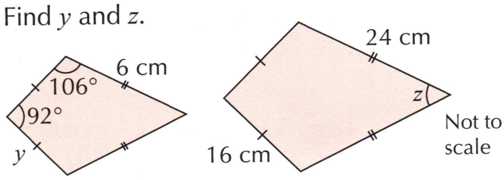
a) *TW* b) *SU* c) *TU*

Q4 Each pair of shapes shown below are similar.

a) Find *x*.



b) Find *y* and *z*.



30.2 Areas and Volumes of Similar Shapes

You can use the scale factor of the side lengths to help you find the areas and volumes of similar shapes. Be careful though — you don't just multiply the area or volume of the smaller shape by this scale factor to find the area or volume of the bigger shape. There's a bit more to it than that...

Areas of Similar Shapes

Learning Objectives — Spec Ref G19/R12:

- Find the areas and surface areas of similar shapes.
- Use ratios to solve similarity problems.

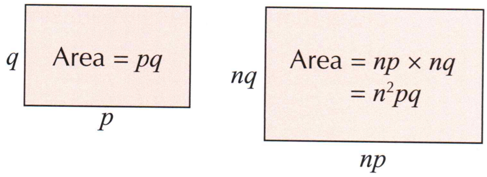
Prior Knowledge Check:
Find areas of 2D shapes and surface areas of 3D shapes — see Sections 27 and 28.
Use ratio notation — see p.42.

For two **similar** shapes, where one has sides that are **twice as long** as the sides of the other, the **area** of the larger shape will be **2^2 times** (i.e. **four** times) the area of the smaller shape. For example, consider a square with side length 3 cm — it has an area of $3^2 = 9$ cm². If that square was enlarged by scale factor 5, the larger square would have an area of $9 \times 5^2 = 225$ cm².

In general, for an enlargement of **scale factor n**, the area of the new shape will be **n^2 times** the area of the original shape.

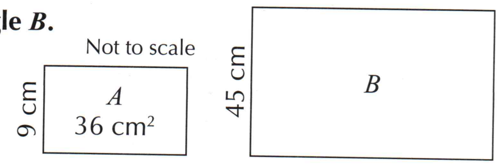
To find the scale factor (n) from the areas, use the formula:

$$n^2 = \frac{\textbf{area of enlarged shape}}{\textbf{area of original shape}}$$



Example 1

Rectangles A and B are similar. Find the area of rectangle B.



1. First find the scale factor by looking at corresponding sides on each shape.

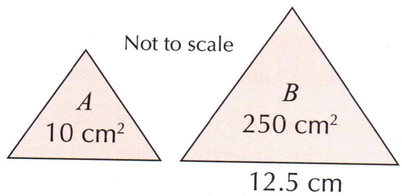
   Scale factor (n) $= \frac{\text{length of } B}{\text{length of } A} = \frac{45}{9} = 5$

2. Multiply the area of A by the scale factor squared to find the area of B.

   Area of $B = 36 \times 5^2 = 36 \times 25 =$ **900 cm²**

Example 2

Triangles A and B are similar. The area of triangle A is 10 cm² and the area of triangle B is 250 cm².



a) What is the scale factor of the enlargement?

1. Use the formula above. $n^2 = \frac{\text{enlarged area}}{\text{original area}} = \frac{250}{10} = 25$

2. Take the square root to find the scale factor of the enlargement. $n = \sqrt{25} = 5$, so the scale factor is **5**

b) The base of triangle B is 12.5 cm. What is the length of the base of triangle A?

Divide the length of the base of triangle B by the scale factor you found in part a).

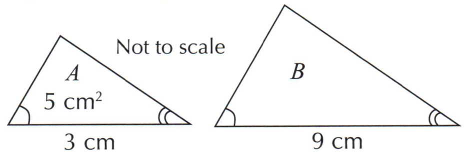
base of triangle A: $12.5 \div 5 =$ **2.5 cm**

Tip: You divide by the scale factor in part b) because you're going from the larger shape to the smaller shape.

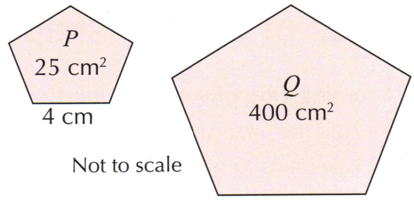
Exercise 1

Q1 Work out the area of the square formed when a square with side length 10 cm is enlarged by a scale factor of 4.

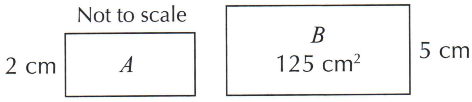
Q2 Triangles A and B are similar. Calculate the area of B.



Q3 The shapes on the right are similar.
Work out the base length of shape Q.

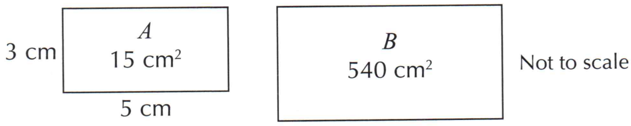


Q4 The rectangles below are similar. Calculate the area of shape A.



Q5 A triangle has perimeter 12 cm and area 6 cm². It is enlarged by a scale factor of 3 to produce a similar triangle. What is the perimeter and area of the new triangle?

Q6 Find the dimensions of rectangle B, given that the rectangles are similar.



Using Ratios

If the ratio of the side lengths is $\boldsymbol{a:b}$, the ratio of the areas is $\boldsymbol{a^2:b^2}$.
So if the ratio of the sides is 1 : 2, the ratio of the areas is $1^2:2^2 = 1:4$.

Example 3

The ratio of the areas of two similar shapes is 49 : 121. One side of the smaller shape measures 2.8 cm. Find the length of the corresponding side on the larger shape.

1. First find the ratio of the side lengths by taking the square root of each side of the area ratio.

   $a^2 : b^2 = 49 : 121$
   so $a : b = \sqrt{49} : \sqrt{121} = 7 : 11$

2. This means that the side of the larger shape is $\frac{11}{7}$ times as big as the corresponding side of the smaller shape — so multiply 2.8 by $\frac{11}{7}$ to find the length you need.

   $2.8 \times \frac{11}{7} = \mathbf{4.4\ cm}$

Exercise 2

Q1 Squares A and B have side lengths given by the ratio 2 : 3. Square A has sides of length 8 cm.
- a) Find the length of one of the sides of B.
- b) Find the area of B.
- c) Find the ratio of the area of A to the area of B.

Q2 The ratio of the areas of two similar shapes is 2.25 : 6.25. Find the ratios of their side lengths.

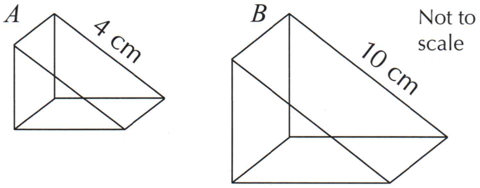
Q3 The ratio of the sides of two similar shapes is 4 : 5. The area of the smaller shape is 20 cm².
Find the area of the larger shape.

Q4 Stuart is drawing a scale model of his workshop. He uses a scale of 1 cm : 50 cm.
The area of the bench on his drawing is 3 cm². What is the area of his bench in real life?

The rules about enlargements and areas of 2D shapes also apply to **surface areas** of **3D shapes**.

Example 4

Shapes A and B on the right are similar triangular prisms. The surface area of shape A is 35 cm².
Find the surface area of shape B.



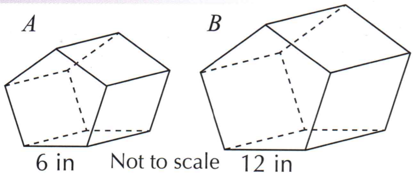
1. Compare corresponding side lengths to find the scale factor of enlargement, n.

   10 cm ÷ 4 cm = 2.5, so the scale factor is $n = 2.5$

2. Multiply the surface area of shape A by n^2 to find the surface area of shape B.

   Surface area of $B = 35 \times 2.5^2 = 35 \times 6.25 =$ **218.75 cm²**

Exercise 3

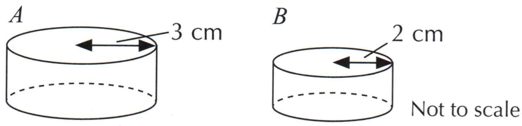


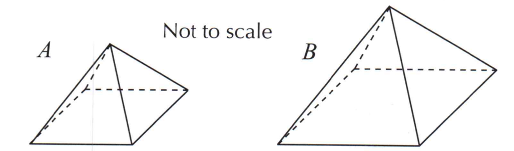
Q1 A and B are similar prisms. B has a surface area of 1440 in². Calculate the surface area of A.

Q2 A sphere has surface area 400π mm². What is the surface area of a similar sphere with a diameter three times as large?

Q3 The 3D solid P has a surface area of 60 cm². Q, a similar 3D solid, has a surface area of 1500 cm². If one side of shape P measures 3 cm, how long is the corresponding side of shape Q?

Q4 Cylinder A has surface area 63π cm². Find the surface area of the similar cylinder B.





Surface area = 32 cm²   Surface area = 72 cm²

Q5 The vertical height of pyramid A is 5 cm. Find the vertical height of the similar pyramid B.

Q6 Similar shapes P, Q and R have surface areas in the ratio 4 : 9 : 12.25. Find the ratio of their side lengths.

Volumes of Similar Shapes

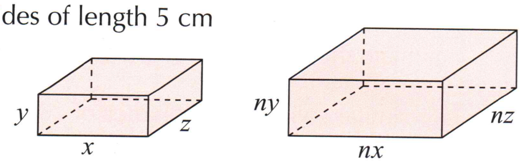
Learning Objectives — Spec Ref G19/R12:

- Find the volumes of similar shapes.
- Use ratios to solve similarity problems.

Prior Knowledge Check:
Find volumes of 3D shapes — see Section 28.
Use ratio notation — see p.42.

For two **similar 3D shapes**, when the side lengths are **doubled**, the volume is multiplied by a factor of **2³** (= **8**). Consider a cube with sides of length 5 cm — it has a volume of $5^3 = 125$ cm³. If that cube was enlarged by scale factor 2, the larger cube would have a volume of $125 \times 2^3 = 1000$ cm³.

In general, if the sides increase by a scale factor of n, the volume increases by a scale factor of n^3.



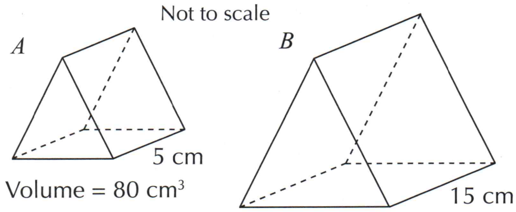
Volume = xyz Volume = $nx \times ny \times nz$
$= n^3xyz$

To find the scale factor (n) from the volumes, use the formula: $n^3 = \frac{\text{volume of enlarged shape}}{\text{volume of original shape}}$

For two similar shapes with side lengths in the ratio $\boldsymbol{a:b}$, their volumes will be in the ratio $\boldsymbol{a^3:b^3}$.

Example 5

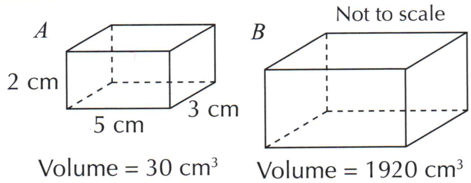
Triangular prisms A and B are similar.
Find the volume of B.



1. Find the scale factor by dividing the side length of B by the corresponding side length of A.
   Scale factor = 15 ÷ 5 = 3
2. Multiply the volume of A by the scale factor cubed.
   Volume of B = volume of A × 3^3 = 80 × 27 = **2160 cm³**

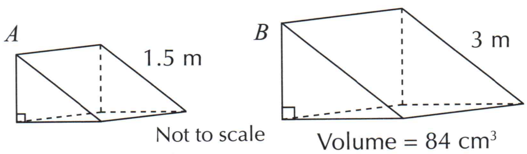
Exercise 4

Q1 Cuboids A and B, shown on the right, are similar. Find the dimensions of B.



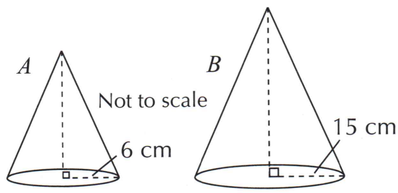
Q2 A 3D solid of volume 18 m³ is enlarged by scale factor 5. What is the volume of the new solid?

Q3 Triangular prisms A and B, shown on the right, are similar. Find the volume of A.



Q4 A cube has sides of length 3 cm. Find the side length of a similar cube whose volume is 3.375 times as big.

Q5 Cone A on the right has a volume of 60π cm³. Find the volume of cone B, given that the shapes are similar.

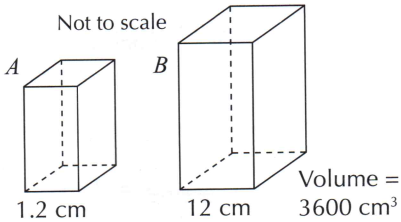


Q6 Two similar solids have side lengths in the ratio 2 : 5.
a) What is the ratio of their volumes?
b) The smaller shape has a volume of 100 mm³. What is the volume of the larger shape?

Q7 A pyramid has volume 32 cm³ and vertical height 8 cm. A similar pyramid has volume 16 384 cm³. What is its vertical height?

Q8 The radius of the planet Uranus is approximately 4 times the size of the radius of Earth. What does this tell you about the volumes of these planets?

Q9 The rectangular prisms on the right are similar. Calculate the volume of shape A.

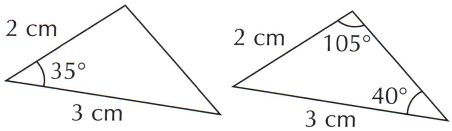


Q10 Two similar solids have volumes of 20 m³ and 1280 m³. James says that the surface area of the larger solid is 16 times the surface area of the smaller solid. Claire says that the surface area is 4 times larger. Who is correct?

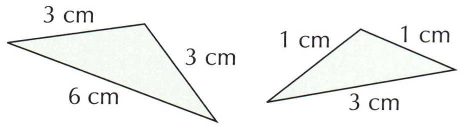
Review Exercise

Q1 For each of the following, decide whether the two triangles are congruent, similar or neither. In each case, explain your answer.

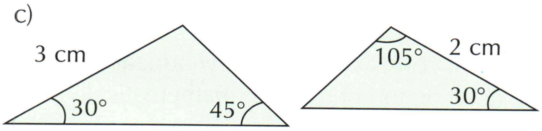
a)



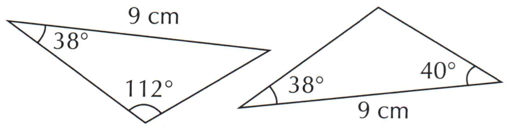
b)



c)



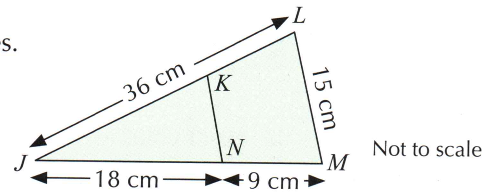
d)

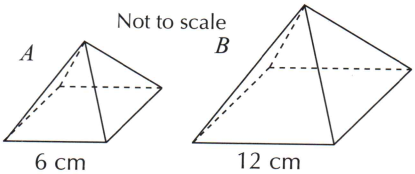


Q2 *JKN* and *JLM* on the right are two similar triangles.

a) Find length *KN*.

b) Find length *KL*.





Q3 The square-based pyramid *A* on the left has surface area 96 cm² and volume 48 cm³.

a) Find the surface area of the similar pyramid *B*.

b) Find the volume of *B*.

Q4 a) Two similar solids have side lengths in the ratio 3 : 5.

(i) Find the ratio of their surface areas. (ii) Find the ratio of their volumes.

b) Two similar solids have surface areas in the ratio 49 : 81.

(i) Find the ratio of their side lengths. (ii) Find the ratio of their volumes.

Q5 Vicky has two similar jewellery boxes in the shape of rectangular prisms. The larger box has dimensions of 9 in × 15 in × 12 in and a volume of 1620 in³. The smaller box has a volume of 60 in³. Find the dimensions of the smaller box.

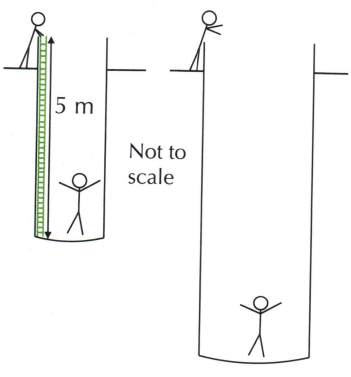
Q6 A solid has surface area 22 cm² and volume 18 cm³. A similar solid has sides that are 1.5 times as long.

a) Calculate its surface area.

b) Calculate its volume.

Q7 Mike has an unfortunate habit of falling down wells. The well he usually falls down has a volume of 1.25π m³. His friends need a ladder that's at least 5 m long to rescue him. One day he falls down a well that is mathematically similar to his usual well. Given that this well has a volume of 10π m³, how long does the ladder need to be this time?

PROBLEM SOLVING



Exam-Style Questions

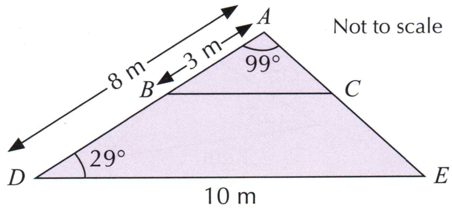
Q1 The diagram on the right shows two similar triangles, *ABC* and *ADE*.

a) Find length *BC*.

[2 marks]

b) Find angle *ACB*.

[1 mark]



Q2 Josephine is making a two-tiered wedding cake. It consists of a small cylindrical cake with diameter 16 cm and height 6 cm placed on top of a larger, mathematically similar cake. The area of the base of the larger cake is 144π cm^2.

a) Calculate the diameter of the larger cake.

[2 marks]

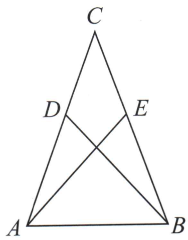
b) Calculate the exact volume of the larger cake.

[3 marks]

Q3 The cross-sectional area of a hexagonal prism is 18 cm^2. The volume of the prism is 270 cm^3.
A larger similar prism has a cross-sectional area of 162 cm^2.
What is the volume of the larger prism?

[3 marks]

Q4 In the diagram, *ABC* is a triangle in which angle *CAB* = angle *CBA*.
D and *E* are points on *AC* and *BC* respectively such that *AD* = *BE*.
Prove that the triangles *AEC* and *BDC* are congruent.



[4 marks]

Q5 An octagonal prism has a surface area of 50 mm^2 and a volume of 88 mm^3.
Another octagonal prism has a surface area of 450 mm^2 and a volume of 792 mm^3.
Are these two shapes similar? Explain your answer.

PROBLEM SOLVING

[3 marks]

Q6 A shop sells bags of bird food in two different sizes. The bags are mathematically similar.
One bag has a height of 3 cm and costs £1.10, while the other has a height of 6 cm and costs £2.50. Is the larger bag better value for money? Explain your answer.

PROBLEM SOLVING

[3 marks]