28.1 Plans, Elevations and Isometric Drawings

It can be difficult to draw 3D shapes accurately on paper — sometimes it's clearer to draw 2D plans and elevations of the 3D shape to show what it looks like from different sides.

Plans and Elevations

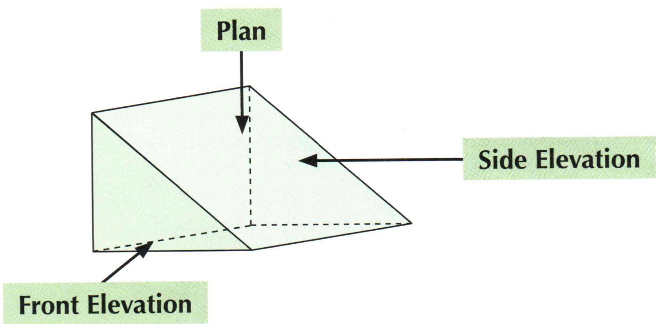
Learning Objective — Spec Ref G13:
Draw plans and elevations of 3D shapes.

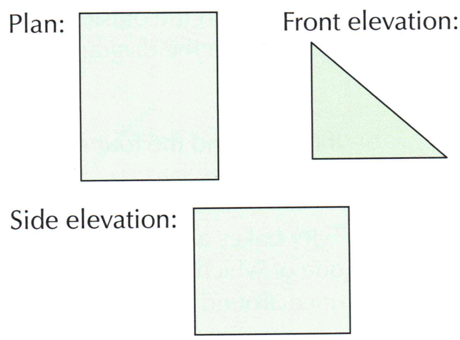
Plans and **elevations**, also known as **projections**, are **2D representations** of **3D objects** viewed from particular directions. There are **three** different projections:

- **Plan** — the 2D view looking **vertically downwards** on the 3D object.
- **Front elevation** — the 2D view looking **horizontally** from the **front** of the 3D object.
- **Side elevation** — the 2D view looking **horizontally** from the **side** of the 3D object.

Tip: The directions for the front and side elevations are usually indicated by arrows.

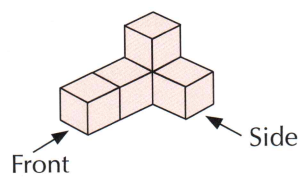
For example, the plan of the triangular prism below would be a **rectangle**, the front elevation would be a **triangle** and the side elevation would be another **rectangle**, as shown.



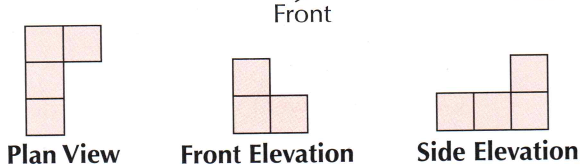


Example 1

Draw the plan and elevations of this 3D shape.

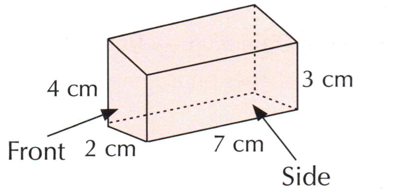


1. Viewed from above, the shape has 4 squares arranged in an L-shape. From this view, you can't tell there's a cube on top of the base layer.
2. Viewed from the front, the shape has 3 squares. You can't see the change in depth.
3. Viewed from the side, the shape has 4 squares in a sideways L-shape. Again, you can't see a change in depth from this elevation.



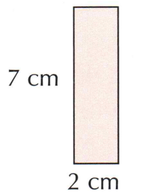
Example 2

Draw the plan and elevations of the prism on the right. Label your drawings with their dimensions.

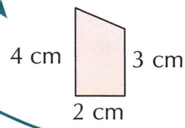


1. Viewed from above, the prism is just a rectangle. So, for the plan, draw a 7 cm × 2 cm rectangle.
2. For the front elevation, you can see the trapezium face, so draw a trapezium with the dimensions given.
3. Viewed from the side, you can see the change of height due to the trapezium. To show this, first draw a 4 cm × 7 cm rectangle. Then, add a straight line 3 cm from the bottom to show the different heights of the top of the trapezium.

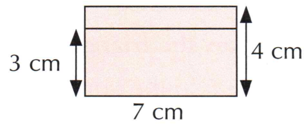
Plan View



Front Elevation

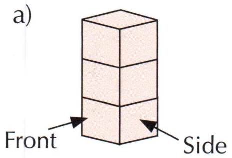


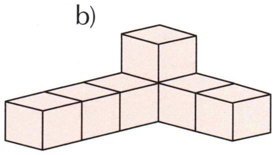
Side Elevation

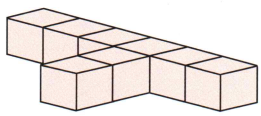


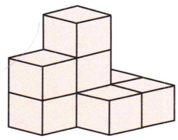
Exercise 1

Q1 For each of the following, draw: (i) the plan view
(ii) the front and side elevations, using the directions shown in a)

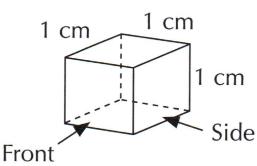
a)


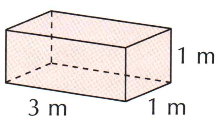
b)


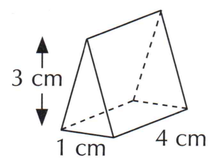
c)


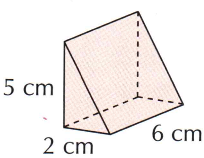
d)


Q2 For each of the following, draw the plan, front elevation and side elevation from the directions indicated in part a). Label your diagrams with their dimensions.

a)


b)


c)


d)


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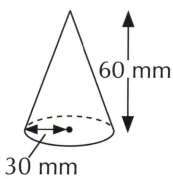
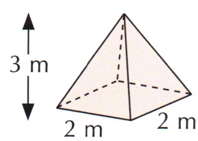


f)


Q3 Draw the plan, front elevation and side elevation for each of the following. Label your diagrams with their dimensions.

a) A cube of side 2 cm

b) A cylinder of radius 3 cm and height 4 cm

c) A 4 cm long prism whose cross-section is an isosceles triangle of height 5 cm and base 3 cm

d) A pyramid of height 5 cm whose base is a square of side 3 cm

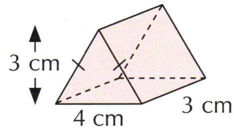
Isometric Drawings

Learning Objective — Spec Ref G13:
Draw 3D shapes on isometric paper.

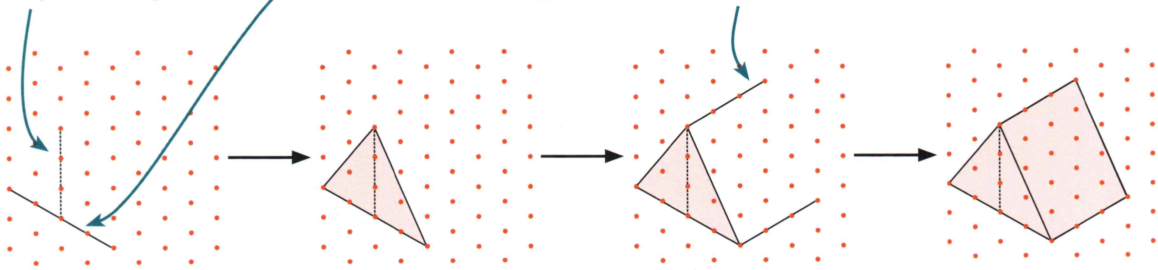
Isometric drawings are drawn on a grid of **dots** or **lines** arranged in a pattern of equilateral triangles. To draw a 3D shape on isometric paper, vertical lines on the shape are shown by **vertical lines** on the isometric paper and horizontal lines are shown by **diagonal lines** on the isometric paper. Each space between the dots in the vertical or diagonal directions represents **one unit** (e.g. 1 cm or 1 m).

Example 3

Draw the triangular prism shown on the right on isometric paper.



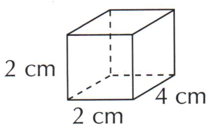
1. Join the dots with vertical lines for the vertical lines in the 3D shape and with diagonal lines for the horizontal lines.
2. Build up the drawing as shown below, using the dots to show the dimensions.
3. The triangle has a vertical height of 3 cm so draw a vertical line 3 spaces long.
4. It has a horizontal width of 4 cm so draw a diagonal line 4 spaces long.
5. The depth of the object is 3 cm so draw a diagonal line 3 spaces long.



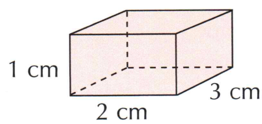
Exercise 2

Q1 Draw the following 3D objects on isometric paper.

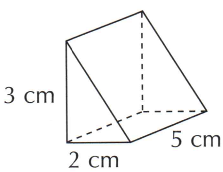
a)



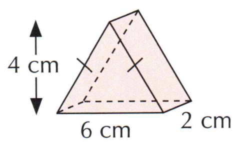
b)



c)



d)



Q2 Draw the following 3D objects on isometric paper.

a) A cube with 2 cm edges

b) A 2 cm × 2 cm × 3 cm cuboid

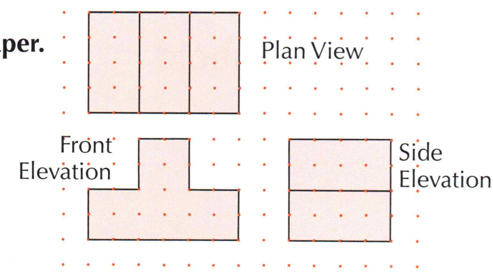
c) A 2 cm long prism whose cross-section is an isosceles triangle with base 4 cm and height 4 cm

d) A 3 cm long prism whose cross-section is an isosceles triangle with base 2 cm and height 4 cm

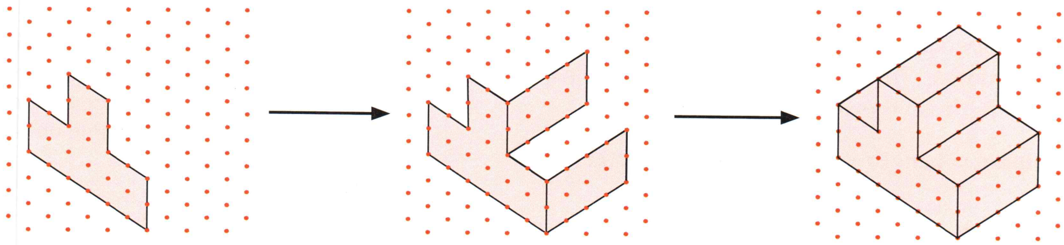
You can also use **projections** to draw shapes on isometric paper. It's often helpful to **picture** or **sketch the shape** first, then use the dimensions to draw it **accurately**.

Example 4

The diagram on the right shows the plan and front and side elevations of a 3D object. Draw the object on isometric paper.



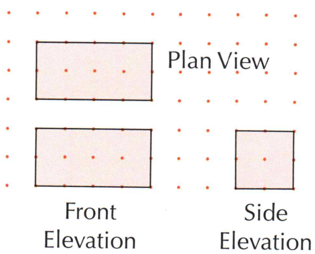
1. Try to picture the shape. It's a prism with the front elevation as its cross-section, which you can see since the plan view and side elevation have the same height all the way across.
2. Draw the cross-section on the isometric paper first, using the dots for the dimensions.
3. Use the side elevation and plan view to complete the drawing.

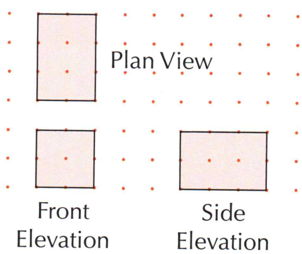


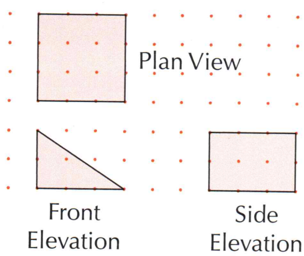
Exercise 3

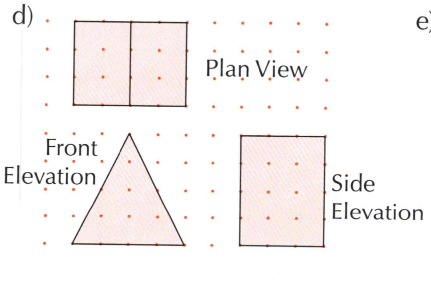
Q1 The following diagrams show the plan and front and side elevations of different 3D objects.

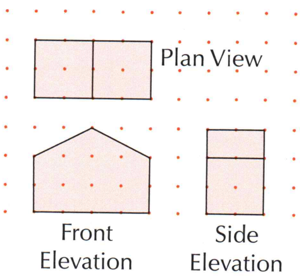
For each of the objects: (i) use the projections to sketch the object and label the dimensions,
(ii) draw the object accurately on isometric paper.

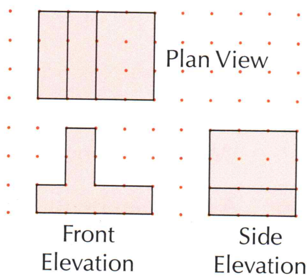
a)


b)


c)


d)


e)


f)


28.2 Volume

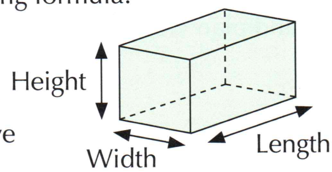
3D shapes have vertices (corners), edges and faces (sides). The volume of a 3D shape is the amount of space inside the shape — it's measured in cubic units, e.g. cm^3 or m^3.

Volume of a Cuboid

Learning Objective — Spec Ref G16:
Calculate the volume of cubes and cuboids.

A **cuboid** is a 3D shape with **six rectangular faces**.
The **volume** of a cuboid is found using the following formula:

Volume = Length × Width × Height



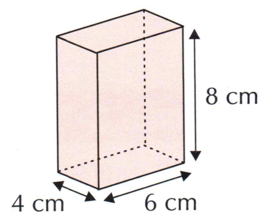
A **cube** is a special type of cuboid — all six of the faces are **squares**. This means all of the edges have the **same length**, so the volume is given by:

Volume = Length × Length × Length = (Length)^3

Tip: The units for volume are cubed because you're multiplying three dimensions — here, length, width, height. Make sure all three are in the same units first.

Example 1

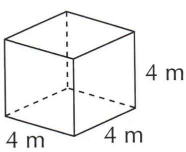
Find the volume of the cuboid shown in the diagram.

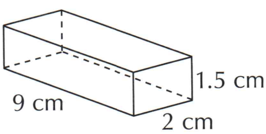


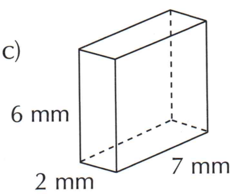
1. Write down the formula for volume. — Volume = length × width × height
2. Substitute the values into the formula. — Volume = 6 cm × 4 cm × 8 cm
3. Calculate the volume — don't forget the units. — = **192 cm^3**

Exercise 1

Q1 Find the volumes of the following cuboids.

a) 

b) 

c) 

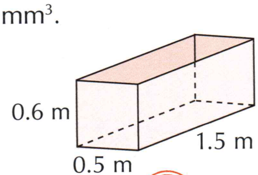
Q2 Find the volume of a cube whose edges are 3.2 mm long.

Q3 Will 3.5 m^3 of sand fit in a cuboid-shaped box with dimensions 1.7 m × 1.8 m × 0.9 m?

Q4 A matchbox is 5 cm long and 3 cm wide. The volume of the matchbox is 18 cm^3. What is its height?

Q5 A cuboid has a height of 9.2 mm and a width of 1.15 cm. Its volume is 793.5 mm^3. What is its length?

Q6 A bath can be modelled as a cuboid with dimensions 1.5 m × 0.5 m × 0.6 m.



a) What is the maximum volume of water that the bath will hold?
b) Find the volume of water needed to fill the bath to a height of 0.3 m.
c) Find the height of the water in the bath if the volume of water in the bath is 0.3 m^3.

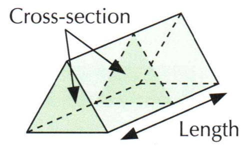
PROBLEM SOLVING

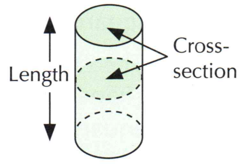
Volume of a Prism

Learning Objective — Spec Ref G16:
Calculate the volume of prisms, including cylinders.

Prior Knowledge Check:
Be able to find the area of a triangle (see p.331 and p.349) and of a circle (see p.354).

A **prism** is a 3D shape which has a **constant cross-section**. This means that if you **slice** the shape anywhere along its length **parallel** to the faces at the end of the length, the new face you produce is **exactly the same** as those faces at the end. For example, a **triangular prism** has a **triangle** as its constant cross-section and a **cylinder** has a **circle**. The **volume** of a prism is given by the following formula:





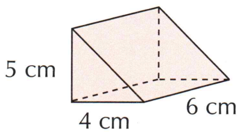
Volume = Area of Cross-Section × Length

The area of the cross-section for a triangular prism is the area of a triangle ($\frac{1}{2}$ **× base × height**), and for a cylinder it's the area of a circle (πr^2).

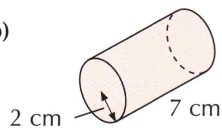
Example 2

Find the volume of each of the shapes shown on the right.

a)



b)



a) 1. Work out the area of the cross-section. Here it's a triangle, so calculate the area of the triangle.

Area of cross-section = area of triangle
$= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 4 \times 5 = 10 \text{ cm}^2$

2. Multiply the cross-sectional area by the length of the prism.

Volume = area of cross-section × length
$= 10 \times 6 =$ **60 cm³**

b) 1. Work out the area of the cross-section. Here it's a circle, so use area = πr^2.

Area of cross-section = area of circle
$= \pi r^2 = \pi \times 2^2 = 4\pi \text{ cm}^2$

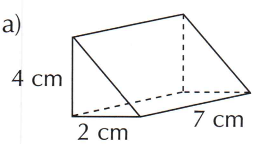
2. Multiply the cross-sectional area by the length of the prism.

Volume = area of cross-section × length
$= 4\pi \times 7 =$ **88.0 cm³** (1 d.p.)

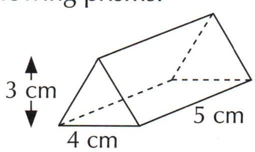
Exercise 2

Q1 Find the volumes of the following prisms.

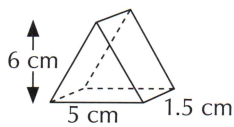
a)



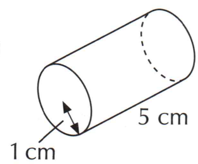
b)



c)



d)



Q2 Find the volumes of the following prisms.

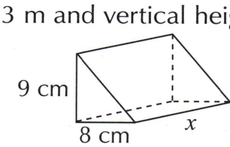
a) a triangular prism of base 4.2 m, vertical height 1.3 m and length 3.1 m

b) a cylinder of radius 4 m and length 18 m

c) a cylinder of radius 6 mm and length 2.5 mm

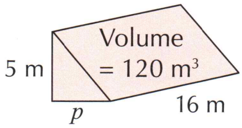
d) a prism with parallelogram cross-section of base 3 m and vertical height 4.2 m, and length 1.5 m

Q3 The triangular prism shown on the left has a volume of 936 cm³. Find x, the length of the prism.

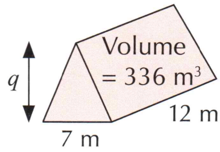


Q4 Find the values of the missing letters in the following prisms.

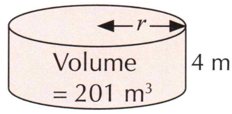
a)



b)

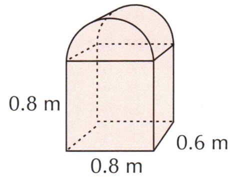


c)



Example 3

The cross-section of a jukebox (shown on the right) is made up of a square and a semicircle. Find the volume of the jukebox.



1. The cross-section is a composite shape, so split the shape up into a square and a semicircle. Find each area and then add them together.

   Area of square = $0.8 \times 0.8 = 0.64\ \text{m}^2$

   Area of semicircle

   $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \times 0.4^2 = 0.2513...\ \text{m}^2$

   Area of cross-section = $0.64 + 0.2513... = 0.8913...\ \text{m}^2$

2. Multiply the area of the cross-section by the length.

   Volume = area of cross-section × length

   $= 0.8913... \times 0.6 =$ **$0.53\ \text{m}^3$** (2 d.p.)

Example 4

A cylindrical tank of radius 28 cm and height 110 cm is half full of water. Find the volume of water in the tank. Give your answer in litres, correct to 1 d.p.

1. Start by working out the capacity of the tank — i.e. its volume.

   Area of cross-section = $\pi r^2 = \pi \times 28^2 = 2463.008...\ \text{cm}^2$

   Volume of tank = $2463.008... \times 110 = 270\,930.950...\ \text{cm}^3$

2. Since the tank is only half full, the volume of water in the tank is half of its total volume.

   Volume of water in the tank = $270\,930.950... \div 2$

   $= 135\,465.475...\ \text{cm}^3$

3. Convert to the unit required. There are $1000\ \text{cm}^3$ in 1 litre. (See Section 22 for more on converting between units.)

   $135\,465.475\ \text{cm}^3 = 135\,465.475 \div 1000$

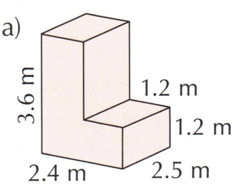
   $= 135.465...$ litres

   = **135.5 litres** (1 d.p.)

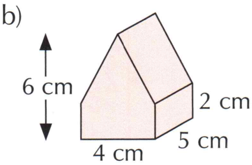
Exercise 3

Q1 By first calculating their cross-sectional areas, find the volumes of the following prisms.

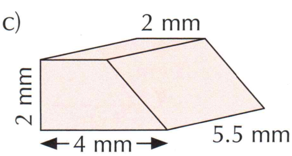
a)



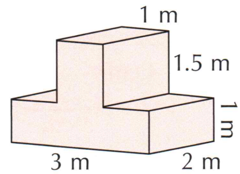
b)



c)



d)



Use the conversion 1 litre = $1000\ \text{cm}^3$ to answer the following questions.

Q2 A cube-shaped box with edges of length 15 cm is filled with water from a glass. The glass has a capacity of 0.125 litres. How many glasses of water will it take to fill the box?

Q3 90 litres of water is pumped into a cylindrical paddling pool, filling it to a depth of 3.5 cm. Find the radius of the paddling pool to 1 decimal place. PROBLEM SOLVING

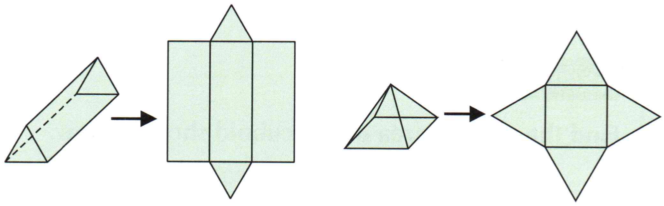
28.3 Nets and Surface Area

Nets are 2D representations of 3D objects. They can be useful for working out the surface area of 3D shapes.

Nets

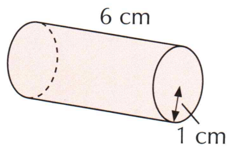
Learning Objective — Spec Ref G17:
Draw the net of a 3D shape.

A **net** is a **2D drawing** of a 3D shape that can be folded up to make the 3D shape. To draw the net, imagine '**unfolding**' the 3D shape so that **all the faces** of the shape are laid out flat. Some examples of nets are shown on the right.

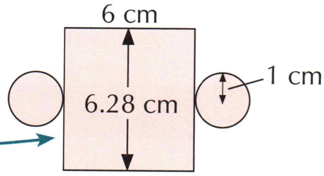


Example 1

Draw a net for the cylinder shown on the right. Label the net with its dimensions.

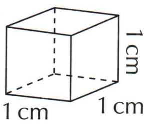


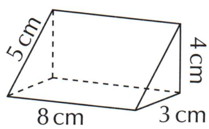
1. The tube can be 'unfolded' to give a rectangle. Its width will be 6 cm but you need to calculate its length, l.
2. The length is the same as the circumference of the circular ends of the cylinder, so use $C = 2\pi r$ (p.353). $l = 2\pi r = 2 \times \pi \times 1$ $= 6.28$ cm (2 d.p.)
3. Draw the rectangle and add on the circular ends to the sides.

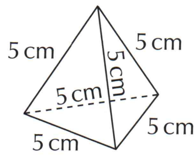


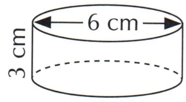
Exercise 1

Q1 Draw a net of each of the following objects. Label each net with its dimensions.

a) 

b) 

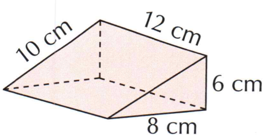
c) 

d) 

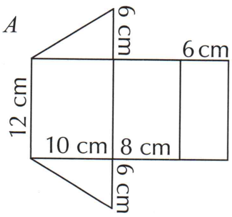
Q2 Draw a net of each of the following objects. Label each net with its dimensions.

a) a cube with 2 cm edges
b) a 1.5 cm × 2 cm × 2.5 cm cuboid
c) a triangular-based pyramid with 3.5 cm edges
d) a cylinder of height 4 cm and radius 2.5 cm
e) a prism of length 3 cm whose cross-section is an equilateral triangle with 2 cm sides
f) a pyramid with square base with 5 cm sides and slanted edge of length 4 cm

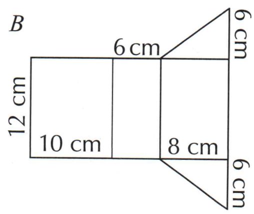
Q3 Which of the nets A, B or C is the net of the triangular prism shown below?



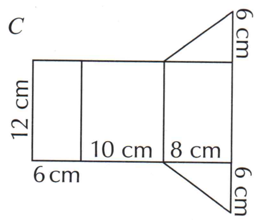
A



B



C



Surface Area

Learning Objective — Spec Ref G17:
Find the surface area of a 3D shape.

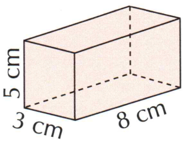
Prior Knowledge Check: Be able to find the area of 2D shapes — see Section 27.

The **surface area** of a 3D shape is the **total area** of all of the **faces** of the shape added together. To find the surface area of a **cuboid** or a **prism**, it's often useful to sketch the **net** of the shape first to make sure that you include **all** of the faces of the shape.

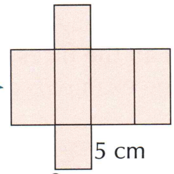
Tip: Spheres and cones have special formulas that give their surface areas. See p.368-369.

Example 2

Find the surface area of the cuboid shown below.



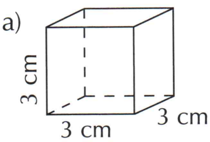
1. Find the area of each face of the shape. Sketch the net first. →

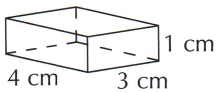


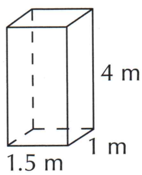
2. You know that a cuboid has six faces, so make sure that each is accounted for.
   2 faces of area $8 \times 5 = 40$ cm²
   2 faces of area $8 \times 3 = 24$ cm²
   2 faces of area $5 \times 3 = 15$ cm²
3. Add together the area of each face to get the surface area.
   Total surface area
   $= (2 \times 40) + (2 \times 24) + (2 \times 15)$
   $= 80 + 48 + 30 =$ **158 cm²**

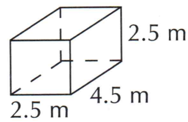
Exercise 2

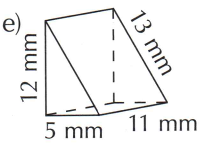
Q1 Find the surface area of the following prisms.

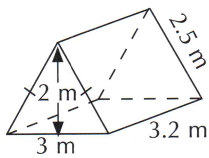
a) 

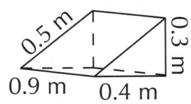
b) 

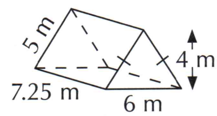
c) 

d) 

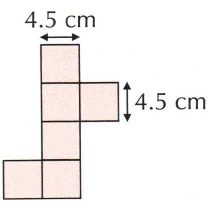
e) 

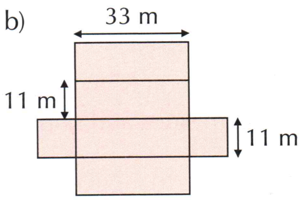
f) 

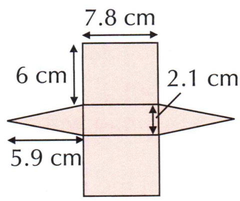
g) 

h) 

Q2 Find the surface area of the following nets and name the shape that the net will make.

a) 

b) 

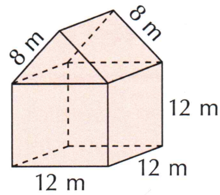
c) 

Q3 Find the surface area of the following prisms.

a) a cube with edges of length 5 m
b) a cube with edges of length 6 mm
c) a 1.5 m × 2 m × 6 m cuboid
d) a 7.5 m × 0.5 m × 8 m cuboid
e) an isosceles triangular prism of height 3 m, slant edges 5 m, base 8 m and length 2.5 m

Q4 The shape shown on the right is made up of a triangular prism and a cube. The bottom face of the prism is attached to the top face of the cube. Find the surface area of the shape.

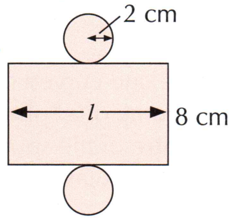
PROBLEM SOLVING



Example 3

By considering its net, find the surface area of a cylinder of radius 2 cm and length 8 cm.

1. Sketch the net of the cylinder.

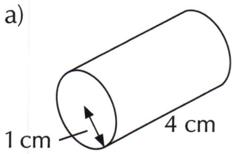


2. Find the length, l, of the rectangle — it's equal to the circumference of the circle.
   $l = 2\pi r = 2\pi \times 2$
   $= 12.566...$ cm
3. Find the area of each face.
   Area of each circle $= \pi r^2 = \pi \times 2^2 = 12.566...$ cm²
   Area of rectangle $= 8 \times 12.566... = 100.530...$ cm²
4. Add the individual areas together. Remember, there are two circular faces.
   Total surface area $= (2 \times 12.566...) + 100.530...$
   $=$ **125.7 cm²** (1 d.p.)

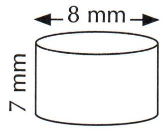
Exercise 3

Q1 Find the surface area of the following cylinders. Give your answers correct to 2 d.p.

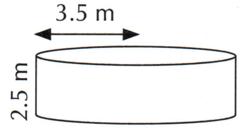
a)



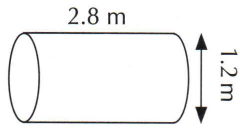
b)



c)



d)



Q2 Find the surface area of the cylinders with the following dimensions. Give your answers correct to 2 d.p.

a) radius = 2 m, length = 7 m
b) radius = 7.5 mm, length = 2.5 mm
c) radius = 12.2 cm, length = 9.9 cm
d) diameter = 22.1 m, length = 11.1 m

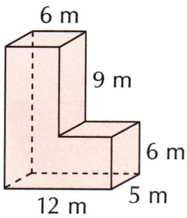
Q3 A cylindrical metal pipe has radius 2.2 m and length 7.1 m. The ends of the pipe are open.

a) Find the curved surface area of the outside of the pipe to 2 d.p.
b) A system of pipes consists of 9 of the pipes described above. What area of metal is required to build the system of pipes? Give your answer correct to 2 d.p.

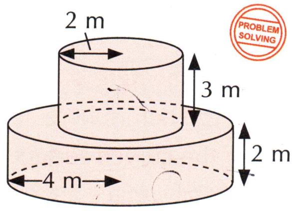
Q4 Ian is painting all the outside surfaces of his cylindrical gas tank. The tank has radius 0.8 m and length 3 m. 1 litre of Ian's paint will cover an area of 14 m². To 2 decimal places, how many litres of paint will Ian need?

Q5 Find the exact surface area of the shapes below.

a)



b)



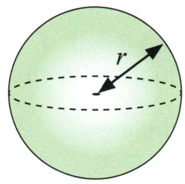
PROBLEM SOLVING

28.4 Spheres, Cones and Pyramids

Spheres, cones and pyramids are 3D shapes which are more complicated than cuboids and prisms. They each have their own formulas for their surface area and their volume.

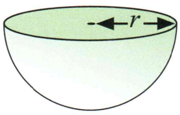
Spheres

Learning Objective — Spec Ref G17:
Find the surface area and volume of a sphere.



A **sphere** has one curved face, no vertices and no edges. To find the surface area and volume of a sphere, you need to know its **radius** — the distance from the **centre** of the sphere to **any point on its surface**. For a sphere with radius r, the formulas for **surface area** and **volume** are as follows:

Surface area = $4\pi r^2$ **Volume = $\frac{4}{3}\pi r^3$**



A **hemisphere** is **half** of a sphere. Its volume is just **half the volume** of a sphere with the same radius: **$V = \frac{2}{3}\pi r^3$**

The surface area is **half of the surface area** of the sphere, **plus** the area of the **flat circular face**: **Surface area = $2\pi r^2 + \pi r^2 = 3\pi r^2$**

Example 1

Find the exact surface area and volume of a sphere with radius 6 cm.

1. Substitute r into the formula for the surface area and work it through.

Surface area $= 4\pi r^2 = 4 \times \pi \times 6^2$
$= \mathbf{144\pi\ cm^2}$

2. Do the same with the formula for the volume.

Volume $= \frac{4}{3}\pi r^3 = \frac{4}{3} \times \pi \times 6^3$
$= \mathbf{288\pi\ cm^3}$

Tip: 'Exact' means you should leave your answer in terms of π.

Exercise 1

Q1 For each of the following spheres with the given radius, r, find:
(i) the exact surface area (ii) the exact volume
a) $r = 5$ cm b) $r = 4$ cm c) $r = 2.5$ m d) $r = 10$ mm

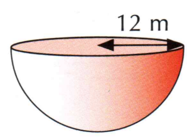
Q2 Find the radius of a sphere with surface area 265.9 cm². Give your answer correct to 1 d.p.

Q3 Find the radius of a sphere with volume 24 429 cm³. Give your answer correct to 1 d.p.

Q4 The surface area of a sphere is 2463 mm². Find the volume of the sphere, correct to 1 d.p.

Q5 The volume of a sphere is 6044 m³. Find the surface area of the sphere, correct to 1 d.p.

Q6 The 3D object shown on the right is a hemisphere.



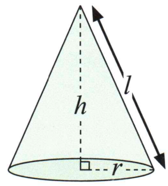
a) Find the volume of the hemisphere, to 1 d.p.
b) (i) Find the area of the curved surface of the hemisphere, to 1 d.p.
(ii) Hence find the total surface area of the hemisphere, to 1 d.p.

Cones

Learning Objective — Spec Ref G17:
Find the surface area and volume of cones and frustums.

Prior Knowledge Check:
Be familiar with Pythagoras' theorem (see p.318) and know how to find the area of a circle (see p.354).

A **cone** is a 3D shape that has a **circular base** and goes up to a **point** at the top. To find the **volume** and **surface area** of a cone, you need to know its base **radius** r, **perpendicular height** h and **slant height** l. The **volume** of a cone is given by:



$$\textbf{Volume} = \frac{1}{3}\pi r^2 h$$

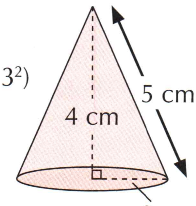
Tip: The perpendicular height goes from the centre of the base to the point. The slant height goes from the edge of the base to the point. Don't mix them up.

A cone has **two faces** — the curved, sloping face and the circular base. The formula for the **surface area** comes from **adding** up the areas of these faces:

$$\textbf{Surface area} = \pi rl + \pi r^2$$

Example 2

Find the exact surface area and volume of the cone shown on the right.



1. Substitute $r = 3$ and $l = 5$ into the formula for the surface area.

   Surface area $= \pi rl + \pi r^2 = (\pi \times 3 \times 5) + (\pi \times 3^2)$
   $= 15\pi + 9\pi = \mathbf{24\pi\ cm^2}$

2. Substitute $r = 3$ and $h = 4$ into the formula for the volume.

   Volume $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 3^2 \times 4$
   $= \mathbf{12\pi\ cm^3}$

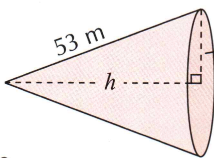
Exercise 2

Q1 Find (i) the exact surface area, and (ii) the exact volume of the cones with the given properties.

a) $r = 5$ m, $h = 12$ m, $l = 13$ m
b) $r = 7$ cm, $h = 24$ cm, $l = 25$ cm
c) $r = 15$ m, $h = 8$ m, $l = 17$ m
d) $r = 30$ mm, $h = 5.5$ mm, $l = 30.5$ mm

Q2 a) Find the exact surface area of the cone on the right.

b) (i) Use Pythagoras' theorem to find h, the perpendicular height of the cone.
(ii) Find the exact volume of the cone.



Q3 A cone has base radius 56 cm and perpendicular height 33 cm.

a) Find the exact volume of the cone.
b) (i) Use Pythagoras' theorem to find the slant height of the cone.
(ii) Find the exact surface area of the cone.

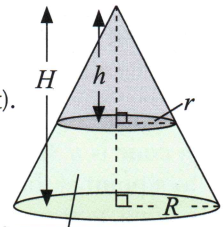
Q4 A cone has perpendicular height 6 mm and volume 39.27 mm³.

a) Find the base radius of the cone, to 1 d.p.
b) (i) Use your answer to part a) to find the slant height, l, of the cone.
(ii) Find the surface area of the cone, to 1 d.p.

Q5 A cone has vertical height 20 cm and volume 9236.28 cm³. PROBLEM SOLVING
Find the surface area of the cone, to 1 d.p.

Frustums

A **frustum** of a cone is the 3D shape left once you **chop** the top bit off a cone **parallel** to its circular base. The smaller, removed cone is always **similar** to the larger, original cone — see Section 30 for more on similarity. To find the **volume** of a frustum, find the volume of the **original cone** and take away the volume of the **removed cone** (the top bit).

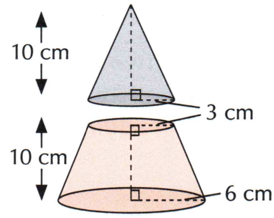


Frustum

Volume of frustum = Volume of original cone − Volume of removed cone = $\frac{1}{3}\pi R^2H - \frac{1}{3}\pi r^2h$

Example 3

Find the exact volume of the frustum shown on the right.



1. Find the volume of the original cone.

   Volume of original cone $= \frac{1}{3}\pi R^2 H$

   $= \frac{1}{3}\pi \times 6^2 \times (10 + 10)$

   $= \frac{1}{3}\pi \times 36 \times 20 = 240\pi$

2. Find the volume of the removed cone.

   Volume of removed cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 3^2 \times 10 = \frac{1}{3}\pi \times 9 \times 10 = 30\pi$

3. Subtract the volume of the removed cone from the volume of the original cone.

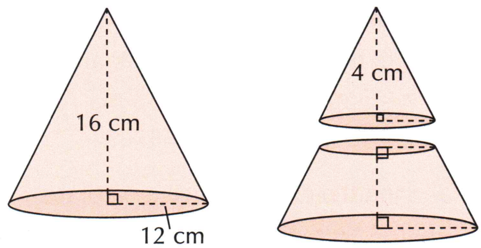
   Volume of frustum = vol. of original cone − vol. of removed cone

   $= 240\pi - 30\pi$

   $= \mathbf{210\pi}$ **cm³**

Exercise 3

Q1 A cone of perpendicular height 4 cm is removed from a cone of perpendicular height 16 cm and base radius 12 cm to leave the frustum shown.



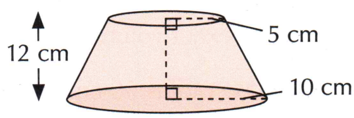
a) By considering the ratio of the heights of the cones, find the base radius of the smaller cone.

b) Find the exact volume of the larger cone.

c) Find the exact volume of the smaller cone.

d) Hence find the exact volume of the frustum.

Q2 Find the exact volume of the frustum shown.

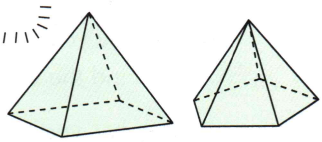


Pyramids

Learning Objective — Spec Ref G17:
Find the surface area and volume of a pyramid.

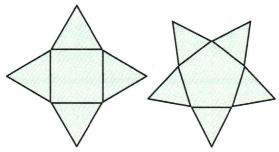
Prior Knowledge Check:
Be able to find the area of a triangle — see pages 331 and 349.

A **pyramid** is a 3D shape that has a **polygon base** (see page 257) which rises to a **point**. A cone is a bit like a pyramid with a circular base.



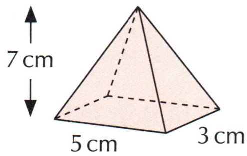
The **volume** of a pyramid can be found by using this formula: **Volume = $\frac{1}{3}$ × base area × height**

The **surface area** of a pyramid can be found by adding up the surface area of each of its **faces** (it might be useful to sketch the **net**). There'll be the **base** (which has, say, n edges), plus the n **triangular faces** on the sides. You might have to use **Pythagoras' theorem** (see p.318) to find the **height** of the triangular faces.



Example 4

Find the volume of the rectangular-based pyramid shown.

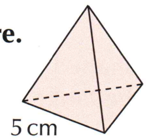


1. Write down the formula for volume of a pyramid.
2. Substitute the values into the formula — since the base is a rectangle, base area = length × width.

$$\text{Volume} = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3} \times (5 \times 3) \times 7$$
$$= \mathbf{35\ cm^3}$$

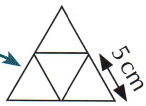
Example 5

Find the surface area of the regular tetrahedron shown here. Give your answer correct to 3 significant figures.



Tip: A tetrahedron is a triangular-based pyramid. 'Regular' means all the triangles are equilateral.

1. Each face is an equilateral triangle with 5 cm sides. Sketch a net to help.



2. Use the formula area $= \frac{1}{2}ab \sin C$ to find the area of each face. All angles are 60° for equilateral triangles.
3. There are four identical faces, so multiply this area by four.

$$\text{Area of one face} = \frac{1}{2} \times 5 \times 5 \times \sin 60°$$
$$= 10.825...$$

$$\text{Total surface area} = 4 \times 10.825...$$
$$= \mathbf{43.3\ cm^2}\ (3 \text{ s.f.})$$

Exercise 4

Q1 A hexagon-based pyramid is 15 cm tall. The area of its base is 18 cm². Calculate its volume.

Q2 Just to be controversial, Pharaoh Tim has decided he wants a pentagon-based pyramid. The area of the pentagon is 27 m² and the perpendicular height of the pyramid is 12 m. What is the volume of Tim's pyramid?

In Questions 3-5, give all answers to 1 decimal place.

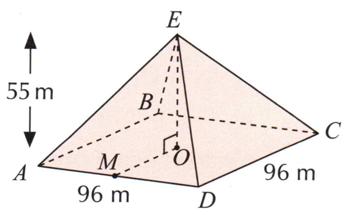
Q3 Find the volume of a pyramid of height 10 cm with rectangular base with dimensions 4 cm × 7 cm.

Q4 Find the surface area of a regular tetrahedron of side 12 mm.

Q5 A square-based pyramid has height 11.5 cm and volume 736 cm³. Find the side length of its base.

Q6 The diagram on the right shows the square-based pyramid $ABCDE$. The side length of the base is 96 m and the pyramid's height is 55 m. O is the centre of the square base, directly below E. M is the midpoint of AD.

PROBLEM SOLVING



a) Find the volume of the pyramid.

b) Use Pythagoras' theorem to find the length EM.

c) Hence find: (i) the area of triangle ADE (ii) the surface area of the pyramid.

28.5 Rates of Flow

Volume can be used to work out the rate of flow of a substance into a 3D container.

Learning Objective — Spec Ref G17:
Calculate rates of flow into and out of 3D shapes.

Prior Knowledge Check:
Be able to convert between different units (Section 22) and between different compound measures (p.289).

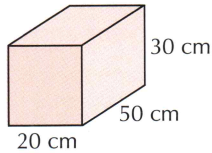
The **rate of flow** tells you **how quickly** a liquid is moving **into**, **out of** or **through** a certain **space**. To work out the rate of flow, you need to know the **total volume** of the space and the **time** it would take to **completely fill** (or empty) the space. The volume **divided** by the time gives the rate of flow.

$$\textbf{Rate of Flow} = \frac{\textbf{Volume of container}}{\textbf{Total time taken}}$$

Rates of flow are measured in **volume per unit time**, e.g. litres per second or m^3 per hour (sometimes written as litres/s or m^3/hr).

Example 1

Water flows into this cuboid-shaped tank at a rate of 150 cm^3 per second. How long does it take to fill the tank?



1. Calculate the volume of the cuboid.

   Volume of cuboid = height × length × width
   = 20 × 30 × 50 = 30 000 cm^3

2. Rearrange the formula and divide the volume by the rate of flow to get the time taken.

   Time to fill tank $= \frac{30\,000}{150} =$ **200 seconds**

Example 2

A spherical water fountain has a radius of 40 cm. Calculate the rate of flow of the water, in litres per second, if it takes 15 minutes for a completely full fountain to empty.

1. Calculate the volume of the fountain in cm^3.

   Volume of sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 40^3 = 268\,082.5...\ cm^3$

2. Convert the volume into litres (1 litre = 1000 cm^3) and the time into seconds (1 minute = 60 seconds).

   268 082.5... cm^3 = 268 082.5... ÷ 1000 = 268.0825... litres
   15 minutes = 15 × 60 = 900 seconds

3. Divide the volume by the time taken to find the rate of flow.

   Rate of flow $= \frac{\text{volume of fountain}}{\text{total time taken}} = \frac{268.0825...}{900}$
   = **0.298 litres per minute** (3 s.f.)

Exercise 1

Q1 Work out the average rates of flow when tanks with these volumes are filled in the given time.
a) Volume = 600 cm^3, Time = 15 seconds b) Volume = 150 litres, Time = 8 hours

Q2 A stream flows at a rate of 3 litres per second. Given that 1 ml is 1 cm^3, convert this rate of flow into:
a) cm^3 per minute b) m^3 per minute c) m^3 per day d) litres per day

Q3 Grain is being poured into an empty cylindrical silo with diameter 9 m and height 20 m. The grain is flowing at a rate of 12 m^3/minute. How long will it take to half fill the silo, to the nearest minute?

Q4 A spherical mould with radius 60 cm is used to make concrete balls.
Concrete flows into the mould at a rate of 2π litres/s. How long does it take to completely fill the mould?

28.6 Symmetry of 3D Shapes

While 2D objects have lines of symmetry, 3D objects have planes of symmetry.

Learning Objective — Spec Ref G12:
Work out the number of planes of symmetry in a 3D shape.

Prior Knowledge Check:
Be able to find lines of symmetry in 2D shapes — see p.261.

A **plane of symmetry** cuts a solid into **two identical halves**.

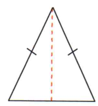
- The number of planes of symmetry of a **prism** is always **one greater** than the number of **lines of symmetry** of its cross-section. This is because a prism also has a plane of symmetry across the prism **parallel** to the cross-section. The only exception to this is a **cube**, which has **nine** planes of symmetry.
- The number of planes of symmetry of a **pyramid** is equal to the number of lines of symmetry of its **base**. The only exception is a **regular tetrahedron**, which has **six** planes of symmetry.

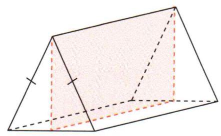
Example 1

How many planes of symmetry does an isosceles triangular prism have?

1. Work out how many lines of symmetry the cross-section has.

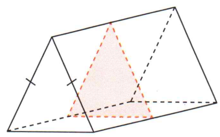
An isosceles triangle has 1 line of symmetry.





2. Add one more to this figure for the plane of symmetry that is parallel to the cross-section.

1 + 1 = 2, so an isosceles triangular prism has **2 planes of symmetry**.

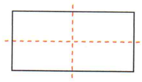


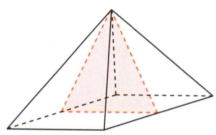
Example 2

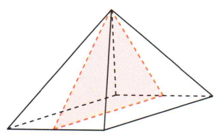
How many planes of symmetry does a rectangular-based pyramid have?

Work out how many lines of symmetry the base has.

base







A rectangle has 2 lines of symmetry, so a rectangle-based pyramid has **2 planes of symmetry**.

Exercise 1

Q1 Write down the number of planes of symmetry of the prisms with the following cross-sections.
a) regular pentagon b) regular octagon c) scalene triangle

Q2 Draw the planes of symmetry of the prisms with the following cross-sections.
a) equilateral triangle b) rectangle c) regular hexagon

Q3 Draw all the planes of symmetry of a cube.

Q4 Draw a prism with 5 planes of symmetry.

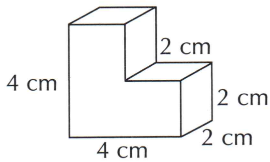
Q5 Write down the number of planes of symmetry of the pyramids with the following bases.
a) regular pentagon b) regular octagon c) square

Q6 Draw all the planes of symmetry of a regular tetrahedron.

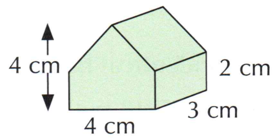
Review Exercise

Q1 Draw the following prisms on isometric paper.

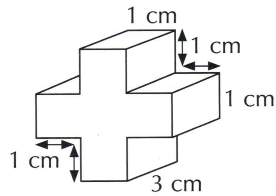
a)



b)



c)



Q2 A cube has a total surface area of 54 cm^2.

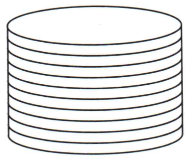
a) Find the area of one face of the cube.

b) Find the length of the edges of the cube.

c) Find the cube's volume.

Q3 A tank is in the shape of a cuboid of length 140 cm and width 85 cm. It is filled with water to a depth of 65 cm. There is room for another 297.5 litres of water in the tank. What is the height of the tank?

Q4 A cylindrical disc has a radius of 4 cm and height of 0.5 cm.



a) Find the surface area of the disc, to 2 d.p.

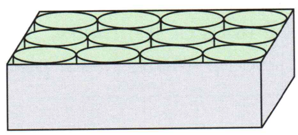
b) Find the surface area of a stack of 10 discs, to 2 d.p., assuming there are no gaps between each disc.

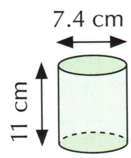
c) Find the volume of the stack of 10 discs, to 2 d.p.

Q5 Beans are sold in cylindrical tins of diameter 7.4 cm and height 11 cm.

a) Find the volume of one tin, to 2 d.p.

The tins are stored in boxes which hold 12 tins in three rows of 4, as shown.



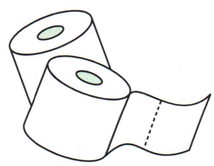


b) Find the dimensions of the box.

c) Find the volume of the box.

d) Calculate the volume of the box that is not taken up by tins when it is fully packed, to 2 d.p.

Q6 Toilet paper is sold in cylindrical rolls of diameter 12 cm and height 11 cm. The card tube at the centre of the roll is 5 cm in diameter. Find the following, giving your answers to 2 decimal places.



a) The total volume of one roll of toilet paper, including the card tube.

b) The volume of the card tube.

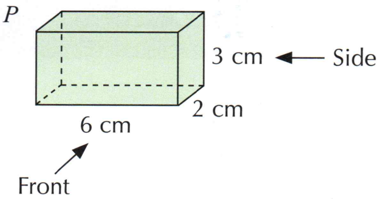
c) The volume of the paper.

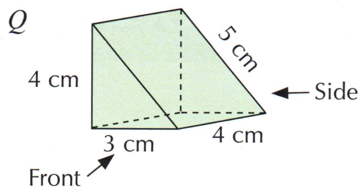
Each rectangular sheet of paper measures 11 cm × 13 cm and is 0.03 cm thick.

d) Find the volume of one sheet of paper.

e) Hence find the number of sheets of paper in one roll, to the nearest sheet.

Q7





For each of the shapes P and Q above:

a) Draw a net of the shape.

b) Draw the plan, and front and side elevations of the shape from the directions indicated.

c) Draw the shape on isometric paper.

d) Calculate the shape's surface area.

e) Calculate the shape's volume.

f) Calculate how long it would take to fill the shape with water flowing at a rate of 2 cm^3 per second.

g) Write down the number of planes of symmetry of the shape.

Q8 The value of the surface area of a sphere (in m^2) is equal to the value of the volume of the sphere (in m^3). What is the sphere's radius?

PROBLEM SOLVING

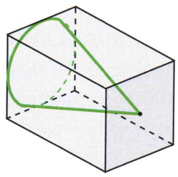
Q9 A cone of diameter 25 mm has volume 7854 mm^3.

PROBLEM SOLVING

a) Find the perpendicular height of the cone to 1 d.p.

b) (i) Find the slant height of the cone to 1 d.p.

(ii) Hence, find the surface area of the cone to 1 d.p.

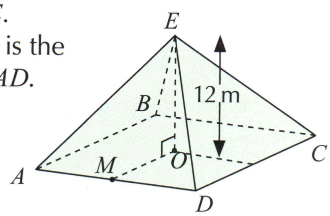


The cone is enclosed in a cuboid, as shown.

c) Find the volume of the cuboid to the nearest mm^3.

d) Find the volume of the empty space between the cone and the cuboid to the nearest mm^3.

Q10 The diagram on the right shows the square-based pyramid $ABCDE$. The pyramid's vertical height is 12 m and its volume is 400 m^3. O is the centre of the square base, directly below E. M is the midpoint of AD.



a) Find the side length of the square base.

b) Find the length EM.

c) Find the surface area of the pyramid.

d) How many planes of symmetry does the pyramid have?

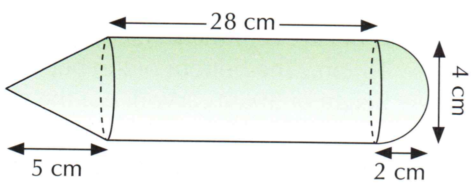
e) Sand enters the pyramid at a rate of 100 litres per minute. Calculate how long it would take to completely fill the pyramid, give your answer in hours and minutes.

Q11 A giant novelty pencil is made up of a cone, a cylinder and a hemisphere, as shown.

a) Find the exact volume of the pencil.

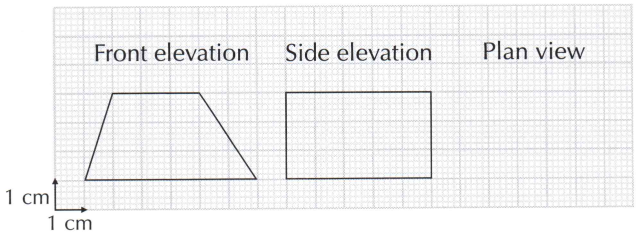
b) (i) Find the exact slant height of the cone.

(ii) Hence find the surface area of the pencil to one decimal place.



Exam-Style Questions

Q1 The diagram shows the front and side elevations of a solid prism drawn accurately on a centimetre square grid.



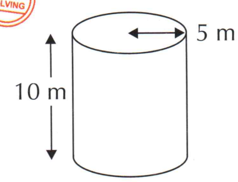
a) On graph paper, accurately draw the plan view of the prism.

[2 marks]

b) Work out the volume of the prism.

[3 marks]

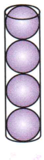
Q2 An industrial oil storage tank is a cylinder with a radius of 5 m and a height of 10 m as shown in the diagram.



Water is being poured into it at a constant rate of 7 cubic metres per minute. The tank has a leak which causes it to lose water at a constant rate of 2 cubic metres per minute. Work out how long it will take for the water to reach halfway up the tank, giving your answer in minutes as a multiple of π.

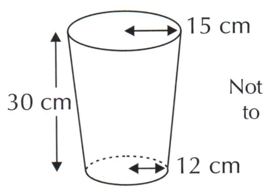
[4 marks]

Q3 Spherical tennis balls are sold in cylindrical tubes with lids. There are 4 balls in each tube. The balls fit snugly inside the tubes. Show that four balls take up $\frac{2}{3}$ of the space inside a tube.



[5 marks]

Q4



A bucket is in the shape of an upturned frustum. The top of the bucket is a circle of radius 15 cm. The bottom of the bucket is a circle of radius 12 cm. The bucket has a perpendicular height of 30 cm. Work out the capacity of the bucket, giving your answer in litres to 3 s.f.

[5 marks]