Section 25 Pythagoras and Trigonometry

25.1 Pythagoras' Theorem

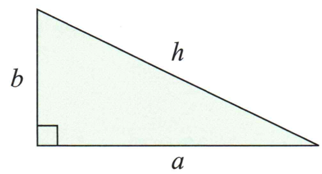
Pythagoras' theorem can be applied to all right-angled triangles. You use it to find the length of one side if you know the lengths of the other two sides.

Learning Objective — Spec Ref G6/G20:
Use Pythagoras' theorem to find missing lengths in right-angled triangles.

Prior Knowledge Check: Be able to square numbers and find square roots (see p.91) and be familiar with surds (see p.99).

The lengths of the sides in a **right-angled triangle** always follow the rule: $h^2 = a^2 + b^2$
h is the **hypotenuse** — this is the **longest** side, which is always **opposite** the right angle. a and b are the **shorter** sides.

This is **Pythagoras' theorem** and it is used to find lengths in right-angled triangles.

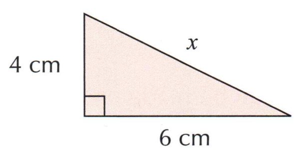


To find the length of the hypotenuse in a right-angled triangle:

- **Square** the lengths of sides a and b.
- **Add** together the squared lengths, a^2 and b^2, to get h^2.
- Take the **square root** of h^2 to find the hypotenuse, h.

Example 1

Find the exact length of x on the triangle shown.



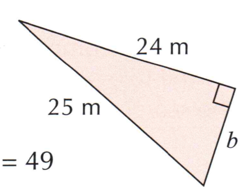
1. Substitute the values from the diagram into the formula. $h^2 = a^2 + b^2$
   $x^2 = 6^2 + 4^2$
2. Add together the squared lengths to get x^2. $x^2 = 36 + 16 = 52$
3. Work out the square root to find x. $x = \sqrt{52} = \sqrt{4 \times 13}$
   $= \mathbf{2\sqrt{13}\ cm}$

Tip: 'Exact length' means you should give your answer as a surd — simplified if possible (see p.99).

You can also use Pythagoras' theorem to find one of the **shorter sides** if you know the hypotenuse and one of the other sides. To do this, **substitute** in the values and then **rearrange** the formula to make the unknown length the subject.

Example 2

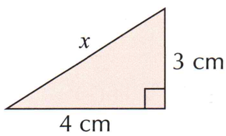
Find the length of b on the triangle shown on the right.

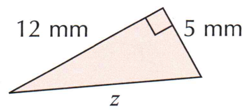


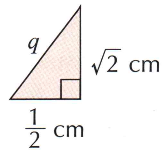
1. Substitute the values from the diagram into the formula. $h^2 = a^2 + b^2$
   $25^2 = 24^2 + b^2$
2. Rearrange to make b^2 the subject. $b^2 = 25^2 - 24^2 = 625 - 576 = 49$
3. Work out the square root to find b. $b = \sqrt{49} = \mathbf{7\ m}$

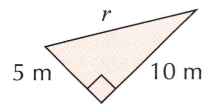
Exercise 1

Q1 Find the exact length of the hypotenuse in each of the triangles below.

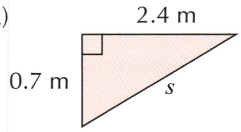
a) 

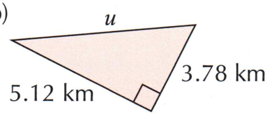
b) 

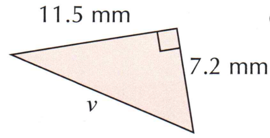
c) 

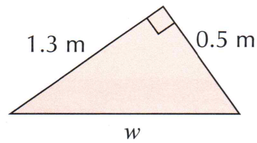
d) 

Q2 Find the length of the hypotenuse in each of the triangles below.
Give your answers correct to 2 decimal places where appropriate.

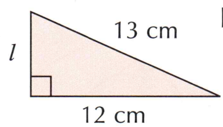
a) 

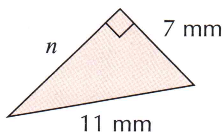
b) 

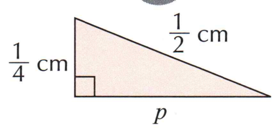
c) 

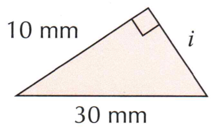
d) 

Q3 Find the exact lengths of the unknown sides in these triangles.

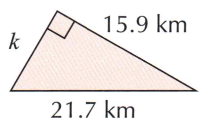
a) 

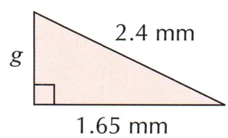
b) 

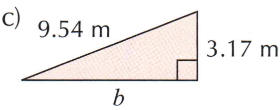
c) 

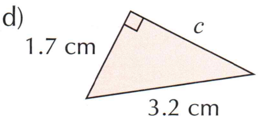
d) 

Q4 Find the lengths of the unknown sides in these triangles. Give your answers correct to 2 d.p.

a) 

b) 

c) 

d) 

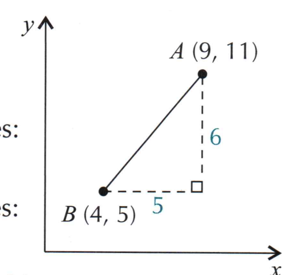
Pythagoras' theorem can be used in lots of other situations too —
you just have to look for ways to **create** a right-angled triangle. For example:

- **Splitting** an **equilateral** or **isosceles triangle** in half can create two identical right-angled triangles.
- The **straight line** between two pairs of **coordinates** forms the hypotenuse of a right-angled triangle with sides equal to the difference in the x-coordinates and the difference in the y-coordinates.

Pythagoras' theorem can also be applied to **real life situations**. The formula is used in the **same way**, you just have to link your answer back to the **context** of the situation.

Example 3

Find the exact distance between points A and B on the grid.



1. Create a right-angled triangle with hypotenuse AB.
2. Find the length of the horizontal side by working out the difference in the x-coordinates. — Difference in x-coordinates: $9 - 4 = 5$
3. Find the length of the vertical side by working out the difference in the y-coordinates. — Difference in y-coordinates: $11 - 5 = 6$
4. Substitute the values into the formula. — $AB^2 = 5^2 + 6^2 = 25 + 36 = 61$
5. Work out the square root to find the distance. Give your answer in surd form. — $AB = \sqrt{61}$

Example 4

A TV has a height of 40 cm and width of w cm.
Its diagonal measures 82 cm. Will the TV fit in a box 75 cm wide?

1. The diagonal and sides form a right-angled triangle, so substitute the values into the formula. $h^2 = a^2 + b^2$
   $82^2 = 40^2 + w^2$
2. Rearrange to make w^2 the subject. $w^2 = 82^2 - 40^2 = 6724 - 1600$
   $= 5124$
3. Work out the square root to find w. $w = \sqrt{5124} = 71.58$ cm (2 d.p.)
4. Use your answer to draw a conclusion. $71.58 < 75$ so **yes, the TV will fit in the box.**

Tip: Sketch a diagram to help you understand the question, if needed.

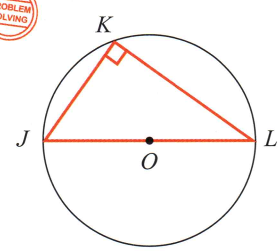
Exercise 2

Unless told otherwise, give your answers correct to 2 decimal places where appropriate.

Q1 An equilateral triangle has sides of length 10 cm. Find the perpendicular distance from a vertex to its opposite side.

Q2 The triangle JKL is drawn inside a circle centred on O, as shown. JK and KL have lengths 4.9 cm and 6.8 cm respectively.

a) Find the length of JL. b) Find the radius of the circle.



Q3 A kite gets stuck at the top of a vertical tree. The kite's 15 m string is taut, and its other end is held on the ground, 8.5 m from the base of the tree. Find the height of the tree.

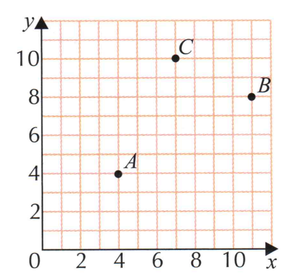
Q4 Newtown is 88 km northwest of Oldtown. Bigton is 142 km from Newtown, and lies northeast of Oldtown. What is the distance from Bigton to Oldtown, to the nearest kilometre?

Q5 A triangle has sides measuring 1.5 m, 2 m and 2.5 m. Show that this is a right-angled triangle.

Q6 A boat is rowed 200 m east and then 150 m south. If it had been rowed to the same point in a straight line instead, how much shorter would the journey have been?

Q7 The points A, B and C are shown on the right. Find the exact lengths of the line segments between the following pairs of points.

a) A and B b) B and C c) A and C

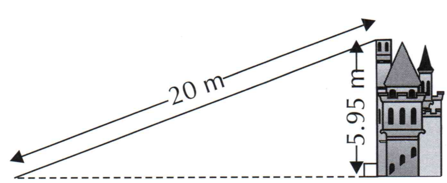


Q8 Find the exact distance between points P and Q with coordinates (11, 1) and (17, 19) respectively.

Q9 Kevin wants to set up a 20 m slide from the top of his 5.95 metre-high tower.

a) How far from the base of the tower should Kevin anchor the slide?

b) A safety inspector shortens the slide and anchors it 1.5 m closer to the tower. What is the new length of the slide?

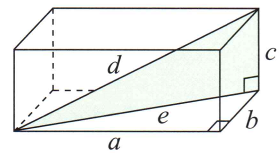


25.2 Pythagoras' Theorem in 3D

You can use Pythagoras' theorem to find dimensions within 3D shapes too.

Learning Objective — Spec Ref G20:
Use Pythagoras' theorem to find lengths in 3D shapes.

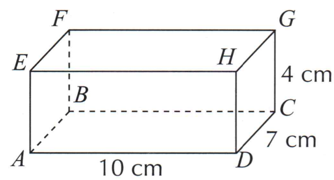
For a **cuboid** of length a, width b and height c, the length of the **longest diagonal**, d, can be found using the formula: $\boldsymbol{d^2 = a^2 + b^2 + c^2}$



This formula comes from using the 2D Pythagoras' theorem **twice**.
In the diagram to the right, a, b and e make a **right-angled triangle** (e is the **diagonal** of one of the cuboid's faces and forms the **hypotenuse**), so $e^2 = a^2 + b^2$.
Side c makes a right-angled triangle with d and e, so $d^2 = e^2 + c^2 = a^2 + b^2 + c^2$.

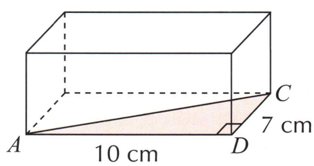
Example 1

Find the exact length AG in the cuboid shown on the right.



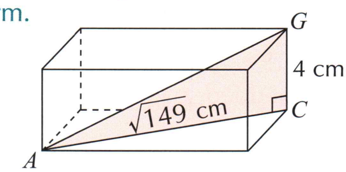
1. Use Pythagoras on the triangle ACD to find the length AC.

   $AC^2 = 10^2 + 7^2 = 149$
   $AC = \sqrt{149}$ cm



2. Now use Pythagoras again on the triangle AGC to find the length AG. Give your answer in surd form.

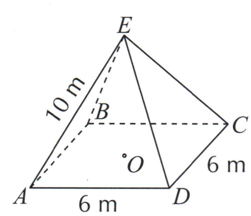
   $AG^2 = (\sqrt{149})^2 + 4^2$
   $= 149 + 16 = 165$
   $AG = \sqrt{165}$ **cm**



Tip: This example showed the step by step working, but you could have done it using the cuboid formula — $d^2 = 10^2 + 7^2 + 4^2 = 165$.

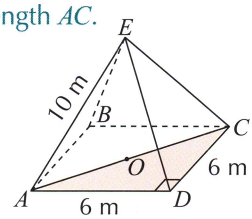
Example 2

A square-based pyramid $ABCDE$ is shown on the right. Point O is the midpoint of the base $ABCD$ and point E lies directly above O. Find the exact length OE.



1. Use Pythagoras on the triangle ACD to find the length AC.

   $AC^2 = 6^2 + 6^2 = 72$
   $AC = \sqrt{72} = 6\sqrt{2}$ m

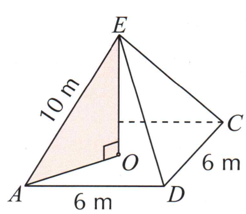


2. O is the midpoint of AC, so halve AC to get AO.

   $AO = 6\sqrt{2} \div 2 = 3\sqrt{2}$ m

3. Use Pythagoras on the triangle AOE to find the length OE. Give your answer in surd form.

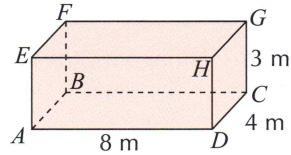
   $AE^2 = AO^2 + OE^2$
   $OE^2 = 10^2 - (3\sqrt{2})^2$
   $= 100 - 18 = 82$
   $OE = \sqrt{82}$ **m**



Tip: You have to do this example in stages — you're not finding the diagonal of a cuboid, so you can't use the formula.

Exercise 1

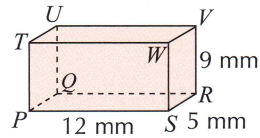
Q1 The cuboid *ABCDEFGH* is shown on the right.



a) By considering the triangle *ABD*, find the exact length *BD*.

b) By considering the triangle *BFD*, find the exact length *FD*.

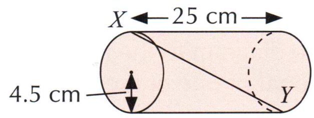
Q2 The cuboid *PQRSTUVW* is shown on the right.



a) Find the exact length *PR*.

b) Find the exact length *RT*.

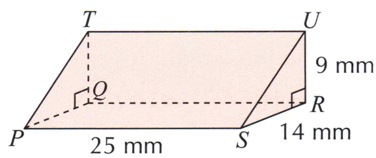
Q3 A cylinder of length 25 cm and radius 4.5 cm is shown on the right. *X* and *Y* are points on opposite edges of the cylinder, such that *XY* is as long as possible. Find the length *XY* to 3 significant figures.



Q4 A cuboid measures 2.5 m × 3.8 m × 9.4 m. Find the length of the diagonal of the cuboid to 2 d.p.

In Questions 5-11, give all answers to 3 significant figures.

Q5 The triangular prism *PQRSTU* is shown on the right.



a) Find the length *QS*.

b) Find the length *ST*.

Q6 Find the length of the diagonal of a cube of side 5 m.

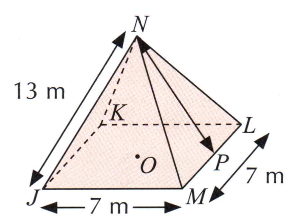
Q7 A pencil case in the shape of a cuboid is 16.5 cm long, 4.8 cm wide and 2 cm deep. What is the length of the longest pencil that will fit in the case? Ignore the thickness of the pencil.

Q8 A spaghetti jar is in the shape of a cylinder. The jar has radius 6 cm and height 28 cm. What is the length of the longest stick of dried spaghetti that will fit inside the jar?

Q9 A square-based pyramid has a base of side 4.8 cm and sloped edges all of length 11.2 cm. Find the vertical height of the pyramid from the centre of the base to the highest point.

Q10 A square-based pyramid has a base of side 3.2 m and a vertical height of 9.2 m. Find the length of the sloped edges of the pyramid, given they are all equal.

Q11 The square-based pyramid *JKLMN* is shown on the right.
O is the centre of the square base, directly below *N*.
P is the midpoint of *LM*.

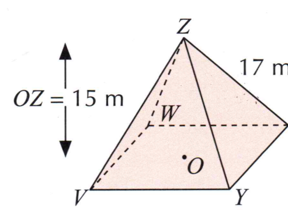


a) Find the length *NP*.

b) Find the length *OJ*.

c) Find the length *ON*.

Q12 The square-based pyramid *VWXYZ* is shown on the right.
Point *O*, the centre of the square *VWXY*, is directly below *Z*.



a) Find the length *VX*.

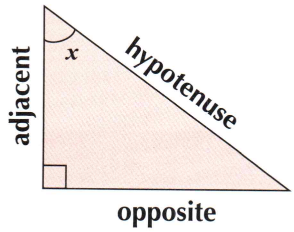
b) Hence find the area of the square *VWXY*.

25.3 Trigonometry — Sin, Cos and Tan

Trigonometry allows you to find missing sides and angles in triangles. For right-angled triangles, you'll need to know the sine (sin), cosine (cos) and tangent (tan) formulas.

The Three Formulas

Learning Objective — Spec Ref G20:
Use trigonometry to find missing lengths and angles in right-angled triangles.



Trigonometry can be used to find lengths or angles in **right-angled triangles**. For a given angle x (as shown on the diagram on the right):

- The side opposite the right angle is the **hypotenuse**.
- The side opposite the given angle is the **opposite**.
- The side between the given angle and the right-angle is the **adjacent**.

The three sides of a right-angled triangle are linked by the following formulas:

$$\sin x = \frac{\text{opp}}{\text{hyp}}, \quad \cos x = \frac{\text{adj}}{\text{hyp}}, \quad \tan x = \frac{\text{opp}}{\text{adj}}$$

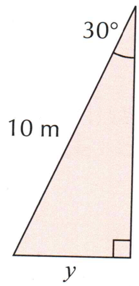
Tip: Remember 'SOH CAH TOA' to help you decide which formula you need to use.

If you're given **one angle** and **one side**, you can use trigonometry to find an **unknown side length**. Look at the side you've been **given** and the side you **want to find** to decide which formula to use — e.g. if you had the **hypotenuse** and wanted to find the **adjacent**, you'd use the **cos** formula as it contains these two sides. **Substitute** the values you know into the appropriate formula and **rearrange** it to find the length you want.

Example 1

Find the length of side y.

1. You're given the hypotenuse and asked to find the opposite, so use the formula for $\sin x$ and substitute in the values you know.
   $\sin x = \frac{\text{opp}}{\text{hyp}}$
   $\sin 30° = \frac{y}{10}$
2. Rearrange the formula to find y.
   $y = 10 \sin 30°$
3. Input '10 sin 30' into your calculator and press '=' to find the value of y.
   $\mathbf{y = 5}$ **m**

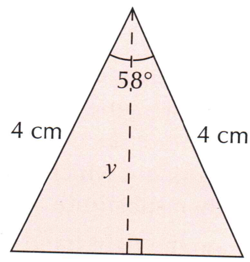


Tip: You might find it helpful to start by labelling the sides O (opposite), A (adjacent) and H (hypotenuse).

Example 2

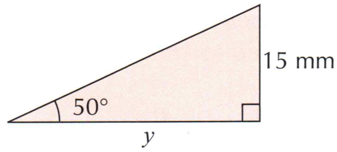
Find the height of the isosceles triangle shown. Give your answer correct to 3 significant figures.

1. Create a right-angled triangle by splitting the triangle in half. Divide the angle by 2 to find the angle in your right-angled triangle.
   $58 \div 2 = 29°$
2. Now you have the hypotenuse, and you need to find the adjacent, so use the formula for $\cos x$.
   $\cos x = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 29° = \frac{y}{4}$
3. Rearrange the formula to find y.
   $y = 4 \cos 29°$
4. Use your calculator to find the value of y.
   $y = 3.498... =$ **3.50 cm** (3 s.f.)



Example 3

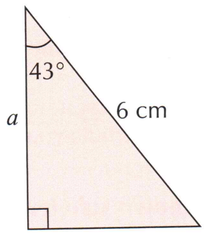
Find the length of side y. Give your answer correct to 3 significant figures.

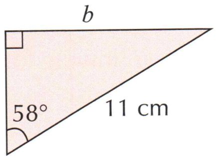


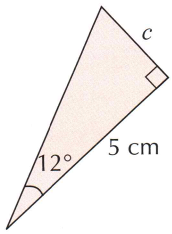
1. You're given the opposite and asked to find the adjacent, so use the formula for tan x.
   $\tan x = \frac{\text{opp}}{\text{adj}}$
2. Rearrange the formula to find y — this time, y is on the bottom of the fraction, so the rearrangement is slightly different.
   $\tan 50° = \frac{15}{y}$
   $y \times \tan 50° = 15 \Rightarrow y = 15 \div \tan 50°$
3. Use your calculator to find the value of y.
   $y = 12.586... =$ **12.6 mm** (3 s.f.)

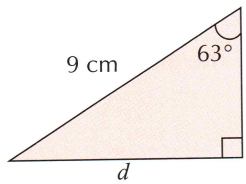
Exercise 1

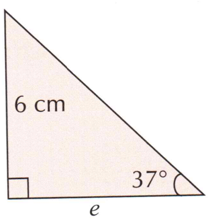
Q1 Find the lengths of the sides marked with letters below. Give your answers to 3 significant figures.

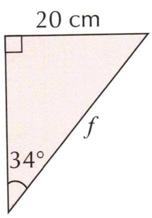
a)


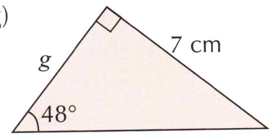
b)


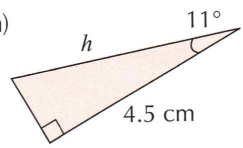
c)


d)


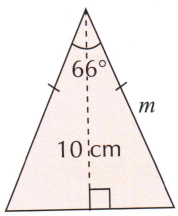
e)


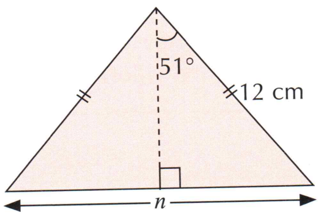
f)


g)


h)


Q2 Find the lengths marked with letters in the triangles below.
Give your answers to 3 significant figures.

a)


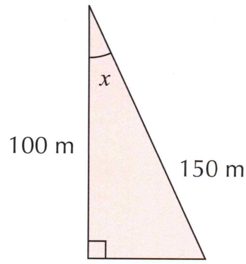
b)


You can also use the trig formulas to find an **angle** if you know two side lengths. You have to use the **inverse functions** of sin, cos and tan (written $\mathbf{sin^{-1}}$, $\mathbf{cos^{-1}}$ and $\mathbf{tan^{-1}}$), which return an **angle**. To find an angle, work out which formula you need from the sides you're given as before, then **substitute** in the known values — this will give you a **fraction** on the right-hand side, e.g. $\sin x = \frac{1}{2}$. Take the **inverse trig function** of the fraction to get the angle — so here you'd do $x = \sin^{-1}\left(\frac{1}{2}\right) = 30°$.

Tip: Inverse trig functions are usually found on a calculator by pressing 'shift' or '2nd' before pressing sin, cos or tan.

Example 4

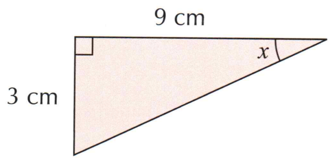
Find the size of angle x. Give your answer correct to 1 decimal place.



1. You're given the adjacent and the hypotenuse, so use the formula for cos x. $\cos x = \frac{\text{adj}}{\text{hyp}} = \frac{100}{150}$
2. Take the inverse of cos to find the angle. $x = \cos^{-1}\left(\frac{100}{150}\right)$
3. Input '$\cos^{-1}(100 \div 150)$' into your calculator and press '=' to find the value of x. $x = 48.189... = $ **48.2°** (1 d.p.)

Example 5

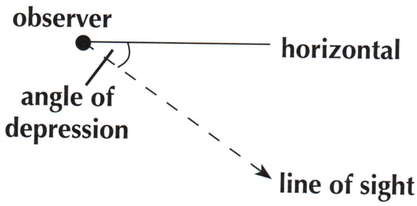
Find the size of angle x. Give your answer correct to 1 decimal place.

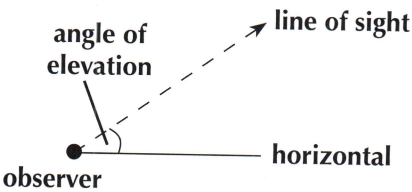


1. You're given the opposite and the adjacent, so use the formula for tan x. $\tan x = \frac{\text{opp}}{\text{adj}} = \frac{3}{9}$
2. Take the inverse of tan to find the angle. $x = \tan^{-1}\left(\frac{3}{9}\right)$
3. Use your calculator to find the value of x. $x = 18.434... = $ **18.4°** (1 d.p.)

Trigonometry can be used to work out angles of **depression** and **elevation**.

The **angle of depression** is the angle between a **horizontal line** and the **line of sight** of an observer at the same level **looking down** — e.g. you could measure the angle of depression of someone looking down from a window.





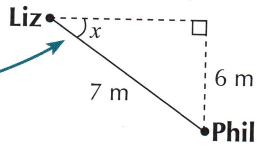
The **angle of elevation** is the angle between a **horizontal line** and the **line of sight** of an observer at the same level **looking up** — e.g. the angle made looking up at a hovering helicopter.

For problems like this, use the information given to **draw** a right-angled triangle and use the formulas in the **same way** as usual. Remember to relate your answer to the **original context** of the problem.

Example 6

Liz holds one end of a 7 m paper chain out of her window. Phil stands in the garden below holding the other end to its full extent. Phil's end of the paper chain is 6 m vertically below Liz's end. Find the size of the angle of depression from Liz to Phil. Give your answer to 1 decimal place.

1. Use the information to draw a right-angled triangle — the angle of depression is the angle below the horizontal.

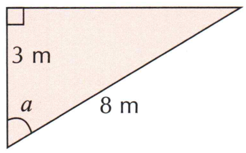


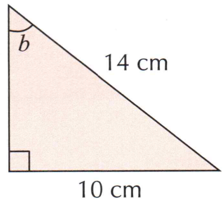
2. You're given the hypotenuse and the opposite, so use the formula for sin x. $\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{6}{7}$
3. Take the inverse of sin to find the angle. $x = \sin^{-1}\left(\frac{6}{7}\right)$
4. Use your calculator to find the value of x. $x = 58.997... = $ **59.0°** (1 d.p.)

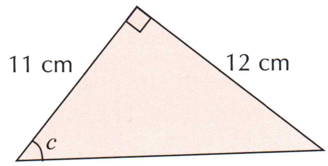
Tip: The angle of elevation from Phil to Liz is also 59.0° because they are allied angles (see p.251).

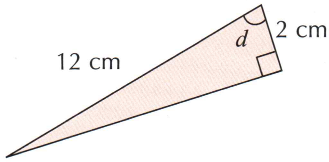
Exercise 2

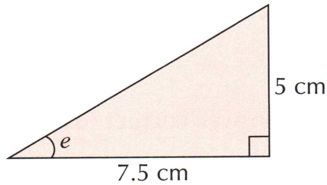
Q1 Find the size of the angles marked with letters. Give your answers to 1 decimal place.

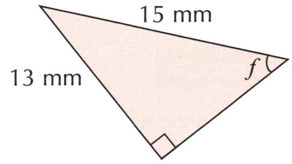
a) 

b) 

c) 

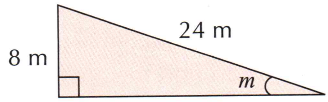
d) 

e) 

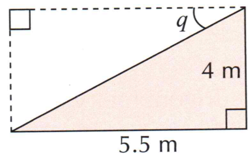
f) 

Q2 Melissa is building slides for an adventure playground.

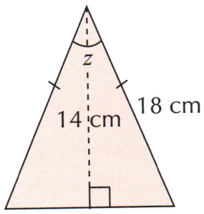
a) The first slide she builds has an 8 m high vertical ladder and a slide of length 24 m. Find m, the slide's angle of elevation, to 1 d.p.

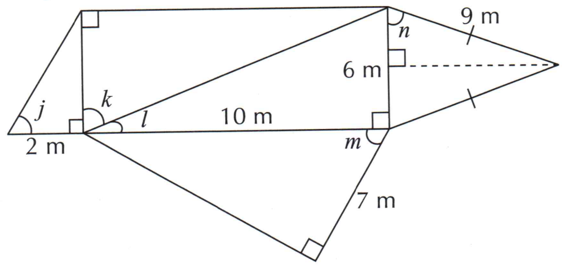


b) A second slide has a 4 m vertical ladder, and the base of the slide reaches the ground 5.5 m from the base of the ladder as shown. Find q, the angle of depression at the top of the second slide, to 1 d.p.



Q3 Find the angles marked with letters in the following diagrams. Give your answers correct to 1 d.p.

a) 

b) 

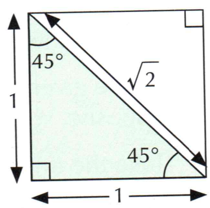
Common Trig Values

Learning Objective — Spec Ref G21:
Know and be able to find trig values for common angles.

Prior Knowledge Check: Be able to simplify surds and rationalise denominators — see p.99-103.

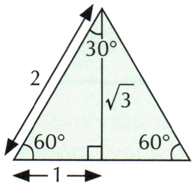
The sin, cos and tan of some angles have **exact values**. You need to either **remember** the values or know how to work them out **without a calculator**. The first of these angles is **45°**.

Start by drawing a **square** with sides of length **1**. Then **split** the square down its **diagonal** to create two **right-angled triangles**. By Pythagoras' theorem, the length of the hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Since the triangle was formed by **bisecting** 90° angles, the two acute angles in the triangle are 90° ÷ 2 = **45°**. Then use the trig formulas to find the sin, cos and tan of 45° — e.g. $\sin 45° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}}$.



$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 45° = 1$$

To find sin, cos and tan of **30°** and **60°**, start with an **equilateral triangle** with sides of length **2**. The interior angles are **60°** (see p.252).
Split the triangle in half using a **perpendicular** line from one vertex to the opposite side, leaving two **right-angled triangles** with acute angles of **60°** and 60° ÷ 2 = **30°**. By Pythagoras' theorem, the perpendicular height is $\sqrt{2^2 - 1^2} = \sqrt{3}$. Then use the trig formulas to get the following results.



$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \sqrt{3}$

$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}}$

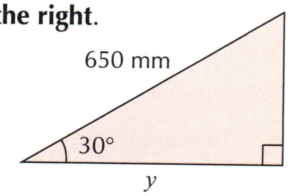
You **can't** use triangles for **0°** or **90°** but you do need to know their trig values:

sin 0° = 0	**cos 0° = 1**	**tan 0° = 0**
sin 90° = 1	**cos 90° = 0**	**tan 90° = undefined**

Tip: The values for 0° and 90° come from the sine, cosine and tangent graphs — see p.204.

Example 7

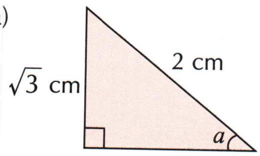
Without using a calculator, find the exact length of side *y* on the triangle on the right.

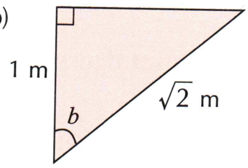


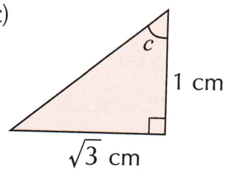
1. You're given the hypotenuse and want to find the adjacent, so use the formula for cos *x*. $\cos x = \frac{\text{adj}}{\text{hyp}}$
2. Substitute in the values you know and rearrange the formula to make *y* the subject. $\cos 30° = \frac{y}{650}$, $y = 650 \times \cos 30°$
3. Replace cos 30° with its exact trig value and work through the formula to find *y*. $= 650 \times \frac{\sqrt{3}}{2} = \mathbf{325\sqrt{3}}$ **mm**

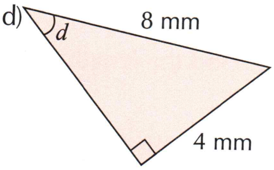
Exercise 3

Q1 Find the size of the angles marked with letters.

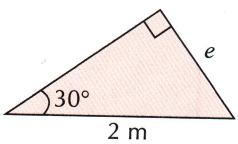
a) 

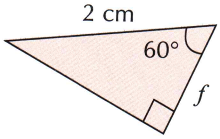
b) 

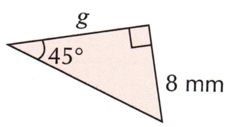
c) 

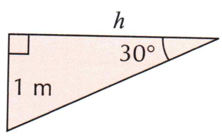
d) 

Q2 Find the exact length of the sides marked with letters.

a) 

b) 

c) 

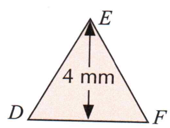
d) 

Q3 Show that:

a) $\tan 45° + \sin 60° = \frac{2+\sqrt{3}}{2}$ b) $\sin 45° + \cos 45° = \sqrt{2}$ c) $\tan 30° + \tan 60° = \frac{4\sqrt{3}}{3}$

Q4 Triangle *ABC* is isosceles. $AC = 7\sqrt{2}$ cm and angle $ABC = 90°$. What is the exact length of side *AB*?

Q5 Triangle *DEF* is an equilateral triangle with a perpendicular height of 4 mm. What is the exact side length of the triangle? Give your answer in its simplest form.

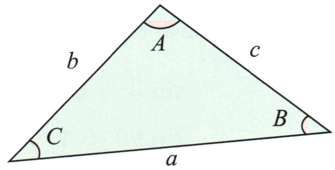


25.4 The Sine and Cosine Rules

You can still use trigonometry to find unknown sides and angles in triangles that aren't right-angled — you just need a few more formulas. And there's also a formula to learn for the area of a triangle too.

The Sine Rule

Learning Objective — Spec Ref G22:
Use the sine rule to find sides and angles in any triangle.



To use trigonometry in triangles that **aren't** right-angled, you must first **label** the sides and angles properly, like in the diagram on the right — side a must always be **opposite** angle A etc. You use **lower case letters** for the **sides** and **upper case letters** for the **angles**.

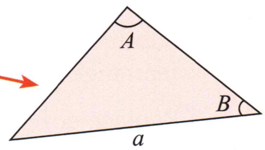
You can then use the **sine rule**, which is: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Tip: You will only need to consider 2 at a time — e.g. $\frac{a}{\sin A} = \frac{b}{\sin B}$.

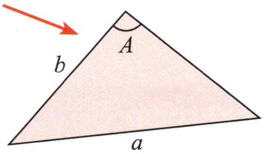
You can also flip the fractions to write it in this form: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

You can use the sine rule if:

- **Two angles and one side** are known (like in the triangle on the right). If you know two angles, you can always find the third by subtracting them from 180°. Then you can use the sine rule to find **either** of the unknown sides.



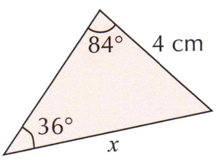
- **One angle, the opposite side and one other side** are known (like in the triangle below right). Here you can find the angle **opposite** the other known side.



To use the sine rule, **substitute** in the values you know and **rearrange** to find the side or angle you want. You'll probably have to use the **inverse sine function** (see p.324) if you're looking for an angle.

Example 1

Find the length of side x. Give your answer to 3 significant figures.

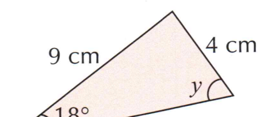


1. Substitute the values from the diagram into the sine rule. $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{4}{\sin 36°} = \frac{x}{\sin 84°}$
2. Rearrange to make x the subject. $x = \frac{4 \sin 84°}{\sin 36°}$
3. Use your calculator to find the value of x. $x =$ **6.77 cm** (3 s.f.)

Tip: Label the triangle using a, A, b and B if you need to.

Example 2

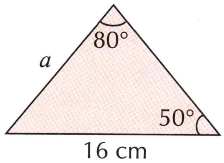
Find the size of the acute angle y below. Give your answer to 1 decimal place.

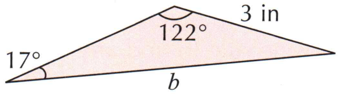


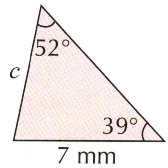
1. Substitute in the values from the diagram. The second version of the formula works best here, but the first would also work. $\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin y}{9} = \frac{\sin 18°}{4}$
2. Rearrange the equation. $\sin y = \frac{9 \sin 18°}{4} \Rightarrow y = \sin^{-1}\left(\frac{9 \sin 18°}{4}\right)$
3. Use the inverse function to find y. $y =$ **44.1°** (1 d.p.)

Exercise 1

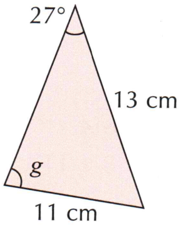
Q1 Find the lengths of the sides marked with letters below. Give your answers to 3 significant figures.

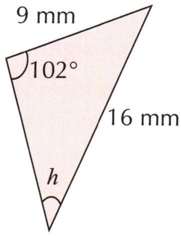
a)


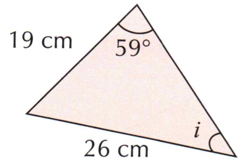
b)


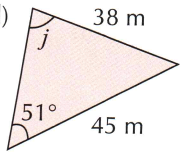
c)


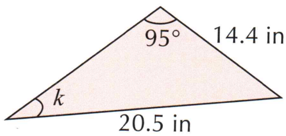
Q2 For each triangle below, find the acute angle marked with a letter. Give your answers to 1 d.p.

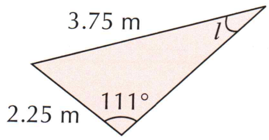
a)


b)


c)


d)


e)


f)


Q3 The triangle XYZ is such that angle $YXZ = 55°$, angle $XYZ = 40°$ and length $YZ = 83$ m. Find:

a) length XZ, to 3 s.f.

b) length XY, to 3 s.f.

Q4 A triangular piece of metal PQR is such that angle $RPQ = 61°$, length $QR = 13.1$ mm and length $PQ = 7.2$ mm. Find the size of the acute angle PQR, correct to 1 decimal place.

Q5 Point B is 13 km north of point A. Point C lies 19 km from point B, on a bearing of 052° from A. Find the bearing of C from B, to the nearest degree. PROBLEM SOLVING

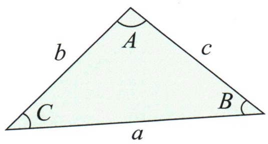
The Cosine Rule

Learning Objective — Spec Ref G22:
Use the cosine rule to find sides and angles in any triangle.

The **cosine rule** can also be used to find unknown angles and sides.

For any triangle labelled as shown in the diagram, the cosine rule is:

$$a^2 = b^2 + c^2 - 2bc \cos A$$



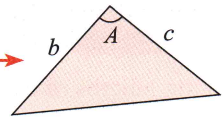
Tip: Whether you use the sine or cosine rule depends on the sides and angles you're given.

This can be rearranged to give $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

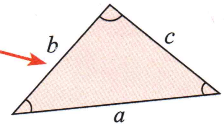
This version is useful when you're trying to find an **angle**.

You can use the cosine rule if:

- **Two sides and the angle between them** are known (like in the triangle on the right). Here, you can find the **third side**.



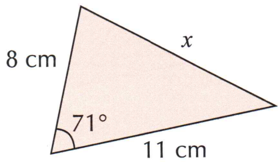
- **All three sides** are known (like in the triangle below right). In this case, you can use the cosine rule to find **any angle**.



To use the cosine rule, **substitute** in the values you know and **rearrange** to find the side or angle you want. You'll probably have to use the **inverse cosine function** if you're looking for an angle (see p.324).

Example 3

Find the length of side x. Give your answer to 3 significant figures.



1. Substitute the values from the diagram into the cosine rule — x is side a since it's opposite the known angle (A).

$a^2 = b^2 + c^2 - 2bc \cos A$

$x^2 = 8^2 + 11^2 - (2 \times 8 \times 11)\cos 71°$

2. Work it through to find x^2.

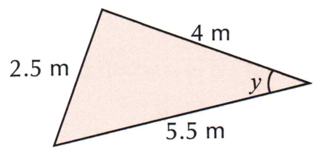
$x^2 = 127.70...$

3. Take the square root to find the value of x.

$x = \sqrt{127.70...}$ = **11.3 cm** (3 s.f.)

Example 4

Find the size of angle y in the triangle on the right. Give your answer to 1 decimal place.



1. Use the second version of the cosine rule to find the angle.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

2. Substitute the values from the diagram into the formula — make the side opposite the angle a.

$$\cos y = \frac{5.5^2 + 4^2 - 2.5^2}{2 \times 5.5 \times 4}$$

3. Use the inverse cos function to find the value of y.

$$y = \cos^{-1}\left(\frac{5.5^2 + 4^2 - 2.5^2}{2 \times 5.5 \times 4}\right)$$

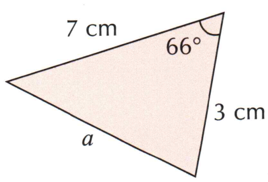
= **24.6°** (1 d.p.)

Tip: Be careful when substituting the values into the formula — the 2.5 m side is side a (and y is angle A).

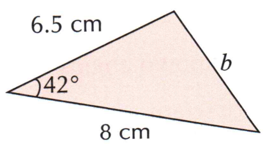
Exercise 2

Q1 Use the cosine rule to find the lengths of the sides marked with letters. Give your answers to 3 s.f.

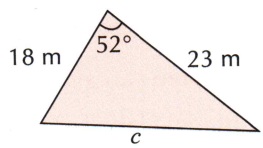
a)



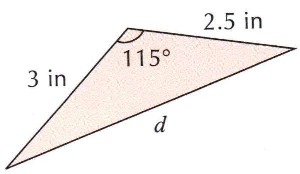
b)



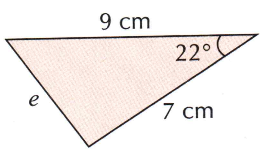
c)



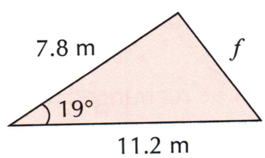
d)



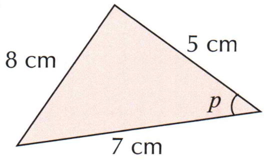
e)

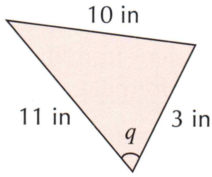


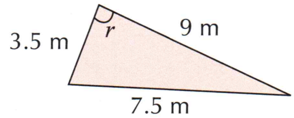
f)

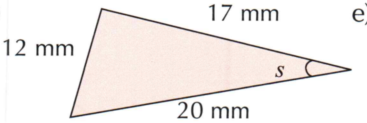


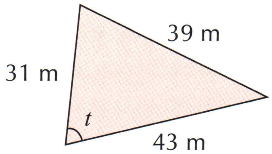
Q2 Use the cosine rule to find the sizes of the angles marked with letters.
Give your answers to 1 d.p.

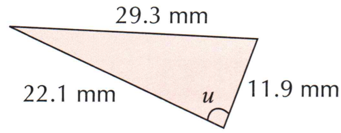
a)


b)


c)


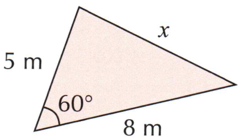
d)


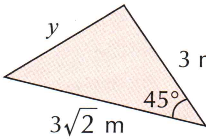
e)


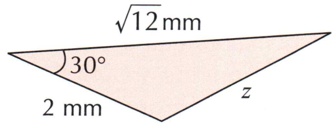
f)


Q3 A triangular sign XYZ is such that $XY = 67$ cm, $YZ = 78$ cm and $XZ = 99$ cm.
Find the size of the angle XYZ, correct to 1 d.p.

Q4 Find the exact values of the letters in each of these triangles.

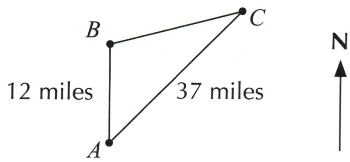
a)


b)


c)


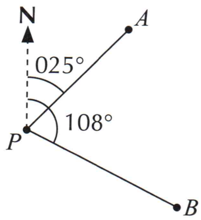
Q5 Village B is 12 miles north of village A. Village C is 37 miles north-east of village A. Find the direct distance between village B and village C, correct to 3 s.f.

PROBLEM SOLVING



Q6 Two ramblers set off walking from point P.
The first rambler walks for 2 km on a bearing of 025° to point A.
The second rambler walks for 3 km on a bearing of 108° to point B.
Find the direct distance between A and B, correct to 3 s.f.

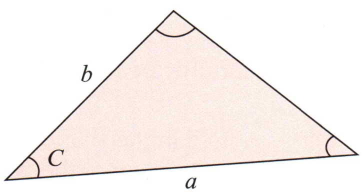
PROBLEM SOLVING



Area of a Triangle

Learning Objective — Spec Ref G23:
Find the area of a triangle using the sine function.

To find the **area** of a triangle, use the formula: **Area = $\frac{1}{2}ab\sin C$**



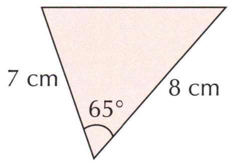
To use this formula, you need to know **two sides and the angle between them**. If you don't, you might have to use the sine and cosine rules first to find the information you need.

You can also use the formula to find the **size of an angle** or **length of a side** when you know the area — just **substitute** in the values you know and **rearrange** the formula to make the angle or side you want the subject.

Tip: If you know the base length and height of a triangle, you can use the simpler formula on p.349 to find the area.

Example 5

Find the area of this triangle. Give your answer to 3 significant figures.

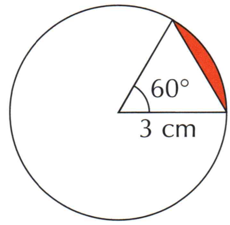


1. Substitute the values from the diagram into the formula.
2. Use your calculator to find the area. Make sure you use the correct units.

$$\text{Area} = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2} \times 7 \times 8 \times \sin 65°$$
$$= \mathbf{25.4\ cm^2}\ \text{(3 s.f.)}$$

Example 6

A segment is formed in a sector with angle 60° in a circle of radius 3 cm. Find the area of the segment. Give your answer to 3 significant figures.



1. Work out the area of the sector. Area of sector $= \frac{60}{360} \times \pi \times 3^2 = \frac{3\pi}{2}$ cm^2
2. The triangle is isosceles (two sides are radii) so you know two sides and the angle between them. Use this information to work out the area. Area of triangle $= \frac{1}{2} \times 3 \times 3 \times \sin 60°$ $= \frac{9\sqrt{3}}{4}$ cm^2
3. Subtract the area of the triangle from the area of the sector to give the area of the segment. Area of segment $= \frac{3\pi}{2} - \frac{9\sqrt{3}}{4}$ $= \mathbf{0.815\ cm^2}$ (3 s.f.)

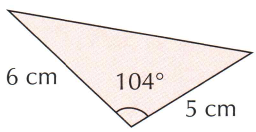
Tip: See p.354 for the formula for the area of a sector.

Exercise 3

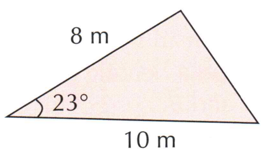
Give all your answers to the questions in this exercise to 3 significant figures.

Q1 Find the area of each of the following triangles.

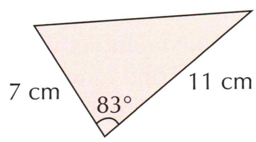
a)



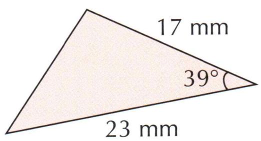
b)



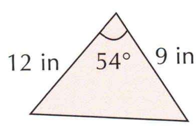
c)



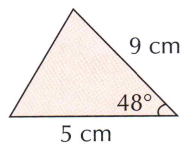
d)



e)



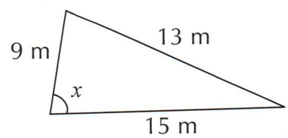
f)

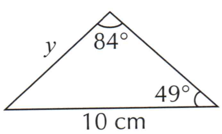


Q2 Find the area of the segment formed in a sector with angle 67° in a circle of radius 4.5 cm.

Q3 A field in the shape of an equilateral triangle has sides of length 32 m. Find the area of the field.

Q4 For the triangle on the right, use the cosine rule to calculate x and hence find the area of the triangle.





Q5 For the triangle on the left, use the sine rule to calculate y and hence find the area of the triangle.

25.5 Trigonometry in 3D

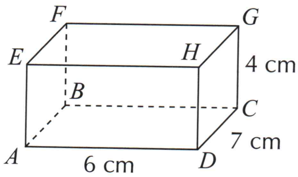
Like Pythagoras' theorem, trigonometry can also be used in 3D to find lengths and angles.

Learning Objective — Spec Ref G20/G22:
Use trigonometry to find lengths and angles in 3D shapes.

You can use **trigonometry** to find lengths and angles in **3D shapes** by creating triangles within them. You can use sin, cos and tan for **right-angled triangles** and the **sine** or **cosine rules** for other triangles. You might need to use **Pythagoras' theorem** too — see p.321. The formulas are used in the same way as for 2D shapes, but you might have to use them **multiple** times to find what you're looking for. It's often a good idea to **sketch** the triangles as you go along to keep track of what you're doing.

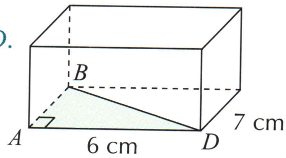
Example 1

Find the angle BDF in the cuboid shown on the right.



1. Use Pythagoras' theorem on the triangle ABD to find the length BD.

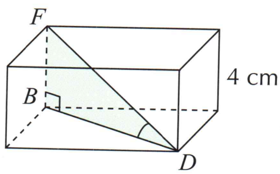
$BD^2 = AD^2 + AB^2$
$= 6^2 + 7^2 = 85$
$BD = \sqrt{85}$ cm



2. BDF forms a right-angled triangle and you know the opposite and adjacent sides to angle BDF, so use the tan formula.

$\tan BDF = \frac{\text{opp}}{\text{adj}} = \frac{FB}{BD} = \frac{4}{\sqrt{85}}$

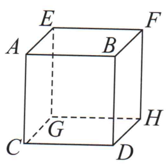
$BDF = \tan^{-1}\left(\frac{4}{\sqrt{85}}\right)$
$= \mathbf{23.5°}$ (1 d.p.)



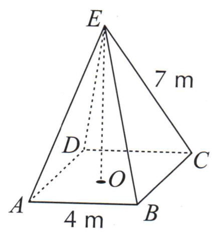
Tip: There may be alternative triangles you can use to find the angle you want.

Exercise 1

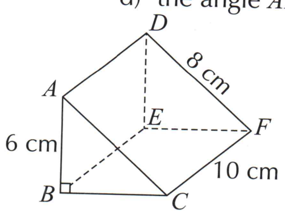
Q1 The cube shown has sides of length 3 m. Find:
a) the exact length AF
b) the exact length FC
c) the angle AHC, to 1 d.p.



Q2 $ABCDE$ is a square-based pyramid. The centre of the base is O, and E lies directly above O. Find:
a) the angle BCE, to 1 d.p
b) the angle AEB, to 1 d.p.
c) the exact vertical height EO
d) the angle AEO, to 1 d.p.



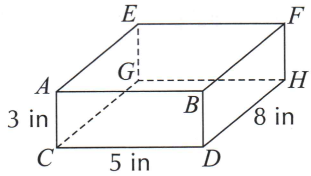
Q3 For the triangular prism shown, find:
a) the angle EDF, to 1 d.p.
b) the exact length DC
c) the angle DCE, to 1 d.p.



Q4 For the cuboid shown, find:

a) the exact length AH

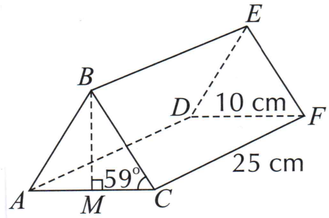
b) the angle EDG, to 1 d.p.



Q5 For the triangular prism shown, where M is the midpoint of AC, find:

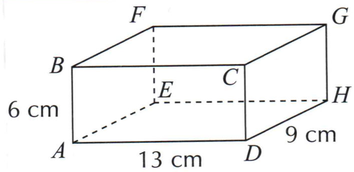
a) the perpendicular height BM, to 3 s.f.

b) the length EM, to 3 s.f.



Example 2

Find the size of angle AFH in the cuboid shown on the right. PROBLEM SOLVING

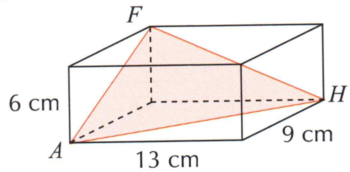


1. Use Pythagoras' theorem to find the lengths AF, AH and FH.

$AF^2 = 6^2 + 9^2 = 117 \Rightarrow AF = \sqrt{117}$

$AH^2 = 13^2 + 9^2 = 250 \Rightarrow AH = \sqrt{250}$

$FH^2 = 6^2 + 13^2 = 205 \Rightarrow FH = \sqrt{205}$



2. AFH isn't a right-angled triangle but you know all 3 sides so use the cosine rule.

$$\cos AFH = \frac{AF^2 + FH^2 - AH^2}{2 \times AF \times FH} = \frac{117 + 205 - 250}{2 \times \sqrt{117} \times \sqrt{205}}$$

3. Take the inverse of cos to find the angle.

$$AFH = \cos^{-1}\left(\frac{117 + 205 - 250}{2 \times \sqrt{117} + \sqrt{205}}\right) = \mathbf{76.6°} \text{ (1 d.p.)}$$

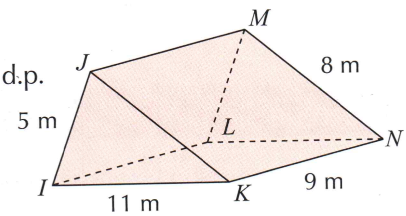
Exercise 2

Q1 The diagram shows the triangular prism $IJKLMN$.

a) Use the cosine rule to find the size of angle JIK, correct to 1 d.p.

b) Hence find the area of triangle IJK, correct to 1 d.p.

c) Hence find the volume of the triangular prism $IJKLMN$, to the nearest m³.



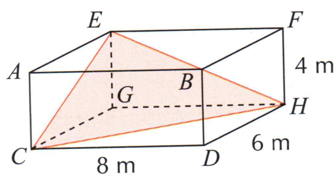
Q2 The diagram shows the triangle CEH drawn inside the cuboid $ABCDEFGH$.

a) Find the exact length CE.

b) Find the exact length CH.

c) Find the exact length EH.

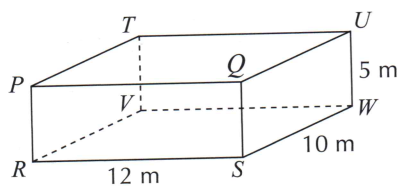
d) Hence use the cosine rule to find the size of angle ECH, correct to 1 d.p.



Q3 The diagram shows the cuboid $PQRSTUVW$.

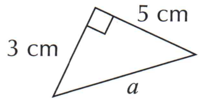
a) Use Pythagoras and the cosine rule to find the size of angle PSU, correct to 1 d.p.

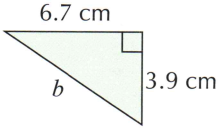
b) Hence find the area of triangle PSU, correct to 1 d.p.

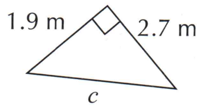


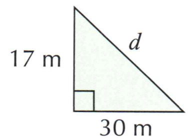
Review Exercise

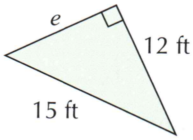
Q1 Find the value of the letter in each triangle below. Give your answers to 3 s.f. where necessary.

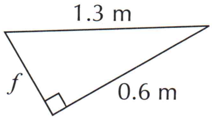
a) 

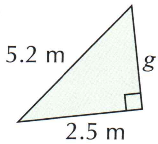
b) 

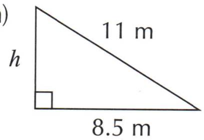
c) 

d) 

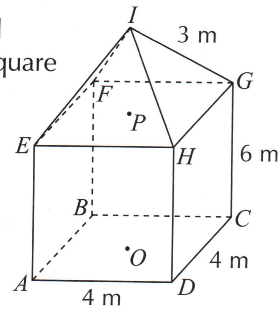
e) 

f) 

g) 

h) 

Q2 The diagram on the right shows a shape made up of the square-based pyramid *EFGHI* and the cuboid *ABCDEFGH*. *P* is the centre of the square *EFGH* and *O* is the centre of *ABCD*. *P* lies on the vertical line *OI*.

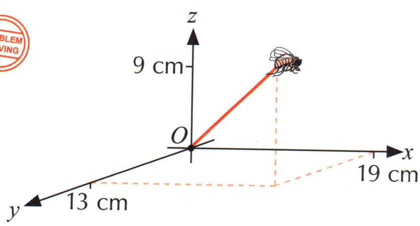


a) (i) Find the exact length *PH*.
(ii) Find the length *PI*.
(iii) Hence find the length *OI*.

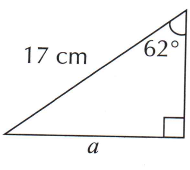
b) (i) Find the exact length *OA*.
(ii) Find the length *AI* to 3 s.f.

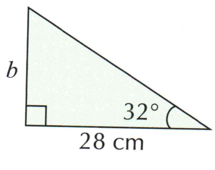
Q3 A fly is tied to one end of a piece of string. The other end of the string is attached to point *O* on a flat, horizontal surface. The string becomes taut when the fly is 19 cm from *O* in the *x*-direction, 13 cm from *O* in the *y*-direction, and 9 cm from *O* in the *z*-direction. Find the length of the piece of string to 3 s.f.

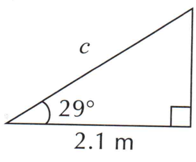
PROBLEM SOLVING

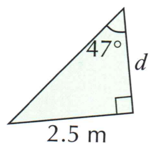


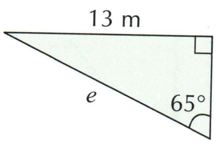
Q4 Find the value of the letter in each triangle below. Give your answers to 3 s.f.

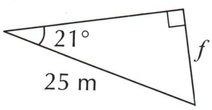
a) 

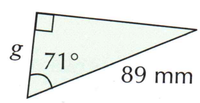
b) 

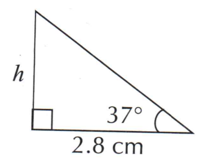
c) 

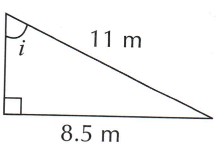
d) 

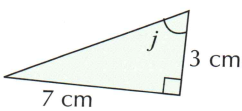
e) 

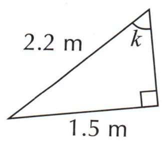
f) 

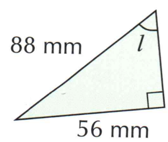
g) 

h) 

i) 

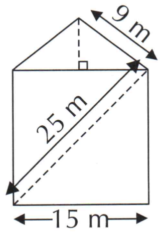
j) 

k) 

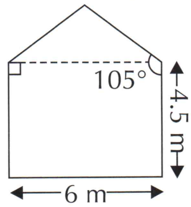
l) 

Q5 The diagrams below show the cross-sections of two houses. Each has a vertical line of symmetry. Find the vertical height of each house.

a)



b)

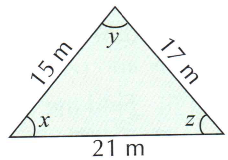


Q6 Points P, Q and R are plotted on a grid of 1 cm squares. P has coordinates (1, 3), Q has coordinates (5, 4) and R has coordinates (7, 1).

a) (i) Find the exact distance PQ. (ii) Find the exact distance PR.

b) Find the bearing of Q from P to the nearest whole number.

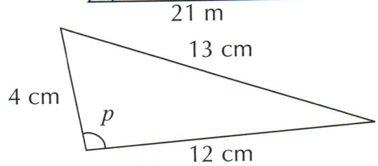
Q7 Using the sine and/or cosine rules, find the angles x, y and z in the triangle shown on the right, correct to 1 d.p.



Q8 Leo has a triangular piece of fabric with dimensions as shown.

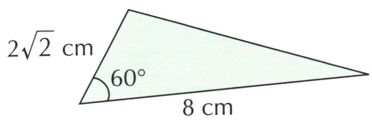
a) Find the size of angle p, to 3 s.f.

b) Hence find the area of the fabric, to 3 s.f.

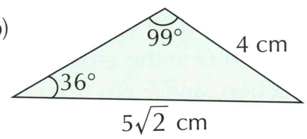


Q9 Find the exact areas of the following triangles.

a)



b)



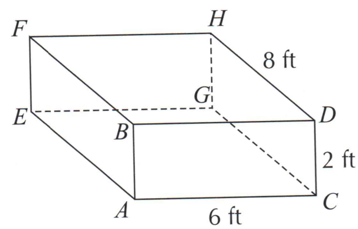
Q10 A wooden crate in the shape of a cuboid is shown on the right.

a) (i) Find the exact length ED.

(ii) Find the angle FDH, to 1 d.p.

(iii) Find the angle CHD, to 1 d.p.

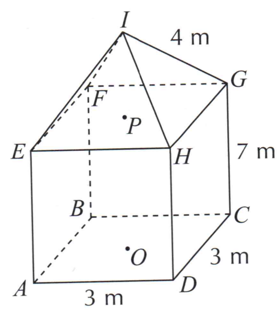
b) Will a 10 ft metal pole fit in the crate?



Q11 The diagram on the right shows a shape made up of the square-based pyramid $EFGHI$ and the cuboid $ABCDEFGH$. P is the centre of the square $EFGH$ and O is the centre of $ABCD$. P lies on the vertical line OI.

a) Find the angle OAI, to 1 d.p.

b) Find the angle OAP, to 1 d.p.

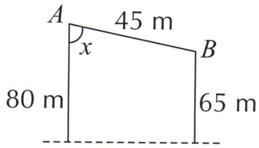


Exam-Style Questions

Q1 A ladder leans against the side of a vertical tower. It reaches a window 20 m above the ground. The base of the ladder is placed 8 m away from the bottom of the tower. What is the angle of elevation? Give your answer correct to 1 decimal place.

[2 marks]

Q2 A taut zip line of length 45 m goes between platforms A and B. Platform A is 80 m above the horizontal ground, and platform B is 65 m above the ground, as shown below.



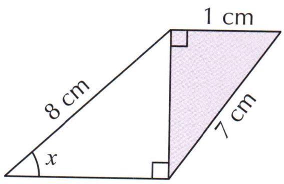
Find x, the angle the wire forms with the vertical at platform A. Give your answer correct to 1 decimal place.

[2 marks]

Q3 The points P and R have coordinates $P(1, 3)$ and $R(7, 8)$. Find the distance between P and R, correct to 3 significant figures.

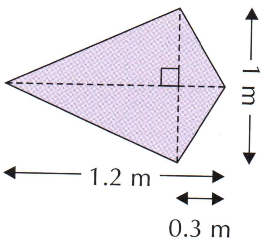
[2 marks]

Q4 The shape shown below is made up of two right-angled triangles. Find the size of angle x.



[3 marks]

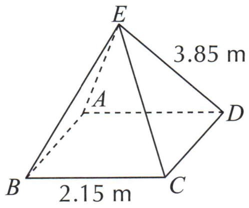
Q5 Rahim wants to put a gold ribbon around the edges of the kite shown below.



Ribbon is sold in lengths of 10 cm. What length of ribbon should he buy?

[3 marks]

Q6 The pyramid $ABCDE$ has a square base of side length 2.15 m. Point E is directly above the centre of the square $ABCD$. The edge ED is 3.85 m long. Work out the angle between ED and the base, giving your answer to 1 decimal place.

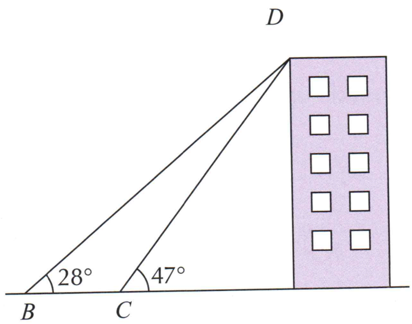


[3 marks]

Q7 An aeroplane takes off from a flat runway and flies for 675 m in a straight line, after which time it is 250 m vertically above the horizontal ground. Work out the angle of elevation of the plane, to 3 significant figures.

[2 marks]

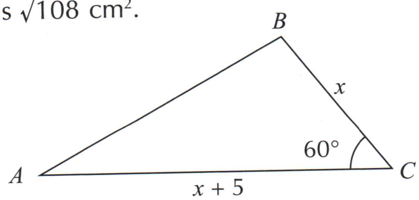
Q8 Bianca (B) and Caleb (C) are standing 25 m apart in a straight line directly facing a vertical tower block, as shown in the diagram. The ground is horizontal. From where Bianca is standing, the angle of elevation from the ground to the top of the tower block is 28°. From where Caleb is standing, the angle of elevation from the ground to the top of the tower block is 47°.



Work out the height of the tower block, giving your answer to three significant figures.

[4 marks]

Q9 The diagram shows a triangle ABC where angle $ACB = 60°$, $AC = x + 5$ cm and $BC = x$ cm. The area of the triangle is $\sqrt{108}$ cm^2.



Find the length of AB.

[6 marks]