# Section 21 Circle Geometry

## 21.1 Circle Theorems 1

*Circle geometry is all about using the properties of circles and lines to find missing angles. Before jumping into the first circle theorems, you'll need to know what all the parts of a circle are called.*

**Learning Objectives — Spec Ref G9/G10:**
- Know the different parts of a circle.
- Be able to apply circle theorems involving radii, tangents, semicircles and chords.

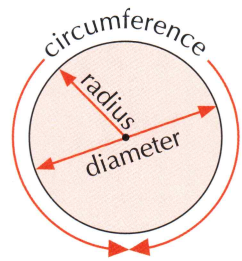
### Parts of a Circle

**Circumference**: the distance around the outside of a circle.

**Radius**: a line from the centre of a circle to the edge.
The circle's centre is the same distance from all points on the edge.

**Diameter**: a line from one side of a circle to the other through the centre.

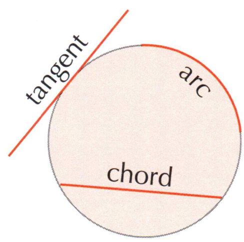
The diameter is **twice** the radius: $d = 2r$




**Tangent**: a straight line that just touches the circle.

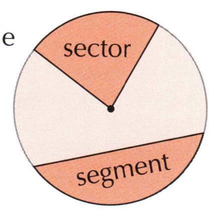
**Arc**: a part of the circumference.

**Chord**: a line between two points on the edge of the circle.




**Sector**: an area of a circle like a "slice of pie".

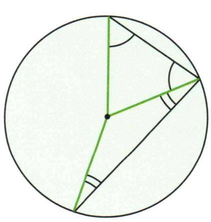
**Segment**: an area of a circle between an arc and a chord.




### Circle Theorems

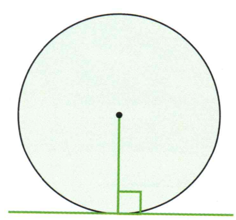
There are **nine circle theorems** (or **rules**) to learn in total — here are the first two:

**Rule 1**: A triangle formed by **two radii** is **isosceles**.



This rule is quite straightforward — all radii are the **same length**, so if two sides of a triangle are radii then it's **isosceles**.

**Rule 2**: A **tangent** to a circle makes a **right angle with the radius** at that point.



**Tip:** Rule 2 is used on page 224 to help you find the equation of the tangent to a circle graph at a particular point.

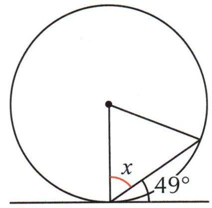
This rule is a bit trickier to get your head around. Draw some **circles**, **tangents** and **radii**, then measure the angle between the tangent and radius to convince yourself that it's true.

## Example 1

**Find the missing angle $x$ in the diagram on the right.**

You have a tangent and a radius so you'll need to use rule 2.

Tangent and radius make an angle of 90°, so $x = 90° - 49° = \mathbf{41°}$



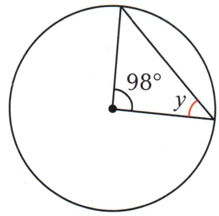

**Tip:** In circle geometry, it's a good idea to give reasons with your calculations so they're easier to follow.

## Example 2

**Find the missing angle $y$ in the diagram on the right.**

Two sides of the triangle are radii, so use rule 1.

The triangle is isosceles, so both angles at the edge of the circle are the same size. So $y = (180° - 98°) \div 2 = \mathbf{41°}$.




# Exercise 1

Q1 Find the value of each letter in the diagrams below.

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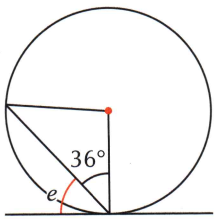


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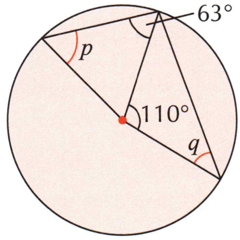


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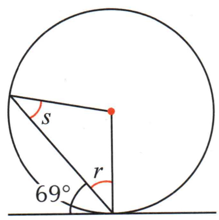


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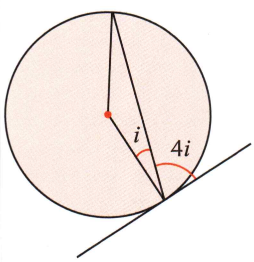


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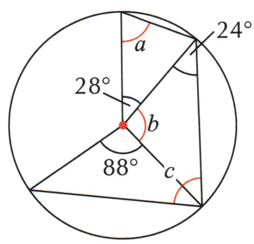
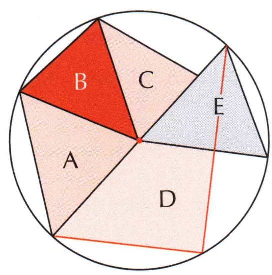
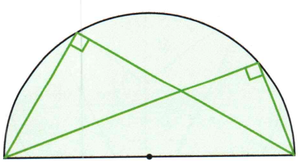


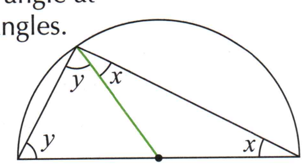

Q2 In the diagram on the right, which of triangles A-E can you be certain are isosceles? Give a reason for your answer.




**Rule 3:** The **angle in a semicircle** is a **right angle**.

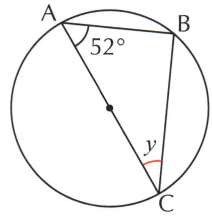


To prove this rule, draw a **radius** going to the angle at the circumference to create two **isosceles** triangles.
The angle at the circumference is $x + y$.
Angles in a triangle add up to 180°,
so $x + y + (x + y) = 180°$
$\Rightarrow\ 2x + 2y = 180°$
$\Rightarrow\ \mathbf{x + y = 90°}$



## Example 3

**Find the missing angle $y$ in the diagram on the right.**




1. You need to find the angle in a semicircle so use rule 3.

   By the 'angles in a semicircle' rule, ∠ABC = 90°.

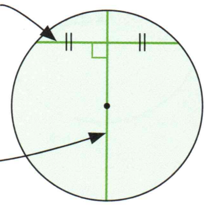
2. You know two angles in a triangle so you can find the third.

   Angles in a triangle add up to 180°, so $y = 180° - 90° - 52° = $ **38°**.

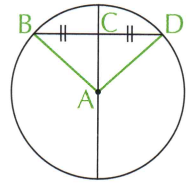
**Rule 4**: A **diameter bisects** a **chord** at right angles.

Bisecting a chord means splitting it in **exactly in half**.

Any chord which is a **perpendicular bisector** of another chord must be a **diameter**.

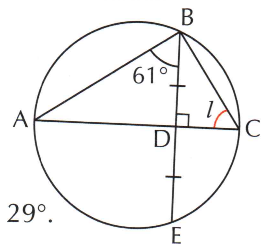


To prove this rule, start with a **diameter** that **bisects a chord**. Draw **two radii** going to either end of the chord. Look at triangles ABC and ADC in the diagram — AC is a common side, AB = AD (both radii) and BC = CD (the chord is bisected). So the triangles are **congruent** (see p.394). ∠ACB and ∠ACD must be **equal** and they lie on a **straight line**, so ∠ACB = ∠ACD = 180° ÷ 2 = **90°**.




## Example 4

**On the diagram on the right, BD = DE. Find the missing angle $l$.**




1. You have a perpendicular bisector of a chord so use rule 4.

   AC is a perpendicular bisector of BE, so AC is a diameter.

2. As AC is a diameter, you can use rule 3.

   By the 'angles in a semicircle' rule, ∠ABC = 90°, so ∠DBC = 90° – 61° = 29°.

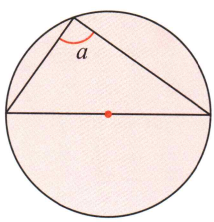
3. You know two angles in triangle BDC, so you can work out angle $l$.

   $l = 180° - 90° - 29° = $ **61°**.

## Exercise 2

Q1 Find the value of each letter in the diagrams below.

a)

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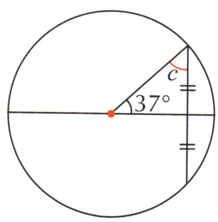


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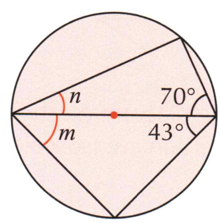


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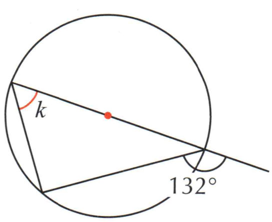


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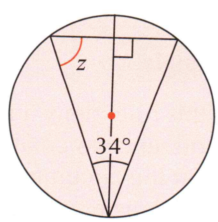
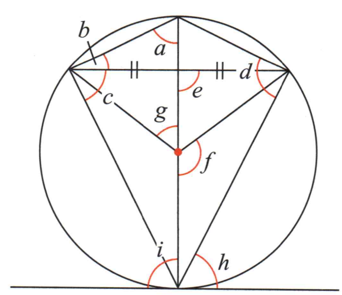


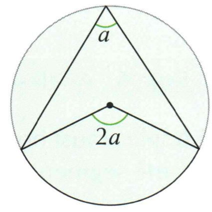




Q2 In the diagram on the right, which of angles $a$-$i$ must be right angles? Explain your answer.

# 21.2 Circle Theorems 2

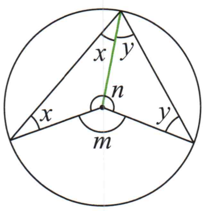
*Circle geometry is all about learning the rules and then trying to spot which one to apply in different situations. Make sure you know the rules on the previous pages before having a go at these ones.*

**Learning Objectives — Spec Ref G10:**
- Be able to find angles at the centre or circumference of a circle.
- Be able to find angles in the same segment.
- Be able to find angles in a cyclic quadrilateral.




__Rule 5__: The angle subtended **at the centre** of a circle is **double** the angle subtended **at the circumference** by the same arc.

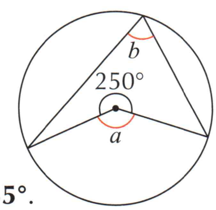
An angle **subtended by an arc** is the angle made where two lines from the ends of the arc meet. The subtended angle is **inside** the shape formed by the arc and the lines.




To prove this rule, draw a **radius** going to the angle at the circumference to create **two isosceles triangles** (see rule 1). The angle at the circumference of the circle is $\boldsymbol{x + y}$.
Angles in a quadrilateral add up to 360°,
so $n = 360° - x - y - (x + y) = 360° - 2(x + y)$.
Angles around a point add up to 360°,
so angle $m = 360° - n = 360° - (360° - 2(x + y)) = \mathbf{2(x + y)}$.
So angle $m$ is **double** the angle at the circumference.

**Example 1**

**Find the missing angles $a$ and $b$ in the diagram on the right.**




1. Angle $a$ and the 250° angle form the angles around a point.
   Angles at a point add up to 360°, so $a = 360° - 250° = \mathbf{110°}$.
2. Now use rule 5 to find angle $b$.
   The angle at the centre is double the angle at the edge, so $b = \frac{a}{2} = \frac{110°}{2} = \mathbf{55°}$.

## Exercise 1

Q1 Find the value of each letter in the diagrams below.

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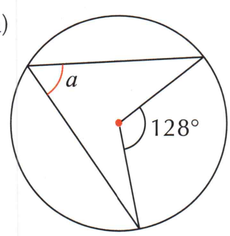


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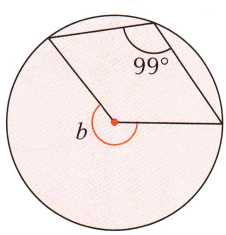
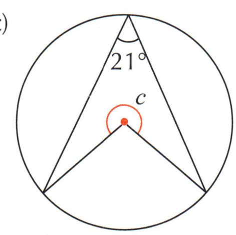



c)
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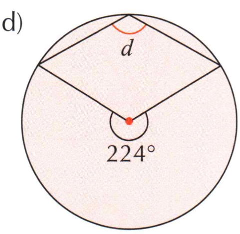


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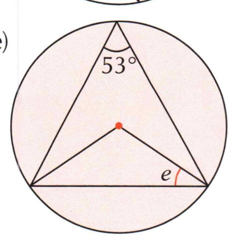


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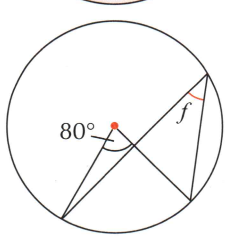


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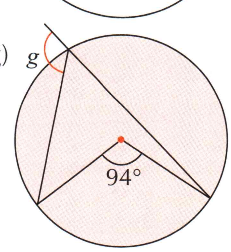


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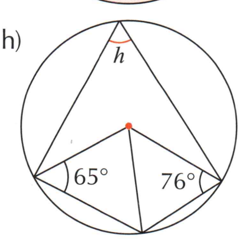

**Q2** Find the value of each letter in the diagrams below.

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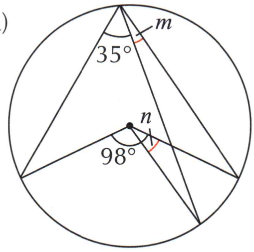


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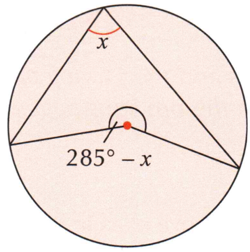


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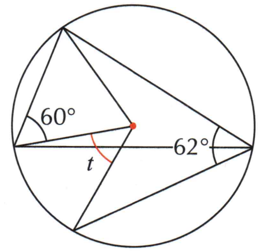
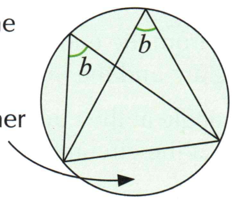
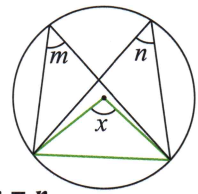



**Rule 6**: Angles subtended by an arc in the **same segment** are **equal**.

Angles must be in the **same segment** — if you drew an angle subtended in the other segment it **wouldn't** be equal to $b$.



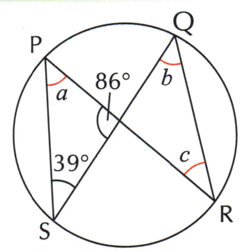

To prove this rule you can just use rule 5. Let $m$ and $n$ be angles subtended by an arc in the **same segment**. Draw angle $x$ subtended at the **centre** of the circle from the **same arc** as $m$ and $n$.
Now using rule 5: $m = \frac{x}{2}$ and $n = \frac{x}{2}$ so $\boldsymbol{m = n}$.




## Example 2

**Find the missing angles $a$, $b$ and $c$ in the diagram on the right.**

1. Angles in a triangle add up to 180°. $a = 180° - 86° - 39° = \mathbf{55°}$
2. Look for angles in the same segment.
   $a$ and $b$ are in the same segment.
   The 39° angle and $c$ are in the same segment.
   Angles in the same segment are equal, so $b = a = \mathbf{55°}$ and $c = \angle PSQ = \mathbf{39°}$.




## Exercise 2

Q1 Find the value of each letter in the diagrams below.

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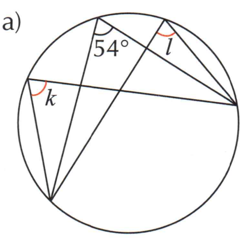


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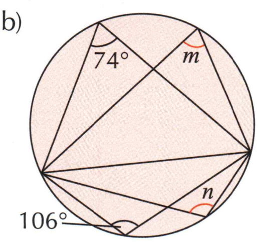


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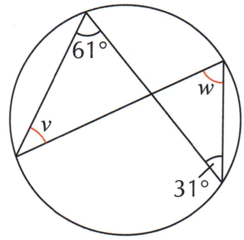


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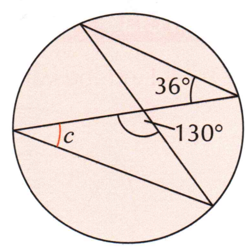


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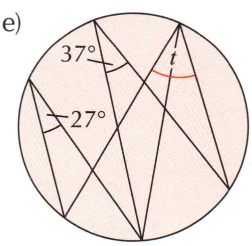


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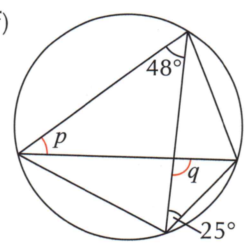


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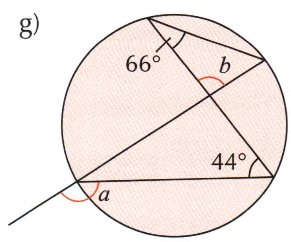


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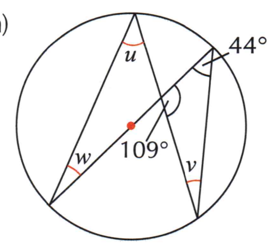
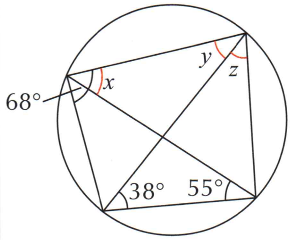


Q2 Find the value of each letter in the diagrams below.

a)

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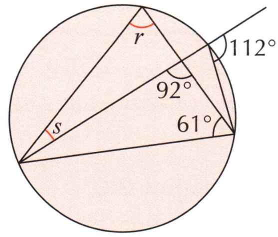
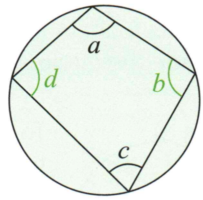


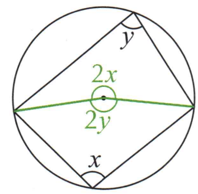

**Rule 7: Opposite angles** in a **cyclic quadrilateral** sum to **180°**.




$a + c = 180°$
$b + d = 180°$

A **cyclic quadrilateral** is any quadrilateral which can be drawn inside a circle with **all four vertices** touching the circumference.

To prove this rule, draw **two radii** going to **opposite corners** of the quadrilateral as shown in the diagram on the right. Now you have **two sets of angles** at the centre and circumference subtended by the same arc, so use rule 5. If the angles at the circumference are $x$ and $y$, the angles in the centre are **twice as big**, so they are $\mathbf{2x}$ and $\mathbf{2y}$. The angles at the **centre** add up to 360° as they are around a point, so $2x + 2y = 360° \Rightarrow \mathbf{x + y = 180°}$



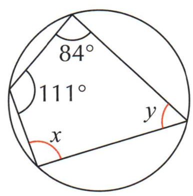

## Example 3

**Find the missing angles $x$ and $y$.**

It's a cyclic quadrilateral so use rule 7:

Opposite angles in a cyclic quadrilateral add up to 180°.

$x = 180° - 84° = \mathbf{96°}$ $\quad$ $y = 180° - 111° = \mathbf{69°}$

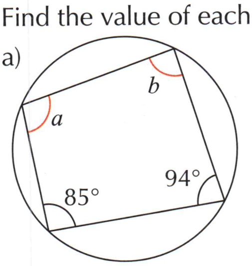



**Tip:** Keep an eye out for cyclic quadrilaterals — they can be hard to spot if there are other lines on the diagram.

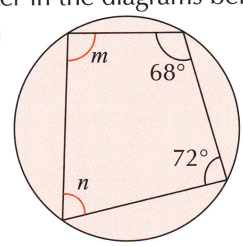
## Exercise 3

Q1 Find the value of each letter in the diagrams below.

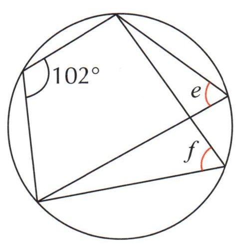
a)



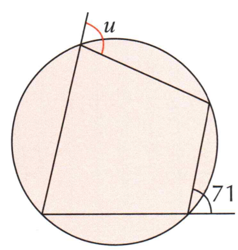

b)



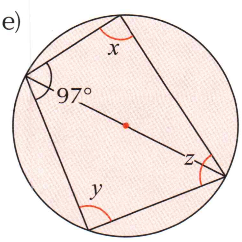

c)



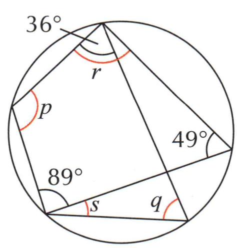

d)



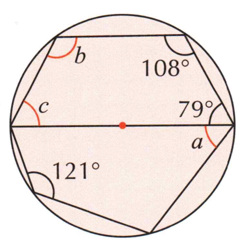

e)



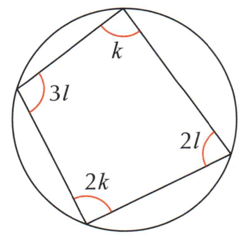

f)




g)




h)

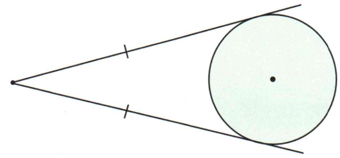


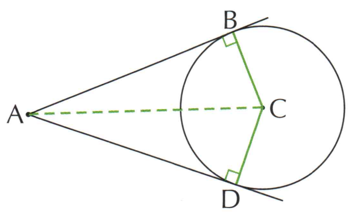
# 21.3 Circle Theorems 3

*Here are the final two circle theorems that you need. Remember that circle geometry often involves using several rules together to find unknown angles, so watch out for places where you can use the other rules.*

**Learning Objectives — Spec Ref G10:**
- Know that two tangents drawn from the same point are equal in length.
- Be able to use the alternate segment theorem.

**Rule 8**: Two **tangents** to a circle drawn from a single point outside the circle are the **same length**.

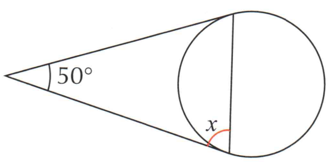




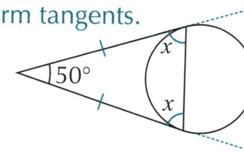

To prove this rule, draw **two radii** going to the points where the **tangents touch the circle**. Then draw another line from the **centre** of the circle to the point **outside** the circle as shown in the diagram on the left. ∠ABC = ∠ADC = **90°** because a tangent meets a radius at 90° (rule 2), so ABC and ADC are **right-angled triangles**. Line AC is the **hypotenuse** of both triangles and BC = DC because they are both **radii**. Condition RHS holds (see p.394), so the triangles are **congruent** and **AB = AD**.

## Example 1

**Find the size of the missing angle $x$.**




1. The two longer edges can be extended to form tangents.
   The tangents and the chord form an isosceles triangle as tangents from the same point are the same length.



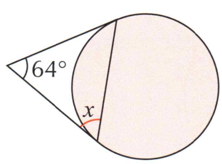

2. The two missing angles are equal as the triangle is isosceles.

$x = (180° - 50°) \div 2 = \mathbf{65°}$

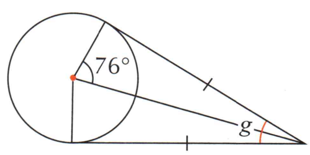
## Exercise 1

Q1 Find the value of each letter in the diagrams below.

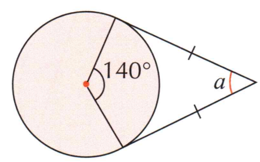
a)



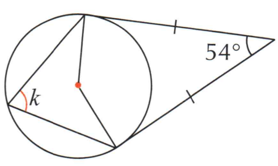

b)



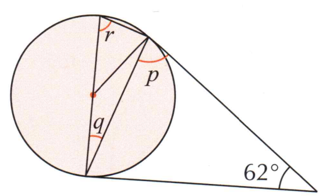

c)



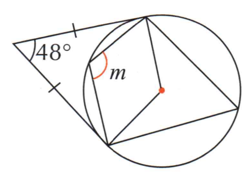

d)



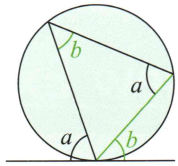

e)



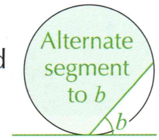

f)



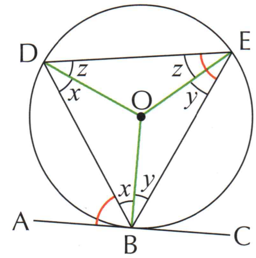
**Rule 9**: The angle between **a tangent and a chord** is equal to the angle subtended from the ends of the chord in the **alternate segment**.




The **alternate segment** to an angle between a tangent and a chord is the segment on the other side of the chord.



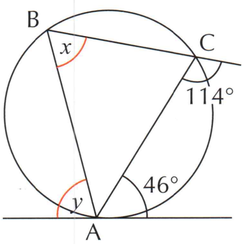

This circle rule is called the **alternate segment theorem**.




To prove it, start with a **triangle** inside a circle with a **tangent** at one of the points of the triangle, as shown in the diagram on the right.

Draw a **radius** to **each corner** of the triangle to create **three isosceles triangles**.
Now, aim to show that ∠**ABD** and ∠**BED** are **equal**.
Using rule 2 you know that ∠ABO = 90° so ∠**ABD = 90° –** $x$
The angles in triangle BDE add up to 180°, so
$x + x + y + y + z + z = 180° \Rightarrow 2(x + y + z) = 180°$
$\Rightarrow x + y + z = 90° \Rightarrow y + z = 90° - x \Rightarrow$ ∠**BED = 90° –** $x$
So ∠**ABD** = ∠**BED**. (You can follow the same method to show ∠**CBE** = ∠**BDE**.)

## Example 2




**Find the size of angles $x$ and $y$.**

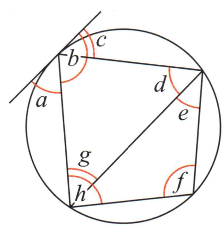
1. You can use the alternate segment theorem to find $x$.
   $x$ is in the alternate segment to the 46° angle, so $x$ = **46°**.
2. ∠ACB is in the alternate segment to $y$.
   Using angles on a straight line, $y$ = ∠ACB = 180° – 114° = **66°**.

## Exercise 2

Q1 According to the alternate segment theorem, which angle in the diagram on the right is:

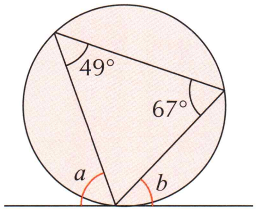
a) equal to angle $a$?

b) equal to angle $c$?

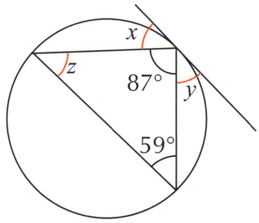



Q2 Find the value of each letter in the diagrams below.

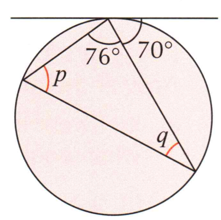
a)



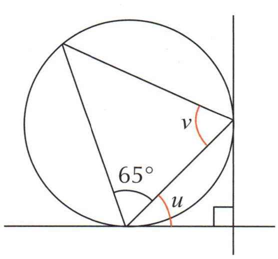

b)



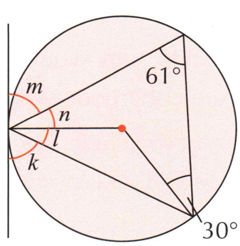

c)



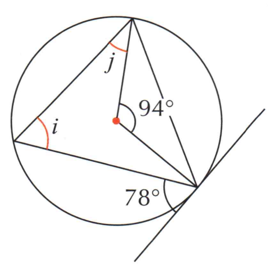

d)




e)



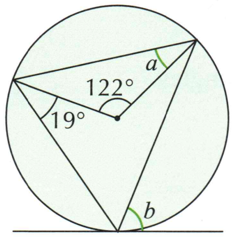

f)



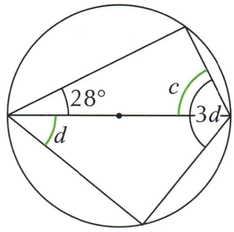
# Review Exercise

**Q1** Find the value of each letter in the diagrams below. Explain your reasoning in each case.

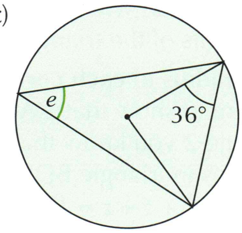
a)



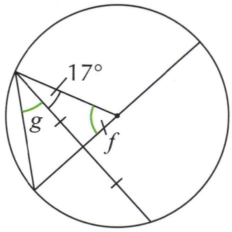

b)



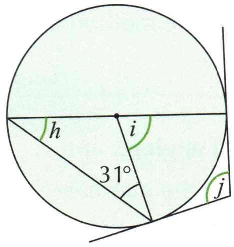

c)



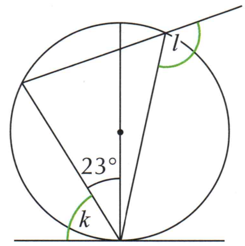

d)



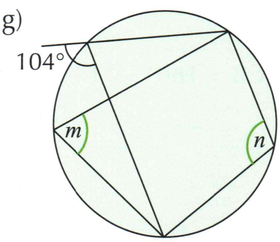

e)



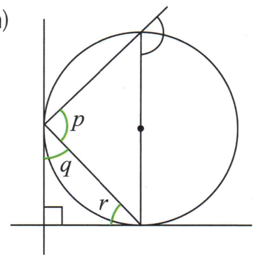

f)



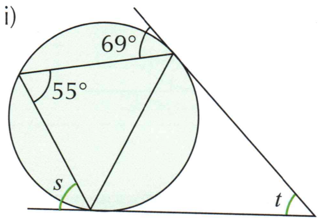

g)




h)




i)

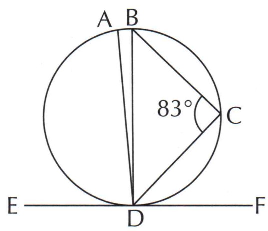



**Q2** Look at the diagram on the right.

PROBLEM SOLVING

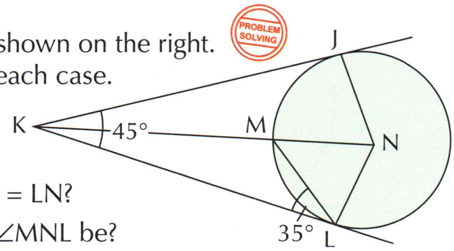
a) What is the size of ∠BDE?

b) How does the answer to part a) prove that AD is not a diameter of the circle?




**Q3** Joel wants to prove that N is not the centre of the circle shown on the right. Answer the questions below, explaining your answer in each case.

PROBLEM SOLVING




a) If N were the centre of the circle, what would the sizes of ∠KLN and ∠MLN be?

b) (i) If N were the centre of the circle, why would MN = LN?

(ii) If MN = LN, what would the sizes of ∠LMN and ∠MNL be?

c) (i) If N were the centre of the circle, why would triangles KJN and KLN be congruent?

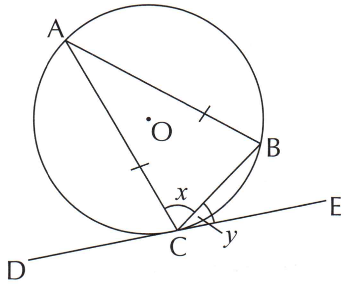
(ii) If triangles KJN and KLN were congruent, what would the size of ∠JNL be?

Joel says that N can't be the centre of the circle, because if it is, then ∠JKL is not 45°.

d) If N were the centre of the circle and ∠KLM = 35°, what would the size of ∠JKL be?

# Exam-Style Questions

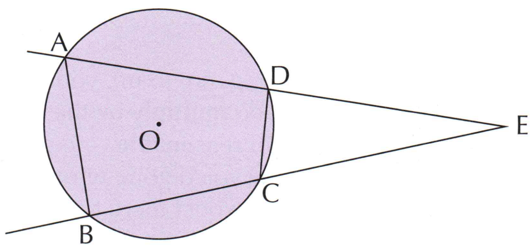
**Q1** The diagram shows a circle with centre O. The points A, B and C are on the circumference such that AB = AC. DE is a tangent to the circle at C.




Given that angle ACB = $x$ and BCE = $y$, prove that $y = 180° - 2x$.
You must give a reason for every statement that you make.

*[3 marks]*

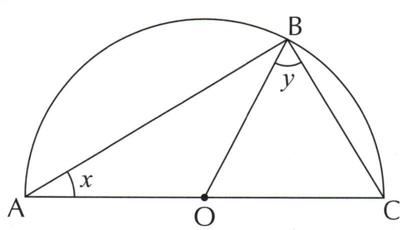
**Q2** In the diagram, a circle with centre O is intersected twice by each of two lines which meet at E.




Prove that the triangles AEB and DEC are similar.

*[3 marks]*

**Q3** The diagram shows a semicircle with centre O and diameter AC.




Using angle OAB = $x$ and angle OBC = $y$, prove that angle ABC = 90°.

*[3 marks]*