18.1 Evaluating Functions

Don't be put off functions by the slightly weird notation — they're actually a really simple concept. Functions are just mathematical rules that can be applied to input values to 'map' them to output values.

Learning Objectives — Spec Ref A7:
- Understand and use function notation.
- Be able to evaluate functions.

Prior Knowledge Check: Be able to substitute numbers into formulas. See p.107.

A **function** is a rule that turns one number (the **input**) into another number (the **output**). For example:

- The function $f(x) = x + 3$ is a rule that **adds 3** to any input value x.
- The same function could also be written as $f: x \rightarrow x + 3$
  You read this as 'the function f **maps** the value of x to the value $x + 3$'.

To **evaluate** a function for a particular value of x, **substitute** this value into the expression for $f(x)$. So for function f above, $f(6) = 6 + 3 = 9$, $f(-2) = (-2) + 3 = 1$, and so on.

Tip: You can show functions on graphs (see Sections 15-16). Each value on the x-axis is 'mapped' to a value on the y-axis. The y-axis is often labelled $y = f(x)$ to show this.

Example 1

For the function $f(x) = 30 - 2x^2$, find: a) f(4), b) f(−2)

a) To find f(4), substitute $x = 4$ into $30 - 2x^2$.

$f(4) = 30 - 2 \times 4^2$
$= 30 - 2 \times 16 = 30 - 32 = \mathbf{-2}$

b) Substitute $x = -2$ into $30 - 2x^2$.
Be careful with the minus signs here.

$f(-2) = 30 - 2 \times (-2)^2$
$= 30 - 2 \times 4 = 30 - 8 = \mathbf{22}$

Functions can be represented by letters other than f — you might see $g(x)$, $h(x)$ etc.

Example 2

For the function $g(x) = \dfrac{2x+1}{x+4}$, find g(3).

To find g(3), substitute $x = 3$ into the function.
Make sure you replace every x with 3.

$$g(3) = \frac{2(3)+1}{3+4} = \frac{6+1}{7} = \frac{7}{7} = \mathbf{1}$$

Exercise 1

Q1	a) $f(x) = x - 5$	Work out:	(i) f(11)	(ii) f(36)	(iii) f(4)
	b) $f(x) = 4x$	Work out:	(i) f(0)	(ii) f(7)	(iii) f(1.5)
	c) $g(x) = 5x - 7$	Calculate:	(i) g(3)	(ii) g(6)	(iii) g(−1)
	d) $f: x \rightarrow 11 - 2x$	Work out:	(i) f(3)	(ii) f(5.5)	(iii) f(−5)

Q2 a) $f(x) = x^2 - 4x$ Work out: (i) f(7) (ii) f(–3) (iii) f(1.5)

b) $g: x \to 20 - 3x^2$ Work out: (i) g(2) (ii) g(3) (iii) g(–1)

c) $h(x) = 2x^2 - x + 4$ Calculate: (i) h(5) (ii) h(–2) (iii) h(8)

d) $f(t) = (4t - 3)^2$ Calculate: (i) f(2) (ii) f(0) (iii) f(0.75)

Q3 a) $g(x) = \frac{20}{x}$ Work out: (i) g(10) (ii) g(1.25) (iii) g(–0.5)

b) $g: x \to \frac{18}{x^2 + 2}$ Work out: (i) g(0) (ii) g(2) (iii) g(–4)

c) $h(x) = \sqrt{\frac{x}{3x + 1}}$ Work out: (i) h(0) (ii) h(1) (iii) h(16)

d) $f(x) = 8^x - 1$ Work out: (i) f(0) (ii) $f\left(\frac{1}{3}\right)$ (iii) $f\left(-\frac{2}{3}\right)$

Example 3

$f(x) = 3x + 4$. Find the value of x for which $f(x) = 22$.

Set up and solve an equation in x:

$22 = 3x + 4$

$18 = 3x$

$\mathbf{x = 6}$

Exercise 2

Q1 For each function below, find the value of x which produces the given output value.

a) $f(x) = 2x - 5$, $f(x) = 35$ b) $g(x) = 8 - 3x$, $g(x) = -10$ c) $h(x) = 9x + 12$, $h(x) = 93$

Q2 For each function below, find the value of x which produces the given output value.

a) $f(x) = x^2$, $f(x) = 225$ b) $g(x) = \sqrt{2x - 1}$, $g(x) = 3$ c) $h(x) = \frac{18}{x + 1}$, $h(x) = 2$

Example 4

For the function $f(x) = 30 - 2x^2$, find an expression for $f(2t)$.

To find $f(2t)$, replace x with $2t$ and expand.

$f(2t) = 30 - 2 \times (2t)^2$

$= 30 - 2 \times 4t^2$

$= \mathbf{30 - 8t^2}$

Tip: This will come in handy for finding composite functions (see next page).

Exercise 3

Q1 a) $f: x \to 2x + 1$. Find expressions for: (i) f(k) (ii) f(2m) (iii) f(3w – 1)

b) $f(x) = 2x^2 + 3x$. Find expressions for: (i) f(u) (ii) f(3a) (iii) $f(t^2)$

c) $f(x) = \frac{4x - 1}{x + 4}$. Find expressions for: (i) f(t) (ii) f(–x) (iii) f(2x)

d) $f(x) = \sqrt{5x - 1}$. Find expressions for: (i) f(w) (ii) f(3x) (iii) f(1 – 2x)

18.2 Composite Functions

You need to be completely happy with function notation before carrying on. Next up, you'll see what happens when you combine two (or more) functions into one new function — called a composite function.

Learning Objective — Spec Ref A7:
Find composite functions.

If f(x) and g(x) are two functions, then the **combined** function gf(x) is called a **composite function**.

gf(x) means 'put x into function f, then put the answer into function g' — you always do the function **closest** to x first. To find the composite function gf(x), write it as g(f(x)), replace f(x) with the expression it represents, then put this into g.

You'll usually find that fg(x) ≠ gf(x) — in other words, applying function g first, then f is **not** the same as applying f then g, so the order in which you combine the functions matters.

Example 1

If f(x) = 3x – 2 and g(x) = x^2 + 1, then: a) Calculate gf(3) b) Find fg(x)

a) Calculate the value of f(3), then use it as the input in g to find gf(3).

$$gf(3) = g(f(3)) = g(3 \times 3 - 2) = g(7) = 7^2 + 1 = \mathbf{50}$$

b) Replace x with g(x) in the expression for f(x), then collect like terms to tidy it up.

$$fg(x) = f(g(x)) = f(x^2 + 1) = 3(x^2 + 1) - 2 = 3x^2 + 3 - 2 = \mathbf{3x^2 + 1}$$

Tip: Here, gf(x) = $9x^2 - 12x + 5$ — which is not the same as fg(x).

Exercise 1

Q1 f(x) = 2x, g(x) = x + 2
a) Find g(1)
b) Use your answer to a) to find fg(1)
c) Find f(5)
d) Use your answer to c) to find gf(5)

Q2
a) f(x) = x – 5, g(x) = 4x Find: (i) fg(7) (ii) gf(8) (iii) gf(x)
b) f(x) = 2x – 1, g(x) = 3x + 1 Find: (i) gf(0) (ii) fg(2) (iii) fg(x)
c) f(x) = 5x – 4, g(x) = $\frac{x}{4}$ Find: (i) fg(4) (ii) gf(4) (iii) gf(x)
d) f(x) = 4x + 3, g(x) = 5x Find: (i) fg(3) (ii) ff(2) (iii) gf(x)
e) f(x) = 2x + 1, g(x) = 11 – x Find: (i) gf(3) (ii) gg(4) (iii) fg(x)
f) f(x) = $\frac{1}{x}$, g(x) = 3x – 7 Find: (i) fg(3) (ii) fg(x) (iii) gf(x)

Q3
a) f(x) = 10 – x, g(x) = x^2 Find: (i) gf(8) (ii) ff(2) (iii) gf(x)
b) f(x) = $\sqrt{x}$, g(x) = 2x + 1 Find: (i) ff(16) (ii) fg(40) (iii) gf(x)
c) f(x) = x^2 + 5, g(x) = 3x – 4 Find: (i) gf(–1) (ii) gf(x) (iii) gg(x)
d) f(x) = $\sqrt{3x+1}$, g(x) = 12 – 2x Find: (i) gf(8) (ii) gg(x) (iii) fg(x)

Example 2

If $f(x) = x^2$ and $g(x) = 7 - 2x$, find $fgf(x)$.

Find gf(x) first, then use that as the input for f.

$$fgf(x) = f(gf(x)) = f(g(x^2)) = f(7 - 2x^2) = \mathbf{(7 - 2x^2)^2}$$

Tip: Do it one step at a time to avoid mistakes.

Exercise 2

Q1 a) $f(x) = \frac{1}{3x+2}$, $g(x) = 2x - 5$, $h(x) = x^2$ Find: (i) $hgf(-1)$ (ii) $fg(x)$ (iii) $hf(x)$

b) $f(x) = x^3$, $g(x) = 6 - x$, $h(x) = 10 - 2x$ Find: (i) $fgh(4)$ (ii) $gh(x)$ (iii) $fh(x)$

c) $f(x) = 2x + 4$, $g(x) = \frac{1}{3x+1}$, $h(x) = \sqrt{x}$ Find: (i) $fh(25)$ (ii) $fg(0.5)$ (iii) $hgf(x)$

d) $f(x) = x^2 + x$, $g(x) = \frac{1}{2x+3}$, $h(x) = \frac{1}{x}$ Find: (i) $ff(3)$ (ii) $gf(-1)$ (iii) $hgf(x)$

Example 3

Let $f(x) = 4x - 7$ and $g(x) = \frac{1}{3x+2}$. Solve the equation $fg(x) = 1$.

1. First find fg(x).

$$fg(x) = f\left(\frac{1}{3x+2}\right) = 4\left(\frac{1}{3x+2}\right) - 7 = \frac{4}{3x+2} - 7$$

2. Now let fg(x) = 1 and solve for x.

$$\frac{4}{3x+2} - 7 = 1$$
$$\frac{4}{3x+2} = 8$$
$$8(3x + 2) = 4$$
$$24x + 16 = 4$$
$$24x = -12$$
$$\mathbf{x = -0.5}$$

Tip: Flick back to Section 9 if you need a reminder about solving equations.

Exercise 3

Q1 $f(x) = 9 - x$, $g(x) = 3x$

a) (i) Find $fg(x)$ (ii) Hence solve $fg(x) = -12$

b) (i) Find $gf(x)$ (ii) Hence solve $gf(x) = 6$

Q2 $f(x) = x + 5$, $g(x) = \frac{x}{2}$. Solve the following equations.

a) $fg(x) = 14$ b) $gf(x) = 8$ c) $gg(x) = 11$

Q3 $f(x) = 6 - x$, $g(x) = 2x^2 - 1$. Solve the following equations.

a) $fg(x) = 7$ b) $gf(x) = 31$ c) $gg(x) = 97$

18.3 Inverse Functions

An inverse operation does the opposite of the original operation — so the inverse of +9 is –9, the inverse of ÷5 is ×5 and so on. Knowing how to use inverse operations will be very helpful in this next topic.

Learning Objective — Spec Ref A7:
Find inverse functions.

Prior Knowledge Check:
Be able to rearrange formulas. See p.109.

An **inverse function** reverses the effect of a function. The inverse of f(x) is written $\mathbf{f^{-1}(x)}$. So if the function f multiplies by 10, the inverse function $f^{-1}(x)$ would divide by 10 — i.e. $f(x) = 10x$, so $f^{-1}(x) = \frac{x}{10}$.

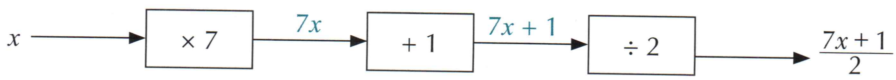
If you apply a function to x, then apply its inverse, you just end up with x. This can be written as $f^{-1}f(x) = x$. This works the other way around as well — applying the inverse, then the function, will get you back to x ($ff^{-1}(x) = x$). Combining these rules gives $\mathbf{ff^{-1}(x) = f^{-1}f(x) = x}$.

There are two methods you can use to find an inverse function — the first one simply involves undoing the operations of the original function, one at a time.

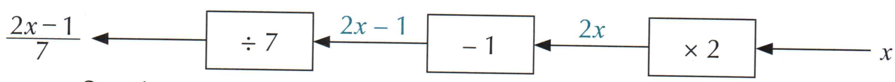
Example 1

Find the inverse of the function $g(x) = \frac{7x+1}{2}$.

1. Write g(x) out as a function machine.



2. Reverse each step in turn to get the inverse.



So $g^{-1}(x) = \mathbf{\frac{2x-1}{7}}$

3. You can check your answer to make sure that the inverse function does reverse the original function.

$g(1) = \frac{7(1)+1}{2} = \frac{8}{2} = 4$ and $g^{-1}(4) = \frac{2(4)-1}{7} = \frac{8-1}{7} = \frac{7}{7} = 1$

This gives the starting number (1), so the inverse looks to be correct.

Tip: Make sure you reverse the **order** of the operations — so the last operation of the original function becomes the first operation to undo in the inverse function.

Exercise 1

In Questions 1-3, find the inverse of each function.

Q1 a) $f(x) = x + 4$ b) $f(x) = x - 3$ c) $f(x) = x - 7$ d) $f(x) = x + 1$
e) $f(x) = 8x$ f) $g(x) = 2x$ g) $g(x) = \frac{x}{3}$ h) $f(x) = \frac{x}{6}$

Q2 a) $f(x) = 4x + 3$ b) $f(t) = 2t - 9$ c) $g(x) = 3x - 5$ d) $f(x) = 8x + 11$
e) $f(x) = \frac{x}{5} - 7$ f) $g(x) = \frac{x}{8} + 1$ g) $f(t) = \frac{t-3}{2}$ h) $h(x) = \frac{x+15}{4}$

Q3 a) $f(x) = \frac{2x+6}{5}$ b) $g(x) = \frac{3x-1}{4}$ c) $f(x) = x^2 - 3$ d) $g(x) = (2x + 7)^2$

For more complicated functions, use the following method:

1. Write out the equation $x = f(y)$ ($f(y)$ is just $f(x)$ with the x's replaced by y's).
2. **Rearrange** the equation to make y the subject.
3. Finally, **replace** y with $f^{-1}(x)$.

Example 2

Find the inverse of the function $f(x) = \frac{\sqrt{4x-1}}{3}$.

1. Write out the equation $x = f(y)$.

$$x = \frac{\sqrt{4y-1}}{3}$$

2. Rearrange the equation to make y the subject.

$$3x = \sqrt{4y-1}$$
$$9x^2 = 4y - 1$$
$$9x^2 + 1 = 4y$$
$$y = \frac{9x^2+1}{4}$$

Tip: Don't rush this step — it's easy to make a mistake.

3. Replace y with $f^{-1}(x)$.

$$\mathbf{f^{-1}(x) = \frac{9x^2+1}{4}}$$

4. Check your inverse function by trying it out on a number.

$$f(2.5) = \frac{\sqrt{4(2.5)-1}}{3} = \frac{\sqrt{10-1}}{3} = \frac{\sqrt{9}}{3} = \frac{3}{3} = 1$$

and $f^{-1}(1) = \frac{9(1)^2+1}{4} = \frac{9+1}{4} = \frac{10}{4} = 2.5$

This gives you the starting number (2.5), so it looks like the inverse is correct.

Exercise 2

In Questions 1-5, find the inverse of each function.

Q1 a) $f(x) = \frac{x}{5} - 8$ b) $f(x) = \frac{3x+1}{5}$ c) $g(x) = \frac{2x}{5} - 7$

Q2 a) $f(x) = 7 - 3x$ b) $f(x) = \frac{9-7x}{4}$ c) $h(x) = \frac{1-6x}{9}$

Q3 a) $g(x) = 4x^2 + 1$ b) $g(x) = (3x-1)^2$ c) $h(x) = \frac{(1-2x)^2}{5}$

Q4 a) $f(x) = 6\sqrt{x} + 1$ b) $g(x) = \sqrt{19-2x}$ c) $f(x) = 25 - 4\sqrt{x}$

Q5 a) $f(x) = \frac{4}{x} - 7$ b) $g(x) = 8 - \frac{2}{\sqrt{x}}$ c) $f(x) = \sqrt{\frac{2}{x-1}}$

Q6 The inverse of the function $g(x) = \frac{1-2x}{3x+5}$ can be found by rearranging the equation $x = \frac{1-2y}{3y+5}$.

a) Show that, if $x = \frac{1-2y}{3y+5}$, then $3xy + 2y = 1 - 5x$.

b) Factorise the expression $3xy + 2y$, and hence find $g^{-1}(x)$.

Q7 Find the inverse of each of the following functions:

a) $f(x) = \frac{2+3x}{x-2}$ b) $g(x) = \frac{4x+1}{2x-7}$ c) $f(x) = \frac{x}{3x+2}$

Review Exercise

Q1 f(x) = $5x - 7$ and g(x) = $2x + 5$

a) Calculate f(8). b) Calculate fg(1). c) Solve f(x) = g(x). d) Find $g^{-1}(x)$.

Q2 f(x) = $4x + 1$ and g(x) = $2x + 5$

a) Solve f(x) = 25 b) Find fg(x) c) Find $f^{-1}(x)$

Q3 f(x) = $\sqrt{x+1}$ and g(x) = $4x$.

a) Find gf(x). b) Find $f^{-1}(x)$.

Q4 f(x) = $2x - 3$

a) Solve f(x) = 5. b) Find $f^{-1}(x)$.

c) Solve the equation f(x) = $f^{-1}(x)$ d) Find ff(x)

Q5 f(x) = $2x - 3$ and g(x) = $18 - 3x$

a) Solve f(x) = g(x). b) Find gf(x). c) Find $g^{-1}(x)$.

Q6 f(x) = $\frac{1}{x} - 5$ and g(x) = $\frac{1}{x+5}$

a) Calculate g(–3). b) Find fg(x). How are functions f and g related?

Q7 f(x) = $2x^2 + 5$ and g(x) = $3x - 1$

a) Work out gg(4). b) Find $g^{-1}(x)$. c) Solve $fg^{-1}(x) = 55$.

Q8 f(x) = $\frac{x}{x-3}$ and g(x) = $x^2 + 3$

a) Calculate f(5). b) Find $f^{-1}(x)$. c) Find fg(x).

Q9 An object is dropped off the top of a tower. The distance, s metres, the object has travelled t seconds after being released is given by the formula s = f(t), where f(t) = $5t^2$.

a) Calculate f(4). b) Solve the equation f(t) = 12.8.

Q10 The function f(t) = $\frac{9t}{5} + 32$ can be used to convert a temperature from °C to °F.

a) Work out f(20). b) Solve the equation f(t) = 60.8.

c) Find $f^{-1}(t)$. d) Explain what your answers to parts a)-c) represent.

Q11 Jill uses the following formula to estimate the temperature T (in degrees Fahrenheit) at height h (in thousands of feet) above sea level: T = f(h) where f(h) = $60 - \frac{7}{2}h$.

a) Calculate f(4).

b) Find the temperature at a height of 7000 feet above sea level.

c) Work out $f^{-1}(32)$. Explain what this answer tells you in the context of this question.

Exam-Style Questions

Q1 f and g are two functions, where $f(x) = 6x + 5$ and $g(x) = \frac{x+3}{2}$.

a) Evaluate g(11).

[1 mark]

b) Find fg(x).

[2 marks]

c) Find an expression for $g^{-1}(x)$.

[3 marks]

Q2 f and g are two functions, where $f(x) = 2x^2$ and $g(x) = 3x - 5$.
Find fg(x), giving your answer in the form $ax^2 + bx + c$.

[3 marks]

Q3 A function f(x) is defined such that:

$$f(x) = \frac{2x+3}{5x-4}, \quad x \neq \frac{4}{5}$$

Find an expression for the inverse function, $f^{-1}(x)$.

[4 marks]

Q4 $f(x) = \frac{4}{5x-15}$ and $g(x) = 2\sqrt{x}$

a) Solve $f(x) = 0.2$.

[2 marks]

b) Work out fg(x).

[2 marks]

Q5 $f(x) = x^2 + 6$, $g(x) = 3x + 4$.
Solve the equation $fg(x) = gf(x)$. PROBLEM SOLVING

[6 marks]