Section 16 Other Types of Graph

16.1 Quadratic Graphs

Quadratics are covered in Sections 6 and 11, so have a look back if you're not familiar with them. You can plot accurate graphs of quadratics by finding coordinates (as you would with most types of graph). You can also sketch them once you know how their equations relate to the shape and features of the graph.

Drawing Quadratic Graphs

Learning Objective — Spec Ref A12:
Be able to recognise, draw and read off quadratic graphs.

Quadratic functions always have x^2 as the highest power of x. The **graphs** of quadratic functions are always the same shape (called a **parabola**) and are **symmetrical** about their lowest (or highest) point.

- If the coefficient of x^2 is **positive** (e.g. $y = 2x^2 - 3x - 1$), the parabola is **u-shaped**.
- If the coefficient of x^2 is **negative** (e.g. $y = -2x^2 + 3x + 1$), the parabola is **n-shaped**.

To **draw the graph** of a quadratic function $y = ax^2 + bx + c$, calculate y-values for a set of x-values in a **table**, plot the pairs of values as **coordinates** on a set of axes, then draw a **smooth curve** through the points. You can then **read off** the graph to estimate the value of x for a given value of y.

Example 1

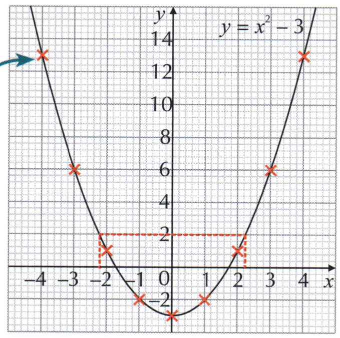
Draw the graph of $y = x^2 - 3$ and use it to estimate x when $y = 2$.

1. Use a table to find the coordinates of points on the graph.

x	–4	–3	–2	–1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
$x^2 - 3$	13	6	1	–2	–3	–2	1	6	13

So the coordinates are (–4, 13), (–3, 6), etc.

2. Plot the coordinates on a suitable set of axes.
Join the points with a smooth curve (not straight lines).



3. Draw a horizontal line at $y = 2$ and read off the values of x: $\boldsymbol{x = \pm 2.2}$ (1 d.p.)

Exercise 1

Q1 For the following quadratic equations, copy and complete the table and draw each graph.

a) $y = 6 - x^2$

x	–4	–3	–2	–1	0	1	2	3	4
x^2			4			1		9	
$6 - x^2$			2			5		–3	

b) $y = 2x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$2x^2$		18				2			

Q2 Draw the graph of $y = x^2 + 5$ for values of x between –4 and 4.

a) Use your graph to find the value of y when: (i) $x = 2.5$ (ii) $x = -0.5$

b) Use your graph to find the values of x when: (i) $y = 6.5$ (ii) $y = 10$

Q3 Draw the graph of $y = 4 - x^2$ for values of x between –4 and 4.
Write down the values of x where the graph crosses the x-axis.

Q4 Draw the graph of $y = 3x^2 - 11$ for values of x between –4 and 4.
Estimate the values of x where the graph crosses the x-axis.

Example 2

Draw the graph of $y = x^2 + 3x - 2$.

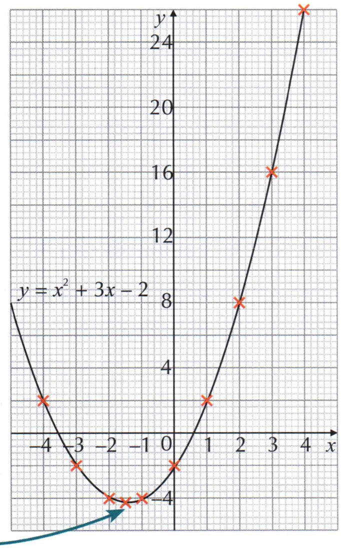
1. Add extra rows to the table to make it easier to work out the y-values.

x	–4	–3	–2	–1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
$+3x$	–12	–9	–6	–3	0	3	6	9	12
–2	–2	–2	–2	–2	–2	–2	–2	–2	–2
$x^2 + 3x - 2$	2	–2	–4	–4	–2	2	8	16	26

2. The table doesn't tell you the lowest point on the curve, so you need to find one more point before you can draw the graph. Quadratic graphs are always symmetrical, so the x-coordinate of the lowest point on the curve is halfway between the two lowest points from the table (or any pair of points with the same y-coordinate).

   So the lowest point of the graph is halfway between $x = -2$ and $x = -1$, when $x = -1.5$ and $y = (-1.5)^2 + (3 \times -1.5) - 2 = -4.25$.

3. Plot the points and join with a smooth curve.



Exercise 2

Q1 Copy and complete the table and draw the graph of $y = 2x^2 + 3x - 7$.

x	–4	–3	–2	–1	0	1	2	3	4
$2x^2$		18						18	
$+3x$		–9						9	
–7		–7						–7	
$2x^2 + 3x - 7$		2						20	

Q2 Draw the graph of $y = x^2 - 5x + 3$ for values of x between –3 and 6.

a) Use your graph to find the value of y when: (i) $x = -1.5$ (ii) $x = 1.5$

b) Use your graph to find the values of x when: (i) $y = 8$ (ii) $y = -2$

Q3 Draw the graph of $y = 11 - 2x^2$ for values of x between –4 and 4.

a) Use your graph to find the value of y when: (i) $x = -2.5$ (ii) $x = 1.25$

b) Use your graph to find the values of x when: (i) $y = 0$ (ii) $y = 11$

Sketching Quadratic Graphs Using Factorising

Learning Objective — Spec Ref A11/A12:
Be able to sketch quadratic graphs by factorising the equation.

Prior Knowledge Check:
Be able to factorise and solve quadratic equations. See p.135-136.

Quadratic graphs are always the same shape, which means you can **sketch** them without a table of coordinates. Draw a u-shaped or n-shaped **parabola** in roughly the correct position and label these **important points**:

- The **y-intercept** is the value of the **constant term** in the equation, i.e. $(0, c)$ for $y = ax^2 + bx + c$, because this is the value of y when $x = 0$.
- The **x-intercepts** can be found by **factorising** the equation. A quadratic $y = (x + p)(x + q)$ crosses the x-axis at $(-p, 0)$ and $(-q, 0)$, because $x = -p$ and $x = -q$ are the solutions to $(x + p)(x + q) = 0$.
- The **turning point** is the **lowest point** (for a u-shaped graph) or **highest point** (for an n-shaped graph) on the curve. Due to the symmetry of the graph, the **x-coordinate** of the turning point is always **halfway** between the **x-intercepts**. Put this value into the equation to find the y-coordinate.

Tip: The turning point lies on the vertical line of symmetry.

Example 3

Sketch the graph of $y = x^2 + x - 6$, and label the turning point and intercepts with their coordinates.

1. Find the y-intercept by putting $x = 0$ in the equation (or just look at the constant term).

   When $x = 0$, $y = 0^2 + 0 - 6 = -6$, so the y-intercept is at $(0, -6)$.

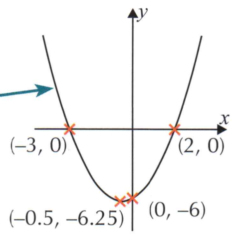
2. Factorise and solve the equation $x^2 + x - 6 = 0$ to find the x-intercepts.

   $x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0 \Rightarrow x = -3$ and $x = 2$, so the x-intercepts are $(-3, 0)$ and $(2, 0)$.

3. Use symmetry to find the turning point. The x-coordinate is halfway between the x-intercepts. Find the y-coordinate by putting the x-coordinate into the equation.

   x-coordinate $= (2 + -3) \div 2 = -0.5$
   y-coordinate $= (-0.5)^2 + (-0.5) - 6 = -6.25$
   So the turning point has coordinates $(-0.5, -6.25)$.

4. Use all this information to sketch and label the graph. The x^2 term is positive so the graph is u-shaped. This means that the turning point will be the lowest point on the graph.



Tip: Sketches don't have to be completely accurate — they just need the correct general shape, and key points labelled and in roughly the correct positions.

Exercise 3

Q1 For each of the following equations, find the coordinates of: (i) the intercepts, (ii) the turning point.

a) $y = (x - 1)(x + 1)$ b) $y = (x + 7)(x + 1)$ c) $y = x^2 + 16x + 60$

Q2 Sketch the graphs of these equations. Label the turning points and intercepts with their coordinates.

a) $y = x^2 - 4$ b) $y = x^2 - 4x - 12$ c) $y = x^2 + 12x + 32$

d) $y = x^2 + x - 20$ e) $y = -x^2 - 2x + 3$ f) $y = -x^2 - 14x - 49$

g) $y = 2x^2 + 4x - 16$ h) $y = 5x^2 - 6x - 8$ i) $y = -2x^2 - x + 6$

Sketching Quadratic Graphs by Completing the Square

Learning Objective — Spec Ref A11/A12:
Be able to sketch quadratic graphs by completing the square.

Prior Knowledge Check:
Be able to complete the square for a quadratic equation. See p.137-139.

You can use the **completed square** form of a quadratic equation to easily find the coordinates of the **turning point** of its graph. For equations of the form $y = (x + p)^2 + q$, the turning point can be found where the expression in **brackets** is **zero**, i.e. where $\boldsymbol{x = -p}$ and $\boldsymbol{y = q}$, at $\boldsymbol{(-p, q)}$.

This also tells you the **number of roots**. For a **positive** quadratic, if the minimum point lies **above** the x-axis (i.e. has a **positive** value of q), the graph won't cross the x-axis and so there are **no real roots**. (The same is true for a **negative** quadratic when the maximum point lies **below** the x-axis with a **negative** value of q.) If the turning point lies **on the x-axis** there is **one repeated root**. Otherwise there are **two real roots**.

Example 4

Find the coordinates of the turning point of the graph with equation $y = 4 + 2x - x^2$.

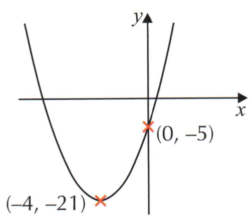
1. Complete the square.
   $y = 4 + 2x - x^2 = -(x^2 - 2x - 4)$
   $= -[(x - 1)^2 - 1 - 4] = 5 - (x - 1)^2$
2. Find the x- and y-values when the expression in brackets is zero.
   $(x - 1) = 0$, so $x = 1$, $y = 5 - 0 = 5$.
   So the turning point is at **(1, 5)**.

Tip: The x^2 term is negative, so the turning point is the highest point on the n-shaped graph.

Example 5

Sketch the graph of $y = x^2 + 8x - 5$. Label the y-intercept and turning point.

1. Complete the square.
   $y = (x + 4)^2 - 16 - 5 = (x + 4)^2 - 21$
2. The turning point occurs when the brackets of the completed square are equal to 0.
   When $(x + 4) = 0$, $x = -4$, $y = 0 - 21 = -21$.
   So the turning point is at $(-4, -21)$.
3. Substitute $x = 0$ into $y = x^2 + 8x - 5$ to find the y-intercept.
   $y = 0^2 + (8 \times 0) - 5 = -5$, so the y-intercept is $(0, -5)$.
4. Sketch the graph through these two points and label their coordinates.
   The x^2 term is positive, so the graph is u-shaped, and the turning point is the lowest (minimum) point on the graph.



Tip: Here, the turning point lies below the x-axis, so the graph crosses the axis twice and there are 2 real roots, at $(x + 4)^2 - 21 = 0$
$\Rightarrow x = -4 \pm \sqrt{21}$.

Exercise 4

Q1 For each of the following equations, find the coordinates of: (i) the y-intercept, (ii) the turning point.

a) $y = (x + 5)^2 - 9$ b) $y = (x - 3)^2 - 30$ c) $y = (x - 4)^2 - 13$

Q2 By completing the square, sketch the graphs of the following equations. Label the y-intercept and turning point of each graph.

a) $y = x^2 - 6x - 5$ b) $y = x^2 - 4x + 2$ c) $y = x^2 + 8x - 6$
d) $y = x^2 + 2x + 8$ e) $y = x^2 - x + 10$ f) $y = -x^2 + 10x - 6$

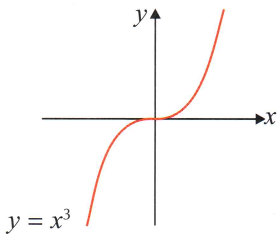
16.2 Cubic Graphs

Just like quadratic graphs, graphs of cubic equations have their own distinctive shape. You can plot them by calculating coordinates as usual, or sketch them using their general shape and any intercepts.

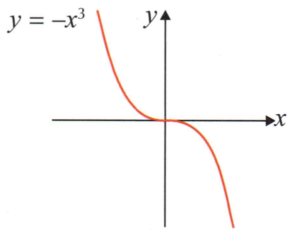
Learning Objective — Spec Ref A12:
Be able to recognise, draw and sketch graphs of cubic functions.

Cubic functions have x^3 as the **highest power** of x — they have equations of the form $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$. **Cubic graphs** all have the same basic shape — a curve with a '**wiggle**' in the middle.

- If the coefficient of x^3 is **positive**, the curve goes **up** from the **bottom left**.



- If the coefficient of x^3 is **negative**, the curve goes **down** from the **top left**.



Tip: Some cubic graphs will have more of a 'wiggle' depending on the values of b and c — like in Example 1 below.

To **draw the graph** of a cubic function $y = f(x)$, find y-values using x-values in a **table**, then **plot** the coordinates and draw a **smooth curve** through the points. You can then **read off** the graph as usual.

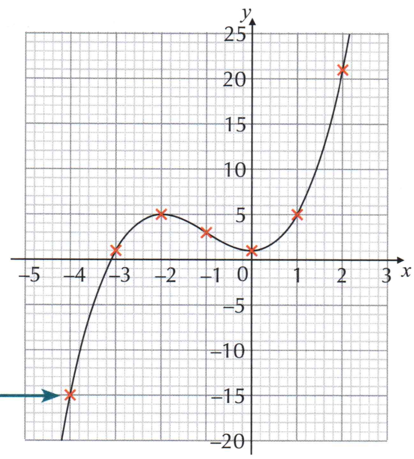
Example 1

a) Draw the graph of $y = x^3 + 3x^2 + 1$.

1. Calculate y-values for a set of x-values in a table. Add extra rows to the table to work out the y-values step by step.

x	–4	–3	–2	–1	0	1	2
x^3	–64	–27	–8	–1	0	1	8
$+3x^2$	48	27	12	3	0	3	12
+1	1	1	1	1	1	1	1
$x^3 + 3x^2 + 1$	–15	1	5	3	1	5	21

2. Plot the coordinates and join the points with a smooth curve. The x^3 term is positive so the curve goes up from the bottom left.



b) Find the coordinates of the x-intercept(s) of $y = x^3 + 3x^2 + 1$.

1. Read off the graph the coordinates where the curve cuts the x-axis.

It cuts the x-axis once at $x = -3.1$ to 1 d.p., so the x-intercept is at **(–3.1, 0)** (1 d.p.).

2. You can check that this value of x fits with the equation.

At the x-intercept, $x = -3.1$, so $y = (-3.1)^3 + (3 \times (-3.1)^2) + 1 = 0.0$ (1 d.p.) ✔

Exercise 1

Q1 For the following cubic equations, copy and complete the table and draw each graph.

a) $y = x^3 + 5$

x	−3	−2	−1	0	1	2	3
x^3	−27					8	
$x^3 + 5$	−22					13	

b) $y = 3x^3 - 4x^2 + 2x - 8$

x	−3	−2	−1	0	1	2	3
$3x^3$	−81				3		
$-4x^2$	−36				−4		
$+2x$	−6				2		
−8	−8				−8		
$3x^3 - 4x^2 + 2x - 8$	−131				−7		

c) $y = 5 - x^3$

x	−3	−2	−1	0	1	2	3
$5 - x^3$		13					−22

Q2 Draw the graph of $y = x^3 + 3$ for values of x between −3 and 3.
Use your graph to estimate the value of y when $x = -2.5$.

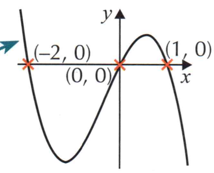
Q3 Draw the graph of $y = x^3 - 6x^2 + 12x - 5$ for values of x between −1 and 5.
Use your graph to estimate the value of x when $y = 0$.

To **sketch** a cubic graph, first look at the sign of the x^3 term (**positive** or **negative**) to decide on the shape. Find the ***y*-intercept** by putting $x = 0$ in the equation. If the equation is in **factorised** form (i.e. $y = (x + p)(x + q)(x + r)$), the solutions to $y = 0$ are the ***x*-intercepts** ($x = -p$, $-q$ and $-r$). You can then draw the '**wiggle**' through these points. If one of the factors is **repeated** (e.g. $y = (x + p)^2(x + q)$), the graph will just **touch** the x-axis at $x = -p$, not cross it.

Example 2

Sketch the cubic graph $y = x(x + 2)(1 - x)$ and label the coordinates of the intercepts.

1. Set $x = 0$ to find the y-intercept. — $y = 0 \times (0 + 2) \times (1 - 0) = 0$, so the y-intercept is at (0, 0).
2. Solve $y = 0$ to find any x-intercepts. — $x(x + 2)(1 - x) = 0 \Rightarrow x = 0$, $x = -2$ or $x = 1$. So the x-intercepts are at (0, 0), (−2, 0) and (1, 0).
3. Expand the brackets to see whether it's a positive or negative cubic. — $y = x(x - x^2 + 2 - 2x) = -x^3 - x^2 + 2x$. The coefficient of x^3 is negative.
4. Sketch a smooth curve with a negative cubic shape (going down from the top left) through the intercepts. Label the coordinates of the intercepts.



Exercise 2

Q1 For the following cubic functions, find the coordinates of the x- and y-intercepts of their graphs.
a) $y = x(x - 1)(x + 2)$ b) $y = (x + 1)(x + 3)(1 - x)$ c) $y = x^2(x + 1)$ d) $y = (x - 10)^3$

Q2 Sketch graphs of each function in Q1, labelling the coordinates of the x- and y-intercepts.

16.3 Reciprocal and Exponential Graphs

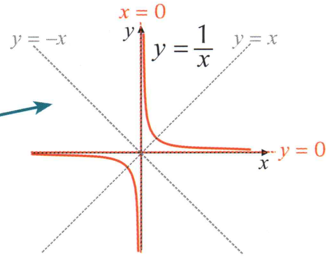
In mathematics, the reciprocal of a number is 'one divided by it' — reciprocal functions always involve 'one divided by a function'. Exponential functions involve something 'to the power of x'.

Reciprocal Graphs

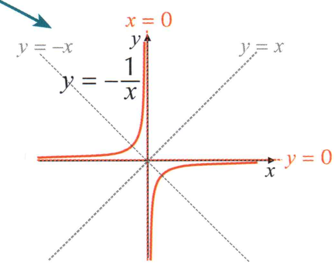
Learning Objective — Spec Ref A12:
Be able to recognise, draw and sketch graphs of reciprocal functions.

The equations $\boldsymbol{y = \frac{1}{x}}$ and $\boldsymbol{y = -\frac{1}{x}}$ are **reciprocal functions** — they can be used to show **inverse proportion** (see p.130). They are **undefined** when $\boldsymbol{x = 0}$ and $\boldsymbol{y = 0}$. This gives their **graphs** their distinctive shape, with horizontal and vertical **asymptotes** at the **axes**. Asymptotes are lines the graph gets very close to, but **never touches**.

The graph of $\boldsymbol{y = \frac{1}{x}}$ lies in the bottom left and top right quadrants.



The graph of $\boldsymbol{y = -\frac{1}{x}}$ lies in the top left and bottom right quadrants.



Both graphs have asymptotes at $\boldsymbol{x = 0}$ and $\boldsymbol{y = 0}$, and **lines of symmetry** at $\boldsymbol{y = x}$ and $\boldsymbol{y = -x}$.

All **general** reciprocal graphs of the form $\boldsymbol{y = \frac{1}{x - a} + b}$ have the **same basic shape** as $\boldsymbol{y = \pm\frac{1}{x}}$, but with different asymptotes. They're just **translations** of the graph of $y = \pm\frac{1}{x}$ (see p.206).

To **sketch** a reciprocal graph, find the **asymptotes**, then draw the curve **between them** in the right position.

A reciprocal graph will have asymptotes where the function is **undefined**.

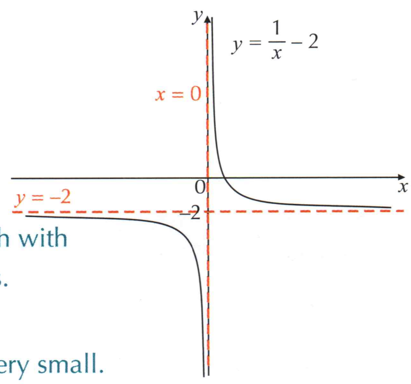
- The function is undefined at $\boldsymbol{x = a}$, as this is the value of x that makes the denominator of the fraction 0, so the graph has a **vertical asymptote** with equation $\boldsymbol{x = a}$.
- As $\frac{1}{x - a}$ is never equal to 0, the function is also undefined at $\boldsymbol{y = b}$, so the graph has a **horizontal asymptote** with equation $\boldsymbol{y = b}$.

Tip: You can also draw a more accurate reciprocal graph using a table of coordinates.

Example 1

By finding the equations of the asymptotes, sketch the graph of $\boldsymbol{y = \frac{1}{x} - 2}$.

1. Find the values of x and y where the function is undefined.
   When $x = 0$, $\frac{1}{x}$ is undefined, so the vertical asymptote is at $x = 0$. Draw this as a dotted line on the axes.
2. As $\frac{1}{x}$ can never equal 0, the function is undefined when $y = -2$. So the horizontal asymptote is at $y = -2$. Draw this as a dotted line on the axes.
3. The coefficient of the reciprocal term is positive, so draw a sketch with the same shape as $y = \frac{1}{x}$ positioned between the two asymptotes.
4. Check the graph does what you'd expect:
   When x is very small, $\frac{1}{x}$ is very large and when x is large, $\frac{1}{x}$ is very small.



Exercise 1

Q1 Copy and complete each table and draw graphs for the following reciprocal functions. Give any rounded numbers to 2 decimal places.

a) $y = \frac{4}{x}$

x	–5	–4	–3	–2	–1	–0.5	–0.1	0.1	0.5	1	2	3	4	5
$\frac{4}{x}$	–0.8			–2					8					

b) $y = \frac{1}{x} + 3$

x	–5	–4	–3	–2	–1	–0.5	–0.1	0.1	0.5	1	2	3	4	5
$\frac{1}{x}$			–0.33			–2					0.5			
+3			3			3					3			
$\frac{1}{x} + 3$			2.67			1					3.5			

Q2 a) For what value of x is the expression $y = \frac{1}{x-2}$ undefined?

b) For what value of y is the expression $y = \frac{1}{x-2}$ undefined?

c) Use your answers to a) and b) to give the equations of the asymptotes of the graph $y = \frac{1}{x-2}$.

d) Sketch the graph of $y = \frac{1}{x-2}$, and give the coordinates of the y-intercept of the graph.

Q3 Find the equations of the asymptotes of the following functions, and use these to sketch their graphs. Find the coordinates of any x- and y-intercepts.

a) $y = \frac{1}{x+5}$ b) $y = \frac{1}{2x} - 1$ c) $y = \frac{1}{3-x} + 10$

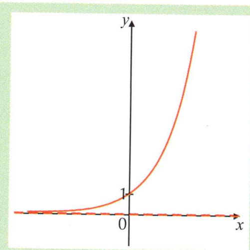
Exponential Graphs

Learning Objective — Spec Ref A12:
Be able to recognise, draw and sketch graphs of exponential functions.

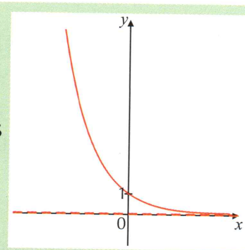
Prior Knowledge Check: Be able to use the laws of indices. See p.93-95.

Exponential functions have the form $y = a^x$, where $a > 0$. **Graphs** of $y = a^x$ always have the same curved shape and lie above the x-axis (i.e. y is positive for all values of x). They have a **horizontal asymptote** at $y = 0$, and a **y-intercept** at **(0, 1)** (as $a^0 = 1$ for any value of a). The bigger the value of a, the **steeper** the graph. **Sketches** of these functions will look like one of the graphs below:

When $a > 1$, as x increases, a^x quickly gets very large. As x gets more negative, a^x gets smaller and smaller but never reaches 0, so the **x-axis** is an **asymptote**.



When $0 < a < 1$, as x increases, a^x gets smaller and smaller but never reaches 0, so the **x-axis** is an **asymptote**. As x gets more negative, a^x quickly gets very large.



The graph of an exponential function of the form $y = a^x + b$ will have the same shape, but the **asymptote** will be at $y = b$, and the **y-intercept** will be at **(0, 1 + b)**.

Tip: This is a vertical translation of $y = a^x$ by $+b$ (see p.206).

Example 2

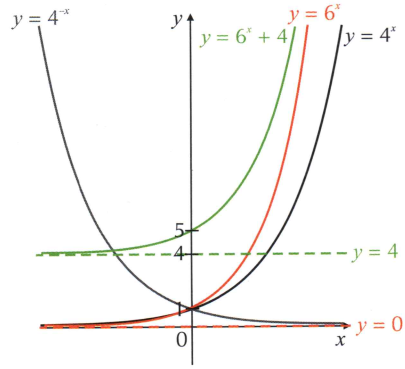
On the same axes, sketch the graphs of $y = 4^x$, $y = 4^{-x}$, $y = 6^x$ and $y = 6^x + 4$.

1. For $y = 4^x$, draw an exponential curve that increases as x increases, has an asymptote at $y = 0$, and a y-intercept at $(0, 1)$.

2. Get $y = 4^{-x}$ in the form $y = a^x$ using power laws:
   $y = 4^{-x} = \frac{1}{4^x} = \left(\frac{1}{4}\right)^x$
   This also has an asymptote at $y = 0$ and a y-intercept at $(0, 1)$, but $a < 1$, so draw an exponential curve that decreases as x increases. It will be the reflection of $y = 4^x$ in the y-axis.

3. The graph of $y = 6^x$ is the same as $y = 4^x$ but steeper.

4. The graph of $y = 6^x + 4$ has the same shape as $y = 6^x$, but with a different asymptote and intercepts:



As x gets more negative, 6^x approaches 0, and so y approaches $0 + 4 = 4$.
So the asymptote is at $y = 4$. The y-intercept is at $x = 0 \Rightarrow y = 6^0 + 4 = 1 + 4 = 5$.

You can also **draw** a more accurate graph using a **table of coordinates**, as usual.
You'll need to draw in the **horizontal asymptote** as you would for a sketch.

Exercise 2

Q1 Copy and complete each table and draw graphs for the following exponential functions.
Give any rounded numbers to 2 decimal places.

a) $y = 3^x$

x	–3	–2	–1	0	1	2	3
3^x		0.11					27

b) $y = 2^{-x}$

x	–3	–2	–1	0	1	2	3
2^{-x}		4		1			

Q2 For each of the following graphs, write down the equation of the asymptote and the point where each graph crosses the y-axis. Use your answers to help you sketch the graphs.

a) $y = 5^x$ b) $y = 2^x - 1$ c) $y = 10^{-x}$

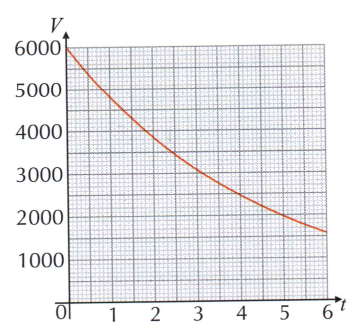
d) $y = 0.1^x$ e) $y = 0.5^x + 3$ f) $y = 10 - 3^x$ PROBLEM SOLVING

Q3 £100 is invested in a bank account that pays 5% interest per year.
The amount of money in the account (in pounds) after x years is equal to 100×1.05^x.
Sketch a graph of $y = 100 \times 1.05^x$ for values of x from 0 to 10, to show how the amount of money in the account changes over 10 years.

Q4 The exponential graph on the right shows how the value of a car in pounds, V, changes with t, the car's age in years. PROBLEM SOLVING

a) If $V = k \times 0.8^t$, use the graph to find the value of the constant k.

b) What does the value of k mean in the context of the question?



16.4 Circle Graphs

Circles can be described by an equation and plotted on coordinate axes. The equation of a circle tells you the fixed distance of the curve from its centre — also known as the radius.

Learning Objective — Spec Ref A16:
Be able to draw graphs of circles from their equation.

The equation of a circle with **radius r** and **centre (0, 0)** is $x^2 + y^2 = r^2$.

To **draw a circle** from its equation, take the **square root** to find the value of r, then draw the circle through $\pm r$ on both axes using a pair of **compasses**, with the **centre at (0, 0)**.

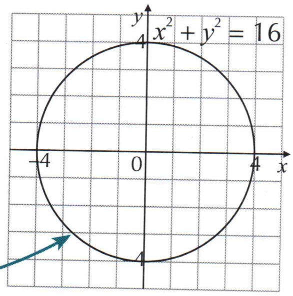
Tip: Tangents to circle graphs are covered on page 224.

To **write** the **equation** of a given circle centred at the origin, put its **radius** into the equation given above.

Example 1

Sketch the graph of the equation $x^2 + y^2 + 1 = 17$.

1. Rearrange the equation to get it into the form $x^2 + y^2 = r^2$. $\quad x^2 + y^2 + 1 = 17 \Rightarrow x^2 + y^2 = 16$
2. Compare it with $x^2 + y^2 = r^2$ to find r^2. $\quad r^2 = 16$
3. Take square roots to find r. $\quad r = \sqrt{16} = 4$
4. Use a pair of compasses to draw the circle with a centre at (0, 0) and a radius of 4.



Exercise 1

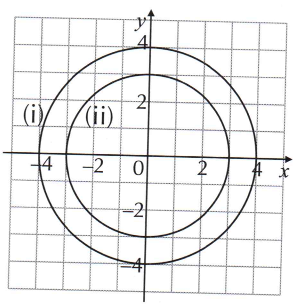
Q1 Sketch the graphs with the following equations. Label the coordinates of the x- and y-intercepts.

a) $x^2 + y^2 = 25$ b) $x^2 + y^2 = 1$ c) $x^2 + y^2 = 6.25$

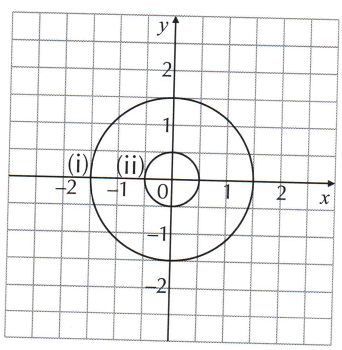
d) $y^2 = 4 - x^2$ e) $2x^2 + 2y^2 = 18$ f) $x^2 + 0.04 = 0.2 - y^2$

Q2 Write down the radius and equation of each of the following circles.

a)



b)



Q3 A circle with centre (0, 0) and diameter 10 is drawn on the same axes as the graph of $y + x = 1$. Draw both graphs and find the coordinates of their points of intersection.

16.5 Trigonometric Graphs

When you see the words sine, cosine and tangent, you'll probably think of right-angled triangles — but if you plot a graph of sin x, cos x or tan x against the angle x, you'll get a very distinctive and useful curve.

Learning Objective — Spec Ref A12:
Be able to recognise and sketch graphs of trigonometric functions.

Prior Knowledge Check:
Be familiar with the three trigonometric ratios and their values for common angles. See p.323-327.

Equations of the form $y = \sin x$, $y = \cos x$ and $y = \tan x$ (where x is an angle) are known as **trigonometric functions**.

The **graphs** of these trigonometric functions are **periodic** — they all have **patterns** that **repeat**. Once you know the **key features** of a pattern, such as the **intercepts**, **asymptotes** and coordinates of **maximum and minimum points**, you can **sketch the graph** for any range of angles by:

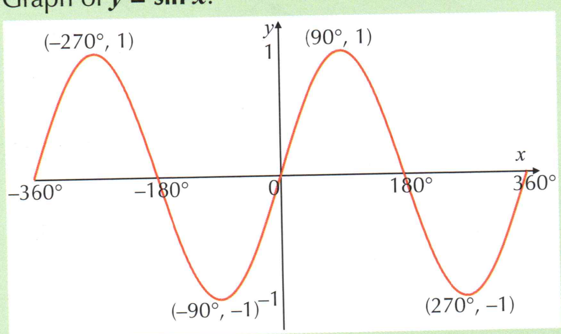
- **Plotting** the **key points** in a pattern and **joining** them with a **smooth curve**,
- Extending as far as necessary in either direction by **repeating** the pattern.

You can also **draw** accurate graphs using a **table of coordinates**, and **read off** other values from the graph.

Sine and Cosine Graphs

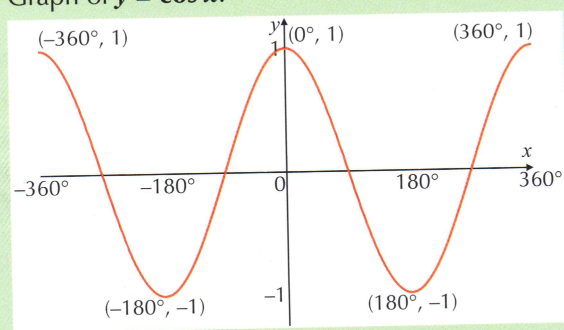
Sine and cosine have similar graphs — often described as a **sine 'wave'** ∿ and a **cos 'bucket'** ⋁.
Each function takes values **between –1 and 1**. They both **repeat every 360°** — they have a **period** of 360°.

Graph of $y = \sin x$:



- Crosses **y-axis** at **(0°, 0)**
- Crosses **x-axis** at **(0°, 0)**, **(180°, 0)**, **(360°, 0)**...
- **Maximum points** at **(–270°, 1)**, **(90°, 1)**...
- **Minimum points** at **(–90°, –1)**, **(270°, –1)**...

Graph of $y = \cos x$:



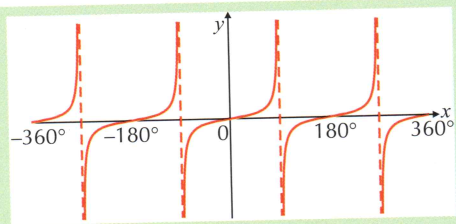
- Crosses **y-axis** at **(0°, 1)**
- Crosses **x-axis** at **(–90°, 0)**, **(90°, 0)**, **(270°, 0)**...
- **Maximum points** at **(–360°, 1)**, **(0°, 1)**, **(360°, 1)**...
- **Minimum points** at **(–180°, –1)**, **(180°, –1)**...

Tangent Graph

The graph of tangent has a **period of 180°** — the pattern repeats every 180°.

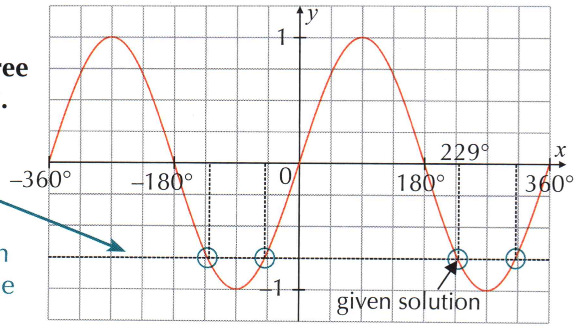
Graph of $y = \tan x$:

- **Vertical asymptotes** at x = **–90°, 90°, 270°**...
  So tan x is **undefined** at these points.
- Crosses **y-axis** at **(0°, 0)**.
- Crosses **x-axis** at **(0°, 0)**, **(180°, 0)**, **(360°, 0)**...
- y can take **any value** from **–∞ to ∞**.



Example 1

Given that one solution to sin x = –0.75 is x = 229° (to the nearest degree), use the graph of y = sin x on the right to find all the values of x to the nearest degree for which sin x = –0.75 in the range –360° ≤ x ≤ 360°.



1. Draw a horizontal line at y = –0.75.
2. The values of x where this line intersects the graph are the solutions to sin x = –0.75.
3. There are four solutions in the range. One of them was given in the question, so use this value and the symmetry of the graph to find the other solutions.
4. The graph between 180° and 360° has a line of symmetry at x = 270°, so the other positive solution will be the same distance from 360° as 229° is from 180°.
5. Using the same method, the negative solutions will be 49° away from 0° and –180°.

229° – 180° = 49° away from 180°
So the other positive solution is at 360° – 49° = **311°**

The negative solutions are
0° – 49° = **–49°** and –180° + 49° = **–131°**

Exercise 1

Q1 Sketch graphs of the following equations for values of x between 0° and 720°. Label the position of the y-intercept, x-intercepts and any turning points and asymptotes.

a) y = sin x b) y = cos x c) y = tan x

In Questions 2-3, give rounded numbers to 2 d.p., and draw graphs for values of x between 0° and 360°.

Q2 a) Copy and complete the table and draw the graph of y = 2 sin x.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
sin x		0.5				0.87			0						–0.71		
2 sin x		1				1.73			0						–1.41		

b) Use your graph to find, to the nearest degree, the values of x between 0° and 360° when:
(i) 2 sin x = 1.2 (ii) 2 sin x = –0.8

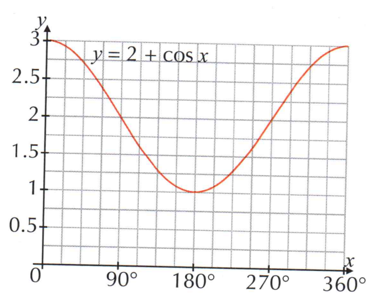
Q3 a) Copy and complete the table and draw the graph of y = sin 2x.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
2x		60°			180°					420°				600°			
sin 2x		0.87			0					0.87				–0.87			

b) Describe the connection between the graph of y = sin 2x and the graph of y = sin x.

Q4 The graph of y = 2 + cos x is shown on the right. One solution to 2 + cos x = 1.75 is x = 104° to the nearest degree. Use the graph to find all the solutions of 2 + cos x = 1.75 in the following intervals:

a) 0° ≤ x ≤ 360° b) 360° ≤ x ≤ 720°



Q5 a) Sketch the graph of y = tan (x + 180°) for values of x between –180° and 180°.

b) Explain how the graph is related to the graph of y = tan x.

16.6 Transforming Graphs

The graph of a function can be transformed to give a new graph that is a translation or a reflection of the original. The equation of the function is also altered (depending on the type of transformation).

Translations

Learning Objective — Spec Ref A13:
Be able to translate graphs horizontally and vertically.

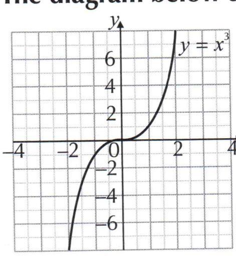
Prior Knowledge Check:
Be able to use function notation (see p.228) and vector notation (see p.339).

Translations can be either **horizontal** (i.e. left or right) or **vertical** (i.e. up or down). When a graph is **translated**, the **shape** of the graph **stays the same** but it **moves** (or slides) on the coordinate grid. If the original graph has equation $y = f(x)$, the **equation** and **coordinates** of the translated graph **change** as follows:

- A **horizontal** translation of ***a*** **units right** has equation $\mathbf{y = f(x - a)}$. This can also be described using a translation vector: $\begin{pmatrix} a \\ 0 \end{pmatrix}$. After a horizontal translation, the ***x*-coordinates** of all points increase by a.
- A **vertical** translation of ***a*** **units up** has equation $\mathbf{y = f(x) + a}$. This can also be described using a translation vector: $\begin{pmatrix} 0 \\ a \end{pmatrix}$. After a vertical translation, the ***y*-coordinates** of all points increase by a.

Example 1

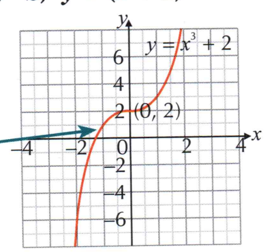
The diagram below shows the graph of $y = x^3$. Draw the graph of: a) $y = x^3 + 2$, b) $y = (x + 2)^3$



a) 1. $y = x^3 + 2$ is a transformation of the form $y = f(x) + a$, where $a = 2$.

2. So translate the graph of $y = x^3$ by 2 units up, i.e. by the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

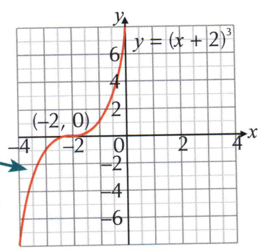
3. Add 2 to the y-coordinates of any key points (such as the y-intercept) and label them.



b) 1. $y = (x + 2)^3$ is a transformation of the form $y = f(x - a)$, where $a = -2$.

2. So translate the graph of $y = x^3$ by 2 units left (or –2 units right), i.e. by the vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$.

3. Add –2 to the x-coordinates of any key points (such as the x-intercept) and label them.



Tip: Take care with the +/– signs for horizontal translations. '$+ a$' in the brackets is a translation of a units left, or $-a$ units right.

Exercise 1

Q1 For each of the functions below:
(i) Describe the translation, stating the translation vector,
(ii) Find the coordinates of the translated point that had coordinates (0, 0) on the graph of $y = f(x)$.

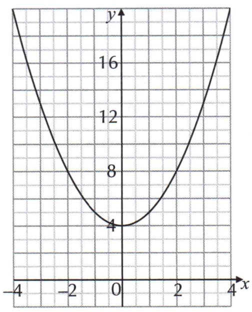
a) $y = f(x) + 3$ b) $y = f(x - 1)$ c) $y = f(x + 2)$ d) $y = f(x) - 6$

Q2 Each of these functions is a translation of the function $y = x^2$.
For each function, describe the translation and sketch the graph.

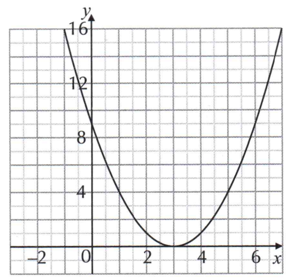
a) $y = x^2 + 1$ b) $y = x^2 - 2$ c) $y = (x - 4)^2$ d) $y = (x + 1)^2$

Q3 These graphs are translations of the graph $y = x^2$. Find the equation of each graph.

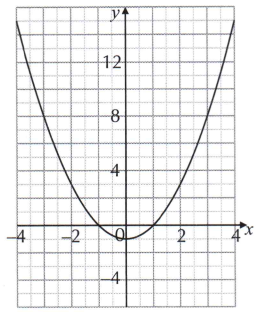
a)



b)



c)

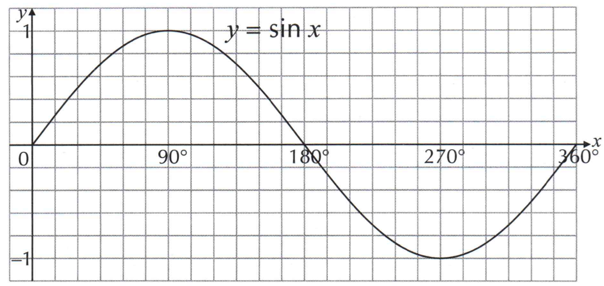


Q4 Each of the following functions is a translation of the function $y = \sin x$. For each one, describe the translation and sketch the graph for $0° \le x \le 360°$.

a) $y = (\sin x) + 1$ b) $y = (\sin x) - 2$

c) $y = \sin (x + 60°)$ d) $y = \sin (x - 90°)$

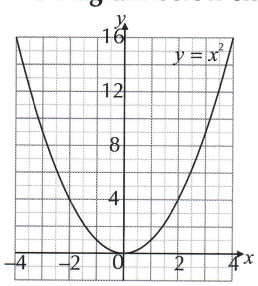
e) $y = \sin (x + 180°)$ f) $y = \sin (x - 360°)$



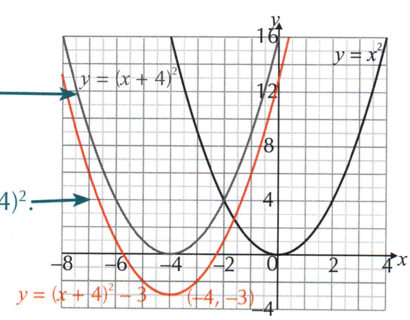
You can also perform a **combination** of translations on a graph. Just do them **one at a time**.

Example 2

The diagram below shows the graph of $y = x^2$. Draw the graph of $y = (x + 4)^2 - 3$.



1. $y = (x + 4)^2$ is a transformation of the form $y = f(x - a)$, where $a = -4$. So it's a translation of 4 units left from $y = x^2$.
2. $y = (x + 4)^2 - 3$ is a transformation of the form $y = f(x) + a$, where $a = -3$. So it's a translation of 3 units down from $y = (x + 4)^2$.
3. In vector form, this is an overall translation of $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.
4. Label any key points, such as the turning point. Subtract 4 from the x-coordinate and subtract 3 from the y-coordinate.



Exercise 2

Q1 For each of the functions below:

(i) Describe the translation, stating the translation vector,

(ii) Find the coordinates of the translated point that had coordinates (0, 0) on the graph of $y = f(x)$.

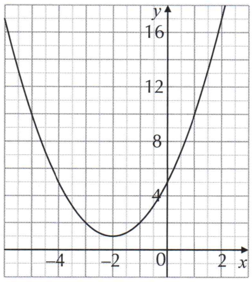
a) $y = f(x - 5) + 6$ b) $y = f(x + 6) - 4$ c) $y = f(x - 3) + 2$ d) $y = f(x + 1) + 9$

Q2 Each of these functions is a translation of the function $y = x^3$. For each function, describe the translation and sketch the graph.

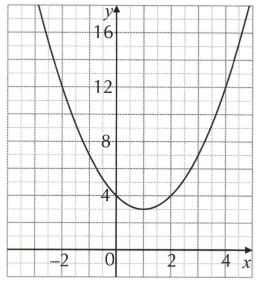
a) $y = (x - 5)^3 + 4$ b) $y = (x + 2)^3 - 3$ c) $y = (x - 4)^3 + 6.5$ d) $y = (x + 2.5)^3 - 3$

Q3 These graphs are translations of the graph $y = x^2$. Find the equation of each graph.

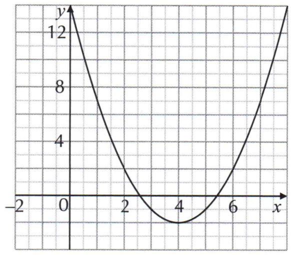
a)



b)



c)



Q4 For each of the following, give another equation that produces the same graph.

a) $y = \tan x$ b) $y = \cos(x + 180°)$ c) $y = \sin(x + 45°)$

PROBLEM SOLVING

Reflections

Learning Objective — Spec Ref A13:
Be able to reflect graphs in the x- or y-axis.

Prior Knowledge Check:
Be able to use function notation.
See p.228.

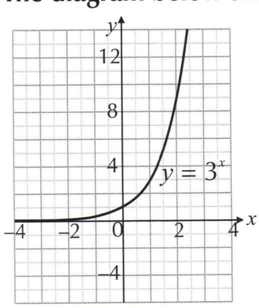
Graphs of functions can be **reflected** in the x- or y-axis. If the original graph has equation $y = f(x)$, the **equation** and **coordinates** of the reflected graph **change** as follows:

- A reflection in the **x-axis** has equation **$y = -f(x)$**. After a reflection in the x-axis, the **y-coordinates** of all the points are **multiplied by –1**.
- A reflection in the **y-axis** has equation **$y = f(-x)$**. After a reflection in the y-axis, the **x-coordinates** of all the points are **multiplied by –1**.

Tip: If a point doesn't change during the transformation, it's called an **invariant point**.

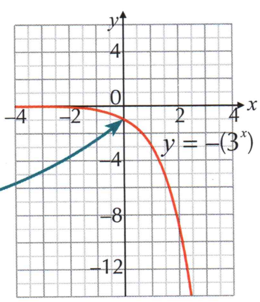
Example 3

The diagram below shows the graph of $y = 3^x$.



a) Draw the graph of $y = -(3^x)$.

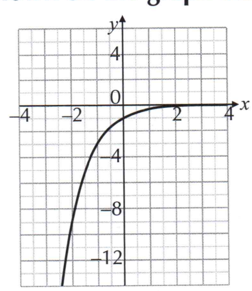
1. $y = -(3^x)$ is a transformation of the form $y = -f(x)$, which is a reflection in the x-axis.
2. The y-intercept has moved from (0, 1) to (0, –1).



b) Find the equation of the transformation of $y = 3^x$ shown on the graph below.

1. The y-intercept is (0, –1) — the same as the graph of $y = -(3^x)$ above. All the other points are reflections of $y = -(3^x)$ in the y-axis.
2. $y = f(-x)$ represents a reflection in the y-axis, so apply this to $y = -(3^x)$ to find the equation of the transformed graph:

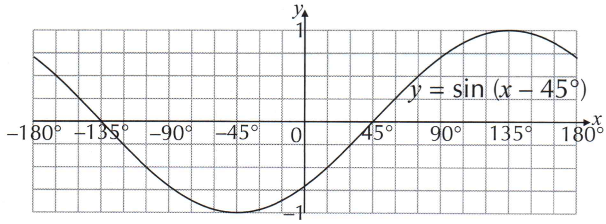
$f(x) = -(3^x)$ so $y = f(-x)$ $\Rightarrow$ $\mathbf{y = -(3^{-x})}$



Tip: It can help to look at a couple of key points, such as intercepts or turning points, and see where they have moved to after the transformation.

Example 4

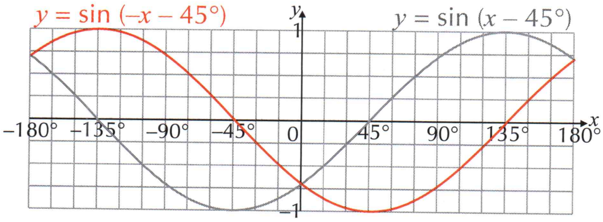
The diagram on the right shows the graph of $y = \sin(x - 45°)$ for $-180° \leq x \leq 180°$.



Use this graph to draw graphs of the following equations:

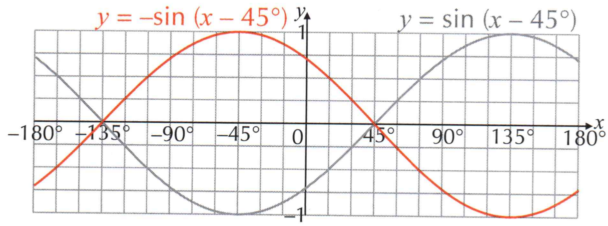
a) $y = \sin(-x - 45°)$

1. $y = \sin(-x - 45°)$ is a transformation of the form $y = f(-x)$, which is a reflection of $y = \sin(x - 45°)$ in the y-axis.
2. The y-intercept will stay the same, but the turning points at (–45°, –1) and (135°, 1) will move to (45°, –1) and (–135°, 1):



b) $y = -\sin(x - 45°)$

1. $y = -\sin(x - 45°)$ is a transformation of the form $y = -f(x)$, which is a reflection of $y = \sin(x - 45°)$ in the x-axis.
2. The x-intercepts will stay the same, but the turning points at (–45°, –1) and (135°, 1) will move to (–45°, 1) and (135°, –1):

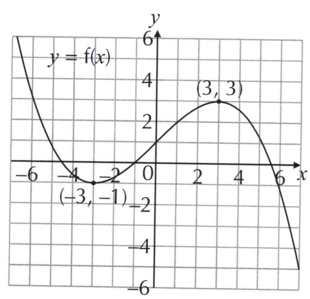


Exercise 3

Q1 The graph on the right shows the function $y = f(x)$.
The graphs of the following functions are reflections of $y = f(x)$.
In each case, describe the reflection and sketch the graph, labelling the new coordinates of the turning points.

a) $y = f(-x)$

b) $y = -f(x)$



Q2 Sketch the graphs of each of the following pairs of functions.
Use a single set of axes for each pair of graphs.

a) $y = \sin x, \ y = -\sin x$

b) $y = \cos x, \ y = -\cos x$

c) $y = x^2, \ y = -x^2$

d) $y = \frac{1}{x}, \ y = -\frac{1}{x}$

e) $y = x - 2, \ y = -x - 2$

f) $y = 2^x, \ y = 2^{-x}$

Q3 Sketch the graphs of $y = x^3$, $y = -x^3$ and $y = (-x)^3$. Explain why two of the graphs are the same.

Q4 The graph of $y = (-x - 2)^2$ is obtained by applying a translation then a reflection to the graph of $y = x^2$.

a) Sketch the graph of $y = x^2$.

b) Apply a translation to the graph of $y = x^2$ to give the graph of $y = (x - 2)^2$.

c) Apply a reflection to your graph from part b) to give the graph of $y = (-x - 2)^2$.

Q5 For the following pairs of functions, describe the transformations that transform the graph of the first function to the graph of the second.

a) $y = x^2, \ y = -(x - 1)^2$

b) $y = x^2, \ y = (-x - 3)^2$

c) $y = \cos x, \ y = -\cos(x + 90°)$

d) $y = \sin x, \ y = \sin(-x) + 1$

Review Exercise

Q1 A stone is dropped from the top of a 55 m-high tower. The distance in metres, h, between the stone and the ground after t seconds is given by the formula $h = 55 - 5t^2$.

a) Draw a graph of the height of the stone for $h \geq 0$ and values of t between 0 and 5 seconds.

b) Use your graph to estimate how long it takes the stone to fall to a height of 20 m above the ground.

c) Find the exact time it takes for the stone to reach the ground.

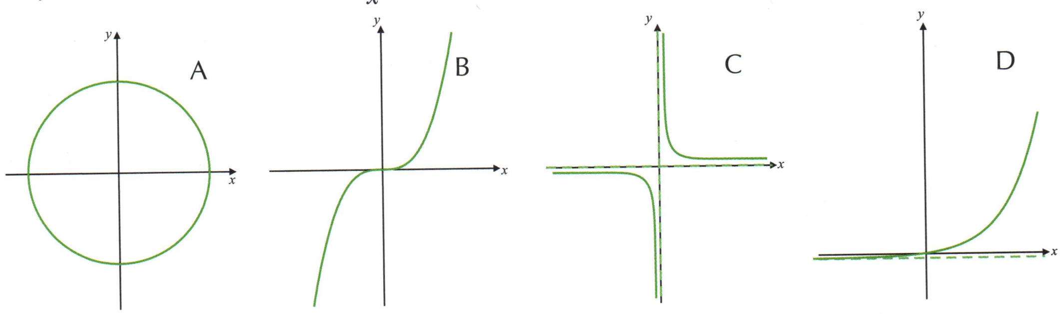
Q2 By completing the square, calculate the position of the y-intercept and turning point of the graph of $y = x^2 - x - 1$.

Q3 Copy and complete the table below and draw a graph of the equation $y = x^3 - 3x^2 - x + 3$.

x	−3	−2	−1	0	1	2	3	4
y		−15					0	

Q4 Match each of these equations to one of the graphs below.

(i) $y = x^3$ (ii) $y = \frac{1}{x}$ (iii) $y = 2^x - 1$ (iv) $x^2 + y^2 = 1$



Q5 Susan uses a credit card to buy a computer that costs £500. The credit card company charges 3% interest per month. If she pays nothing back, the amount in pounds Susan will owe after t months is given by the formula $b = 500 \times (1.03)^t$.

a) Explain what the numbers 500 and 1.03 represent in the formula.

b) Draw a graph to show how this debt will increase during one year if Susan doesn't repay any money.

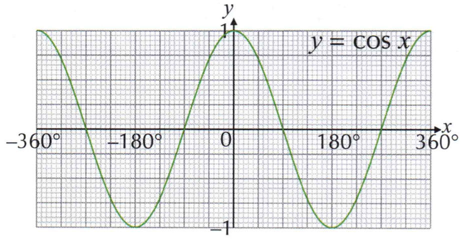
c) Use your graph to estimate how long the interest on the credit card will take to reach £100.

Q6 Point P has coordinates (3, 4).

a) Use Pythagoras' theorem to find the distance of P from the origin.

b) Use your answer to a) to write the equation of the circle with centre (0, 0) that passes through point P.

Q7 The graph of $y = \cos x$ is shown on the right.



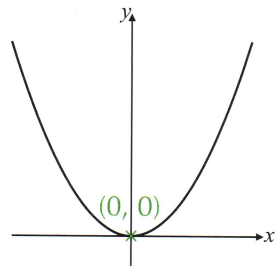
a) Use the graph to find the values of x between –360° and 360° for which:

(i) $\cos x = 0.25$ (ii) $\cos x = -0.25$

b) If $\cos 50° = 0.643$, use the graph to write down all the values of x between –360° and 360° for which:

(i) $\cos x = 0.643$ (ii) $\cos x = -0.643$

Q8 The graph of $y = x^2$ has a minimum point at (0, 0).



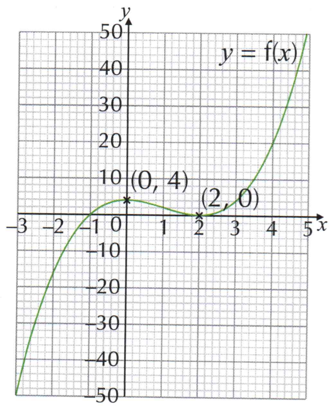
a) (i) The graph of $y = x^2$ is transformed to give the graph of $y = x^2 + 3$. Describe what this transformation does to the coordinates of each point on the original graph.

(ii) What are the coordinates of the minimum point of the graph of $y = x^2 + 3$?

b) Write down the coordinates of the minimum point of:

(i) $y = (x - 1)^2$ (ii) $y = (x + 3)^2 - 2$

Q9 A turning point is a place where the gradient of a graph changes from positive to negative, or from negative to positive. The graph of $y = f(x)$ shown on the right has two turning points, (0, 4) and (2, 0). Use the graph of $f(x)$ to sketch the graphs of the following functions. On each graph, mark the coordinates of the two turning points.



a) $f(x) - 1$ b) $f(x + 2)$

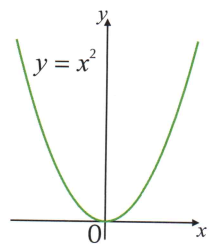
c) $f(-x)$ d) $-f(x)$

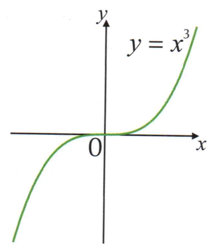
e) $f(x + 1) - 2$ f) $f(x - 2) - 3$

g) $-f(x + 1)$ h) $f(-x) - 1$

Q10 Even functions are functions with graphs that have the y-axis as a line of symmetry. $y = x^2$ is an example of an even function.

Odd functions are functions with graphs that have rotational symmetry of order 2 about the origin. $y = x^3$ is an example of an odd function.





Sketch the following graphs and say whether the functions are even, odd or neither.

PROBLEM SOLVING

a) $y = x^2 + 5$ b) $y = -x^3$ c) $y = \sin x$ d) $y = (x - 3)^2$

e) $y = \cos x$ f) $y = x^3 - 4$ g) $y = \tan x$ h) $y = \frac{1}{x}$

Exam-Style Questions

Q1 Sketch graphs of the following equations.
Label the coordinates of any intersections with the axes.

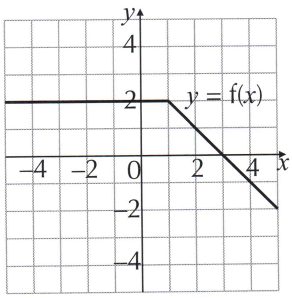
a) $y = x^3$

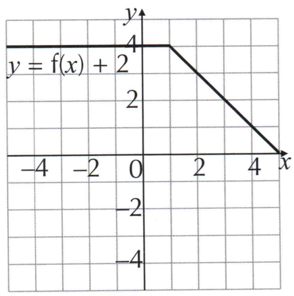
[1 mark]

b) $y = 1 - x^2$

[3 marks]

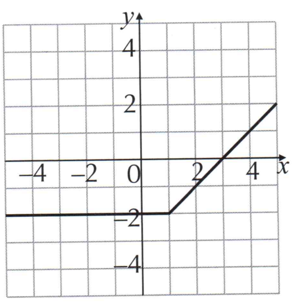
Q2 A function f(x) has the graph $y = f(x)$ as shown below left. The graph of $y = f(x)$ can be transformed to make the graph of $y = f(x) + 2$, as shown below right.





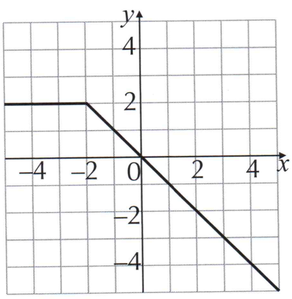
The following two graphs are two other transformations of $y = f(x)$.
For each graph, write its equation in function notation.

a)



[1 mark]

b)



[1 mark]

Q3 A circle with equation $x^2 + y^2 = 9$ is translated 2 units in the positive y-direction.

a) Sketch the translated graph, labelling the y-intercepts.

[2 marks]

b) Find the exact coordinates of the points where the translated graph crosses the x-axis.

[2 marks]

Q4 The table below shows pairs of values satisfying the equation $y = \frac{4}{x + 0.5}$.

x	0	0.5	1.5	3.5	7.5
y	p	4	q	1	r

a) Work out the values of p, q and r.

[1 mark]

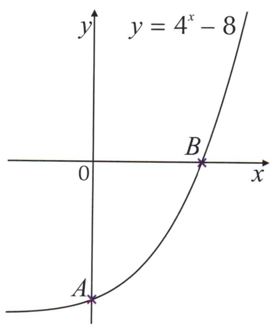
b) Use your values for p, q, r and the other values in the table to plot the graph of $y = \frac{4}{x + 0.5}$ for $0 \le x \le 7.5$ on suitable axes.

[2 marks]

c) By plotting the graph of $y = 6 - x$ on the same set of axes, explain how the graphs show that the equation $\frac{4}{x + 0.5} = 6 - x$ has 2 solutions.

[2 marks]

Q5 The graph of $y = 4^x - 8$ is sketched on the axes below.



Not drawn accurately

The graph crosses the y-axis at point A and the x-axis at point B.

a) Work out the coordinates of point A.

[1 mark]

b) Work out the coordinates of point B.

[2 marks]

Q6 Point P is on the graph of $y = \tan x°$ and has an x-coordinate of 60.

a) State the exact y-coordinate of P, giving your answer as a surd.

[1 mark]

The graph is transformed by the translation $\begin{pmatrix} 20 \\ 0 \end{pmatrix}$.

b) Find the equation of the transformed graph.

[1 mark]