14.1 Term to Term Rules

A sequence is a list of numbers or shapes which follows a particular rule. Each number or shape in the sequence is called a term. Term to term rules tell you how to get from one term to the next.

Number Sequences

Learning Objectives — Spec Ref A23/A24:

- Find rules for number sequences and use them to find terms in a sequence.
- Recognise and use sequences of square, cube and triangular numbers.

There are different types of number sequences which follow different rules to get from one term to the next. You'll come across the following types: **arithmetic**, **geometric**, **Fibonacci-type** and **quadratic** sequences, and sequences of **triangular**, **square** and **cube** numbers.

Arithmetic Sequences

Arithmetic sequences (also known as **linear** sequences or arithmetic progressions) are ones where the terms **increase** or **decrease** by the **same value** each time — known as the **common difference**. The sequences below are both arithmetic:

1, 7, 13, 19, 25 (+6, +6, +6, +6)

The rule is 'add 6 to the previous term'.

75.1, 74.5, 73.9, 73.3, 72.7 (−0.6, −0.6, −0.6, −0.6)

The rule is 'subtract 0.6 from the previous term'.

Geometric Sequences

Geometric sequences (or geometric progressions) are ones where consecutive terms are found by **multiplying** by the **same value** each time, known as the **common ratio**.

2, 8, 32, 128 (×4, ×4, ×4)

The rule is 'multiply the previous term by 4'. The common ratio is 4.

100, 50, 25, 12.5 (÷2, ÷2, ÷2)

The rule is 'divide the previous term by 2'. The common ratio is 0.5 since this is what you're multiplying by.

Tip: Terms in a geometric sequence are of the form ar^n, where n is integer (≥ 0) and r is the common ratio.

Example 1

Consider the sequence that starts 2, 6, 18, 54...

a) Describe the rule for finding the next term in the sequence, and name this type of sequence.

The difference between terms changes each time, so the rule for finding the next term involves multiplication.

$2 \xrightarrow{\times 3} 6 \xrightarrow{\times 3} 18 \xrightarrow{\times 3} 54...$

Consecutive terms are found by multiplying by the common ratio (which is $6 \div 2 = 3$, as you've just found), so...

So the rule is:
Multiply the previous term by 3.

It's a **geometric sequence**.

b) Write down the next three terms in the sequence.

Starting with 54, keep multiplying by 3.

54, **162**, **486**, **1458** (×3, ×3, ×3)

Exercise 1

Q1 Write down the first 5 terms of the sequence with:

a) first term = 14; term to term rule 'multiply the previous term by 4, then subtract 1'.

b) first term = 11; term to term rule 'multiply the previous term by –2, then add 1'.

Q2 The first four terms of a sequence are 3, 6, 12, 24.

a) Write down what you multiply each term in the sequence by to find the next term.

b) Write down the next three terms in the sequence.

Q3 For each of the following: (i) explain the rule for finding the next term in the sequence,
(ii) name the type of sequence,
(iii) find the next three terms in the sequence.

a) 3, 5, 7, 9...

b) 4, 12, 36, 108...

c) 192, 96, 48, 24...

d) 0, –4, –8, –12...

e) 16, 4, 1, $\frac{1}{4}$...

f) 1, –2, 4, –8...

Q4 Copy the following sequences and fill in the blanks.

a) 7, 13, 19, 25, ☐, 37

b) 9, 5, ☐, –3, –7, –11

c) ☐, 0.8, 3.2, 12.8, ☐

Q5 The first four terms of a sequence are –5, –2, 1, 4. Work out the 54th term of the sequence.

Q6 The first four terms of a sequence are 1, $2\sqrt{3}$, 12, $24\sqrt{3}$.
Find the 7th term of the sequence.

Fibonacci-Type Sequences

For **Fibonacci-type** sequences, the rule is always '**add together** the **two previous terms**'.

2, 2, 4, 6, 10, 16, 26
2 + 2, 2 + 4, 4 + 6, 6 + 10, 10 + 16

Quadratic Sequences

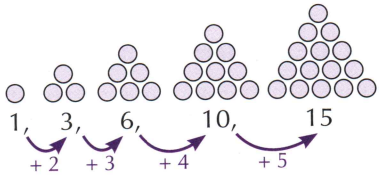
In a **quadratic** sequence, the **difference** between terms **changes** by the **same amount** each time.

1, 2, 4, 7, 11, 16, 22
+1, +2, +3, +4, +5, +6
+1, +1, +1, +1, +1

Sequences of Triangular, Square and Cube Numbers

Some sequences are harder to spot by their term to term rules, but you might recognise them as the **square numbers** (1, 4, 9...), **cube** numbers (1, 8, 27...) or **triangular numbers** (shown below):

To make the sequence of triangular numbers, start at 1 and then add 2, then 3, then 4, then 5 etc. to each new term.



Tip: These types of sequences are often linked to patterns in shapes, such as the areas or volumes of a sequence of squares or cubes. There's more about shape sequences on p.165.

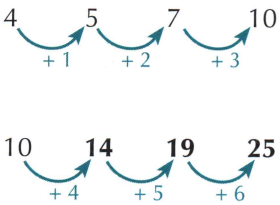
Other Sequences

Other sequences might have term to term rules that use a **combination** of different types of rules. You'll need to try and spot any **patterns** in the **differences** or **ratios** between the terms. If each term in a sequence is **greater** than the one before, it's an **ascending** sequence, and if each term is **less** than the one before, it's a **descending** sequence. Sequences can be **finite** (i.e. they stop) or **infinite** (never-ending).

Example 2

a) Find the next three terms in the sequence 4, 5, 7, 10...

1. Try finding the difference between neighbouring terms.
2. Here, the difference is increasing by 1 each time. (This means it's a quadratic sequence.)
3. Use this to find the next three terms in the sequence. Start with 10, and add 4, then add 5, then add 6.



b) The first four terms in a sequence are 8, 27, 64 and 125. What is the 9th term?

1. All the numbers in the sequence are cube numbers. So the 9th term will also be a cube number.
2. The first term is 2^3, the second is 3^3, and so on.
3. So the 9th term will be 10^3.

$10^3 = 10 \times 10 \times 10 =$ **1000**

Once you're familiar with the different ways that sequences can be generated, you can solve **algebraic** sequence problems. If you need to, look back at pages 113-115 for a recap on solving equations.

Example 3

The rule for finding the next term in a sequence is 'multiply the previous term by 3, then add on x'. The 1st term is 2 and the 4th term is 119. Find the value of x.

1. Use the rule to write down an expression for each term until you reach the 4th term.

$$\begin{aligned}\text{1st term} &= 2\\ \text{2nd term} &= (3 \times 2) + x = 6 + x\\ \text{3rd term} &= 3(6 + x) + x = 18 + 3x + x = 18 + 4x\\ \text{4th term} &= 3(18 + 4x) + x = 54 + 12x + x = 54 + 13x\end{aligned}$$

2. You know that the 4th term is 119, so you can write down an equation for the 4th term. Then just solve the equation to find x.

$$\begin{aligned}54 + 13x &= 119\\ 13x &= 119 - 54 = 65\\ x &= 65 \div 13 = \mathbf{5}\end{aligned}$$

Exercise 2

Q1 For each sequence below, find:
(i) the difference between consecutive terms for the first 5 terms,
(ii) the next three terms in the sequence.

a) 3, 4, 6, 9, 13...

b) 20, 18, 15, 11, 6...

c) 1, 2, 0, 3, –1...

Q2 The sequence 1, 1, 2, 3, 5... is a Fibonacci-type sequence, usually referred to as 'the Fibonacci sequence'. Find the next three terms in the sequence.

Q3 The first 5 terms of a sequence are 1, 1, 2, 6, 24.

a) For the first 5 terms, find the number you multiply by each time to get the next term.

b) Find the next two terms in the sequence.

Q4 Find the 10th term in the sequence of triangular numbers.

Q5 A sequence begins 0, 3, 8, 15, 24...

a) What type of sequence is this?

b) How does it relate to the sequence of square numbers?

c) Find the 100th term in the sequence. PROBLEM SOLVING

Q6 Find b for each of the sequences described below.

a) 1st term = 1, 4th term = 43; term to term rule = multiply the previous term by 4, then subtract b.

b) 1st term = 9, 3rd term = –135; term to term rule = multiply the previous term by b, then subtract 54.

Q7 The first three terms of a quadratic sequence are 5, $5 + x$ and $7 + 2x$.
Find a factorised expression, in terms of x, for the 6th term in the sequence. PROBLEM SOLVING

Using Sequence Notation

Sequences and their term to term rules can be given in **algebraic notation**.

- A term can be written x_n or u_n, where n tells you the **position** of the term in the sequence — so x_1 is the first term, x_2 is the second term, etc. Sometimes the first term is called x_0 (instead of x_1), in which case x_1 is the second term, etc.
- The term to term rule can then be given as a **formula** for x_{n+1} in terms of x_n.
  In other words, it tells you how to get from one term (x_n) to the next (x_{n+1}).
  For example, the formula for the rule 'add two to the previous term' is $x_{n+1} = x_n + 2$.
- You're often given a **starting value** (e.g. x_1) and from there you can work out x_2, x_3 etc.

Example 4

The rule for finding the next term in a sequence is $x_{n+1} = 2x_n + 3$ where $x_1 = 4$.
What is the value of x_5?

1. Substitute x_1 into the formula for x_n to get x_2. $x_2 = 2x_1 + 3 = (2 \times 4) + 3 = 11$
2. Substitute x_2 into the formula for x_n to get x_3. $x_3 = 2x_2 + 3 = (2 \times 11) + 3 = 25$
3. Substitute x_3 to get x_4... $x_4 = 2x_3 + 3 = (2 \times 25) + 3 = 53$
4. ...and then x_4 to get x_5. $x_5 = 2x_4 + 3 = (2 \times 53) + 3 = \mathbf{109}$

Exercise 3

Q1 In each of the following, use the sequence rules and the values of x_1 to find the value of x_6.

a) $x_{n+1} = x_n + 5$ where $x_1 = 3$ b) $x_{n+1} = x_n - 7$ where $x_1 = 20$ c) $x_{n+1} = 3x_n$ where $x_1 = 2$

d) $x_{n+1} = 2x_n + 3$ where $x_1 = 3$ e) $x_{n+1} = \frac{1}{2}x_n + 2$ where $x_1 = 4$ f) $x_{n+1} = \frac{1}{2}x_n + 8$ where $x_1 = 8$

Q2 A sequence is generated using the rule $x_{n+1} = 2x_n - 6$ where $x_1 = 8$. Find the following:

a) x_4 b) x_6 c) $x_3 + x_5$

Q3 A sequence with the rule $u_{n+1} = 3u_n - 1$ has term $u_5 = 14$. Find the exact value of u_1. PROBLEM SOLVING

Shape Sequences

Learning Objective — Spec Ref A23/A24:
Find rules for patterns of shapes and use them to find patterns in a sequence.

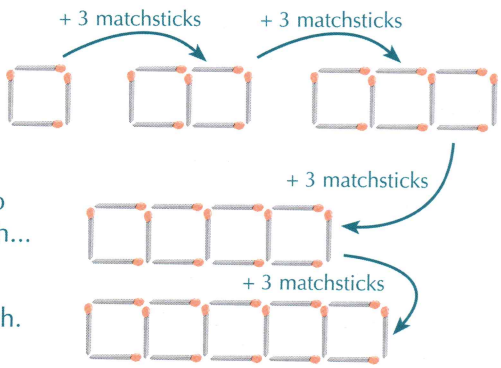
Patterns of **shapes** can form **sequences**. Just as with number sequences, you need to find the **rule** to get from one pattern to the next in the sequence. You can then use that rule to find other patterns.

Example 5

The matchstick shapes on the right form the first three patterns in a sequence.

a) Draw the fourth and fifth patterns in the sequence.

1. First work out how to get from one pattern to the next. You have to add 3 matchsticks to add an extra square to the previous pattern.
2. The fourth pattern is the next one in the sequence, so add 3 matchsticks to the third pattern to get the fourth...
3. ...and add another 3 matchsticks to that to get the fifth.



b) How many matchsticks are needed to make the eighth pattern in the sequence?

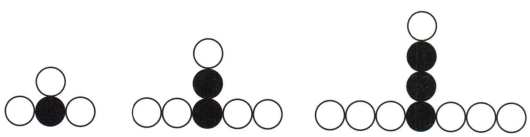
1. You don't need to draw all the patterns — just work with the numbers of matchsticks.
2. There are 16 matchsticks in the 5th pattern. You need to add on 3 lots of 3 matchsticks to get the 8th pattern.

The sequence of numbers of matchsticks is 4, 7, 10, 13, 16...

16 + 3 + 3 + 3 = **25 matchsticks**

Example 6

The black and white circles on the right form the first 3 patterns in a sequence. Find the number of black circles and the number of white circles in the 9th pattern.



1. Write the numbers of each colour of circle in each pattern as two number sequences.

   Black circles: 1, 2, 3
   White circles: 3, 5, 7

2. Find the term to term rules for each sequence.

   Rule for black circles: +1 each time.
   Rule for white circles: +2 each time.

3. Apply these rules 6 times, starting at the 3rd term, to get the 9th term.
   Black: add on 6 lots of 1 to the 3rd term.
   White: add on 6 lots of 2 to the 3rd term.

   3 + (6 × 1) = **9 black circles**
   7 + (6 × 2) = **19 white circles**

Tip: It can be easier to pick up on patterns in the numbers than patterns in the shapes.

Exercise 4

Q1 The first three patterns of some sequences are shown below. For each of the sequences:

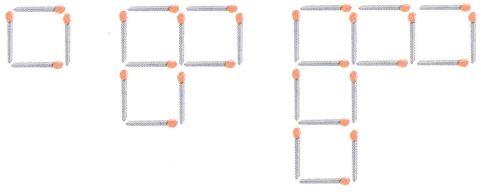
(i) Explain the rule for making the next pattern.

(ii) Draw the fourth and fifth patterns in the sequence.

(iii) Find the number of matches needed to make the sixth pattern.

a)

b)



c)

Q2 For each of the sequences below:

(i) Draw the next three patterns in the sequence.

(ii) Explain the rules for generating the number of circles of each colour in the next pattern.

(iii) Work out how many green circles there are in the 7th pattern.

(iv) Work out how many orange circles there are in the 10th pattern.

a)

b)

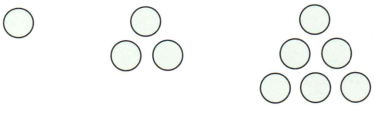
Q3 For each of the sequences below:

(i) Explain the rules for generating the number of circles in the next pattern.

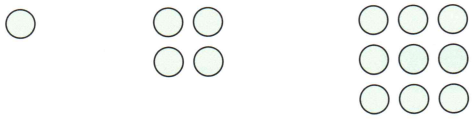
(ii) Name the special number sequence that the number of circles follows.

(iii) Draw the 6th pattern in the sequence.

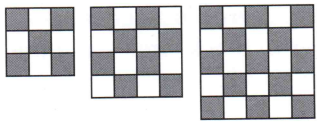
a)



b)



Q4 The designs shown on the right form the first three patterns in a sequence.



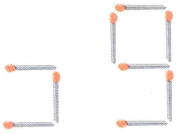
a) Draw the next pattern in the sequence.

b) Work out the number of white squares in the 6th pattern in the sequence.

c) Work out the number of grey squares in the pattern which has 100 squares in total.

PROBLEM SOLVING

Q5 The matchstick shapes below form the first two patterns in a shape sequence.



Describe two different ways that the sequence could be continued, and draw the next pattern in the sequence in each case.

14.2 Using the nth Term

Using a term to term rule is fine for the first few terms, but it can be a chore if you want the hundredth term. Luckily, there's a way to get from the position, n, to the term itself by using a formula for the 'nth term'.

Learning Objective — Spec Ref A23:
Use the nth term to find terms in a sequence.

Prior Knowledge Check:
Be able to substitute values into a given formula (p.107) and solve equations (p.113-115).

You can work out the value of a term in a sequence by using its **position** (n) in the sequence, and the **nth term rule** for that sequence. The 1st term (x_1) has $n = 1$, the 2nd term (x_2) has $n = 2$, the 10th term (x_{10}) has $n = 10$, and so on. The nth term formula tells you what to do to n to get the value of that term. For example, if the nth term was $2n + 1$, you'd multiply n by 2 then add 1 to get the term.

Example 1

The nth term of a sequence is $7n - 1$. Find the first four terms of the sequence.

To find the 1st, 2nd, 3rd and 4th terms of the sequence, substitute the values $n = 1$, $n = 2$, $n = 3$ and $n = 4$ into the formula.

$(7 \times 1) - 1 = 6$
$(7 \times 2) - 1 = 13$
$(7 \times 3) - 1 = 20$
$(7 \times 4) - 1 = 27$

So the first four terms are **6**, **13**, **20** and **27**.

Exercise 1

Q1 Find the first five terms of a sequence if the nth term is given by:

a) $n + 5$ b) $4n - 2$ c) $10 - n$ d) $10n - 8$

Q2 Find the first five terms of a sequence if the nth term is given by:

a) $n^2 + 1$ b) $2n^2 + 1$ c) $3n^2 - 1$ d) $n(n - 1)$

Q3 The nth term of a sequence is $100 - 3n$. Find the value of:

a) the 3rd term b) the 10th term c) the 30th term d) the 40th term

Q4 The nth term of a sequence can be found using the formula $x_n = 35 + 5n$. Find the value of:

a) x_1 b) x_5 c) x_{10} d) x_{100}

Q5 Each of the following gives the nth term of a different sequence.
For each sequence, find: (i) the 5th term, (ii) the 10th term, (iii) the 100th term.

a) $2n + 3$ b) $4n + 12$ c) $30 - 3n$ d) $-20 + 2n$

Q6 Each of the following rules generates a different sequence.
For each sequence, find: (i) x_4, (ii) x_{10}, (iii) x_{20}.

a) $x_n = \frac{1}{2}n + 30$ b) $x_n = 2n^2 + 8$ c) $x_n = 5n^3 - 50$ d) $x_n = 3n - n^2 + 100$

Q7 Each of the following gives the nth term of a different sequence.
For each sequence, find: (i) the 2nd term, (ii) the 5th term, (iii) the 20th term.

a) $2n^2 + 3$ b) $3(n^2 + 2)$ c) $n(n + 1)$ d) $n^3 + 2$

If you know the value of a term in a sequence, but not its position, set up and solve an **equation** using the nth term rule to find the position, n.

Example 2

The nth term of a sequence is $4n + 5$.

a) Which term has the value 41?

1. Make the nth term equal to 41.
2. Solve the equation to find n.

$4n + 5 = 41$
$4n = 36$
$n = 9$
So 41 is the **9th term**.

b) Which is the first term in this sequence to have a value greater than 100?

1. Find the value of n which would give a value of 100. As before, set up and solve an equation for n.
2. As long as the sequence is increasing, the first term that will give a value over 100 will be the next whole number value of n.
3. Check your answer by working out some terms in the sequence.

$4n + 5 = 100$
$4n = 95$
$n = 23.75$

So the first term with a value greater than 100 must be the **24th term**.

Check: 23rd term = $(4 \times 23) + 5 = 97$ (<100)
24th term = $(4 \times 24) + 5 = 101$ (>100) ✔

Tip: If you're confident using inequalities (see Section 13) you could solve $4n + 5 > 100$ to get the same answer.

Exercise 2

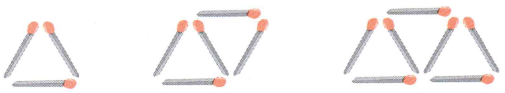
Q1 a) The nth term of a sequence is $7n + 4$. Which term of the sequence has the value 53?
b) The nth term of a sequence is $5n - 8$. Which term of the sequence has the value 37?

Q2 The nth term of a sequence is $17 - 2n$. Which term has the value: a) 9? b) –3?

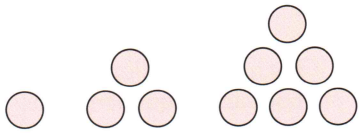
Q3 The nth term of a sequence is $n^2 - 1$. Which term has the value: a) 8? b) 99?

Q4 The nth term of a sequence is $4n - 10$.
Which term in the sequence is the first to have a value greater than 50?

Q5 The formula for the number of matches in the nth pattern of this 'matchstick sequence' is $2n + 1$.
Which pattern in the sequence is made using:
a) 17 matches? b) 55 matches?



Q6 The formula for the number of circles in the nth triangle in the sequence shown on the right is $\frac{n(n+1)}{2}$.
a) Find the number of circles needed to make the 58th triangle.
b) Which term in the sequence will be the first term to be made from over 200 circles?



Q7 The nth term of a sequence is $-n(n + 2)$.
a) Which term has the value –48?
b) Which is the first term to have a value less than –20?

You can use the nth term to **check** if a given value is a term in that sequence. As before, set up and solve an equation to find the value of n for a given (suspected) term. The term is **only** part of the sequence if n is a **whole number**. If not, then you know the value is **not** part of the sequence.

Example 3

A sequence has nth term $3n + 2$.

a) Is 37 a term in the sequence?

1. Form an equation by setting the formula for the nth term equal to 37. $3n + 2 = 37$
2. Solve your equation to find n. $3n = 35$, $n = 11.666...$
3. Since n is not a whole number, 37 is not a term in the sequence. So 37 is **not a term in the sequence**.

Tip: Don't just answer 'yes' or 'no' — you need to show whether or not n is a whole number.

b) Find the term in the sequence that is closest to 37.

1. From part a) you know that to get 37, $n = 11.666...$ So the term closest to 37 in the sequence will be where n is the closest whole number to 11.666... The closest whole number to 11.666... is 12.
2. Substitute $n = 12$ into the nth term formula. $3 \times 12 + 2 = 38$. The term in the sequence closest to 37 is **38**.

Exercise 3

Q1 Show that 80 is a term in the sequence with nth term equal to $3n - 1$.

Q2 A sequence has nth term $21 - 2n$.

a) Show that -1 is a term in this sequence.

b) Write down the position of -1 in the sequence.

Q3 a) Find the first four terms of the sequence with nth term $6(n + 1)$.

b) Determine whether or not 64 is a term in this sequence.

c) If 64 is a term in the sequence, write down its position. If not, find the number closest to 64 that is a term in the sequence.

Q4 A sequence has nth term $17 + 3n$.
Determine whether or not each of the following is a term in this sequence.

a) 52 b) 98 c) 105

d) 248 e) 996 f) $20n$

14.3 Finding the nth Term

You've seen how useful nth terms can be for working out the values of terms in a sequence. It's even more useful to be able to find the nth term for yourself. How you go about this depends on the type of sequence — there are different methods for arithmetic and quadratic sequences.

Arithmetic Sequences

Learning Objective — Spec Ref A25:
Find the nth term of an arithmetic sequence.

Prior Knowledge Check:
Be able to substitute values into a given formula (p.107), solve equations (p.113-115) and solve simultaneous equations (p.144-146).

As you saw on page 161, **arithmetic (linear) sequences** are ones where the terms increase or decrease by the same value each time (the **common difference**, d).

The nth term of an **arithmetic** sequence is given by $dn + c$.

To find c, look at the value produced by $d \times n$ and work out what you need to **add** or **subtract** to get from dn to the corresponding term in the sequence.

Example 1

a) Find the nth term of the sequence 45, 42, 39, 36...

Tip: If the terms are decreasing then d will be negative.

1. First, find the term to term rule to work out what type of sequence it is.

45 $\xrightarrow{-3}$ 42 $\xrightarrow{-3}$ 39 $\xrightarrow{-3}$ 36

2. It's an arithmetic sequence, so find the common difference, d, between the terms.

It decreases by the same amount (3) each time, so it's an arithmetic sequence.
Common difference $d = -3$

3. Work out what you need to add or subtract to get from $d \times n$ ($= -3n$) to the term in the sequence.

$-3n$:	-3	-6	-9	-12
	$\downarrow +48$	$\downarrow +48$	$\downarrow +48$	$\downarrow +48$
Term:	45	42	39	36

4. To get from $-3n$ to each term in the sequence, you have to add 48, so $c = +48$.

So the nth term is $\mathbf{-3n + 48}$.

5. Check that it works by substituting in a value of n.

Check: 2nd term ($n = 2$) is
$(-3 \times 2) + 48 = 42$ ✔

b) Find the 100th term in the sequence.

1. You've just worked out the nth term formula, so you can now use it to find the 100th term.

The nth term is $-3n + 48$.

2. Substitute $n = 100$ into $-3n + 48$.

$(-3 \times 100) + 48 = -300 + 48 = \mathbf{-252}$

Exercise 1

Q1 Find the formula for the nth term of each of the following sequences.

a) 7, 13, 19, 25... b) 4, 8, 12, 16... c) 41, 81, 121, 161... d) $-9, -5, -1, 3$...

Q2 For each of the following sequences: (i) Find the nth term formula. (ii) Find the 70th term.

a) 40, 37, 34, 31... b) 78, 69, 60, 51... c) 100, 92, 84, 76... d) $-10, -25, -40, -55$...

Q3 a) Find an expression for the nth term of sequence A, which starts 4, 7, 10, 13...

b) Find an expression for the nth term of sequence B, which starts 5, 8, 11, 14...

c) (i) How does each term in sequence B compare with the corresponding term in sequence A?

(ii) What do you notice about the formulas giving the nth term for these two sequences?

Q4 a) Find an expression for the nth term of the sequence 13, 9, 5, 1...

b) Use your answer to part a) to write down an expression for the nth term of the sequence 11, 7, 3, –1...

PROBLEM SOLVING

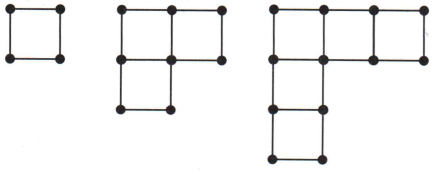
Q5 In the shape sequence on the right, the horizontal and vertical lines between each pair of dots are 1 unit long. Find:

PROBLEM SOLVING

a) the number of dots in the nth pattern in the sequence.

b) the area of the nth pattern in the sequence.

c) the number of dots and area of the 23rd pattern in the sequence.



Example 2

Find the nth term of the sequence $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}...$

1. Find the nth term for the numerator and denominator of the fractions separately.
2. The sequence in the numerator is easy to see — the nth term is just n.
3. Find the sequence in the denominator in the same way as any other arithmetic sequence. Find the common difference, d, and compare the sequence with dn to find c.
4. Combine the nth terms for the numerator and denominator to find the nth term of the sequence.

Numerator:

1 (+1) 2 (+1) 3 (+1) 4

nth term of the numerator = n

Denominator:

3 (+2) 5 (+2) 7 (+2) 9

Common difference $d = 2$

$2n$: 2 (+1) 4 (+1) 6 (+1) 8 (+1)

Term: 3 5 7 9

So nth term of the denominator = $2n + 1$

nth term of the sequence = $\frac{n}{2n+1}$

Tip: When you have a sequence of fractions, always check for separate rules in the top and bottom numbers.

Exercise 2

Q1 Find the nth term of each of the following sequences.

a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}...$ b) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}...$ c) $\frac{5}{2}, \frac{6}{3}, \frac{7}{4}, \frac{8}{5}...$

Q2 Find the nth term of each of the following sequences.

a) $\frac{1}{5}, \frac{4}{10}, \frac{7}{15}, \frac{10}{20}...$ b) $\frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{2}{1}...$ c) $\frac{5}{9}, \frac{8}{8}, \frac{11}{7}, \frac{14}{6}...$

Q3 Find the 100th term of the following sequence: $\frac{1}{30}, \frac{-1}{40}, \frac{-3}{50}, \frac{-5}{60}...$

PROBLEM SOLVING

There's another way of finding the ***n*th term** of an **arithmetic** sequence that uses the **first term** of the sequence, ***a***, as well as the **common difference**, ***d***:

The nth term of an **arithmetic** sequence is $\boldsymbol{a + (n - 1)d}$.

If you're given the first few terms of a sequence, **identify** ***a*** and ***d***, put them in the **formula**, then **simplify**.

Example 3

Find the *n*th term of the arithmetic sequence that begins –2, 5, 12, 19...

1. It's an arithmetic sequence, so find the value of the first term (a) and the common difference (d).

First term: $a = -2$
Difference: $d = +7$

2. Substitute a and d into the formula $a + (n - 1)d$.

$a + (n - 1)d = -2 + (n - 1) \times 7$

3. Simplify the expression.

$= -2 + 7n - 7 = 7n - 9$

4. Write down your formula.

So the nth term is $\boldsymbol{7n - 9}$.

5. Check that it works.

Check: 2nd term ($n = 2$) is $(7 \times 2) - 9 = 5$ ✔

If you're given just two terms and their positions in a sequence, you can still work out the nth term of the sequence using the formula $a + (n - 1)d$. You can either use **simultaneous equations** (as shown in Example 4 below), or take a more **problem-solving** approach (as shown in Example 5 on the next page).

Example 4

The 2nd term of an arithmetic sequence is 7 and the 5th term is 19.

a) Find the values of *a* and *d*.

1. Write equations for the 2nd and 5th terms using nth term $= a + (n - 1)d$.

2nd term ($n = 2$): $a + (2 - 1)d = 7$
$a + d = 7$ ①

5th term ($n = 5$): $a + (5 - 1)d = 19$
$a + 4d = 19$ ②

2. Solve the simultaneous equations. Subtract equation ① from equation ② to eliminate a.

$$\begin{array}{r} a + 4d = 19 \\ -(a + \ \ d = \ \ 7) \\ \hline 3d = 12 \end{array}$$

3. Solve the equation for d.

$d = 4$

4. Put $d = 4$ into one of the original equations and solve for a.

Using equation ①: $a + 4 = 7$
$a = 3$

So $\boldsymbol{a = 3}$ and $\boldsymbol{d = 4}$.

5. Put these values into equation ② to check.

Check: $a + 4d = 3 + 4 \times 4 = 19$ ✔

b) Find the *n*th term of the sequence.

1. Substitute the values of a and d you found above into the formula $a + (n - 1)d$.

$a + (n - 1)d = 3 + (n - 1) \times 4$

2. Simplify the expression.

$= 3 + 4n - 4$
$= 4n - 1$

3. Write down your formula.

So the nth term is $\boldsymbol{4n - 1}$.

4. Check it works for a term that you know.

Check: 2nd term ($n = 2$) is $(4 \times 2) - 1 = 7$ ✔

Example 5

a) Find a and d for an arithmetic sequence where the 1st term is 11 and the 6th term is 51.

1. You know a straight away — it's the 1st term. $a = 11$
2. Write an equation for the 6th term using nth term $= a + (n - 1)d$. $11 + (6 - 1)d = 51$
3. Solve the equation to find d. $11 + 5d = 51$, $5d = 51 - 11 = 40$, $d = 40 \div 5 = 8$

So $\boldsymbol{a = 11}$ and $\boldsymbol{d = 8}$.

b) Find a and d for an arithmetic sequence where the 3rd term is 8 and the 4th term is 5.

1. d is the difference between the 3rd and 4th terms, because they are consecutive. $d =$ 4th term $-$ 3rd term $= 5 - 8 = -3$
2. Write an equation for one of the terms using nth term $= a + (n - 1)d$. $a + (3 - 1) \times -3 = 8$
3. Solve the equation to find a. $a + 2 \times -3 = 8$, $a - 6 = 8$, $a = 8 + 6 = 14$

So $\boldsymbol{a = 14}$ and $\boldsymbol{d = -3}$.

Exercise 3

Q1 For each of the following arithmetic sequences, find:
(i) a and d, (ii) the nth term of the sequence.

a) 1st term = 3, 4th term = 9

b) 5th term = 17, 6th term = 21

c) 4th term = –8, 5th term = 2

Q2 In each of the following arithmetic sequences, you are given two of the terms in the form x_n. For each sequence, find the nth term of the sequence.

a) $x_3 = 15$, $x_6 = 30$ b) $x_2 = 12$, $x_5 = 33$ c) $x_4 = 25$, $x_7 = 49$

d) $x_2 = 13$, $x_6 = -7$ e) $x_3 = -3$, $x_6 = 24$ f) $x_6 = 61$, $x_9 = 97$

Q3 Given that the third term of an arithmetic sequence is 4 and the twelfth term is 1, find:

a) a and d

b) the nth term of the sequence

Q4 The 61st term of an arithmetic sequence is 546 and the 81st term is 726. Is 42 a term in this sequence?

Quadratic Sequences

Learning Objective — Spec Ref A25:
Find the nth term of a quadratic sequence.

Prior Knowledge Check:
Be able to find the nth term of an arithmetic sequence. See p.170.

In **quadratic sequences**, the difference between terms changes by the same amount each time (see p.162). It's this **constant**, **second difference** that allows you to find the ***n*th term** of a quadratic sequence.

The nth term of a **quadratic sequence** is $\boldsymbol{an^2 + bn + c}$
where a, b and c are constants and $a \neq 0$.

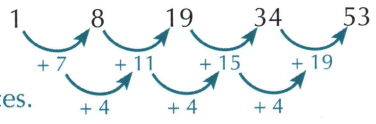
There are more steps involved in working out the nth term than for an arithmetic sequence. You need to:

- Find the **constant second difference**.
- **Divide** it by **2** to get $\boldsymbol{a}$.
- **Subtract** $\boldsymbol{an^2}$ from each term in the sequence.
- Find the ***n*th term** of the remaining **linear** (arithmetic) sequence — this is $\boldsymbol{bn + c}$.

Example 6

a) Find the *n*th term of the sequence that begins 1, 8, 19, 34, 53...

1. Find the (first) difference between each pair of terms. It's not constant, so the sequence isn't arithmetic.



2. Work out the (second) difference between the differences. It's constant (+4), so the sequence must be quadratic.

3. Divide the constant difference by 2 to get a, the coefficient of the n^2 term.

$a = 4 \div 2 = 2$
So the rule contains a $2n^2$ term.

4. For each term in the sequence, subtract the corresponding $2n^2$ term. This leaves a linear sequence.

Term:	1	8	19	34	53
$2n^2$:	2	8	18	32	50
Term $- 2n^2$:	-1	0	1	2	3

5. Work out the rule for the remaining linear sequence (using either the method on p.172 as shown here, or p.170) and add it to the $2n^2$ term.

The rule for the linear sequence is
$-1 + (n - 1) \times 1 = n - 2$.
So the rule for the sequence is $\boldsymbol{2n^2 + n - 2}$.

b) Find the 10th term of the sequence above.

1. Once you've found the nth term, substitute $n = 10$ into the formula.

10th term ($n = 10$) $= 2 \times 10^2 + 10 - 2$
$= 200 + 10 - 2 = 208$

2. This gives you the value of the 10th term.

So the 10th term is **208**.

Exercise 4

Q1 For each of the following quadratic sequences, find: (i) the nth term, (ii) the 10th term.

a) 1, 5, 11, 19, 29...
b) 5, 12, 21, 32, 45...
c) 5, 12, 23, 38, 57...
d) 7, 20, 39, 64, 95...
e) 6, 24, 52, 90, 138...
f) 10, 23, 44, 73, 110...

Review Exercise

Q1 For each of the following: (i) Explain the rule for finding the next term in the sequence.
(ii) Find the next three terms in the sequence.

a) 1, 2, 4, 8... b) 4, 7, 10, 13... c) 5, 3, 1, –1...

d) 1, 1.5, 2, 2.5... e) 0.01, 0.1, 1, 10... f) –2, –6, –18, –54...

Q2 Copy the following sequences and fill in the blanks.

a) □, –4, –16, –64, □, –1024 b) –72, □, –18, –9, □, –2.25 c) –63, –55, □, □, □, –23

Q3 The matchstick shapes on the right form the first three patterns in a sequence. Find the number of matchsticks in the 10th pattern.

Q4 Find the first five terms of a sequence if the nth term is given by:

a) $3n + 2$ b) $5n - 1$ c) $3 - 4n$ d) $-7 - 3n$

Q5 Each of the following rules generates a different sequence.
For each sequence, find: (i) x_4 (ii) x_{10} (iii) x_{20}

a) $x_n = 8n + 4$ b) $x_n = 450 - 5n$ c) $x_n = 3n^2$ d) $x_n = n^2 + n$

Q6 Each of the following gives the nth term of a different sequence.
For each sequence, find: (i) the 2nd term (ii) the 5th term (iii) the 20th term

a) $2n^2$ b) $4n^2 - 5$ c) $\frac{1}{2}n^2 + 20$ d) $400 - n^2$

Q7 For each of the following sequences:
(i) Find the formula for the nth term. (ii) Find the value of the 70th term.

a) 10, 8, 6, 4... b) 70, 60, 50, 40... c) 60, 55, 50, 45... d) 6, 3, 0, –3...

Q8 A sequence starts –5, –1, 3, 7. Determine whether each of the following is a term in this sequence. For those that are, state the position of the term in the sequence.

PROBLEM SOLVING

a) 43 b) 138 c) 384 d) 879

Q9 Find the nth term of the following sequence: $\frac{16}{3}, \frac{36}{7}, \frac{56}{11}, \frac{76}{15}...$

Q10 Given that the second term of an arithmetic sequence is –7 and the fifteenth term is 32, find the nth term of the sequence.

Q11 For each quadratic sequence below:
(i) Find the formula for the nth term. (ii) Find the value of the 10th term.

a) 7, 8, 10, 13, 17... b) 5, 7, 11, 17, 25... c) 3, 5, 9, 15, 23...

Exam-Style Questions

Q1 An arithmetic sequence starts with the value –5.
The 19th term in the sequence is –95.

a) Find an expression in terms of n for the nth term of the sequence.

[2 marks]

b) Hence explain why –113 is not a term in the sequence.

[1 mark]

Q2 A sequence is made using the following formula: $u_{n+1} = 2u_n^2 - 7$, for $n \geq 1$
The second term in the sequence is $u_2 = -6.28$.

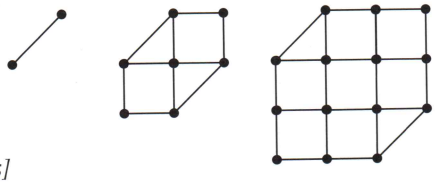
a) Find u_3.

[2 marks]

b) Find the value of u_1 given that it is positive.

[2 marks]

Q3 In the sequence on the right, the horizontal and vertical lines between each pair of dots are 2 cm long.



a) Find the number of dots in the nth pattern in the sequence.

[3 marks]

b) Find the area of the 100th pattern in the sequence.

[5 marks]

Q4 The first five terms of a number sequence are 7, 14, 33, 70 and 131.

a) Is the sequence geometric? Explain your answer.

[1 mark]

b) Is the sequence quadratic? Explain your answer.

[1 mark]

Q5 A Fibonacci-type number sequence is made by following the rules below:

Rule	Example
1. Choose any two numbers.	3, 8
2. To get the next number, add the previous two.	3 + 8 = 11
3. Repeat rule 2 to produce further terms.	3, 8, 11, 19, ...

A Fibonacci-type sequence has a fourth term of 22 and a sixth term of 57.
Find the first two numbers of the sequence.

[4 marks]