12.1 Simultaneous Linear Equations

In simultaneous equations you'll be given a pair of equations with two unknowns (e.g. two equations which contain both x and y). The solution will be a pair of values for the unknowns that make both equations true.

Learning Objectives — Spec Ref A19/A21:
- Solve two linear equations simultaneously.
- Form simultaneous equations from real-life situations.

Prior Knowledge Check: Be able to solve equations — see p.113-115

There are **two** main algebraic methods for solving **simultaneous equations** — elimination and substitution.

- For the **elimination method**, you **add** the two equations together (or **subtract** one from the other) so that one variable is **eliminated**. To follow this method, you might need to **rearrange** both equations into the form $ax + by = c$ first.
- For the **substitution method**, you **rearrange** one equation to give one **variable** in terms of the **other** (e.g. $2x - y = -3$ would become $y = 2x + 3$). You then **substitute** this expression into the other equation, so only one variable is left (e.g. you would replace y with $2x + 3$, then collect like terms).

The **final step** in both methods is to **substitute** the value you've just worked out for one variable into either equation to find the value of the other variable.

The **solutions** are where **graphs** of the equations would **cross** (see p.219).

Tip: Both methods involve getting rid of one variable first — you'll be left with an equation in terms of the other variable, which you can then solve.

Example 1 — Elimination Method

Solve the simultaneous equations: **(1) $11 - x = y$**
(2) $3y = x - 7$

1. Rearrange the equations so that they're both in the form $ax + by = c$.

   (1) $x + y = 11$
   (2) $x - 3y = 7$

2. You've got $+x$ in both equations so subtract equation (2) from equation (1) to eliminate x.

$$\begin{array}{r} x + y = 11 \\ -\,(x - 3y = 7) \\ \hline 4y = 4 \end{array}$$

3. Solve the resulting equation for y.

   $y = 1$

4. Put $y = 1$ into one of the original equations and solve for x.

   $x + 1 = 11$
   $x = 10$

5. Use the other equation to check the answer.

   $x - 3y = 10 - 3(1) = 7$ ✔
   So $\boldsymbol{x = 10,\ y = 1}$

Tip: At step 2, if the coefficients of the variable you're eliminating have the same sign (both +ve or both –ve), subtract one equation from the other. If the coefficients have opposite signs (one +ve and one –ve), add the two equations.

Example 2 — Substitution Method

Solve the simultaneous equations: **(1) $7x - y = 16$**
(2) $y - 2 = x$

1. You've got y in both equations so rearrange equation (2) to get y on its own. $y = x + 2$
2. Substitute $y = x + 2$ into equation (1). $7x - (x + 2) = 16$
3. Solve for x. $6x - 2 = 16$, $6x = 18$, $x = 3$
4. Put $x = 3$ into one of the original equations and solve for y. $y - 2 = 3$, $y = 5$
5. Use the other equation to check the answer. $7x - y = 7(3) - 5 = 16$ ✔
   So $x = 3$, $y = 5$

Tip: You could have used the elimination method here — you'd add the equations together to eliminate y.

Exercise 1

Solve each of the following pairs of simultaneous equations.

Q1 a) $x + 3y = 13$, $x - y = 5$
b) $2x - y = 7$, $4x + y = 23$
c) $x + 2y = 6$, $x + y = 2$
d) $3x - 2y = 16$, $2x + 2y = 14$
e) $x - y = 8$, $x + 2y = -7$
f) $2x + 4y = 16$, $3x + 4y = 24$
g) $4x - y = -1$, $4x - 3y = -7$
h) $3x + y = 11$, $6x - y = -8$
i) $6x + y = 9$, $2x - y = 7$

If **neither** variable has the same coefficient, you'll have to **multiply** one or both equations to make one set of coefficients match.
For example, if you had the equations $2x + y = 4$ and $3x + 2y = 7$, you'd multiply the first equation by 2 to make the y-coefficients match.
You then add or subtract using the **elimination method** to find the solutions.

Tip: Pick the variable that needs the least amount of work to match up its coefficients in each equation.

Example 3

Solve the simultaneous equations: **(1) $5x - 4y = 23$**
(2) $2x + 6y = -25$

1. Multiply equation (1) by 3 and equation (2) by 2 to match the y-coefficients.
$$\begin{aligned} 3 \times (1):&\quad 15x - 12y = 69 \\ 2 \times (2):&\quad +\ 4x + 12y = -50 \end{aligned}$$
2. Add the resulting equations to eliminate y and solve the equation for x.
$$19x = 19$$
$$x = 1$$
3. Put $x = 1$ into one of the original equations and solve for y. $5(1) - 4y = 23 \Rightarrow 5 - 4y = 23 \Rightarrow -4y = 18 \Rightarrow y = -4.5$
4. Use the other equation to check the answer. $2x + 6y = 2(1) + 6(-4.5) = 2 - 27 = -25$ ✔
   So $x = 1$, $y = -4.5$

Tip: You can label the new equations as (3) and (4) to help you keep track of things. Then Step 2 in this example is (3) + (4).

Exercise 2

Solve each of the following pairs of simultaneous equations.

Q1 a) $3x + 2y = 16$, $2x + y = 9$
b) $4x + 3y = 16$, $5x - y = 1$
c) $4x - y = 22$, $3x + 4y = 26$
d) $2x + 3y = 10$, $x - y = 5$
e) $4x - 2y = 8$, $x - 3y = -3$
f) $3e - 5r = 17$, $9e + 2r = -17$

Q2 a) $3x - 2y = 8$, $5x - 3y = 14$
b) $4p + 3q = 17$, $3p - 4q = 19$
c) $4u + 7v = 15$, $5u - 2v = 8$
d) $2c + 6d = 19$, $3c + 8d = 28$
e) $3r - 4s = -22$, $8r + 3s = -4$
f) $3m + 5n = 14$, $7m + 2n = 23$

You could also be given **real-life contexts** and have to set up simultaneous equations for yourself.

Example 4

Sue buys 4 dining chairs and 1 table for £142. Ken buys 6 of the same chairs and 2 of the same tables for £254. What is the price of one chair? What is the price of one table? PROBLEM SOLVING

1. Choose some variables, then write the question as two simultaneous equations.

Cost of one chair = £c. Cost of one table = £t.
(1) $4c + t = 142$
(2) $6c + 2t = 254$

2. Multiply equation (1) by 2 to give the same coefficients of t and label it (3). Subtract equation (2) from (3).

$2 \times (1) = (3)$: $8c + 2t = 284$
(2) $-(6c + 2t = 254)$
$2c = 30$

3. Solve the resulting equation for c.

$c = 15$

4. Put $c = 15$ into one of the original equations and solve for t.

$4(15) + t = 142$
$60 + t = 142$
$t = 82$

5. Use the other equation to check the answer, then write it in the context of the original question.

$6c + 2t = 6(15) + 2(82) = 90 + 164 = 254$ ✔
Chairs cost **£15** each. Tables cost **£82** each.

Exercise 3

PROBLEM SOLVING

Q1 The sum of two numbers, x and y, is 58, and the difference between them is 22. Given that x is greater than y, use simultaneous equations to find both numbers.

Q2 A grandfather with 7 grandchildren bought 4 sherbet dips and 3 chocolate bars for £1.91 last week and 3 sherbet dips and 4 chocolate bars for £1.73 the week before. Calculate the price of each item.

Q3 Three friends have just finished a computer game. At the end of the game, Zoe, with 7 yellow aliens and 5 blue spiders scored 85 points; James, with 6 yellow aliens and 11 blue spiders scored 93 points. Hal had 8 yellow aliens and 1 blue spider. How many points did Hal score?

Q4 The lengths of the sides of an equilateral triangle are $3(x + y)$ cm, $(5x + 2y - 1)$ cm and $4x + 4 + y$ cm. Find the side length of the triangle.

12.2 Simultaneous Linear and Quadratic Equations

A set of simultaneous equations including a quadratic term could have more than one pair of solutions.

Learning Objective — Spec Ref A19:
Solve one linear and one quadratic equation simultaneously.

Prior Knowledge Check: Be able to solve quadratic equations. See Section 11.

To solve simultaneous equations with one **linear equation** and one **quadratic** equation, use the **substitution method**.

- First **rearrange** the **linear equation** to get it in terms of **one variable** (e.g. $2x - y = -3$ would become $y = 2x + 3$).
- **Substitute** this into the **quadratic equation** — you'll be left with a quadratic equation in one variable.
- **Solve** the resulting quadratic equation — you'll probably get **2 solutions**.
- **Substitute** each of these values into one of the original equations to find the values of the other variable — the **linear equation** is often the easiest to use.

Tip: If there's a variable with a coefficient of 1, it's usually best to make that the subject so it can be easily substituted into the other equation.

The **solutions** to these pairs of simultaneous equations correspond to the **points on a graph** where a straight line and a quadratic curve **cross**. If the straight line crosses the curve **twice** there will be **2 pairs** of solutions to the simultaneous equations. The line could also cross the curve only **once** or **not at all**.

Example 1

Solve the simultaneous equations: **(1)** $x + y = 5$
(2) $y = x^2 - 3x - 30$

1. Rearrange equation (1) to get y on its own. $y = 5 - x$
2. Substitute $y = 5 - x$ into $y = x^2 - 3x - 30$. $5 - x = x^2 - 3x - 30$
3. Rearrange to get zero on the right hand side. $x^2 - 3x + x - 30 - 5 = 0$
   $x^2 - 2x - 35 = 0$
4. Solve the quadratic. $(x + 5)(x - 7) = 0$
   $x + 5 = 0$ or $x - 7 = 0$
   so $x = -5$ or $x = 7$
5. Substitute into equation (1) to find a y-value for each value of x. If $x = -5$, then $-5 + y = 5$, so $y = 10$.
   If $x = 7$, then $7 + y = 5$, so $y = -2$.
6. Write the solutions in pairs. $\mathbf{x = -5, y = 10}$ **and** $\mathbf{x = 7, y = 2}$

Tip: If you'd rearranged (1) to make x the subject, you'd have to do a bit more work at step 2.

Exercise 1

Q1 Find the solutions to each of the following pairs of simultaneous equations.

a) $y = x^2 - 4x + 8$
$y = 2x$

b) $y = x^2 - x - 1$
$3x = 2 - y$

c) $y = x^2 - 4x - 28$
$y = 3x + 2$

d) $y = x^2 + 3x - 2$
$y + x = 3$

e) $y = x^2 - 4x + 2$
$y = 2x - 6$

f) $y = x^2 - 2x - 3$
$y = 3x + 11$

g) $y = x^2 - x - 5$
$y = 2x + 5$

h) $y = x^2 - 4x + 8$
$4 = y - x$

i) $y = 2x^2 + x - 2$
$y = 8x - 5$

Q2 The line $2y = x + 3$ and the curve $y = x^2 - 2x - 2$ cross at points M and N. Find the coordinates of M and N by solving the simultaneous equations.

Q3 Find where the line $y = 4 - 3x$ crosses the curve $y = 6x^2 + 10x - 1$ by solving the equations simultaneously.

Q4 Use simultaneous equations to find the coordinates where the line $y = 5x$ meets the curve $y = x^2 + 3x + 1$. What can you say about the line and the curve?

Q5 Solve these equations simultaneously: $x - 4y = 2$ and $y^2 + xy = 0$

Q6 Solve these pairs of simultaneous equations.

a) $x + y = 7$
$x^2 - xy = 4$

b) $x + y = 5$
$x + xy + 2y^2 = 2$

c) $x + y = 2$
$y^2 - x = 0$

d) $x - y = 4$
$x^2 + y = 2$

e) $x + y = 4$
$x^2 + 3xy = 16$

f) $4y + x = 10$
$xy + x = -8$

Example 2

Solve the simultaneous equations: **(1) $x^2 + y^2 = 10$**
(2) $2x + y = 5$

1. Rearrange equation (2) to get y on its own. $y = 5 - 2x$
2. Substitute $y = 5 - 2x$ into $x^2 + y^2 = 10$.
$x^2 + (5 - 2x)^2 = 10$
$x^2 + 25 - 20x + 4x^2 = 10$
3. Rearrange to get zero on the right hand side.
$5x^2 - 20x + 15 = 0$
$x^2 - 4x + 3 = 0$
4. Solve the quadratic.
$(x - 1)(x - 3) = 0$
$x - 1 = 0$ or $x - 3 = 0$
so $x = 1$ or $x = 3$
5. Substitute into equation (2) to find a y-value for each value of x.
If $x = 1$, then $2(1) + y = 5$, so $y = 3$
If $x = 3$, then $2(3) + y = 5$, so $y = -1$
6. Write the solutions in pairs. **$x = 1, y = 3$ and $x = 3, y = -1$**

Tip: Equation (1) is the equation of a circle (see p.203).

Exercise 2

Q1 Solve these pairs of simultaneous equations.

a) $2x + y = 3$
$y^2 - x^2 = 0$

b) $3x + y = 4$
$x^2 + 3xy + y^2 = -16$

c) $x - y = -4$
$x^2 + y^2 - x = 20$

Q2 The equations $x - y = -3$ and $3x^2 + 7x + y^2 = 21$ are plotted on a graph.

a) Show that at the points of intersection, $4x^2 + 13x - 12 = 0$.

b) Find the exact values of x when $4x^2 + 13x - 12 = 0$.

c) Find the exact coordinates of the points of intersection.

Q3 Find the exact coordinates where the graphs of the following pairs of equations intersect.

a) $x = 3y + 4$
$x^2 + y^2 = 34$

b) $2x + 2y = 1$
$x^2 + y^2 = 1$

c) $\sqrt{5}y - x = 6$
$x^2 + y^2 = 36$

Review Exercise

Q1 Solve each of the following pairs of simultaneous equations.

a) $e + 2f = 7$
$6e + 2f = 10$

b) $3g + h = 5\frac{1}{5}$
$3g - 2h = -6\frac{4}{5}$

c) $4j - 2i = 8$
$4j + 7i = -37$

Q2 Solve each of the following pairs of simultaneous equations.

a) $5x - 3y = 12$
$2x - y = 5$

b) $2x - y = 11$
$-4x - 7y = 5$

c) $5k + 3l = 4$
$3k + 2l = 3$

d) $2c + 6d = 19$
$3c + 8d = 28$

e) $3r - 4s = -22$
$8r + 3s = -4$

f) $3e - 5f = 8\frac{1}{2}$
$7e - 3f = 15\frac{1}{2}$

Q3 A teacher with a back problem is concerned about the weight of the books she carries home. She knows that 2 textbooks and 30 exercise books weigh a total of 6.9 kg and that 1 textbook and 20 exercise books weigh a total of 4.2 kg. The doctor has suggested she does not carry over 5 kg at a time.

PROBLEM SOLVING

a) Calculate the mass of one exercise book and the mass of one textbook.

b) Can she carry 1 textbook and 25 exercise books?

c) If she needs to carry 2 textbooks, how many exercise books could she carry?

Q4 An interior designer recently spent £1359.55 buying 25 kettles and 20 toasters for a new development. She spent £641.79 on 12 kettles and 9 toasters for a smaller group of new houses. For her current assignment she needs 80 kettles and 56 toasters. If the price of each toaster and each kettle is the same in all three cases, how much should she allow for this cost on her current assignment?

PROBLEM SOLVING

Q5 Find the solution to each of these pairs of simultaneous equations.

a) $y = 2x^2 + 9x + 30$
$y = 9 - 8x$

b) $y = 4x^2 - 5x + 2$
$y = 2x - 1$

c) $y = 7x^2 - 12x - 1$
$y = 8x + 2$

d) $5 = y - 2x$
$2x^2 - y + 3 = x$

e) $3y^2 = x - 8y$
$x + 3y = 4$

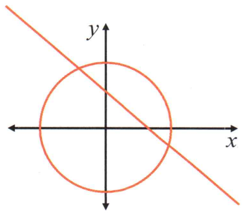
f) $3y - x = 2$
$y^2 + xy + x = 1$

g) $x - y + 5x^2 = 1$
$y + 2x = 1$

h) $4y - x = 2$
$x^2 - 3xy = 88$

i) $x + y = 0$
$x^3 - 3x^2y + y^2 = 0$

Q6 As shown on the graph, the line $x + y = 3$ and the circle $x^2 + y^2 = 17$ cross at two points. Find the coordinates for the points where the line meets the circle.



Exam-Style Questions

Q1 Solve the simultaneous equations.

$$7m + 2n = 23$$
$$3m + 5n = 14$$

[3 marks]

Q2 Solve the simultaneous equations.

$$2c + 4v = 580$$
$$3c + 2v = 542$$

[3 marks]

Q3 The equation of line A is $y = 2x - 2$, and the equation of line B is $5y = 20 - 2x$.
Find the coordinates of the point where lines A and B cross.

[3 marks]

Q4 3 kg of organic apples and 2 kg of organic pears cost £19.80,
while 2 kg of these apples and 3 kg of these pears cost £20.70.
Work out the price of 1 kg of the apples and the price of 1 kg of the pears.

PROBLEM SOLVING

[4 marks]

Q5 Find the coordinates of the points where the curve $y = x^2$ and the line $y = 6x - 8$ intersect.

[4 marks]

Q6 A rectangle has an area of 35 cm^3 and a perimeter of 24 cm^2.
Use simultaneous equations to find the lengths of sides x and y, given that $x > y$.

PROBLEM SOLVING

[5 marks]

Q7 Solve the simultaneous equations.

$$y = 2x + 2$$
$$x^2 + y^2 = 8$$

[5 marks]