10.1 Direct Proportion

Proportion is all about how quantities change in relation to one another. You might have seen proportion before in Section 4 — here you'll see how to tackle these questions algebraically.

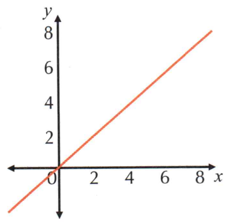
Direct Proportion

Learning Objective — Spec Ref R10:
Understand and use direct proportion.

Two quantities are **directly proportional** if they are always in the **same ratio** — so if one quantity **doubles**, the other also **doubles**, if one **triples**, the other also **triples**, and so on.

A **proportional relationship** can be written as a **proportionality statement**: '$y \propto x$', which is read as '**y is proportional to x**' or '**as x varies, y varies directly**'.

You can write this as an equation: $y = kx$, where k is the **constant of proportionality** (you just replace '$\propto$' with '$= k$' to make the equation).



The graph of a **proportional relationship** is a **straight line** through the **origin**. The **gradient** of the graph (see p.180) is equal to the constant of proportionality.

Example 1

y is directly proportional to x. Fill in the gaps in the table.

x	3	5	10	12	
y			25		100

1. Write the proportionality statement and make it into an equation: $y \propto x$, so $y = kx$
2. The table shows that when $x = 10$, $y = 25$. Use this to find k. $25 = k \times 10$
   $k = 25 \div 10 = 2.5$
   So $y = 2.5x$
3. Use the equation to complete the table:

x	3	5	10	12	$100 \div 2.5 =$ **40**
y	$2.5 \times 3 =$ **7.5**	$2.5 \times 5 =$ **12.5**	25	$2.5 \times 12 =$ **30**	100

Tip: Rearrange the equation to find the missing x-value (use $x = y \div 2.5$).

Example 2

m is directly proportional to e. Given that $m = 72$ when $e = 6$, calculate the value of e when $m = 36$.

1. Write the proportionality statement and make it into an equation. $m \propto e$, so $m = ke$
2. Use the given values to find k. $72 = k \times 6$
   so $k = 72 \div 6 = 12$
3. Put the value of k into the equation $m = ke$. $m = 12e$
4. Substitute $m = 36$ into the equation and solve for e. $36 = 12e$
   $e = 36 \div 12 =$ **3**

Exercise 1

Q1 In each of the following tables, y is directly proportional to x.
Use this information to fill in the gaps in each table.

a)

x	22	33
y	2	

b)

x	24	
y	18	24

c)

x	2	7	10	21	
y			15		36

d)

x	–4	0		12
y	–14		21	

e)

x		8
y	12	9

f)

x	–27	78		
y		104	272	980

Q2 j is directly proportional to h. When $j = 15$, $h = 5$. What is the value of j when $h = 40$?

Q3 r is directly proportional to t. When $r = 9$, $t = 6$. What is the value of r when $t = 7.5$?

Q4 p is directly proportional to q. When $p = 11$, $q = 3$. What is the value of q when $p = 82.5$?

Q5 Given that $b = 142$ when $s = 16$ and that b is directly proportional to s, find the value of:
a) b when $s = 18$, b) s when $b = 200$.

Other Types of Direct Proportion

Learning Objective — Spec Ref R10:
Understand and use other types of direct proportion.

There are also other types of direct proportion, where one quantity **varies proportionally** with a **function** of another quantity (rather than with the quantity itself).
These relationships can also be written as **proportionality statements**.

Here are some **examples** of other types of **direct proportion**:

- If y is directly proportional to the **square** of x then $y \propto x^2$, so $y = kx^2$.
- If y is directly proportional to the **cube** of x then $y \propto x^3$, so $y = kx^3$.
- If y is directly proportional to the **square root** of x then $y \propto \sqrt{x}$, so $y = k\sqrt{x}$.

Tip: A graph showing direct proportion will always go through the origin.

Example 3

p is directly proportional to the square of q.

a) Given that $p = 125$ when $q = 5$, find p when $q = 7$.

1. Write the proportionality statement and make it into an equation. — $p \propto q^2$, so $p = kq^2$
2. Use the given values to find k. — $125 = k \times 5^2 = 25k$, so $k = 125 \div 25 = 5$
3. Substitute $k = 5$ into $p = kq^2$ — $p = 5q^2$
4. Substitute in $q = 7$ and solve for p. — $p = 5 \times 7^2 = \mathbf{245}$

b) Hence find q when $p = 3125$.

Substitute $p = 3125$ into the equation and solve for q.

$3125 = 5q^2$
$q^2 = 3125 \div 5 = 625$
$q = \pm\sqrt{625} = \mathbf{\pm 25}$

Example 4

p is directly proportional to the cube of q.

a) Given that $p = 81$ when $q = 3$, find p when $q = 5$.

1. Write the proportionality statement and make it into an equation. — $p \propto q^3$, so $p = kq^3$
2. Use the given values to find k. — $81 = k \times 3^3 = 27k$, so $k = 81 \div 27 = 3$
3. Substitute $k = 3$ into $p = kq^3$ — $p = 3q^3$
4. Substitute in $q = 5$ and solve for p. — $p = 3 \times 5^3 = \mathbf{375}$

b) Hence find q when $p = 3000$.

Substitute $p = 3000$ into the equation and solve for q.

$3000 = 3q^3$
$q^3 = 3000 \div 3 = 1000$
$q = \sqrt[3]{1000} = \mathbf{10}$

Exercise 2

Q1 In each of the following cases, y is directly proportional to the square of x.

a) If $y = 64$ when $x = 2$, find y when $x = 5$.

b) If $y = 539$ when $x = 7$, find x when $y = 1331$.

Q2 y is directly proportional to the square root of x. Complete the table.

x	1	9	16	
y		84		560

Q3 f is directly proportional to the square of g. It is found that $g = 100$ when $f = 200$.

a) Find f when $g = 61.5$.

b) Given that $g > 0$, find the exact value of g when $f = 14$.

Q4 The time taken for a ball to drop from its maximum height is directly proportional to the square root of the distance fallen. Given that a ball takes 3 seconds to drop 34.1 m, find the time taken for the ball to drop 15 m. PROBLEM SOLVING

Q5 The volume of a sphere is directly proportional to the cube of its radius. The volume of a sphere of radius 12 cm is 2304π cm³. PROBLEM SOLVING

a) Find the constant of proportionality in terms of π.
Use this to write an equation for the volume of a sphere in terms of its radius.

b) Find the volume of a sphere of radius 21 cm, in terms of π.

c) Find the radius of a sphere of volume 1000 cm³, correct to 1 decimal place.

10.2 Inverse Proportion

If one quantity decreases as the other increases, you might be dealing with inverse proportion.

Inverse Proportion

Learning Objective — Spec Ref R10:
Understand and use inverse proportion.

If two quantities are **inversely proportional**, as one **increases** the other **decreases** proportionally — so if one quantity **doubles**, the other is **halved**. The **product** of the two quantities remains **constant**.

An **inverse proportional relationship** can be written as a **proportionality statement**: '$y \propto \frac{1}{x}$' which is read as either '**y is inversely proportional to x**' or '**y varies inversely with x**'.

As with **direct proportion**, you can write this as an **equation** using a **constant of proportionality**: $y = \frac{k}{x}$

A **graph** showing an inverse proportion relationship will be a **reciprocal graph** (see p.200).

Example 1

y is inversely proportional to x. Fill in the gaps in the table.

x	1		5	10
y		100		20

1. Write the proportionality statement and make it into an equation. $y \propto \frac{1}{x}$, so $y = \frac{k}{x}$
2. The table shows that when $x = 10$, $y = 20$. Use this to find k. $20 = \frac{k}{10}$

   $k = 20 \times 10 = 200$

   So $y = \frac{200}{x}$
3. Use the equation to complete the table:

x	1	200 ÷ 100 = **2**	5	10
y	200 ÷ 1 = **200**	100	200 ÷ 5 = **40**	20

Tip: Rearrange the equation to find the missing x-value (use $x = \frac{200}{y}$).

Example 2

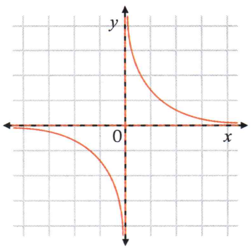
a) y is inversely proportional to x, and $x = 4$ when $y = 15$. Find y when $x = 10$.

1. Write the proportionality statement and make it into an equation. $y \propto \frac{1}{x}$, so $y = \frac{k}{x}$
2. Use the given values to find k. $15 = k \div 4$, so $k = 15 \times 4 = 60$
3. Put $k = 60$ into the equation. $y = \frac{60}{x}$
4. Substitute $x = 10$ into the equation and solve for y. $y = \frac{60}{x} = \frac{60}{10} = \mathbf{6}$

b) Sketch the graph of the relationship.

The curve never touches or crosses the x- or y-axes. Both axes are asymptotes to the curve.

The graph is a reciprocal graph.



Tip: An asymptote is a straight line that a curve gets closer and closer to but never touches.

Exercise 1

Q1 In each of the following tables, y is inversely proportional to x.
Use this information to fill in the gaps in each table.

a)

x	12	
y	15	12

b)

x	11	22
y	4	

c)

x	1	3	6	20	
y			15		270

Q2 p is inversely proportional to q. When $p = 7$, $q = 4$. What is the value of p when $q = 56$?

Q3 s is inversely proportional to t. When $s = 5$, $t = 16$. What is the value of t when $s = 48$?

Q4 Given that w is inversely proportional to z and $w = 15$ when $z = 4$,

a) find z when $w = 25$,

b) explain what happens to z when w is doubled.

Other Types of Inverse Proportion

Learning Objective — Spec Ref R10:
Understand and use other types of inverse proportion.

There are other types of inverse proportion where a quantity is **inversely proportional** to a **function** of another. As for normal inverse proportion, you can also write these as proportionality statements.

Here are some **examples** of other types of **inverse proportion**:

- If y is inversely proportional to the **square** of x then $y \propto \frac{1}{x^2}$, so $y = \frac{k}{x^2}$.
- If y is inversely proportional to the **cube** of x then $y \propto \frac{1}{x^3}$, so $y = \frac{k}{x^3}$.
- If y is inversely proportional to the **square root** of x then $y \propto \frac{1}{\sqrt{x}}$, so $y = \frac{k}{\sqrt{x}}$.

Example 3

y is inversely proportional to the cube of x, and when $x = 4$, $y = 10$.
Find the value of x when $y = 50$.

1. Write the proportionality statement and make it into an equation. $y \propto \frac{1}{x^3}$, so $y = \frac{k}{x^3}$
2. Use the given values to find k. $10 = \frac{k}{4^3} = \frac{k}{64}$
so $k = 10 \times 64 = 640$
3. Substitute $k = 640$ into the equation for y. $y = \frac{640}{x^3}$
4. Substitute $y = 50$ into the equation and solve for x. $50 = \frac{640}{x^3}$
$x^3 = 640 \div 50 = 12.8$
$x = \sqrt[3]{12.8} = \mathbf{2.34}$ (3 s.f.)

Example 4

y is inversely proportional to the square root of x, and when $x = 4$, $y = 4$.
Find the value of x when $y = 2$, given that k is a positive integer.

1. Write the proportionality statement and make it into an equation. $y \propto \frac{1}{\sqrt{x}}$, so $y = \frac{k}{\sqrt{x}}$
2. Use the given values to find k. $4 = k \div \sqrt{4} = k \div 2$
   so $k = 4 \times 2 = 8$
3. Substitute $k = 8$ into the equation for y. $y = \frac{8}{\sqrt{x}}$
4. Substitute $y = 2$ into the equation and solve for x. $2 = \frac{8}{\sqrt{x}}$
   $\sqrt{x} = 8 \div 2 = 4$
   $x = 4^2 = \mathbf{16}$

Exercise 2

Q1 Complete these tables.

a) y is inversely proportional to the square of x.

x	2	5		0.4
y	8		2	

b) y is inversely proportional to the square root of x and k is positive.

x		9	100	
y	6	8		$\frac{1}{3}$

Q2 h is inversely proportional to the cube of f. It is known that $h = 12.5$ when $f = 2$. Find the value of h when $f = 5$.

Q3 a is inversely proportional to the square of c, and when $c = 6$, $a = 3$. Find the two possible values of c when $a = 12$.

Q4 The air pressure from an electric pump is inversely proportional to the square of the radius of the tube to the pump. A tube with radius 10 mm creates 20 units of air pressure.

a) How much pressure will a tube of radius 15 mm create?

b) If an air-bed is to be pumped up using a maximum of 30 units of air pressure, what radius of tube should be used to achieve the quickest fill?

Q5 b is inversely proportional to the square of c. When $c = 1$, $b = 64$. Find values of b and c such that $b = c$.

Q6 The quantities u and v are related by the equation $v = \frac{k}{u^2}$.

a) Decide which of the following statements are true and which are false:

(i) u is proportional to the square of v.
(ii) v multiplied by the square of u is equal to a constant.
(iii) If you double v, you halve u.
(iv) If you double u, you divide v by 4.

b) If $k = 900$ and u and v are both positive integers, find at least 3 sets of possible values for u and v.

Review Exercise

Q1 Copy and complete the table on the right if:

a) y is directly proportional to x,

b) y is inversely proportional to x.

x	2	3	9	
y		8		100

Q2 p is inversely proportional to the cube of g and when $g = 1.5$, $p = 10$.

a) Find p when $g = 2.1$.

b) Find g when $p = 15$.

Q3 A person's reach with an upstretched arm is roughly proportional to their height. On average, statistics show that a person can reach 1.3 times their height.

a) Write down both a proportionality statement and an equation for this situation.

b) Would you expect a person of height 1.75 m to be able to touch a ceiling 2.5 m high? Show working to justify your answer.

Q4 Match each of the following statements to one of the sketch graphs below.

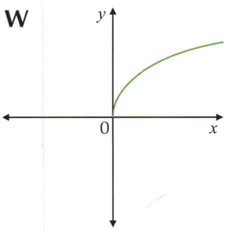
A: y is directly proportional to x

B: y is directly proportional to x^2

C: y is directly proportional to the cube of x

D: y is directly proportional to $\sqrt{x}$

W



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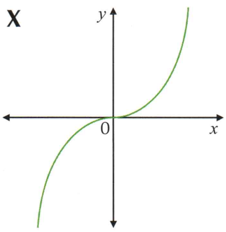

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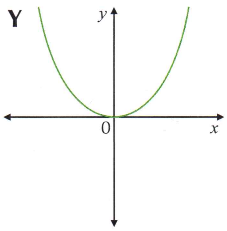

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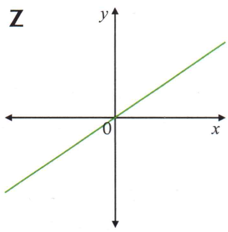

Q5 a) Copy and complete the table on the right if:

(i) y is inversely proportional to the square of x

(ii) y is inversely proportional to the cube of x

x	−6	−4	−2	2	4	6
y				6		

b) Using your tables, sketch graphs of these two types of inverse proportionality.

Q6 Coulomb's inverse square law states that the force of attraction or repulsion between two point charges is directly proportional to the product of the sizes of the charges and inversely proportional to the square of the distance between them.

PROBLEM SOLVING

This can be written as: $F = k\dfrac{Q_1 \times Q_2}{d^2}$, where Q_1 and Q_2 represent the sizes of the point charges, d is the distance between the point charges, and k is a constant.

If the force is found to be 10^6 units when the sizes of two point charges, Q_1 and Q_2, are each equal to 10^{-2} units and the points are 3 metres apart, find the force when these same point charges are 8 metres apart.

Exam-Style Questions

Q1 b is directly proportional to the square root of d.
Given that $b = 5$ when $d = 2.2$, find b (to 3 s.f.) when $d = 0.5$.

[1 mark]

Q2 y is inversely proportional to the square of x.

x		10		30
y	9000	90	40	

Fill in the missing values in the table, given that all values of x are positive.

[3 marks]

Q3 When a mass is attached to the bottom of a 20 cm long vertically held spring, the length that the spring extends is directly proportional to the magnitude of the mass. When a mass of 15 g is attached, the spring has a length of 20.9 cm. Work out how long the spring will be if an additional mass of 10 g is also attached.

[3 marks]

Q4 The maximum possible air pressure, p, from a mountain bike pump is inversely proportional to the square of the diameter, d, of the pump's cylinder. If a maximum pressure of 8.5 bars is possible from a pump whose cylinder has a diameter of 32 mm, find a formula for p in terms of d.

[2 marks]

Q5 In the table of values, $q \propto p^n$.

p	1	4
q	8	0.5

Find the value of n.

[3 marks]

Q6 Z is directly proportional to x^3. Z is also directly proportional to y^2. When $y = 8$, $x = 5$.
Find the value of x when y is 27.

[4 marks]