Exam-style practice

Mathematics
A Level
Paper 1: Pure Mathematics

Time: 2 hours
You must have: Mathematical Formulae and Statistical Tables, Calculator

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1 A curve C has parametric equations $x = \sin^2 t$, $y = 2\tan t$, $0 \leqslant t < \frac{\pi}{2}$

Find $\frac{dy}{dx}$ in terms of t. **(4)**

2 Find the set of values of x for which

a $2(7x - 5) - 6x < 10x - 7$ **(2)**

b $|2x + 5| - 3 > 0$ **(4)**

c **both** $2(7x - 5) - 6x < 10x - 7$ **and** $|2x + 5| - 3 > 0$. **(1)**

3 The line with equation $2x + y - 3 = 0$ does not intersect the circle with equation $x^2 + kx + y^2 + 4y = 4$

a Show that $5x^2 + (k - 20)x + 17 > 0$. **(4)**

b Find the range of possible values of k. Write your answer in exact form. **(3)**

4 Prove, for an angle θ measured in radians, that the derivative of $\cos\theta$ is $-\sin\theta$.
You may assume the compound angle formula for $\cos(A \pm B)$, and that

$$\lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 1 \text{ and } \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) = 0.$$ **(5)**

5 $f(x) = (3 + px)^6$, $x \in \mathbb{R}$

Given that the coefficient of x^2 is 19440,

a find two possible values of p. **(4)**

Given further that the coefficient of x^5 is negative,

b find the coefficient of x^5. **(2)**

6 The point R with x-coordinate 2 lies on the curve with equation $y = x^2 + 4x - 2$. The normal to the curve at R intersects the curve again at a point T. Find the coordinates of T, giving your answers in their simplest form. **(6)**

7 A geometric series has first term a and common ratio r. The second term of the series is 96 and the sum to infinity of the series is 600.

a Show that $25r^2 - 25r + 4 = 0$. **(4)**

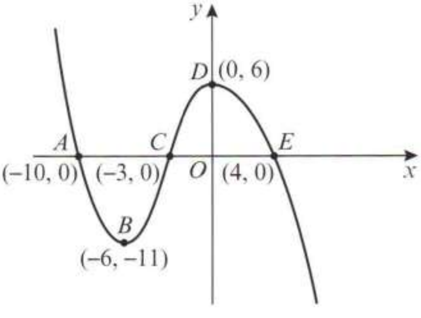
b Find the two possible values of r. **(2)**

For the larger value of r:

c find the corresponding value of a. **(1)**

d find the smallest value of n for which S_n exceeds 599.9. **(3)**

8 The diagram shows the graph of $f(x)$. The points B and D are stationary points of the graph.



Sketch, on separate diagrams, the graphs of:

a $y = |f(x)|$ **(3)**

b $y = -f(x) + 5$ **(3)**

c $y = 2f(x - 3)$ **(3)**

9 Find all the solutions, in the interval $0 \leqslant x \leqslant 2\pi$, to the equation $31 - 25\cos x = 19 - 12\sin^2 x$, giving each solution to 2 decimal places. **(5)**

10 The value, V, of a car decreases over time, t, measured in years. The rate of decrease in value of the car is proportional to the value of the car at that time.

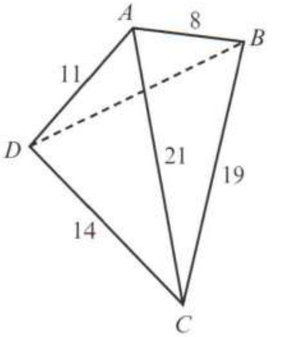
a Given that the initial value of the car is V_0, show that $V = V_0 e^{-kt}$ **(4)**

The value of the car after 2 years is £25 000 and after 5 years is £15 000.

b Find the exact value of k and the value of V_0 to the nearest hundred pounds. **(3)**

c Find the age of the car when its value is £5000. **(3)**

11 The diagram shows the positions of 4 cities: A, B, C and D. The distances, in miles, between each pair of cities, as measured in a straight-line, are labelled on the diagram. A new road is to be built between cities B and D.



a What is the minimum possible length of this road? Give your answer to 1 decimal place. **(7)**

b Explain why your answer to part **a** is a minimum. **(1)**

12 A footballer takes a free-kick. The path of the ball towards the goal can be modelled by the equation $y = -0.01x^2 + 0.22x + 1.58$, $x \geqslant 0$, where x is the horizontal distance from the goal in metres and y is the height of the ball in metres. The goal is 2.44 m high.

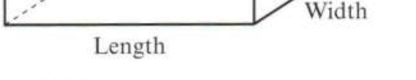
a Rewrite y in the form $A - B(x + C)^2$, where A, B and C are constants to be found. **(3)**

b Using your answer to part **a**, state the distance from goal at which the ball is at the greatest height and its height at this point. **(2)**

c How far from the goal is the football when it is kicked? **(2)**

d The football is headed towards the goal. The keeper can save any ball that would cross the goal line at a height of up to 1.5 m. Explain with a reason whether the free kick will result in a goal. **(2)**

13 A box in the shape of a rectangular prism has a lid that overlaps the box by 3 cm, as shown. The width of the box is x cm, and the length of the box is double the width. The height of the box is h cm. The box and lid can be created exactly from a piece of cardboard of area 5356 cm^2. The box has volume, V cm^3.



a Show that $V = \frac{2}{3}(2678x - 9x^2 - 2x^3)$ **(5)**

Given that x can vary

b use differentiation to find the positive value of x, to 2 decimal places, for which V is stationary. **(4)**

c Prove that this value of x gives a maximum value of V. **(2)**

d Find this maximum value of V. **(1)**

Given that V takes its maximum value,

e determine the percentage of the area of cardboard that is used in the lid. **(2)**

Exam-style practice

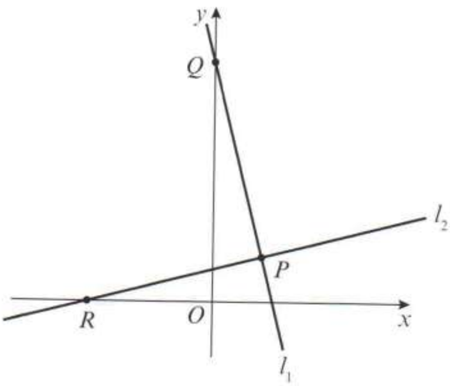
Mathematics
A Level
Paper 2: Pure Mathematics

Time: 2 hours

You must have: Mathematical Formulae and Statistical Tables, Calculator

1 The graph of $y = ax^2 + bx + c$ has a maximum at $(-2, 8)$ and passes through $(-4, 4)$. Find the values of a, b and c. **(3)**

2 The points $P(6, 4)$ and $Q(0, 28)$ lie on the straight line l_1 as shown.



a Work out an equation for the straight line l_1. **(2)**

The straight line l_2 is perpendicular to l_1 and passes through the point P.

b Work out an equation for the straight line l_2. **(2)**

c Work out the coordinates of R. **(2)**

d Work out the area of $\triangle PQR$. **(3)**

3 The function f is defined by $f: x \rightarrow e^{3x} - 1, \quad x \in \mathbb{R}$.
Find $f^{-1}(x)$ and state its domain. **(4)**

4 A student is asked to solve the equation $\log_4(x + 3) + \log_4(x + 4) = \frac{1}{2}$

The student's attempt is shown.

$$\log_4(x + 3) + \log_4(x + 4) = \tfrac{1}{2}$$
$$(x + 3) + (x + 4) = 2$$
$$2x + 7 = 2$$
$$2x = -5$$
$$x = -\tfrac{5}{2}$$

a Identify the error made by the student. **(1)**

b Solve the equation correctly. **(4)**

5 The function p has domain $-14 \leqslant x \leqslant 10$, and is linear from $(-14, 18)$ to $(-6, -6)$ and from $(-6, -6)$ to $(10, 2)$.

a Sketch $y = \text{p}(x)$. **(2)**

b Write down the range of $\text{p}(x)$. **(1)**

c Find the values of a, such that $\text{p}(a) = -3$. **(2)**

6 $\text{f}(x) = x^3 - kx^2 - 10x + k$

a Given that $(x + 2)$ is a factor of $\text{f}(x)$, find the value of k. **(2)**

b Hence, or otherwise, find all the solutions to the equation $\text{f}(x) = 0$, leaving your answers in the form $p \pm \sqrt{q}$ when necessary. **(4)**

7 In $\triangle DEF$, $DE = x - 3$ cm, $DF = x - 10$ cm and $\angle EDF = 30°$. Given that the area of the triangle is 11 cm^2,

a show that x satisfies the equation $x^2 - 13x - 14 = 0$ **(3)**

b calculate the value of x. **(2)**

8 The curve C has parametric equations $x = 6\sin t + 5$, $y = 6\cos t - 2$, $-\frac{\pi}{3} \leqslant t \leqslant \frac{3\pi}{4}$

a Show that the Cartesian equation of C can be written as $(x + h)^2 + (y + k)^2 = c$, where h, k and c are integers to be determined. **(4)**

b Find the length of C. Write your answer in the form $p\pi$, where p is a rational number to be found. **(3)**

9 $\dfrac{4x^2 + 7x}{(x - 2)(x + 4)} \equiv A + \dfrac{B}{x - 2} + \dfrac{C}{x + 4}$

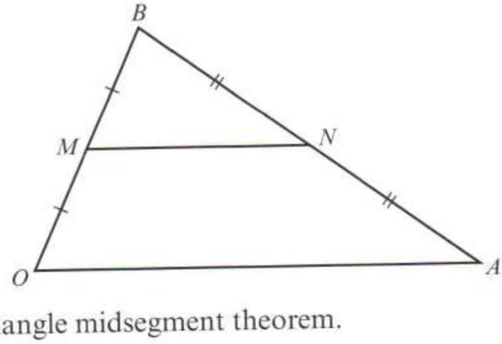
a Find the values of the constants A, B and C. **(4)**

b Hence, or otherwise, expand $\dfrac{4x^2 + 7x}{(x - 2)(x + 4)}$ in ascending powers of x, as far as the term in x^2.

Give each coefficient as a simplified fraction. **(6)**

10 OAB is a triangle. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The points M and N are midpoints of OB and BA respectively.

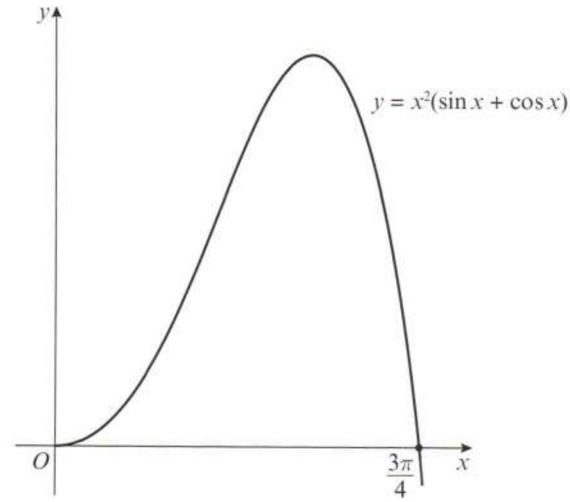
The triangle midsegment theorem states that 'In a triangle, the line joining the midpoints of any two sides will be parallel to the third side and half its length.'



Use vectors to prove the triangle midsegment theorem. **(4)**

11 The diagram shows the region R bounded by the x-axis and the curve with equation

$y = x^2(\sin x + \cos x)$, $0 \leqslant x \leqslant \frac{3\pi}{4}$



The table shows corresponding values of x and y for $y = x^2(\sin x + \cos x)$.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$
y	0	0.20149	0.87239	1.81340		2.08648	0

a Copy and complete the table giving the missing value for y to 5 decimal places. **(1)**

b Using the trapezium rule, with all the values for y in the completed table, find an approximation for the area of R, giving your answer to 3 decimal places. **(4)**

c Use integration to find the exact area of R, giving your answer to 3 decimal places. **(6)**

d Calculate, to one decimal place, the percentage error in your approximation in part **b**. **(1)**

12 Ruth wants to save money for her newborn daughter to pay for university costs. In the first year she saves £1000. Each year she plans to save £150 more, so that she will save £1150 in the second year, £1300 in the third year, and so on.

a Find the amount Ruth will save in the 18th year. **(2)**

b Find the total amount that Ruth will have saved over the 18 years. **(3)**

Ruth decides instead to increase the amount she saves by 10% each year.

c Calculate the total amount Ruth will have saved after 18 years under this scheme. **(4)**

13 **a** Express $0.09\cos x + 0.4\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$
Give the value of α to 4 decimal places. **(4)**

The height of a swing above the ground can be modelled using the equation

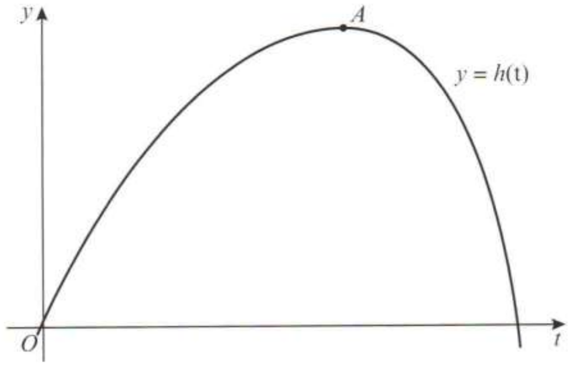
$h = \frac{16.4}{0.09\cos\left(\frac{t}{2}\right) + 0.4\sin\left(\frac{t}{2}\right)}$, $0 \leqslant t \leqslant 5.4$, where h is the height of the swing, in cm, and

t is the time, in seconds, since the swing was initially at its greatest height.

b Calculate the minimum value of h predicted by this model, and the value of t, to 2 decimal places, when this minimum value occurs. **(3)**

c Calculate, to the nearest hundredth of a second, the times when the swing is at a height of exactly 100 cm. **(4)**

14 The diagram shows the height, h, in metres of a rollercoaster during the first few seconds of the ride. The graph is $y = h(t)$, where $h(t) = -10e^{-0.3(t-6.4)} - 10e^{0.8(t-6.4)} + 70$.



a Find $h'(t)$. **(3)**

b Show that when $h'(t) = 0$, $t = \frac{5}{4}\ln\left(\frac{3e^{-0.3(t-6.4)}}{8}\right) + 6.4$ **(2)**

To find an approximation for the t-coordinate of A, the iterative formula

$$t_{n+1} = \frac{5}{4}\ln\left(\frac{3e^{-0.3(t_n-6.4)}}{8}\right) + 6.4 \text{ is used.}$$

c Let $t_0 = 5$. Find the values of t_1, t_2, t_3 and t_4. Give your answers to 4 decimal places. **(3)**

d By choosing a suitable interval, show that the t-coordinate of A is 5.508, correct to 3 decimal places. **(2)**

Answers

CHAPTER 1

Prior knowledge 1

1 a $(x-1)(x-5)$ b $(x+4)(x-4)$ c $(3x-5)(3x+5)$

2 a $\frac{x-3}{x+6}$ b $\frac{x+4}{3x+1}$ c $-\frac{x+5}{x+3}$

3 a even b either c either d odd

Exercise 1A

1 B At least one multiple of three is odd.

2 a At least one rich person is not happy.
 b There is at least one prime number between 10 million and 11 million.
 c If p and q are prime numbers there exists a number of the form $(pq+1)$ that is not prime.
 d There is a number of the form $2^n - 1$ that is neither a prime nor a multiple of 3.
 e None of the above statements are true.

3 a There exists a number n such that n^2 is odd but n is even.
 b n is even so write $n = 2k$
 $n^2 = (2k)^2 = 4k^2 = 2(2k^2) \Rightarrow n^2$ is even.
 This contradicts the assumption that n^2 is odd.
 Therefore if n^2 is odd then n must be odd.

4 a Assumption: there is a greatest even integer $2n$.
 $2(n+1)$ is also an integer and $2(n+1) > 2n$
 $2n + 2 = \text{even} + \text{even} = \text{even}$
 So there exists an even integer greater that $2n$.
 This contradicts the assumption.
 Therefore there is no greatest even integer.
 b Assumption: there exists a number n such that n^3 is even but n is odd.
 n is odd so write $n = 2k + 1$
 $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1$
 $= 2(4k^3 + 6k^2 + 3k) + 1 \Rightarrow n^3$ is odd.
 This contradicts the assumption that n^3 is even.
 Therefore, if n^3 is even then n must be even.
 c Assumption: if pq is even then neither p nor q is even.
 p is odd, $p = 2k + 1$
 q is odd, $q = 2m + 1$
 $pq = (2k+1)(2m+1) = 4km + 2k + 2m + 1$
 $= 2(2km + k + m) + 1 \Rightarrow pq$ is odd.
 This contradicts the assumption that pq is even.
 Therefore, if pq is even then at least one of p and q is even.
 d Assumption: if $p + q$ is odd then neither p nor q is odd
 p is even, $p = 2k$
 q is even, $q = 2m$
 $p + q = 2k + 2m = 2(k+m) \Rightarrow p + q$ is even
 This contradicts the assumption then $p + q$ is odd.
 Therefore, if $p + q$ is odd then at least one of p and q is odd.

5 a Assumption: if ab is an irrational number then neither a nor b is irrational.
 a is rational, $a = \frac{c}{d}$ where c and d are integers.
 b is rational, $b = \frac{e}{f}$ where e and f are integers.
 $ab = \frac{ce}{df}$, ce is an integer, df is an integer.
 Therefore ab is a rational number.
 This contradicts assumption then ab is irrational.
 Therefore if ab is an irrational number then at least one of a and b is an irrational number.
 b Assumption: neither a nor b is irrational.
 a is rational, $a = \frac{c}{d}$ where c and d are integers.
 b is rational, $b = \frac{e}{f}$ where e and f are integers.
 $a + b = \frac{cf + de}{df}$
 cf, de and df are integers.
 So $a + b$ is rational. This contradicts the assumption that $a + b$ is irrational.
 Therefore if $a + b$ is irrational then at least one of a and b is irrational.
 c Many possible answers e.g. $a = 2 - \sqrt{2}$, $b = \sqrt{2}$.

6 Assumption: there exists integers a and b such that $21a + 14b = 1$.
 Since 21 and 14 are multiples of 7, divide both sides by 7.
 So now $3a + 2b = \frac{1}{7}$
 $3a$ is also an integer. $2b$ is also an integer.
 The sum of two integers will always be an integer, so $3a + 2b$ = 'an integer'.
 This contradicts the statement that $3a + 2b = \frac{1}{7}$.
 Therefore there exists no integers a and b for which $21a + 14b = 1$.

7 a Assumption: There exists a number n such that n^2 is a multiple of 3, but n is not a multiple of 3.
 We know that all multiples of 3 can be written in the form $n = 3k$, therefore $3k + 1$ and $3k + 2$ are not multiples of 3.
 Let $n = 3k + 1$
 $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$
 In this case n^2 is not a multiple of 3.
 Let $m = 3k + 2$
 $m^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$
 In this case m^2 is also not a multiple of 3.
 This contradicts the assumption that n^2 is a multiple of 3.
 Therefore if n^2 is a multiple of 3, n is a multiple of 3.
 b Assumption: $\sqrt{3}$ is a rational number.
 Then $\sqrt{3} = \frac{a}{b}$ for some integers a and b.
 Further assume that this fraction is in its simplest terms: there are no common factors between a and b.
 So $3 = \frac{a^2}{b^2}$ or $a^2 = 3b^2$.
 Therefore a^2 must be a multiple of 3.
 We know from part **a** that this means a must also be a multiple of 3.
 Write $a = 3c$, which means $a^2 = (3c)^2 = 9c^2$.
 Now $9c^2 = 3b^2$, or $3c^2 = b^2$.
 Therefore b^2 must be a multiple of 3, which implies b is also a multiple of 3.
 If a and b are both multiples of 3, this contradicts the statement that there are no common factors between a and b.
 Therefore, $\sqrt{3}$ is an irrational number.

8 Assumption: there is an integer solution to the equation $x^2 - y^2 = 2$.
Remember that $x^2 - y^2 = (x - y)(x + y) = 2$
To make a product of 2 using integers, the possible pairs are: (2, 1), (1, 2), (−2, −1) and (−1, −2).
Consider each possibility in turn.
$x - y = 2$ and $x + y = 1 \Rightarrow x = \frac{3}{2}, y = -\frac{1}{2}$.
$x - y = 1$ and $x + y = 2 \Rightarrow x = \frac{3}{2}, y = \frac{1}{2}$.
$x - y = -2$ and $x + y = -1 \Rightarrow x = -\frac{3}{2}, y = \frac{1}{2}$.
$x - y = -1$ and $x + y = -2 \Rightarrow x = -\frac{3}{2}, y = -\frac{1}{2}$.
This contradicts the statement that there is an integer solution to the equation $x^2 - y^2 = 2$.
Therefore the original statement must be true: There are no integer solutions to the equation $x^2 - y^2 = 2$.

9 Assumption: $\sqrt[3]{2}$ is rational and can be written in the form $\sqrt[3]{2} = \frac{a}{b}$ and there are no common factors between a and b.
$2 = \frac{a^3}{b^3}$ or $a^3 = 2b^3$
This means that a^3 is even, so a must also be even.
If a is even, $a = 2n$.
So $a^3 = 2b^3$ becomes $(2n)^3 = 2b^3$ which means $8n^3 = 2b^3$ or $4n^3 = b^3$ or $2(2n^3) = b^3$.
This means that b^3 must be even, so b is also even.
If a and b are both even, they will have a common factor of 2.
This contradicts the statement that a and b have no common factors.
We can conclude the original statement is true: $\sqrt[3]{2}$ is an irrational number.

10 a $n - 1$ could be non-positive, e.g. if $n = \frac{1}{2}$

b Assumption: There is a least positive rational number, n.
$n = \frac{a}{b}$ where a and b are integers.
Let $m = \frac{a}{2b}$. Since a and b are integers, m is rational and $m < n$.
This contradicts the statement that n is the least positive rational number.
Therefore, there is no least positive rational number.

Exercise 1B

1 a $\frac{a^2}{cd}$ b a c $\frac{1}{2}$ d $\frac{1}{2}$ e $\frac{4}{x^2}$ f $\frac{r^5}{10}$

2 a $\frac{1}{x-2}$ b $\frac{a-3}{2(a+3)}$ c $\frac{x-3}{y}$ d $\frac{y+1}{y}$
e $\frac{x}{6}$ f 4 g $\frac{1}{x+5}$ h $\frac{3y-2}{2}$ i $\frac{2(x+y)^2}{(x-y)^2}$

3 All factors cancel exactly except $\frac{x-8}{8-x} = \frac{x-8}{-(x-8)} = -1$

4 $a = 5, b = 12$

5 a $\frac{x-4}{2x+10}$ b $x = \frac{10e^2+4}{1-2e^2}$

6 a $\frac{2x^2-3x-2}{6x-8} \div \frac{x-2}{3x^2+14x-24} = \frac{2x^2-3x-2}{6x-8} \times \frac{3x^2+14x-24}{x-2} = \frac{(2x+1)(x-2)}{2(3x-4)} \times \frac{(3x-4)(x+6)}{x-2}$
$= \frac{(2x+1)(x+6)}{2} = \frac{2x^2+13x+6}{2}$

b $f'(x) = 2x + \frac{13}{2}$; $f'(4) = \frac{29}{2}$

Exercise 1C

1 a $\frac{7}{12}$ b $\frac{7}{20}$ c $\frac{p+q}{pq}$ d $\frac{7}{8x}$ e $\frac{3-x}{x^2}$ f $\frac{2a-15}{10b}$

2 a $\frac{x+3}{x(x+1)}$ b $\frac{-x+7}{(x-1)(x+2)}$ c $\frac{8x-2}{(2x+1)(x-1)}$
d $\frac{-x-5}{6}$ e $\frac{2x-4}{(x+4)^2}$ f $\frac{23x+9}{6(x+3)(x-1)}$

3 a $\frac{x+3}{(x+1)^2}$ b $\frac{3x+1}{(x-2)(x+2)}$ c $\frac{-x-7}{(x+1)(x+3)^2}$
d $\frac{3x+3y+2}{(y-x)(y+x)}$ e $\frac{2x+5}{(x+2)^2(x+1)}$ f $\frac{7x+8}{(x+2)(x+3)(x-4)}$

4 $\frac{2x-19}{(x+5)(x-3)}$

5 a $\frac{6x^2+14x+6}{x(x+1)(x+2)}$ b $\frac{-x^2-24x-8}{3x(x-2)(2x+1)}$
c $\frac{9x^2-14x-7}{(x-1)(x+1)(x-3)}$

6 $\frac{50x+3}{(6x+1)(6x-1)}$

7 a $x + \frac{6}{x+2} + \frac{36}{x^2-2x-8}$
$= \frac{x(x+2)(x-4)}{(x+2)(x-4)} + \frac{6(x-4)}{(x+2)(x-4)} + \frac{36}{(x+2)(x-4)}$
$= \frac{x^3-2x^2-2x+12}{(x+2)(x-4)}$

b Divide $x^3 - 2x^2 - 2x + 12$ by $(x + 2)$ to give $x^2 - 4x + 6$

Exercise 1D

1 a $\frac{4}{x+3} + \frac{2}{x-2}$ b $\frac{3}{x+1} - \frac{1}{x+4}$
c $\frac{3}{2x} - \frac{5}{x-4}$ d $\frac{4}{2x+1} - \frac{1}{x-3}$
e $\frac{2}{x+3} + \frac{4}{x-3}$ f $-\frac{2}{x+1} - \frac{1}{x-4}$
g $\frac{2}{x} - \frac{3}{x+4}$ h $\frac{3}{x+5} - \frac{1}{x-3}$

2 $A = \frac{1}{2}, B = -\frac{3}{2}$

3 $A = 24, B = -2$

4 $A = 1, B = -2, C = 3$

5 $D = -1, E = 2, F = -5$

6 $\frac{3}{x+1} - \frac{2}{x+2} - \frac{6}{x-5}$

7 a $\frac{3}{x} - \frac{2}{x+1} + \frac{5}{x-1}$
b $\frac{-1}{5x+4} + \frac{2}{2x-1}$

Challenge

$\frac{6}{x-2} + \frac{1}{x+1} - \frac{2}{x-3}$

Online Full worked solutions are available in SolutionBank.

Exercise 1E

1 $A = 0, B = 1, C = 3$
2 $D = 3, E = -2, F = -4$
3 $P = -2, Q = 4, R = 2$
4 $C = 3, D = 1, E = 2$
5 $A = 2, B = -4$
6 $A = 2, B = 4, C = 11$
7 $A = 4, B = 1$ and $C = 12$.
8 **a** $\frac{4}{x+5} - \frac{19}{(x+5)^2}$ **b** $\frac{2}{x} - \frac{1}{2x-1} + \frac{6}{(2x-1)^2}$

Exercise 1F

1 $A = 1, B = 1, C = 2, D = -6$
2 $a = 2, b = -3, c = 5, d = -10$
3 $p = 1, q = 2, r = 4$
4 $m = 2, n = 4, p = 7$
5 $A = 4, B = 1, C = -8$ and $D = 3$.
6 $A = 4, B = -13, C = 33$ and $D = -27$
7 $p = 1, q = 0, r = 2, s = 0$ and $t = -6$
8 $a = 2, b = 1, c = 1, d = 5$ and $e = -4$
9 $A = 3, B = -4, C = 1, D = 4, E = 1$
10 **a** $(x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$
b $(x - 1)(x^2 + 1)$, $a = 1, b = -1, c = 1, d = 0$ and $e = 1$.

Exercise 1G

1 $A = 1, B = -2, C = 8$
2 $A = 1, B = -2, C = 3$
3 $A = 1, B = 0, C = 3, D = -4$
4 $A = 2, B = -3, C = 5, D = 1$
5 $A = 1, B = 5, C = -5$
6 $A = 2, B = -4, C = 1$
7 **a** $4 + \frac{2}{(x-1)} + \frac{3}{(x+4)}$ **b** $x + \frac{3}{x} + \frac{2}{(x-2)} - \frac{1}{(x-2)^2}$
8 $A = 2, B = -3, C = \frac{34}{11}, D = \frac{73}{11}$
9 $A = 2, B = 2, C = 3, D = 2$.
10 $A = 1, B = -1, C = 5, D = -\frac{38}{3}, E = \frac{8}{3}$.

Mixed exercise 1

1 Assume $\sqrt{\frac{1}{2}}$ is a rational number.
Then $\sqrt{\frac{1}{2}} = \frac{a}{b}$ for some integers a and b.
Further assume that this fraction is in its simplest terms: there are no common factors between a and b.
So $0.5 = \frac{a^2}{b^2}$ or $2a^2 = b^2$.
Therefore b^2 must be a multiple of 2.
We know that this means b must also be a multiple of 2.
Write $b = 2c$, which means $b^2 = (2c)^2 = 4c^2$.
Now $4c^2 = 2a^2$, or $2c^2 = a^2$.
Therefore a^2 must be a multiple of 2, which implies a is also a multiple of 2.
If a and b are both multiples of 2, this contradicts the statement that there are no common factors between a and b.
Therefore, $\sqrt{\frac{1}{2}}$ is an irrational number.

2 Assume there exists a rational number q such that q^2 is irrational.
So write $q = \frac{a}{b}$ where a and b are integers.
$q^2 = \frac{a^2}{b^2}$
As a and b are integers a^2 and b^2 are integers.
So q^2 is rational.
This contradicts assumption that q^2 is irrational.
Therefore if q^2 is irrational then q is irrational.

3 **a** $\frac{1}{3}$ **b** $\frac{2(x^2+4)(x-5)}{(x^2-7)(x+4)}$ **c** $\frac{2x+3}{x}$

4 **a** $\frac{2x-4}{x-4}$ **b** $\frac{4(e^6-1)}{e^6-2}$

5 **a** $a = \frac{3}{4}, b = -\frac{13}{8}, c = -\frac{5}{8}$
b $g'(x) = \frac{3}{2}x - \frac{13}{8}$, $g'(-2) = -\frac{37}{8}$

6 $\frac{6x^2 + 18x + 5}{x^2 - 3x - 10}$

7 $x + \frac{3}{x-1} - \frac{12}{x^2+2x-3}$
$= \frac{x(x+3)(x-1)}{(x+3)(x-1)} + \frac{3(x+3)}{(x+3)(x-1)} - \frac{12}{(x+3)(x-1)}$
$= \frac{(x^2+3x+3)(x-1)}{(x+3)(x-1)} = \frac{x^2+3x+3}{x+3}$

8 $A = 3, B = -2$
9 $P = 1, Q = 2, R = -3$
10 $D = 5, E = 2$
11 $A = 4, B = -2, C = 3$
12 $D = 2, E = 1, F = -2$
13 $A = 1, B = -4, C = 3, D = 8$
14 $A = 2, B = -4, C = 6, D = -11$
15 $A = 1, B = 0, C = 1, D = 3$
16 $A = 1, B = 2, C = 3, D = 4, E = 1$.
17 $A = 2, B = -\frac{9}{4}, C = \frac{1}{4}$
18 $P = 1, Q = -\frac{1}{2}, R = \frac{5}{2}$
19 **a** $f(-3) = 0$ or $f(x) = (x + 3)(2x^2 + 3x + 1)$
b $\frac{1}{(x+3)} + \frac{8}{(2x+1)} - \frac{5}{(x+1)}$

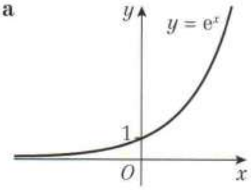
Challenge
Assume L is not perpendicular to OA. Draw the line through O which is perpendicular to L. This line meets L at a point B, outside the circle. Triangle OBA is right-angled at B, so OA is the hypotenuse of this triangle, so $OA > OB$. This gives a contradiction, as B is outside the circle, so $OA < OB$. Therefore L is perpendicular to OA.

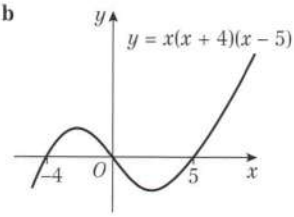
CHAPTER 2

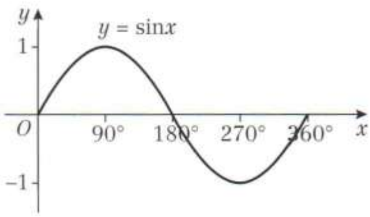
Prior knowledge 2

1 **a** $y = \frac{9-5x}{7}$ **b** $y = \frac{5p-8x}{2}$ **c** $y = \frac{5x-4}{8+9x}$

2 **a** $25x^2 - 30x + 5$
b $\frac{1}{6x-14}$ **c** $\frac{3x+7}{-x-1}$

3 **a**


b


c


4 a 28 b 0 c 18

Exercise 2A

1 a $\frac{3}{4}$ b 0.28 c 8 d $\frac{19}{56}$ e 4 f 11

2 a 5 b 46 c 40

3 a 16 b 65 c 0

4 a Positive $|x|$ graph with vertex at (1, 0), y-intercept at (0, 1)

b Positive $|x|$ graph with vertex at $(-1\frac{1}{2}, 0)$, y-intercept at (0, 3)

c Positive $|x|$ graph with vertex at $(\frac{7}{4}, 0)$, y-intercept at (0, 7)

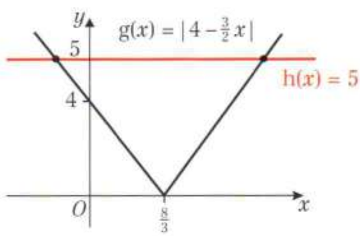
d Positive $|x|$ graph with vertex at (10, 0), y-intercept at (0, 5)

e Positive $|x|$ graph with vertex at (7, 0), y-intercept at (0, 7)

f Positive $|x|$ graph with vertex at $(\frac{3}{2}, 0)$, y-intercept at (0, 6)

g Negative $|x|$ graph with vertex and y-intercept at (0, 0)

h Negative $|x|$ graph with vertex at $(\frac{1}{3}, 0)$, y-intercept at (0, −1)

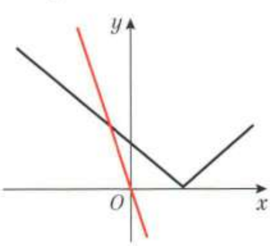
5 a


b $x = -\frac{2}{3}$ and $x = 6$

6 a $x = 2$ and $x = -\frac{4}{3}$ b $x = 7$ or $x = 3$

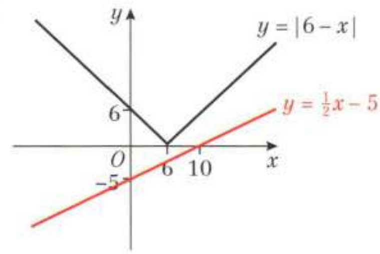
c No solution d $x = 1$ and $x = -\frac{1}{7}$

e $x = -\frac{2}{5}$ or $x = 2$ f $x = 24$ or $x = -12$

7 a


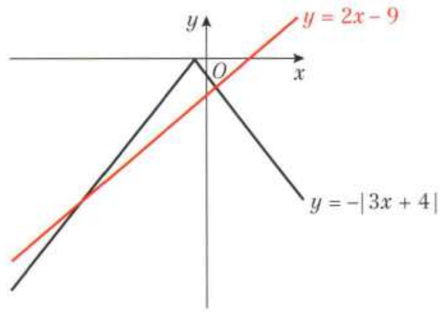
b $x = -\frac{4}{3}$

8 $x = -3$, $x = 4$

9 a


b The two graphs do not intersect, therefore there are no solutions to the equation $|6 - x| = \frac{1}{2}x - 5$.

10 Value for x cannot be negative as it equals a modulus.

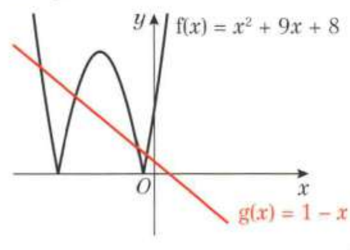
11 a


b $x < -13$ and $x > 1$

12 $-23 < x < \frac{5}{3}$

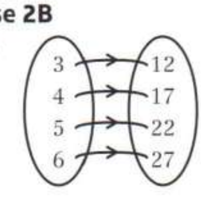
13 a $k = -3$ b Solution is $x = 6$.

Challenge

a


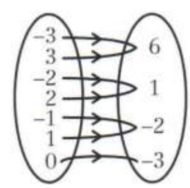
b There are 4 solutions: $x = -5 \pm 3\sqrt{2}$ and $x = -4 \pm \sqrt{7}$

Exercise 2B

1 a i


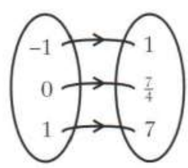
ii one-to-one

iii $\{f(x) = 12, 17, 22, 27\}$

b i


ii many-to-one

iii $\{g(x) = -3, -2, 1, 6\}$

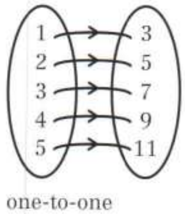
c i


ii one-to-one

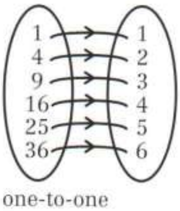
iii $\{h(x) = 1, \frac{7}{4}, 7\}$

2 a i one-to-one ii function
b i one-to-one ii function
c i one-to-many ii not a function
d i one to many ii not a function
e i many-to-one
ii not valid at the asymptote, so not a function.
f i many to one ii function

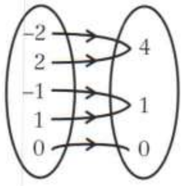
3 a 6 b $\pm 2\sqrt{5}$ c 4 d 2, −3

4 a i


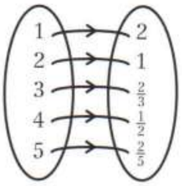
ii one-to-one

b i


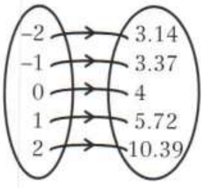
ii one-to-one

c i


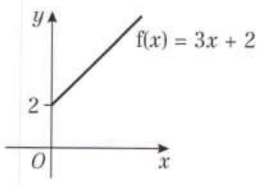
ii many-to-one

d i


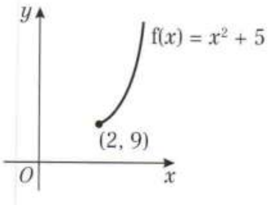
ii one-to-one

e i


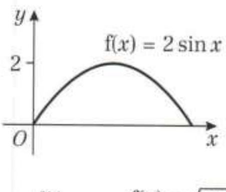
ii one-to-one

5 a i


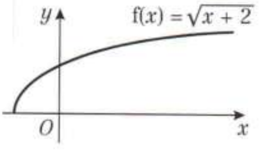
ii $f(x) \geqslant 2$
iii one-to-one

b i


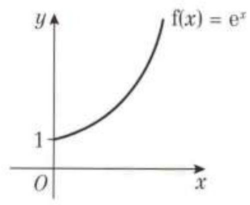
ii $f(x) \geqslant 9$
iii one-to-one

c i


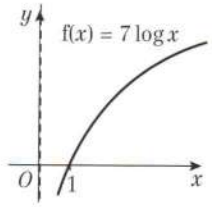
ii $0 \leqslant f(x) \leqslant 2$
iii many-to-one

d i


ii $f(x) \geqslant 0$
iii one-to-one

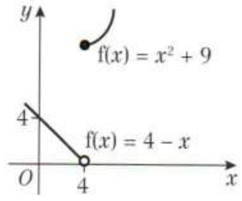
e i


ii $f(x) \geqslant 1$
iii one-to-one

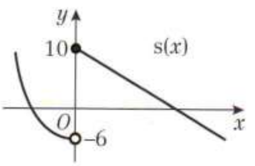
f i


ii $f(x) \in \mathbb{R}$
iii one-to-one

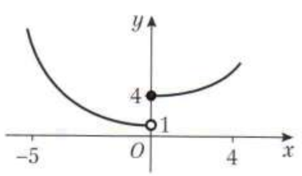
6 a $g(x)$ is not a function because it is not defined for $x = 4$

b


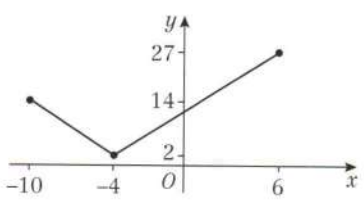
c i 1 ii 109
d $a = -86$ or $a = 9$

7 a


b −7
c −2 and 5

8 a


b $a = -3.91$ or $a = 3.58$

9 a


b Range $\{2 \leqslant h(x) \leqslant 27\}$ c $a = -9, a = 0$

10 $c = \frac{2}{5}, d = \frac{44}{5}$ 11 $a = 2, b = -1$ 12 $a = 3$

Exercise 2C

1 a 7 b $\frac{9}{4}$ or 2.25 c 0.25 d −47 e −26

2 a $4x^2 - 15$ b $16x^2 + 8x - 3$ c $\frac{1}{x^2} - 4$
d $\frac{4}{x} + 1$ e $16x + 5$

3 a $fg(x) = 3x^2 - 2$ b $x = 1$

4 a $qp(x) = \frac{4x - 5}{x - 2}$ b $x = \frac{9}{4}$

5 a 23 b $x = \frac{13}{7}$ and $x = \frac{13}{5}$

6 **a** $f^2(x) = f\left(\frac{1}{x+1}\right) = \frac{1}{\left(\frac{1}{x+1}\right)+1} = \frac{x+1}{x+2}$

b $f^3(x) = \frac{x+2}{2x+3}$

7 **a** 2^{x+3} **b** $2^x + 3$ **c** $\frac{\ln\left(\frac{3}{7}\right)}{\ln(2)}$

8 **a** $20x$ **b** x^{20}

9 **a** $(x+3)^3 - 1$, $qp(x) > -1$

b 999 **c** $x = 2$

10 $3 \pm \frac{\sqrt{6}}{2}$

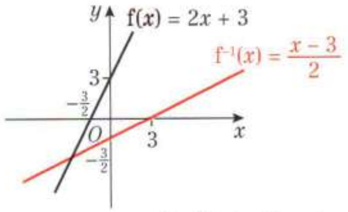
11 **a** $-8 \leqslant g(x) \leqslant 12$ **b** 6 **c** 10.5

Exercise 2D

1 **a** **i** $\{y \in \mathbb{R}\}$ **ii** $f^{-1}(x) = \frac{x-3}{2}$

iii Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$

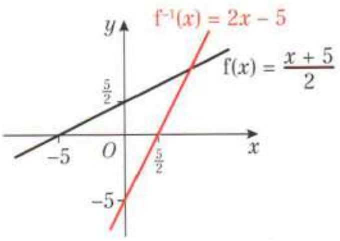
iv



b **i** $\{y \in \mathbb{R}\}$ **ii** $f^{-1}(x) = 2x - 5$

iii Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$

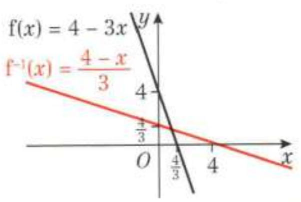
iv



c **i** $\{y \in \mathbb{R}\}$ **ii** $f^{-1}(x) = \frac{4-x}{3}$

iii Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$

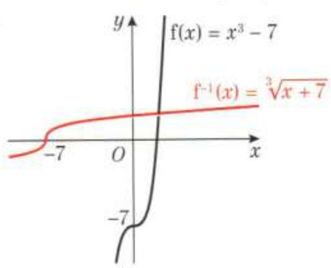
iv



d **i** $\{y \in \mathbb{R}\}$ **ii** $f^{-1}(x) = \sqrt[3]{x+7}$

iii Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$

iv



2 **a** $f^{-1}(x) = 10 - x$, $\{x \in \mathbb{R}\}$ **b** $g^{-1}(x) = 5x$, $\{x \in \mathbb{R}\}$

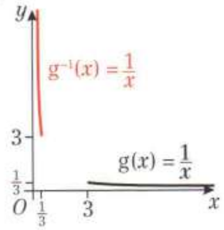
c $h^{-1}(x) = \frac{3}{x}$, $\{x \neq 0\}$ **d** $k^{-1}(x) = x + 8$, $\{x \in \mathbb{R}\}$

3 Domain becomes $x < 4$

4 **a** **i** $0 < g(x) \leqslant \frac{1}{3}$ **ii** $g^{-1}(x) = \frac{1}{x}$

iii $\{x \in \mathbb{R}, 0 < x \leqslant \frac{1}{3}\}$, $g^{-1}(x) \geqslant 3$

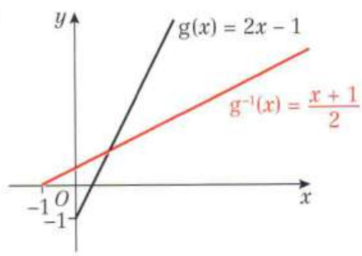
iv



b **i** $g(x) \geqslant -1$ **ii** $g^{-1}(x) = \frac{x+1}{2}$

iii $\{x \in \mathbb{R}, x \geqslant -1\}$, $g^{-1}(x) \geqslant 0$

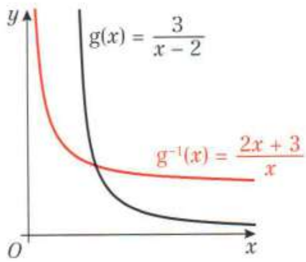
iv



c **i** $g(x) > 0$ **ii** $g^{-1}(x) = \frac{2x+3}{x}$

iii $\{x \in \mathbb{R}, x > 0\}$, $g^{-1}(x) > 2$

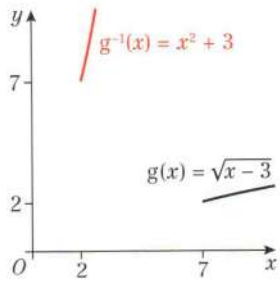
iv



d **i** $g(x) \geqslant 2$ **ii** $g^{-1}(x) = x^2 + 3$

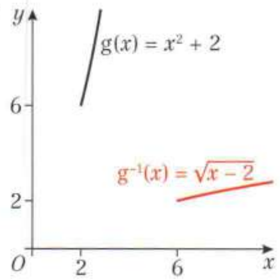
iii $\{x \in \mathbb{R}, x \geqslant 2\}$, $g^{-1}(x) \geqslant 7$

iv



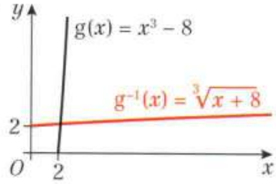
e **i** $g(x) > 6$ **ii** $g^{-1}(x) = \sqrt{x-2}$

iii $\{x \in \mathbb{R}, x > 6\}$, $g^{-1}(x) > 2$

iv


f **i** $g(x) \geqslant 0$ **ii** $g^{-1}(x) = \sqrt[3]{x+8}$

iii $\{x \in \mathbb{R}, x \geqslant 0\}$, $g^{-1}(x) \geqslant 2$

iv


5 $t^{-1}(x) = \sqrt{x+4} + 3$, $\{x \in \mathbb{R}, x \geqslant 0\}$

6 **a** -2 **b** $m^{-1}(x) = \sqrt{x-5} - 2$ **c** $x > 5$

7 **a** tends to $\pm\infty$

b 7

c $h^{-1}(x) = \dfrac{2x+1}{x-2}$ $\{x \in \mathbb{R}, x \neq 2\}$

d $2 + \sqrt{5}$, $2 - \sqrt{5}$

8 **a** $nm(x) = x$

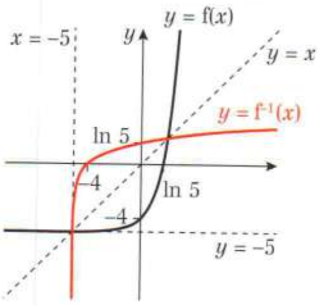
b The functions m and n are inverse of one another as $mn(x) = nm(x) = x$.

9 $st(x) = \dfrac{3}{\frac{3-x}{x} + 1} = x$, $ts(x) = \dfrac{3 - \frac{3}{x-1}}{\frac{3}{x+1}} = x$

10 **a** $f^{-1}(x) = -\sqrt{\dfrac{x+3}{2}}$ $\{x \in \mathbb{R}, x > -3\}$

b $a = -1$

11 **a** $f(x) > -5$ **b** $f^{-1}(x) = \ln(x+5)$ $\{x \in \mathbb{R}, x > -5\}$

c


d $g^{-1}(x) = e^x + 4$, $x \in \mathbb{R}$ **e** $x = 1.95$

12 **a** $f(x) = \dfrac{3(x+2)}{x^2 + x - 20} - \dfrac{2}{x-4}$

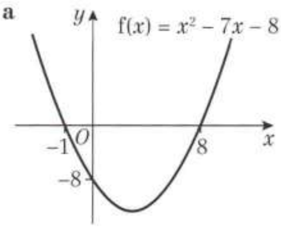
$= \dfrac{3(x+2)}{(x+5)(x-4)} - \dfrac{2(x+5)}{(x+5)(x-4)} = \dfrac{x-4}{(x+5)(x-4)}$

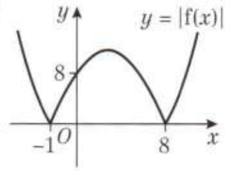
$= \dfrac{1}{x+5}$

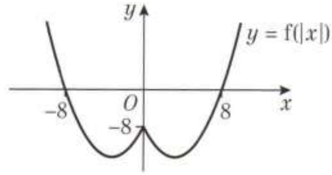
b $\left\{y \in \mathbb{R}, 0 < y < \frac{1}{9}\right\}$

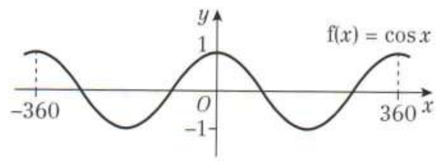
c $f^{-1}: x \to \dfrac{1}{x} - 5$. Domain is $\left\{x \in \mathbb{R}, 0 < x < \frac{1}{9} \text{ and } x \neq 0\right\}$

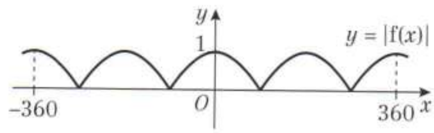
Exercise 2E

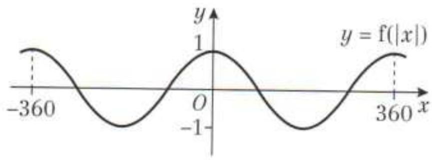
1 **a**


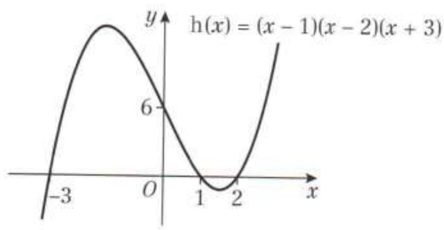
b


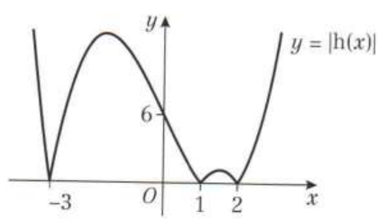
c


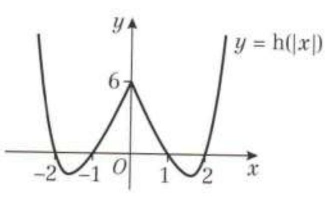
2 **a**


b


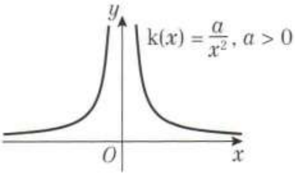
c


3 **a**


b


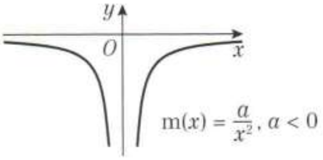
c


4 a



b Both these graphs would match the original graph.

c

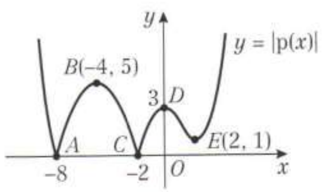


d i True, $|k(x)| = \left|\frac{a}{x^2}\right| = \left|\frac{-a}{x^2}\right| = |m(x)|$

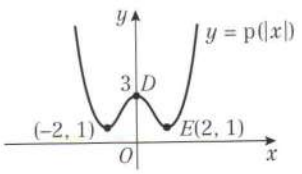
ii False, $k(|x|) = \frac{a}{|x|^2} \neq \frac{-a}{|x|^2} = m(|x|)$

iii True, $m(|x|) = \frac{-a}{|x|^2} = \frac{-a}{x^2} = m(x)$

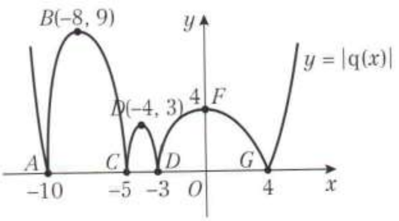
5 a



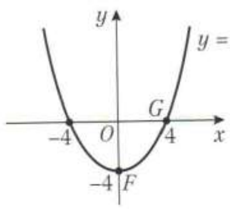
b



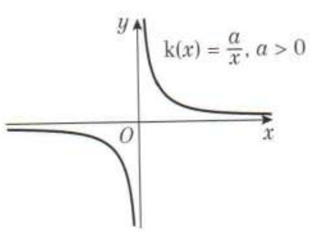
6 a



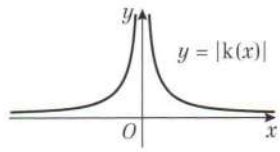
b



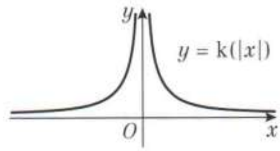
7 a



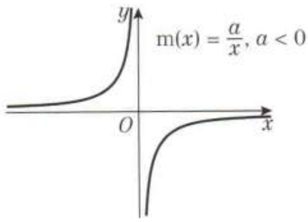
b



c

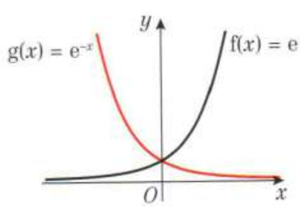


8 a



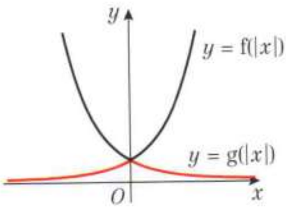
b They are reflections of each other in the x-axis. $|m(x)| = -m(|x|)$

9 a

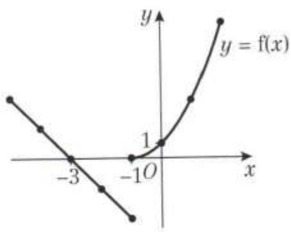


b They would be the same as the original graph.

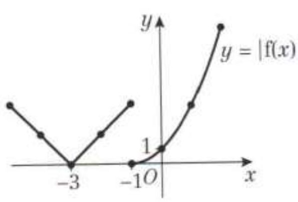
c



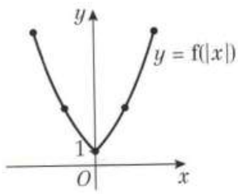
10 a $-4 < f(x) \leqslant 9$



b

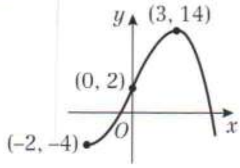


c

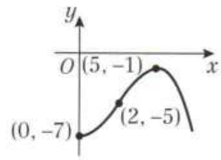


Exercise 2F

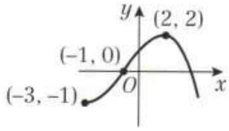
1 a



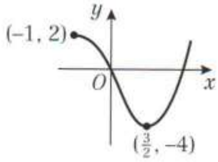
b



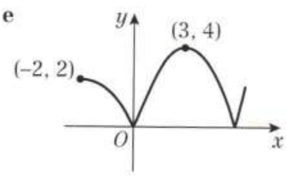
c



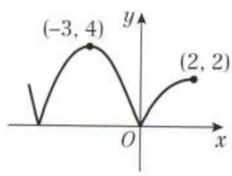
d



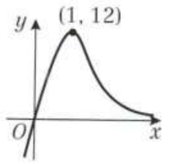
e



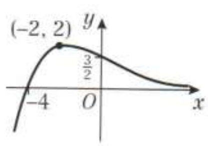
f



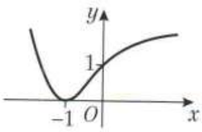
2 a



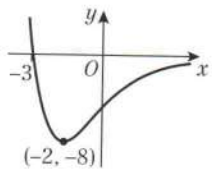
b



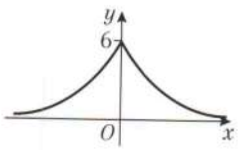
c



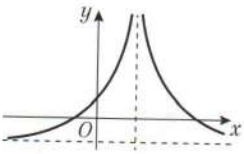
d



e

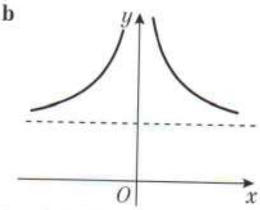


3 a



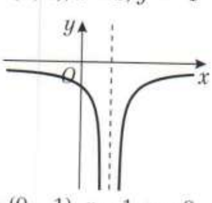
$A = (0, 2)$, $x = 2$, $y = -1$

b



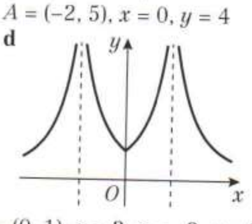
$A = (-2, 5)$, $x = 0$, $y = 4$

c



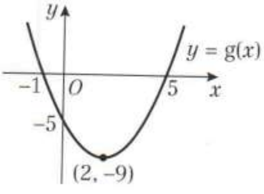
$A = (0, -1)$, $x = 1$, $y = 0$

d



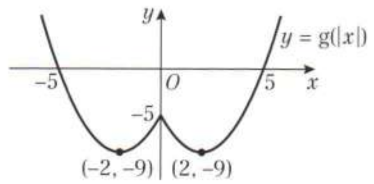
$A = (0, 1)$, $x = 2$, $x = -2$, $y = 0$

4 a

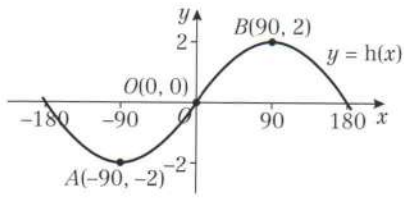


b **i** $(6, -18)$ **ii** $(1, -9)$ **iii** $(2, 9)$

c

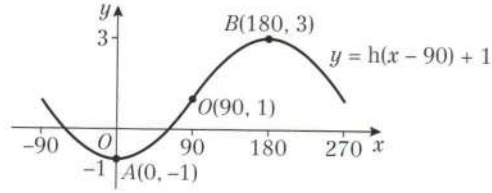


5 a

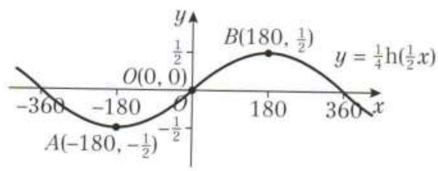


b $A(-90, -2)$ and $B(90, 2)$

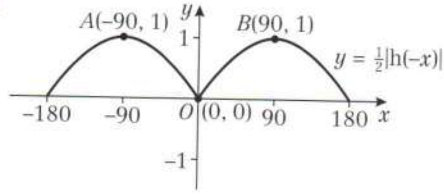
c i



ii

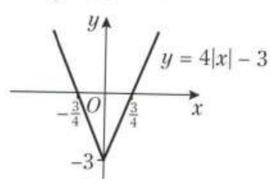


iii

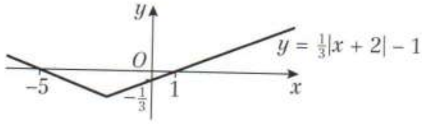


Exercise 2G

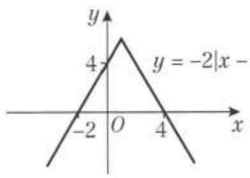
1 a Range $f(x) \geqslant -3$



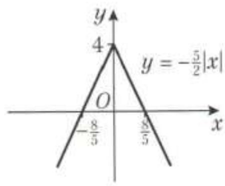
b Range $f(x) \geqslant -1$



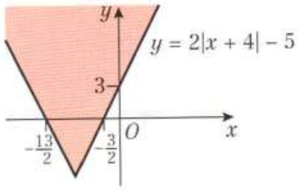
c Range $f(x) \leqslant 6$



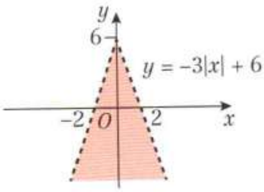
d Range $f(x) \leqslant 4$



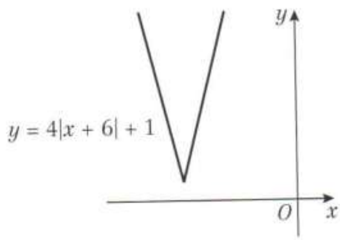
2 a, b



3 a, b

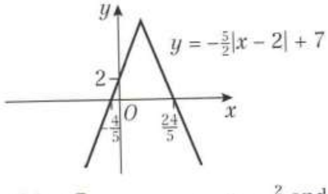


4 a



b $f(x) \geqslant 1$ **c** $x = -\frac{16}{3}$ and $x = -\frac{48}{7}$

5 a



b $g(x) \leqslant 7$ **c** $x = -\frac{2}{3}$ and $x = \frac{22}{7}$

6 $k < 14$

7 $b = 2$

8 a $h(x) \geqslant -7$

b Original function is many-to-one, therefore the inverse is one-to-many, which is not a function.

c $-\frac{1}{2} < x < \frac{5}{2}$ **d** $k < -\frac{23}{3}$

9 a $a = 10$ **b** $P(-3, 10)$ and $Q(2, 0)$

c $x = -\frac{6}{7}$ and $x = -6$

10 a $m(x) \leqslant 7$ **b** $x = -\frac{35}{23}$ and $x = -5$

c $k < 7$

Challenge

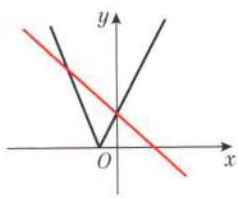
1 a $A(3, -6)$ and $B(7, -2)$

b 6 units^2.

2 Graphs intersect at $x = \frac{1}{3}$ and $x = \frac{17}{3}$.
Maximum point of $f(x)$ is $(3, 10)$. Minimum point of $g(x)$ is $(3, 2)$. Using area of a kite, area $= \frac{64}{3}$

Mixed exercise 2

1 a

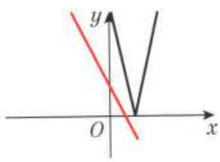


b $x = 0, x = -4$

2 $k > -\frac{11}{4}$

3 $x = -\frac{24}{19}$ and $x = \frac{40}{21}$

4 a



b The graphs do not intersect, so there are no solutions.

5 a i one-to-many **ii** not a function

b i one-to-one **ii** function

c i many-to-one **ii** function

d i many-to-one **ii** function

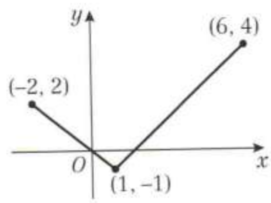
e i one-to-one

ii not a function

f i one-to-one

ii not a function

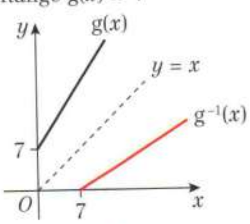
6 a



b $\frac{1}{2}$ and $1\frac{1}{2}$

7 a $pq(x) = 4x^2 + 10x$ **b** $x = \dfrac{-3 \pm \sqrt{21}}{4}$

8 a Range $g(x) \geqslant 7$



b $g^{-1}(x) = \dfrac{x - 7}{2}$, $\{x \in \mathbb{R}, x \geqslant 7\}$

c $g^{-1}(x)$ is a reflection of $g(x)$ in the line $y = x$

9 a $f^{-1}(x) = \dfrac{x + 3}{x - 2}$, $\{x \in \mathbb{R}, x > 2\}$

b i Range $f^{-1}(x) > 1$ **ii** $\{x \in \mathbb{R}, x > 2\}$

10 a $f(x) = \frac{x}{x^2-1} - \frac{1}{x+1} = \frac{x}{(x-1)(x+1)} - \frac{1}{x+1}$

$= \frac{x}{(x-1)(x+1)} - \frac{x-1}{(x-1)(x+1)} = \frac{1}{(x-1)(x+1)}$

b $f(x) > 0$ **c** $x = 6$

11 a $20, 28, \frac{1}{9}$ **b** $f(x) \geqslant -8, g(x) \in \mathbb{R}$

c $g^{-1}(x) = \sqrt[3]{x-1}, \{x \in \mathbb{R}\}$

d $4(x^3 - 1)$ **e** $a = \frac{5}{3}$

12 a $a = -3$ **b** $f^{-1}: x \mapsto \sqrt{x+13} - 3, x > -4$

13 a $f^{-1}(x) = \frac{x+1}{4}, \{x \in \mathbb{R}\}$

b $gf(x) = \frac{3}{8x-3}, \{x \in \mathbb{R}, x \neq \frac{3}{8}\}$

c -0.076 and 0.826 (3 d.p.)

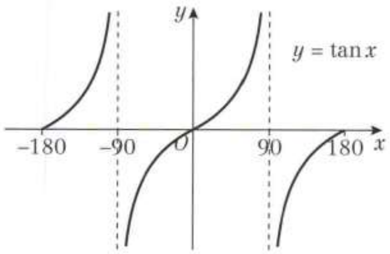
14 a $f^{-1}(x) = \frac{2x}{x-1}, \{x \in \mathbb{R}, x \neq 1\}$

b Range $f^{-1}(x) \in \mathbb{R}, f^{-1}(x) \neq 2$

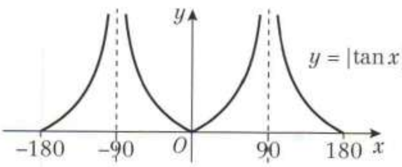
c -1 **d** $1, \frac{6}{5}$

15 a $8, 9$ **b** -45 and $5\sqrt{2}$

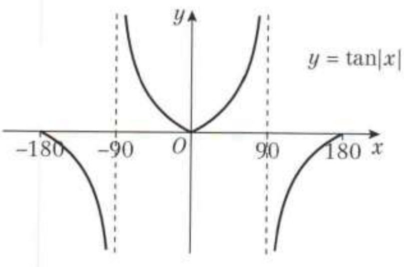
16 a



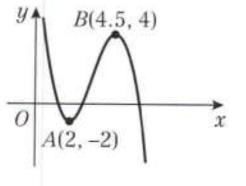
b



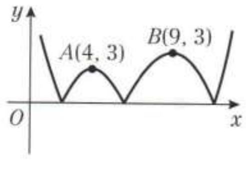
c



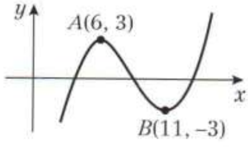
17 a



b

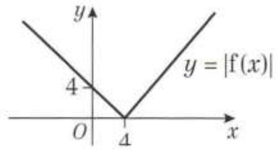


c



18 a $g(x) \geqslant 0$ **b** $x = 0, x = 8$

c



19 a Positive $|x|$ graph with vertex at $\left(\frac{a}{2}, 0\right)$ and y-intercept at $(0, a)$.

b Positive $|x|$ graph with vertex at $\left(\frac{a}{4}, 0\right)$ and y-intercept at $(0, a)$.

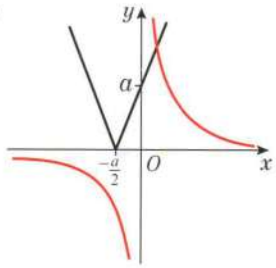
c $a = 6, a = 10$

20 a Positive $|x|$ graph with vertex at $(2a, 0)$ and y-intercept at $(0, 2a)$.

b $x = \frac{3a}{2}, x = 3a$

c Negative $|x|$ graph with x-intercepts at $(a, 0)$ and $(3a, 0)$ and y-intercept at $(0, -a)$.

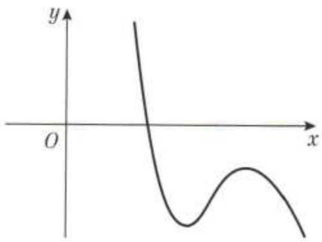
21 a, b



c One intersection point **d** $x = \frac{-a + \sqrt{(a^2 + 8)}}{4}$

22 a $(1, 2), (\frac{5}{2}, 5\ln\frac{5}{2} - \frac{13}{4})$

b

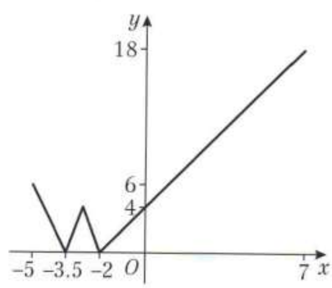


c $(3, -6)$, Minimum

$(\frac{9}{2}, \frac{39}{4} - 15\ln\frac{5}{2})$, Maximum

23 a $-2 \leqslant f(x) \leqslant 18$ **b** 0

c



d $x = \frac{7 \pm \sqrt{5}}{2}$

24 a $p(x) \leqslant 10$

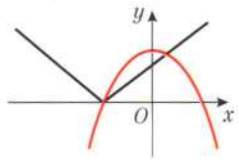
b Original function is many-to-one, therefore the inverse is one-to-many, which is not a function.

c $-11 < x < 3$

d $k > 8$

Challenge

a



b $(-a, 0), (a, 0), (0, a^2)$ **c** $a = 5$

CHAPTER 3

Prior knowledge 3

1 **a** 22, 27, 32 **b** −1, −4, −7 **c** 9, 15, 21
d 48, 96, 192 **e** $\frac{1}{32}, \frac{1}{64}, \frac{1}{128}$ **f** −16, 64, −256

2 **a** $x = 5.64$ **b** $x = 3.51$ **c** $x = 9.00$

Exercise 3A

1 **a** **i** 7, 12, 17, 22 **ii** $a = 7, d = 5$
b **i** 7, 5, 3, 1 **ii** $a = 7, d = -2$
c **i** 7.5, 8, 8.5, 9 **ii** $a = 7.5, d = 0.5$
d **i** −9, −8, −7, −6 **ii** $a = -9, d = 1$

2 **a** $2n + 3$, 23 **b** $3n + 2$, 32
c $27 - 3n$, −3 **d** $4n - 5$, 35
e nx, $10x$ **f** $a + (n - 1)d$, $a + 9d$

3 **a** 22 **b** 40 **c** 39 **d** 46 **e** 18 **f** n

4 $d = 6$ **5** $p = \frac{2}{3}, q = 5$ **6** −1.5

7 24 **8** −70 **9** $k = \frac{1}{2}, k = 8$

10 $-2 + 3\sqrt{5}$

Challenge

$a = 4, b = 2$

Exercise 3B

1 **a** 820 **b** 450 **c** −1140
d −294 **e** 1440 **f** 1425
g −1155 **h** $231x + 21$

2 **a** 20 **b** 25 **c** 65 **d** 4 or 14

3 2550 **4** 20

5 $d = -\frac{1}{2}$, 20th term = −5.5 **6** $a = 6, d = -2$

7 $S_{50} = 1 + 2 + 3 + \ldots + 50$
$S_{50} = 50 + 49 + 48 + \ldots + 1$
$2 \times S_{50} = 50(51) \Rightarrow S_{50} = 1275$

8 $S_{2n} = 1 + 2 + 3 + \ldots + 2n$
$S_{2n} = 2n + (2n - 1) + (2n - 2) + \ldots + 1$
$2 \times S_n = 2n(2n + 1) \Rightarrow S_n = n(2n + 1)$

9 $S_n = 1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1)$
$S_n = (2n - 1) + (2n - 3) + \ldots + 5 + 3 + 1$
$2 \times S_n = n(2n) \Rightarrow S_n = n^2$

10 **a** $a + 4d = 33, a + 9d = 68$

$d = 7, a = 5$ so $S_n = \frac{n}{2}[2(5) + (n - 1)7]$

$\Rightarrow 2225 = \frac{n}{2}(7n + 3) \Rightarrow 7n^2 + 3n - 4450 = 0$

b 25

11 **a** $\frac{304}{k + 2}$

b $S_n = \frac{152}{k + 2}(k + 1 + 303) = \frac{152k + 46\,208}{k + 2}$

c 17

12 **a** 1683

b **i** $\frac{100}{p}$

ii $S_{\frac{100}{p}} = \frac{50}{p}\left[8p + \left(\frac{100 - p}{p}\right)4p\right]$

$S_{\frac{100}{p}} = \frac{50}{p}[4p + 400] = 200\left[1 + \frac{100}{p}\right]$

c $161p + 81$

13 **a** $5n + 1$ **b** 285

c $S_k = \frac{k}{2}[2(6) + (k - 1)5] = \frac{k}{2}(5k + 7)$

$\frac{k}{2}(5k + 7) \leqslant 1029$

$5k^2 + 7k - 2058 \leqslant 0$

$(5k - 98)(k + 21) \leqslant 0$

d $k = 19$

Challenge

$S_n = \frac{n}{2}(2\ln 9 + (n - 1)\ln 3) = \frac{n}{2}(\ln 81 - \ln 3 + n\ln 3)$

$= \frac{n}{2}(\ln 27 + n\ln 3) = \frac{n}{2}(\ln 3^3 + \ln 3^n)$

$= \frac{n}{2}(\ln 3^{n+3}) = \frac{1}{2}(\ln 3^{n^2+3n}) \Rightarrow a = \frac{1}{2}$

Exercise 3C

1 **a** Geometric, $r = 2$ **b** Not geometric
c Not geometric **d** Geometric, $r = 3$
e Geometric, $r = \frac{1}{2}$ **f** Geometric, $r = -1$
g Geometric, $r = 1$ **h** Geometric, $r = -\frac{1}{4}$

2 **a** 135, 405, 1215 **b** −32, 64, −128
c 7.5, 3.75, 1.875 **d** $\frac{1}{64}, \frac{1}{256}, \frac{1}{1024}$
e p^3, p^4, p^5 **f** $-8x^4, 16x^5, -32x^6$

3 **a** $x = 3\sqrt{3}$ **b** $9\sqrt{3}$

4 **a** $486, 2 \times 3^{n-1}$ **b** $\frac{25}{8}, 100 \times \left(\frac{1}{2}\right)^{n-1}$
c $-32, (-2)^{n-1}$ **d** $1.61051, (1.1)^{n-1}$

5 10, 6250 **6** $a = 1, r = 2$ **7** $\frac{1}{8}, -\frac{1}{8}$

8 **a** $\frac{x^2}{2x} = \frac{2x}{8 - x} \Rightarrow x^2(8 - x) = 4x^2 \Rightarrow x^3 - 4x^2 = 0$
b 2 097 152
c Yes, 4096 is in sequence as n is integer, $n = 11$

9 **a** $ar^5 = 40 \Rightarrow 200p^5 = 40$
$\Rightarrow p^5 = \frac{1}{5} \Rightarrow \log p^5 = \log\left(\frac{1}{5}\right)$
$\Rightarrow 5\log p = \log 1 - \log 5 \Rightarrow 5\log p + \log 5 = 0$
b $p = 0.725$

10 $k = 12$

11 $n = 8.69$, so not a sequence as n not an integer

12 No, −49152 is in sequence

13 $n = 11$, 3 145 728

Exercise 3D

1 **a** 255 **b** 63.938 **c** 1.110
d −728 **e** $546\frac{2}{3}$ **f** −1.667

2 4.9995 **3** 14.4147 **4** $\frac{5}{4}, -\frac{9}{4}$

5 19 terms **6** 22 terms

7 a $\frac{25\left(1-\left(\frac{3}{5}\right)^k\right)}{\left(1-\frac{3}{5}\right)} > 61 \Rightarrow 1-\left(\frac{3}{5}\right)^k > \frac{122}{125} \Rightarrow \left(\frac{3}{5}\right)^k < \frac{3}{125}$

$\Rightarrow k\log\left(\frac{3}{5}\right) < \log\left(\frac{3}{125}\right) \Rightarrow k > \frac{\log(0.024)}{\log(0.6)}$

b $k = 8$

8 $r = \pm 0.4$

9 $S_{10} = \frac{a[(\sqrt{3})^{10} - 1]}{\sqrt{3}-1} = \frac{a(243-1)}{\sqrt{3}-1}$

$= \frac{242a(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = 121a(\sqrt{3}+1)$

10 $\frac{a(2^4-1)}{1} = \frac{b(3^4-1)}{2}$

$15a = 40b \Rightarrow a = \frac{8}{3}b$

11 a $\frac{2k+5}{k} = \frac{k}{k-6} \Rightarrow (k-6)(2k+5) = k^2$

$k^2 - 7k - 30 = 0$

b $k = 10$ c 2.5 d 25429

Exercise 3E

1 a Yes as $|r| < 1$, $\frac{10}{9}$ b No as $|r| \geqslant 1$

c Yes as $|r| < 1$, $6\frac{2}{3}$

d No; arithmetic series does not converge.

e No as $|r| \geqslant 1$ f Yes as $|r| < 1$, $4\frac{1}{2}$

g No; arithmetic series does not converge.

h Yes as $|r| < 1$, 90

2 $\frac{2}{3}$ 3 $-\frac{2}{3}$ 4 20 5 $13\frac{1}{3}$

6 $\frac{23}{99}$ 7 $r = -\frac{1}{2}$, $a = 12$

8 a $-\frac{1}{2} < x < \frac{1}{2}$ b $S_\infty = \frac{1}{1+2x}$

9 a 0.9787 b 1.875

10 a $\frac{30}{1-r} = 240 \Rightarrow 1 - r = \frac{1}{8} \Rightarrow r = \frac{7}{8}$

b 2.51 c 99.3 d 11

11 a $ar = \frac{15}{8} \Rightarrow a = \frac{15}{8r}$

$\frac{a}{1-r} = 8 \Rightarrow a = 8(1-r)$

$\frac{15}{8r} = 8(1-r) \Rightarrow 15 = 64r - 64r^2$

$\Rightarrow 64r^2 - 64r + 15 = 0$

b $\frac{3}{8}, \frac{5}{8}$ c 5, 3 d 7

Challenge

a First series: $a + ar + ar^2 + ar^3 + \dots$
Second series: $a^2 + a^2r^2 + a^2r^4 + a^2r^6 + \dots$
Second series is geometric with common ratio is r^2 and first term a^2.

b $\frac{a}{1-r} = 7 \Rightarrow a = 7(1-r) \Rightarrow a^2 = 49(1-r)(1-r)$

$\frac{a^2}{1-r^2} = 35 \Rightarrow \frac{49(1-r)(1-r)}{(1-r)(1+r)} = 35$

$49(1-r) = 35(1+r) \Rightarrow 49 - 49r = 35 + 35r \Rightarrow r = \frac{1}{6}$

Exercise 3F

1 a i $4 + 7 + 10 + 13 + 16$ ii 50

b i $3 + 12 + 27 + 48 + 75 + 108$ ii 273

c i $1 + 0 + (-1) + 0 + 1$ ii 1

d i $-\frac{2}{243} + \frac{2}{729} - \frac{2}{2187} + \frac{2}{6561}$ ii $-\frac{40}{6561}$

2 a i $\sum_{r=1}^{4} 2r$ ii 20

b i $\sum_{r=1}^{5}(2 \times 3^{r-1})$ ii 242

c i $\sum_{r=1}^{6}\left(-\frac{3}{2}r + \frac{15}{2}\right)$ ii 13.5

3 a i 26 ii $\sum_{r=1}^{26}(6r+1)$

b i 7 ii $\sum_{r=1}^{7}\left(\frac{1}{3} \times \left(\frac{2}{5}\right)^{r-1}\right)$

c i 16 ii $\sum_{r=1}^{16}(17 - 9r)$

4 a −280 b 4 194 300

c 9300 d $-\frac{7}{4}$

5 a 2134 b 45854 c $\frac{3}{16}$ d 96

6 $\sum_{r=1}^{n} 2r = 2 + 4 + 6 + \dots + 2n$; $a = 2$, $d = 2$

$S_n = \frac{n}{2}(4 + (n-1)2) = \frac{n}{2}(2 + 2n) = n + n^2$

7 $\sum_{r=1}^{n} 2r = n + n^2$

$\sum_{r=1}^{n}(2r-1) = \frac{n}{2}(2 + (n-1)2) = \frac{n}{2}(2n) = n^2$

$\sum_{r=1}^{n} 2r - \sum_{r=1}^{n}(2r-1) = n + n^2 - n^2 = n$

8 a $\frac{8}{3}((-2)^k - 1)$ b $99k - k^2$

c $6k - k^2 + 27$

9 $\frac{25}{98304}$

10 a $a = 11$, $d = 3$

$377 = \frac{k}{2}(2(11) + (k-1)(3)) = \frac{k}{2}(19 + 3k)$

$3k^2 + 19k - 754 = 0 \Rightarrow (3k + 58)(k - 13) = 0$

b $k = 13$

11 a $a = 6$, $r = 3$; $S_k = \frac{6(3^k - 1)}{3 - 1} = 3(3^k - 1)$

$\Rightarrow 3(3^k - 1) = 59046 \Rightarrow 3^k = 19683$

$\Rightarrow k\log 3 = \log 19683 \Rightarrow k = \frac{\log 19683}{\log 3}$

b 4723920

12 a $|x| < \frac{1}{3}$ b $\frac{1}{6}$

Challenge

$\sum_{r=1}^{10}[a + (r-1)d]$

$S_{10} = 5(2a + 9d)$

$\sum_{r=11}^{14}[a + (r-1)d] = \sum_{r=1}^{14}[a + (r-1)d] - \sum_{r=1}^{10}[a + (r-1)d]$

$= [7(2a + 13d) - 5(2a + 9d)] = 4a + 46d$

$4a + 46d = 10a + 45d \Rightarrow 6a = d$

Exercise 3G

1 a 1, 4, 7, 10 b 9, 4, −1, −6
c 3, 6, 12, 24 d 2, 5, 11, 23
e 10, 5, 2.5, 1.25 f 2, 3, 8, 63

2 a $u_{n+1} = u_n + 2, u_1 = 3$ b $u_{n+1} = u_n - 3, u_1 = 20$
c $u_{n+1} = 2u_n, u_1 = 1$ d $u_{n+1} = \frac{u_n}{4}, u_1 = 100$
e $u_{n+1} = -1 \times u_n, u_1 = 1$ f $u_{n+1} = 2u_n + 1, u_1 = 3$
g $u_{n+1} = (u_n)^2 + 1, u_1 = 0$ h $u_{n+1} = \frac{u_n + 2}{2}, u_1 = 26$

3 a $u_{n+1} = u_n + 2, u_1 = 1$ b $u_{n+1} = u_n + 3, u_1 = 5$
c $u_{n+1} = u_n + 1, u_1 = 3$ d $u_{n+1} = u_n + \frac{1}{2}, u_1 = 1$
e $u_{n+1} = u_n + 2n + 1, u_1 = 1$ f $u_{n+1} = 3u_n + 2, u_1 = 2$

4 a $3k + 2$ b $3k^2 + 2k + 2$ c $\frac{10}{3}, -4$

5 $p = -4, q = 7$

6 a $x_2 = x_1(p - 3x_1) = 2(p - 3(2)) = 2p - 12$
$x_3 = (2p - 12)(p - 3(2p - 12)) = (2p - 12)(-5p + 36)$
$= -10p^2 + 132p - 432$
b 12 c −252288

7 a $16k + 25$
b $a_4 = 4(16k + 25) + 5 = 64k + 105$
$\sum_{r=1}^{4} a_r = k + 4k + 5 + 16k + 25 + 64k + 105$
$= 85k + 135 = 5(17k + 27)$

Exercise 3H

1 a i increasing
b i decreasing
c i increasing
d i periodic ii 2

2 a i 17, 14, 11, 8, 5 ii decreasing
b i 1, 2, 4, 8, 16 ii increasing
c i −1, 1, −1, 1, −1 ii periodic
iii 2
d i −1, 1, −1, 1, −1 ii periodic
iii 2
e i 20, 15, 10, 5, 0 ii decreasing
f i 20, −15, 20, −15, 20 ii periodic
iii 2
g i $k, \frac{2k}{3}, \frac{4k}{9}, \frac{8k}{27}, \frac{16k}{81}$
ii dependent on value of k

3 $0 < k < 1$ 4 $p = -1$

5 a 4 b 0

Challenge

$u_3 = \frac{1 + b}{a}, u_4 = \frac{a + b + 1}{ab}, u_5 = \frac{a + 1}{b}, u_6 = a, u_7 = b$

Order is 5 as $u_6 = u_1$ and $u_7 = u_2$

Exercise 3I

1 a £5800 b £(3800 + 200m)

2 a £222 500 b £347 500
c It is unlikely her salary will rise by the same amount each year.

3 a £9.03 b 141 days

4 a 220 b 242 c 266 d 519

5 57.7, 83.2

6 a £18 000 b after 7.88 years

7 a £13 780
b Let a denote term of first year and u denote term of second year
$a_{52} = 10 + 51(10) = 520$
$u_1 = 520 + 11$
$u_2 = 531 + 11 = 542$
c £42 198

8 a 500 m is 10 terms,
$S_{10} = \frac{10}{2}(1000 + 9(140)) = 11\,300$
b 1500 m

9 a £2450 b £59 000 c $d = 30$

10 59 days 11 20.15 years

12 11.2 years 13 $2^{64} - 1 = 1.84 \times 10^{19}$

14 a 2.401 m b 48.8234 m

15 a 26 days b 98.5miles on 25th day

16 25 years

Mixed exercise 3

1 a $ar^2 = 27, ar^5 = 8 \Rightarrow r^3 = \frac{8}{27} \Rightarrow r = \frac{2}{3}$
b 60.75 c 182.25 d 3.16

2 a $ar = 80, ar^4 = 5.12$
$\Rightarrow r^3 = \frac{8}{125} \Rightarrow r = \frac{2}{5} = 0.4$
b 200 c $333\frac{1}{3}$ d 8.95×10^{-4}

3 a 76, 60.8 b 0.876 c 367 d 380

4 a $1, \frac{1}{3}, -\frac{1}{9}$
b $\sum_{n=1}^{15}\left(3\left(\frac{2}{3}\right)^n - 1\right) = \sum_{n=1}^{15} 3\left(\frac{2}{3}\right)^n - \sum_{r=1}^{15} 1$
$\sum_{n=1}^{15} 3\left(\frac{2}{3}\right)^n = \frac{2\left(1 - \left(\frac{2}{3}\right)^{15}\right)}{\frac{1}{3}} = 5.9863$
$\sum_{r=1}^{15} 1 = 15$
$5.9863 - 15 = -9.014$
c $u_{n+1} = 3\left(\frac{2}{3}\right)^{n+1} - 1 = 3 \times \frac{2}{3}\left(\frac{2}{3}\right)^n - 1 = \frac{1}{3}\left(2 \times 3\left(\frac{2}{3}\right)^n - 3\right)$
$= \frac{2_{u_n} - 1}{3}$

5 a 0.8 b 10 c 50 d 0.189

6 a £8874.11 b after 9.9 years

7 a $\frac{p(2q + 2)}{p(3q + 1)} = \frac{p(2q - 1)}{p(2q + 2)}$
$(2q + 2)^2 = (2q - 1)(3q + 1)$
$4q^2 + 8q + 4 = 6q^2 - q - 1$
$0 = 2q^2 - 9q - 5 = (q - 5)(2q + 1) \Rightarrow q = 5$ or $-\frac{1}{2}$
b 867.62

8 a $S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 2)d) + (a + (n - 1)d)$ (1)
$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \ldots (a + 2d) + (a + d) + a$ (2)
Adding (1) and (2):
$2 \times S_n = n(2a + (n - 1)d) \Rightarrow S_n = \frac{n}{2}(2a + (n - 1)d)$
b 5050

9 32

10 a $a = 25, d = -3$ b −3810

11 a 26733 b 53467

12 45 cm

13 $S_{2n} = \frac{2n}{2}(2(4) + (2n - 1)4) = n(4 + 8n) = 4n(2n + 1)$

14 a $u_2 = 2k - 4, u_3 = 2k^2 - 4k - 4$ b 5, −3

15 a $a + 4d = 14, \frac{3}{2}(2a + 2d) = -3$
$3a + 3d = -3, 3a + 12d = 42$
$9d = 45 \Rightarrow d = 5 \Rightarrow a = -6$

b 59

16 a $a + 3d = 3k, 3(2a + 5d) = 7k + 9 \Rightarrow$
$6a + 15d = 7k + 9$
$6a + 15\left(\frac{3k - a}{3}\right) = 7k + 9$
$6a + 15k - 5a = 7k + 9 \Rightarrow a = 9 - 8k$

b $\frac{11k - 9}{3}$ **c** 1.5 **d** 415

17 a $a_1 = p, a_2 = \frac{1}{p}, a_3 = \frac{1}{\frac{1}{p}} = 1 \times \frac{p}{1} = p$
$a_1 = a_3 \Rightarrow$ Sequence is periodic, order 2

b $500\left(p + \frac{1}{p}\right)$

18 a $a_1 = k, a_2 = 2k + 6, a_3 = 2(2k + 6) + 6 = 4k + 18$
$a_1 < a_2 < a_3 \Rightarrow k < 2k + 6 < 4k + 18 \Rightarrow k > -6$

b $a_4 = 8k + 42$

c $a_4 = 8k + 42$
$\sum_{r=1}^{4} a_r = k + 2k + 6 + 4k + 18 + 8k + 42$
$= 15k + 66 = 3(5k + 22)$
therefore divisible by 3

19 a $a = 130$
$S_\infty = \frac{130}{1 - r} = 650 \Rightarrow 130 = 650 - 650r$
$-520 = -650r \Rightarrow r = \frac{-520}{-650} = \frac{4}{5}$

b 6.82

c 513.69 (2 d.p.)

d $\frac{130(1 - (0.8)^n)}{0.2} > 600 \Rightarrow 1 - (0.8)^n > \frac{12}{13}$
$(0.8)^n < \frac{1}{13} \Rightarrow n\log(0.8) < -\log 13 \Rightarrow n > \frac{-\log 13}{\log 0.8}$

20 a $25000 \times 1.02^2 = 26010$

b $25000 \times 1.02^n > 50000$
$1.02^n > 2 \Rightarrow n\log 1.02 > \log 2 \Rightarrow n > \frac{\log 2}{\log 1.02}$

c 2047

d 214574

e People may visit the doctor more frequently than once a year, some may not visit at all, depends on state of health

21 a $2n + 1$ **b** 150

c i $S_q = \frac{q}{2}(2(3) + (q - 1)2) = 2q + q^2$
$S_q = p \Rightarrow q^2 + 2q - p = 0$

ii 39

22 a $ar = -3, \frac{a}{1 - r} = 6.75$
$\Rightarrow -\frac{3}{r} \times \frac{1}{1 - r} = 6.75 \Rightarrow \frac{-3}{r - r^2} = 6.75$
$6.75r - 6.75r^2 + 3 = 0$
$27r^2 - 27r - 12 = 0$

b $-\frac{1}{3}$ series is convergent so $|r| < 1$

c 6.78

Challenge

a $u_{n+2} = 5u_{n+1} - 6u_n$.
$= 5[p(3^{n+1}) + q(2^{n+1})] - 6[p(3^n) + q(2^n)]$
$= 5\left[p\left(\frac{1}{3}\right)(3^{n+2}) + q\left(\frac{1}{2}\right)(2^{n+2})\right]$
$-6\left[p\left(\frac{1}{3}\right)^2(3^{n+2}) + q\left(\frac{1}{2}\right)^2(2^{n+2})\right]$
$= \left(\frac{5}{3}p - \frac{6}{9}p\right)(3^{n+2}) + \left(\frac{5}{2}q - \frac{6}{4}q\right)(2^{n+2})$
$= p(3^{n+2}) + q(2^{n+2})$

b $u_n = \left(\frac{2}{3}\right)(3^n) + \left(\frac{3}{2}\right)(2^n)$ or e.g. $u_n = 2(3^{n-1}) + 3(2^{n-1})$

c $u_{100} = 3.436 \times 10^{47}$ (4 s.f.) so contains 48 digits.

CHAPTER 4

Prior knowledge 4

1 a $1 + 35x + 525x^2 + 4375x^3$

b $9765625 - 39062500x + 70312500x^2 - 75000000x^3$

c $64 + 128x + 48x^2 - 80x^3$

2 a $\frac{4}{1 + 2x} + \frac{3}{1 - 5x}$ **b** $\frac{12}{1 + 2x} - \frac{13}{(1 + 2x)^2}$

c $\frac{8}{3x - 4} + \frac{56}{(3x - 4)^2}$

Exercise 4A

1 a i $1 - 4x + 10x^2 - 20x^3 \dots$ **ii** $|x| < 1$

b i $1 - 6x + 21x^2 - 56x^3 \dots$ **ii** $|x| < 1$

c i $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \dots$ **ii** $|x| < 1$

d i $1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} \dots$ **ii** $|x| < 1$

e i $1 - \frac{x}{4} + \frac{5x^2}{32} - \frac{15x^3}{128} \dots$ **ii** $|x| < 1$

f i $1 - \frac{3x}{2} + \frac{15x^2}{8} - \frac{35x^3}{16} \dots$ **ii** $|x| < 1$

2 a i $1 - 9x + 54x^2 - 270x^3 \dots$ **ii** $|x| < \frac{1}{3}$

b i $1 - \frac{5x}{2} + \frac{15x^2}{4} - \frac{35x^3}{8} \dots$ **ii** $|x| < 2$

c i $1 + \frac{3x}{2} - \frac{3x^2}{8} + \frac{5x^3}{16} \dots$ **ii** $|x| < \frac{1}{2}$

d i $1 - \frac{35x}{3} + \frac{350x^2}{9} - \frac{1750x^3}{81} \dots$ **ii** $|x| < \frac{1}{5}$

e i $1 - 4x + 20x^2 - \frac{320x^3}{3} \dots$ **ii** $|x| < \frac{1}{6}$

f i $1 + \frac{5x}{4} + \frac{5x^2}{4} + \frac{55x^3}{48} \dots$ **ii** $|x| < \frac{4}{3}$

3 a i $1 - 2x + 3x^2 - 4x^3 \dots$ **ii** $|x| < 1$

b i $1 - 12x + 90x^2 - 540x^3 \dots$ **ii** $|x| < \frac{1}{3}$

c i $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} \dots$ **ii** $|x| < 1$

d i $1 - x - x^2 - \frac{5x^3}{3} \dots$ **ii** $|x| < \frac{1}{3}$

e i $1 - \frac{x}{4} + \frac{3x^2}{32} - \frac{5x^3}{128} \dots$ **ii** $|x| < 2$

f i $1 + \frac{4x}{3} + \frac{20x^2}{9} + \frac{320x^3}{81} \dots$ **ii** $|x| < \frac{1}{2}$

4 a Expansion of $(1-2x)^{-1} = 1 + 2x + 4x^2 + 8x^3 + \ldots$
Multiply by $(1+x) = 1 + 3x + 6x^2 + 12x^3 + \ldots$

b $|x| < \frac{1}{2}$

5 a $1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3$

b $f(x) = \sqrt{\frac{103}{100}} = \frac{\sqrt{103}}{\sqrt{100}} = \frac{\sqrt{103}}{10}$

c $3.1 \times 10^{-6}\%$

6 a $a = \pm 8$ b ± 160

7 For small values of x ignore powers of x^3 and higher.

$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots$, $(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \ldots$

$\sqrt{\frac{1+x}{1-x}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x}{2} + \frac{x^2}{4} + \frac{3x^2}{8} + \ldots = 1 + x + \frac{x^2}{2}$

8 a $2 - 42x + 114x^2$

b 0.052%

c The expansion is only valid for $|x| < \frac{1}{5}$. $|0.5|$ is not less than $\frac{1}{5}$.

9 a $1 - \frac{9}{2}x + \frac{27}{8}x^2 + \frac{27}{16}x^3$

b $0.97^{\frac{3}{2}} = \left(\frac{\sqrt{97}}{10}\right)^3 = \frac{97\sqrt{97}}{1000}$

c 9.84886

Challenge

a $1 - \frac{1}{2x} + \frac{3}{8x^2}$

b $h(x) = \left(\frac{10}{9}\right)^{-\frac{1}{2}} = \left(\frac{9}{10}\right)^{\frac{1}{2}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$

c 3.16

Exercise 4B

1 a i $2 + \frac{x}{2} - \frac{x^2}{16} + \frac{x^3}{64}$ ii $|x| < 2$

b i $\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$ ii $|x| < 2$

c i $\frac{1}{16} + \frac{x}{32} + \frac{3x^2}{256} + \frac{x^3}{256}$ ii $|x| < 4$

d i $3 + \frac{x}{6} - \frac{x^2}{216} + \frac{x^3}{3888}$ ii $|x| < 9$

e i $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8}x + \frac{3\sqrt{2}}{64}x^2 - \frac{5\sqrt{2}}{256}x^3$ ii $|x| < 2$

f $\frac{5}{3} - \frac{10}{9}x + \frac{20}{27}x^2 - \frac{40}{81}x^3$ ii $|x| < \frac{3}{2}$

g i $\frac{1}{2} + \frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$ ii $|x| < 2$

h i $\sqrt{2} + \frac{3\sqrt{2}}{4}x + \frac{15\sqrt{2}}{32}x^2 + \frac{51\sqrt{2}}{128}x^3$ ii $|x| < 1$

2 $\frac{1}{25} - \frac{8}{125}x + \frac{48}{625}x^2 - \frac{256}{3125}x^3$

3 a $2 - \frac{x}{4} - \frac{x^2}{64}$

b $m(x) = \sqrt{\frac{35}{9}} = \frac{\sqrt{35}}{\sqrt{9}} = \frac{\sqrt{35}}{3}$

c 5.91609 (correct to 5 decimal places), % error $= 1.38 \times 10^{-4}\%$

4 a $a = \frac{1}{9}, b = -\frac{2}{81}$ b $\frac{5}{486}$

5 For small values of x ignore powers of x^3 and higher.

$(4-x)^{-1} = \frac{1}{4} + \frac{x}{16} + \frac{x^2}{64} + \ldots$

Multiply by $(3 + 2x - x^2) = \frac{3}{4} + \frac{x}{2} - \frac{x^2}{4} + \frac{3x}{16} + \frac{x^2}{8} + \frac{3x^2}{64}$

$= \frac{3}{4} + \frac{11}{16}x - \frac{5}{64}x^2$

6 a $\frac{1}{\sqrt{5}} - \frac{x}{5\sqrt{5}} + \frac{3x^2}{50\sqrt{5}}$ b $-\frac{1}{\sqrt{5}} + \frac{11x}{5\sqrt{5}} - \frac{23x^2}{50\sqrt{5}}$

7 a $2 - \frac{3}{32}x - \frac{27}{4096}x^2$ b 1.991

8 a $\frac{3}{4-2x} = \frac{3}{4} + \frac{3x}{8} + \frac{3x^2}{16}$, $\frac{2}{3+5x} = \frac{2}{3} - \frac{10x}{9} + \frac{50x^2}{27}$

$\frac{3}{4-2x} - \frac{2}{3+5x} = \frac{1}{12} + \frac{107}{72}x - \frac{719}{432}x^2$

b 0.0980311

c 0.0032%

Exercise 4C

1 a $\frac{4}{1-x} - \frac{4}{2+x}$ b $2 + 5x + \frac{7}{2}x^2$

c valid $|x| < 1$

2 a $-\frac{2}{2+x} + \frac{4}{(2+x)^2}$ b $B = \frac{1}{2}, C = -\frac{3}{8}$

c $|x| < 2$

3 a $\frac{2}{1+x} + \frac{3}{1-x} - \frac{4}{2+x}$ b $3 + 2x + \frac{9}{2}x^2 + \frac{5}{4}x^3$

c $|x| < 1$

4 a $A = -\frac{14}{5}$ and $B = \frac{9}{5}$ b $-1 + 11x + 5x^2$

5 a $2 - \frac{1}{x+5} + \frac{6}{x-4}$ b $\frac{3}{10} - \frac{67}{200}x - \frac{407}{4000}x^2$

c $|x| < 4$

6 a $A = 3, B = -2$ and $C = 3$ b $\frac{5}{6} - \frac{19}{36}x - \frac{97}{216}x^2$

7 a $A = -\frac{7}{9}, B = \frac{28}{3}$ and $C = \frac{8}{9}$ b $11 + 38x + 116x^2$

c 0.33%

Mixed exercise 4

1 a i $1 - 12x + 48x^2 - 64x^3$ ii all x

b i $4 + \frac{x}{8} - \frac{x^2}{512} + \frac{x^3}{16384}$ ii $|x| < 16$

c i $1 + 2x + 4x^2 + 8x^3$ ii $|x| < \frac{1}{2}$

d i $2 - 3x + \frac{9x^2}{2} - \frac{27x^3}{4}$ ii $|x| < \frac{2}{3}$

e i $2 + \frac{x}{4} + \frac{3x^2}{64} + \frac{5x^3}{512}$ ii $|x| < 4$

f i $1 - 2x + 6x^2 - 18x^3$ ii $|x| < \frac{1}{3}$

g i $1 + 4x + 8x^2 + 12x^3$ ii $|x| < 1$

h i $-3 - 8x - 18x^2 - 38x^3$ ii $|x| < \frac{1}{2}$

2 $1 - \frac{x}{4} - \frac{x^2}{32} - \frac{x^3}{128}$

3 a $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$ b $\frac{1145}{512}$

4 a $c = -9, d = 36$ b 1.282

c calculator = 1.28108713, approximation is correct to 2 decimal places.

5 a $a = 4$ or $a = -4$

b coefficient of $x^3 = 4$, coefficient of $x^3 = -4$.

6 **a** $1 - 3x + 9x^2 - 27x^3$

b $(1 + x)(1 - 3x + 9x^2 - 27x^3)$
$= 1 - 3x + 9x^2 - 27x^3 + x - 3x^2 + 9x^3$
$= 1 - 2x + 6x^2 - 18x^3$

c $x = 0.01$, $0.980\,58$

7 **a** $n = -2$, $a = 3$ **b** -108

c $|x| < \frac{1}{3}$

8 For small values of x ignore powers of x^3 and higher.
$\frac{1}{\sqrt{4+x}} = \frac{1}{2} - \frac{x}{16} + \frac{3x^2}{256}$, $\frac{3}{\sqrt{4+x}} = \frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$

9 **a** $\frac{1}{2} + \frac{x}{16} + \frac{3}{256}x^2$ **b** $\frac{1}{2} + \frac{17}{16}x + \frac{35}{256}x^2$

10 **a** $\frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3$ **b** $\frac{1}{2} - \frac{x}{4} + \frac{3}{8}x^2 - \frac{9}{16}x^3$

11 **a** $\frac{1}{2} - \frac{x}{16} + \frac{3x^2}{256} - \frac{5x^3}{2048}$ **b** 0.6914

12 $\frac{1}{27} - \frac{4}{27}x + \frac{32}{81}x^2$

13 **a** $A = 1$, $B = -4$, $C = 3$ **b** $-\frac{3}{8} - \frac{51}{64}x + \frac{477}{512}x^2$

14 **a** $A = 3$ and $B = 2$ **b** $5 - 28x + 144x^2$

15 **a** $10 - 2x + \frac{5}{2}x^2 - \frac{11}{4}x^3$, so $B = \frac{5}{2}$ and $C = -\frac{11}{4}$

b Percent error = 0.0027%

Challenge

$1 - \frac{3x^2}{2} + \frac{27x^4}{8} - \frac{135x^6}{16}$

Review exercise 1

1 Assumption: there are finitely many prime numbers, p_1, p_2, p_3 up to p_n. Let $X = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$
None of the prime numbers $p_1, p_2, \ldots p_n$ can be a factor of X as they all leave a remainder of 1 when X is divided by them. But X must have at least one prime factor. This is a contradiction.
So there are infinitely many prime numbers.

2 Assumption: $x = \frac{a}{b}$ is a solution to the equation, where a and b are integers with no common factors.
$\left(\frac{a}{b}\right)^2 - 2 = 0 \Rightarrow \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$
So a^2 is even, which implies that a is even.
Write $a = 2n$ for some integer n.
$(2n)^2 = 2b^2 \Rightarrow 4n^2 = 2b^2 \Rightarrow 2n^2 = b^2$
So b^2 is even, which implies that b is even.
This contradicts the assumption that a and b have no common factor.
Hence there are no rational solutions to the equation.

3 $\frac{4x - 3}{x(x - 3)}$

4 **a** $f(x) = \frac{(x+2)^2 - 3(x+2) + 3}{(x+2)^2} = \frac{x^2 + x + 1}{(x+2)^2}$

b $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$

c $x^2 + x + 1 > 0$ from **b** and $(x + 2)^2 > 0$ as $x \neq -2$

$\frac{-1}{x-1} + \frac{4}{2x-3}$

$P = 2$, $Q = -1$, $R = -1$

$A = \frac{2}{9}$, $B = \frac{2}{9}$, $C = \frac{2}{3}$

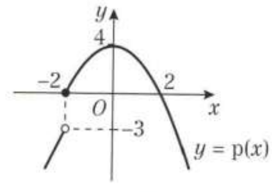
8 $A = 3$, $B = 1$, $C = -2$

9 $d = 3$, $e = 6$, $f = -14$

10 $p(x) = 6 - \frac{2}{1-x} + \frac{5}{1+2x}$

11 $x > \frac{2}{3}$ or $x < -5$

12 **a** Range: $p(x) \leqslant 4$



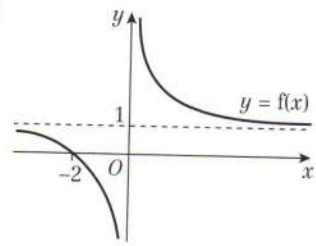
b $a = -\frac{25}{4}$ or $a = 2\sqrt{6}$

13 **a** $qp(x) = \frac{-5x - 18}{x + 4}$
$a = -5$, $b = -18$, $c = 1$, $d = 4$

b $x = -\frac{39}{10}$

c $r^{-1}(x) = \frac{-4x - 18}{x + 5}$, $x \in \mathbb{R}$, $x \neq -5$

14 **a**



b $\frac{\left(\frac{x+2}{x}\right) + 2}{\left(\frac{x+2}{x}\right)} = \frac{x + 2 + 2x}{x + 2} = \frac{3x + 2}{x + 2}$

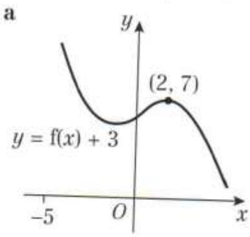
c ln 13 **d** $g^{-1}(x) = \frac{e^x + 5}{2}$, $x \in \mathbb{R}$

15 **a** $3(1 - 2x) + b = 1 - 2(3x + b)$, $b = -\frac{2}{3}$

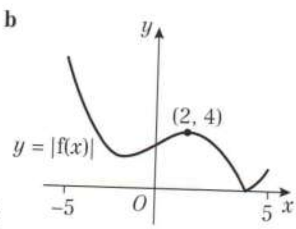
b $p^{-1}(x) = \frac{3x + 2}{9}$, $q^{-1}(x) = \frac{1 - x}{2}$

c $p^{-1}(x)q^{-1}(x) = q^{-1}(x)p^{-1}(x) = \frac{-3x + 7}{18}$,
$a = -3$, $b = 7$, $c = 18$

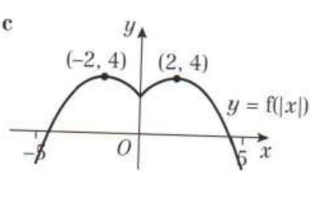
16 **a**



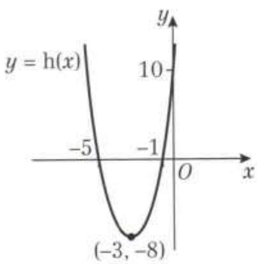
b



c

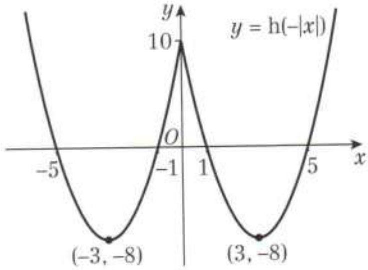


17 a

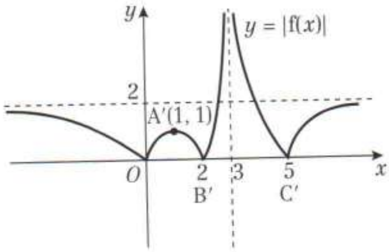


b **i** (−5, −24) **ii** (3, −8) **iii** (−3, 8)

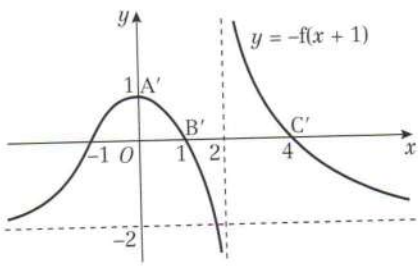
c



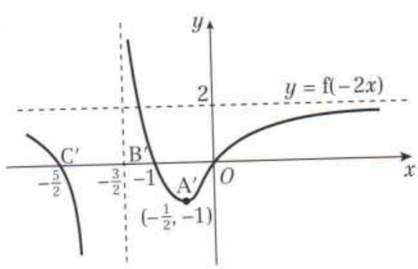
18 a i



ii



iii



b **i** 6 **ii** 4

19 a $b = -9$ **b** $A(9, -3)$, $B(15, 0)$

c $x = 15$, $x = -21$

20 a $f(x) \leqslant 8$

b The function is not one-to-one.

c $-\frac{32}{3} < x < -\frac{8}{7}$

d $k > \frac{44}{3}$

21 a $k = 0.6$, $k = -4$ **b** $a = 16$, $d = 8$

22 a Solve $a + 3d = 72$, $a + 10d = 51$ simultaneously to obtain $a = 81$, $d = -3$

$1125 = \frac{n}{2}(162 + (n - 1)(-3))$

$2250 = 165n - 3n^2$

Therefore $3n^2 - 165n + 2250 = 0$

b $n = 25$, $n = 30$

23 a $a = 19p - 18$, $d = 10 - 2p$, 30th term $= 272 - 39p$

b $p = 12$

24 a $r^6 = \frac{225}{64} \Rightarrow \ln r^6 = \ln\left(\frac{225}{64}\right) \Rightarrow 6\ln r - \ln\left(\frac{225}{64}\right) = 0$

$\Rightarrow 6\ln r + \ln\left(\frac{64}{225}\right) = 0$

b $r = 1.23$

25 a 60

b $a = 10$, $r = \frac{5}{6}$

$$\frac{10\left(1 - \left(\frac{5}{6}\right)^k\right)}{1 - \frac{5}{6}} > 55 \Rightarrow 1 - \left(\frac{5}{6}\right)^k > \frac{11}{12}$$

$$\Rightarrow \frac{1}{12} > \left(\frac{5}{6}\right)^k \Rightarrow \log\left(\frac{1}{12}\right) > \log\left(\left(\frac{5}{6}\right)^k\right)$$

$$\Rightarrow \log\left(\frac{1}{12}\right) > k\log\left(\frac{5}{6}\right) \Rightarrow \frac{\log\left(\frac{1}{12}\right)}{\log\left(\frac{5}{6}\right)} < k$$

c $k = 14$

26 a $4 + 4r + 4r^2 = 7 \Rightarrow 4r^2 + 4r - 3 = 0$

b $r = \frac{1}{2}$ or $r = -\frac{3}{2}$ **c** 8

27 a $x = 1$, $r = 3$ and $x = -9$, $r = -\frac{1}{3}$

b 243 **c** 182.25

28 a $a_1 = k$, $a_2 = 3k + 5$

b $a_3 = 3a_2 + 5 = 9k + 20$

c **i** $40k + 90$ **ii** $10(4k + 9)$

29 a 2860

b $2400 \times 1.06^{N-1} > 6000 \Rightarrow 1.06^{N-1} > 2.5$

$\Rightarrow \log 1.06^{N-1} > \log 2.5 \Rightarrow (N - 1)\log 1.06 > \log 2.5$

c $N = 16.7...$, therefore $N = 17$

d $S_n = \frac{2400(1.06^{10} - 1)}{1.06 - 1} = 31633.90...$ donations.

Total value of donations is 5 times this, so £158,000 over the 10-year period.

30 a $|x| < \frac{1}{4}$

b $\frac{6}{1 + 4x} = \frac{24}{5} \Rightarrow x = \frac{1}{16}$

31 a $(1 - x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2$

$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3 + ...$

$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + ...$

b $|x| < 1$. Accept $-1 < x < 1$.

32 a $a = 9$ $n = -\frac{36}{54} = -\frac{2}{3}$

b −360

c $-\frac{1}{9} < x < \frac{1}{9}$

33 a $1 + 6x + 6x^2 - 4x^3$

b $\left(1 + 4\left(\frac{3}{100}\right)\right)^{\frac{3}{2}} = \left(\frac{112}{100}\right)^{\frac{3}{2}} = \left(\sqrt{\frac{112}{100}}\right)^3 = \left(\frac{\sqrt{112}}{10}\right)^3$

$= \frac{112\sqrt{112}}{1000}$

c 10.58296 **d** 0.00039%

34 $\frac{1}{27} - \frac{x}{27} + \frac{2x^2}{81} - \frac{8x^3}{729}$

35 a $(4 - 9x)^{\frac{1}{2}} = 2\left(1 - \frac{9}{4}x\right)^{\frac{1}{2}} = 2 - \frac{9}{4}x - \frac{81}{64}x^2$

b $\sqrt{4 - 9\left(\frac{1}{100}\right)} = \sqrt{\frac{391}{100}} = \frac{\sqrt{391}}{10}$

c Approximate: 1.97737 correct to 5 decimal places.

36 a $a = 2, b = -1, c = \frac{3}{16}$ **b** $\frac{1}{8}$

$a = -2, b = 1, c = \frac{3}{16}$

37 a $A = 1, B = 2$ **b** $3 - x + 11x^2 - \ldots$

38 a $A = -\frac{3}{2}, B = \frac{1}{2}$ **b** $-1 - x + 4x^3 + \ldots$

39 a $A = 2, B = 10, C = 1$ **b** $\frac{25}{9} - \frac{25}{27}x + \frac{25}{9}x^2 + \ldots$

40 a $A = 2, B = 5, C = -2$

b $2 + 5(4 + x)^{-1} - 2(3 + 2x)^{-1}$

$= 2 + \frac{5}{4}\left(1 + \frac{x}{4}\right)^{-1} - \frac{2}{3}\left(1 + \frac{2}{3}x\right)^{-1}$

$= 2 + \frac{5}{4}\left(1 - \frac{x}{4} + \frac{x^2}{16}\right) - \frac{2}{3}\left(1 - \frac{2}{3}x + \frac{4}{9}x^2\right)$

$= \frac{31}{12} + \frac{19}{144}x - \frac{377}{1728}x^2$

Challenge

1 a $(x + 2)^2 + (y - 3)^2 = 25$ **b** 15

2 $a_1 = m, a_2 = m + k, a_3 = m + 2k, \ldots$

$6m + 45k = 4m + 50k \Rightarrow 2m = 5k \Rightarrow m = \frac{5}{2}k$

3 A: $x = \frac{19 - \sqrt{41}}{4}$, B: $x = \frac{16}{3}$, C: $x = \frac{19 + \sqrt{41}}{4}$

CHAPTER 5

Prior knowledge 5

1 a $-\frac{1}{2}$ **b** $-\frac{\sqrt{2}}{2}$ **c** $\sqrt{3}$ **d** $-\frac{\sqrt{3}}{2}$

2 a 1 **b** $-\tan^2\theta$ **c** $|\sin\theta|$

3 a $(\sin 2\theta + \cos 2\theta)^2 = \sin^2 2\theta + 2\sin 2\theta\cos 2\theta + \cos^2 2\theta$

$= 1 + 2\sin 2\theta\cos 2\theta$

b $\frac{2}{\sin\theta} - 2\sin\theta = \frac{2 - 2\sin^2\theta}{\sin\theta} = \frac{2(1 - \sin^2\theta)}{\sin\theta} = \frac{2\cos^2\theta}{\sin\theta}$

4 a 75.5°, 284° **b** 15°, 75°, 195°, 255°

c 19.7°, 118°, 200°, 298° **d** 75.5°, 132°, 228°, 284°

Exercise 5A

a 9° **b** 12° **c** 75° **d** 225°

e 270° **f** 540°

a 26.4° **b** 57.3° **c** 65.0° **d** 99.2°

a 0.479 **b** 0.156 **c** 1.74 **d** 0.909

e −0.443

4 a $\frac{2\pi}{45}$ **b** $\frac{\pi}{18}$ **c** $\frac{\pi}{8}$ **d** $\frac{\pi}{6}$

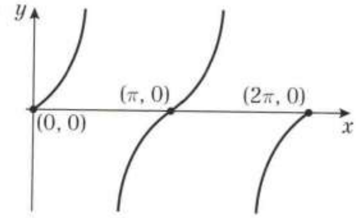
e $\frac{5\pi}{8}$ **f** $\frac{4\pi}{3}$ **g** $\frac{3\pi}{2}$ **h** $\frac{7\pi}{4}$

i $\frac{11\pi}{6}$

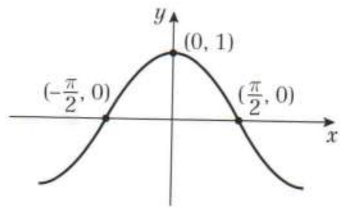
5 a 0.873 rad **b** 1.31 rad **c** 1.75 rad **d** 2.79 rad

e 4.01 rad **f** 5.59 rad

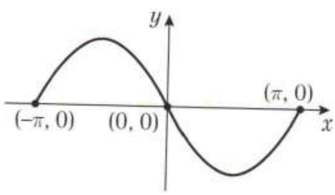
6 a



b



7 a



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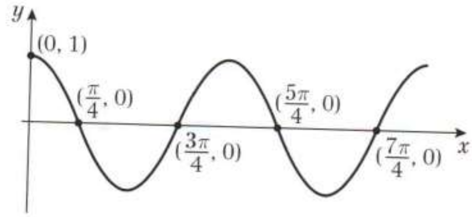

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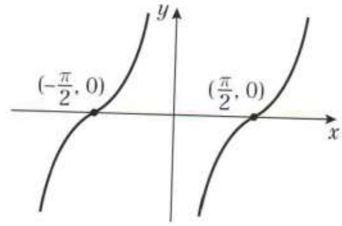
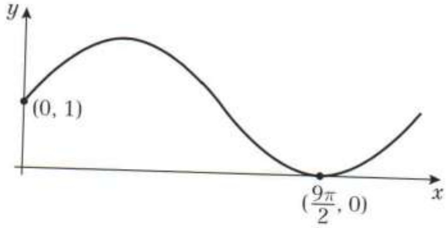


d



8 $(0, -0.5)$

$\left(-\frac{11\pi}{6}, 0\right), \left(-\frac{5\pi}{6}, 0\right), \left(\frac{\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right)$

Challenge

a $2\pi n, n \in \mathbb{Z}$ b $\frac{3\pi}{2} + 2\pi n, n \in \mathbb{Z}$ c $\frac{\pi}{2} + \pi n, n \in \mathbb{Z}$

Exercise 5B

1 a $\sin\left(\pi - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ b $-\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

c $\sin\left(2\pi - \frac{\pi}{6}\right) = -\frac{1}{2}$ d $\cos\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{2}$

e $\cos\left(2\pi - \frac{\pi}{3}\right) = \frac{1}{2}$ f $\cos\left(\pi + \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

g $\tan\left(\pi - \frac{\pi}{4}\right) = -1$ h $-\tan\left(\pi + \frac{\pi}{4}\right) = -1$

i $\tan\left(\pi + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

2 a $\frac{\sqrt{3}}{2}$ b $\frac{\sqrt{3}}{2}$ c $-\frac{\sqrt{3}}{2}$

d $-\frac{\sqrt{2}}{2}$ e $-\sqrt{3}$ f $\sqrt{3}$

3 $AC = \frac{2}{\sin\left(\frac{\pi}{3}\right)} = \frac{4\sqrt{3}}{3}$

$DC^2 = AD^2 + AC^2 = \left(\frac{2\sqrt{6}}{3}\right)^2 + \left(\frac{4\sqrt{3}}{3}\right)^2 = 8$

$DC = 2\sqrt{2}$

Exercise 5C

1 a i 2.7 ii 2.025 iii 7.5π

b i $\frac{50}{3}$ ii 1.8 iii 3.6

c i $\frac{4}{3}$ ii 0.8 iii 2

2 $\frac{10}{3}\pi$ cm 3 2π 4 $5\sqrt{2}$ cm

5 a 10.4 cm b 1.25 rad

6 7.5 7 0.8

8 a $\frac{1}{3}\pi$ b $6 + \frac{4}{3}\pi$ cm

9 6.8 cm

10 a $R - r$

b $\sin\theta = \frac{r}{R - r} \Rightarrow (R - r)\sin\theta = r \Rightarrow (R\sin\theta - r\sin\theta) = r$

$\Rightarrow R\sin\theta = r + r\sin\theta \Rightarrow R\sin\theta = r(1 + \sin\theta).$

c 2.43 cm

11 2 rad

12 a 36 m b 13.6 km/h

13 a 3.5 m b 15.3 m

14 a 2.59 rad b 44 mm

Exercise 5D

1 a 19.2 cm^2 b $\frac{27}{4}\pi\text{ cm}^2$ c $\frac{162}{125}\pi\text{ cm}^2$

d 25.1 cm^2 e $6\pi - 9\sqrt{3}\text{ cm}^2$ f $\frac{63}{2}\pi + 9\sqrt{2}\text{ cm}^2$

2 a $\frac{16}{3}\pi\text{ cm}^2$ b 5 cm^2

3 a 4.47 b 3.96 c 1.98

4 12 cm^2

5 a $\cos\theta = \frac{10^2 + 10^2 - 18.65^2}{2 \times 10 \times 10} = -0.739 \ldots$

b 120 cm^2

6 $40\frac{2}{3}$ cm

7 a 12

b $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 12^2 \times 0.5 = 36\text{ cm}^2$

c 1.48 cm^2

8 a $l = r\theta = \frac{x\pi}{12}, x = \frac{12l}{\pi}$

$A = \frac{1}{2}r^2\theta = \frac{1}{2}\left(\frac{12l}{\pi}\right)^2\frac{\pi}{12} = \frac{\pi}{24}\left(\frac{144l^2}{\pi^2}\right) = \frac{6l^2}{\pi}$

b 5π cm c 60

9 $\triangle COB = \frac{1}{2}r^2\sin\theta$

Shaded area $= \frac{1}{2}r^2(\pi - \theta) - \frac{1}{2}r^2\sin(\pi - \theta)$

$= \frac{1}{2}r^2\pi - \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

Since $\triangle$COB = shaded area,

$\frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2\pi - \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$\sin\theta = \pi - \theta - \sin\theta$

$\theta + 2\sin\theta = \pi$

10 38.7 cm^2

11 8.88 cm^2

12 a $OAD = \frac{1}{2}r^2\theta, OBC = \frac{1}{2}(r + 8)^2\theta$

$ABCD = \frac{1}{2}(r + 8)^2\theta - \frac{1}{2}r^2\theta = 48$

$\frac{1}{2}(r^2 + 16r + 64)\theta - \frac{1}{2}r^2\theta = 48$

$(r^2 + 16r + 64)\theta - r^2\theta = 96$

$16r + 64 = \frac{96}{\theta} \Rightarrow r = \frac{6}{\theta} - 4$

b 28 cm

13 78.4 ($\theta = 0.8$)

14 a $14^2 = 12^2 + 10^2 - 2 \times 12 \times 10\cos A$

$196 = 144 + 100 - 240\cos A$

$-48 = -240\cos A$

$0.2 = \cos A$

$A = \cos^{-1}(0.2) = 1.369438406\ldots = 1.37$ (3 s.f.)

b 34.1 m^2

15 a 18.1 cm b 11.3 cm^2

16 a 98.79 cm^2 b 33.24 cm

17 4.62 cm^2

Challenge

Area $= \frac{1}{2}r^2\theta$, arc length, $l = r\theta$

Area $= \frac{1}{2}rl$

Exercise 5E

1 a 0.795, 5.49 b 3.34, 6.08

c 1.37, 4.51 d π

2 a 0.848, 2.29 b 0.142, 3.28

c 1.08, 4.22 d 0.886, 5.40

3 a 1.16, 5.12 b 3.61, 5.82

c 0.896, 4.04 d 0.421, 5.86

4 a $-\frac{5\pi}{6}, \frac{\pi}{6}$ b 0.201, 2.94

c −5.39, −0.896, 0.896, 5.39

d −1.22, 1.22, 5.06, 7.51

e 1.77, 4.91, 8.05, 11.2

f 4.89

5 a 0.322, 2.82, 3.46, 5.96

b 1.18, 1.96, 3.28, 4.05, 5.37, 6.15

c $\frac{\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}, \frac{19\pi}{24}, \frac{25\pi}{24}, \frac{31\pi}{24}, \frac{37\pi}{24}, \frac{43\pi}{24}$

d 0.232, 2.91, 3.37, 6.05

6 a $-\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}$

b $-\frac{5\pi}{6}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$

Online Full worked solutions are available in SolutionBank.

c $-5.92, -4.35, -2.78, -1.21, 0.359, 1.93, 3.50, 5.07$
d $-2.46, -0.685, 0.685, 2.46, 3.83, 5.60, 6.97, 8.74$

7 a $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ b $0, 2.82, \pi, 5.96, 2\pi$
c π d $0.440, 2.70, 3.58, 5.84$

8 a π b $0.501, 2.64, 3.64, 5.78$
c No solutions d $1.10, 5.18$

9 a $\frac{\pi}{3}, \frac{11\pi}{6}$ b $\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}$
c $-0.986, 0.786$ d $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

10 a $-\frac{\pi}{4}, \frac{3\pi}{4}, 0.412, 2.73$ b $0, 0.644, \pi, 5.64$

11 $0.3, 0.5, 2.6, 2.9$

12 $0.7, 2.4, 3.9, 5.6$

13 $8\sin^2 x + 4\sin x - 20 = 4$
$8\sin^2 x + 4\sin x - 24 = 0$
$2\sin^2 x + \sin x - 6 = 0$
Let $Y = \sin x \Rightarrow 2Y^2 + Y - 6 = 0$
$\Rightarrow (2Y-3)(Y+2) = 0 \Rightarrow$ So $Y = 1.5$ or $Y = -2$
Since $Y = \sin x$, $\sin x = 1.5 \rightarrow$ No Solutions,
$\sin x = -2 \rightarrow$ No Solutions

14 a Using the quadratic formula with $a = 1$, $b = -2$ and $c = -6$ (can complete the square as well)
$$\tan x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times (-6)}}{2 \times 1}$$
$$\tan x = \frac{2 \pm \sqrt{4 + 24}}{2} = \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}$$
b $1.3, 2.1, 4.4, 5.3, 7.6, 8.4$

15 a $\sin x = 0.599$ (3 d.p.)
b $0.64, 2.50$

Exercise 5F

1 a $\frac{2}{3}$ b 1 c 1

2 a $\frac{\sin 3\theta}{\theta \sin 4\theta} \approx \frac{3\theta}{\theta \times 4\theta} = \frac{3\theta}{4\theta^2} = \frac{3}{4\theta}$

b $\frac{\cos\theta - 1}{\tan 2\theta} \approx \frac{1 - \frac{\theta^2}{2} - 1}{2\theta} = \frac{-\frac{\theta^2}{2}}{2\theta} = -\frac{\theta}{4}$

c $\frac{\tan 4\theta + \theta^2}{3\theta - \sin 2\theta} \approx \frac{4\theta + \theta^2}{3\theta - 2\theta} = \frac{4\theta + \theta^2}{\theta} = 4 + \theta$

3 a 0.970379 b 0.970232
c -0.015% d -1.77%
e The larger the value of θ the less accurate the approximation is.

$\frac{\theta - \sin\theta}{\sin\theta} \times 100 = 1 \Rightarrow (\theta - \sin\theta) \times 100 = \sin\theta$
$\Rightarrow 100\theta - 100\sin\theta = \sin\theta \Rightarrow 100\theta = 101\sin\theta$.

a $\frac{4\cos 3\theta - 2 + 5\sin\theta}{1 - \sin 2\theta} \approx \frac{4\left(1 - \frac{(3\theta)^2}{2}\right) - 2 + 5\theta}{1 - 2\theta}$
$= \frac{4\left(1 - \frac{9\theta^2}{2}\right) - 2 + 5\theta}{1 - 2\theta} = \frac{4 - 18\theta^2 - 2 + 5\theta}{1 - 2\theta}$
$= \frac{(1 - 2\theta)(9\theta + 2)}{1 - 2\theta} = 9\theta + 2$

b 2

Challenge

1 a $CD = AC\theta$
b $\sin\theta \approx \frac{CD}{AD} = \frac{r\theta}{r} = \theta$
$\tan\theta \approx \frac{CD}{AC} = \frac{r\theta}{r} = \theta$

2 a $1 - \frac{x^2}{2}$
b $\cos\theta = \sqrt{1 - \sin^2\theta} \approx 1 - \frac{\sin^2\theta}{2}$, if $\sin\theta \approx \theta$ then this becomes $\cos\theta \approx 1 - \frac{\theta^2}{2}$

Mixed exercise 5

1 a $\frac{\pi}{3}$ b $8.56\,\text{cm}^2$

2 a $120\,\text{cm}^2$ b $161.07\,\text{cm}^2$

3 a 1.839 b $11.03\,\text{cm}$

4 a $\frac{p}{r}$
b Area $= \frac{1}{2}r^2\theta = \frac{1}{2}r^2\frac{p}{r} = \frac{1}{2}pr\,\text{cm}^2$
c $12.206\,\text{cm}^2$
d $1.105 \leq \theta \leq 1.151$ (3 d.p.)

5 a 1.28 b 16 c $1:3.91$

6 a Area of shape $X = 2d^2 + \frac{1}{2}d^2\pi$
Area of shape $Y = \frac{1}{2}(2d)^2\theta$
$2d^2 + \frac{1}{2}d^2\pi = \frac{1}{2}(2d)^2\theta$
$2d^2 + \frac{1}{2}d^2\pi = 2d^2\theta \Rightarrow 1 + \frac{1}{4}\pi = \theta$
b $(3\pi + 12)\,\text{cm}$ c $\left(18 + \frac{3\pi}{2}\right)\text{cm}$ d $12.9\,\text{mm}$

7 a $A_1 = \frac{1}{2} \times 6^2 \times \theta - \frac{1}{2} \times 6^2 \times \sin\theta = 18(\theta - \sin\theta)$
b $A_2 = \pi \times 6^2 - 18(\theta - \sin\theta) = 36\pi - 18(\theta - \sin\theta)$
Since $A_2 = 3A_1$
$36\pi - 18(\theta - \sin\theta) = 3 \times 18(\theta - \sin\theta)$
$36\pi - 18(\theta - \sin\theta) = 54(\theta - \sin\theta)$
$36\pi = 72(\theta - \sin\theta)$
$\frac{1}{2}\pi = \theta - \sin\theta$
$\sin\theta = \theta - \frac{\pi}{2}$

8 a $10^2 = 5^2 + 9^2 - 2 \times 5 \times 9\cos A$
$100 = 25 + 81 - 90\cos A$
$-6 = -90\cos A$
$\frac{1}{15} = \cos A$
$A = \cos^{-1}\left(\frac{1}{15}\right) = 1.504$
b i $6.77\,\text{cm}^2$ ii $15.7\,\text{cm}^2$ iii $22.5\,\text{cm}$

9 a $\frac{1}{2}r^2 \times 1.5 = 15 \Rightarrow r^2 = 20$
$r = \sqrt{20} = 2\sqrt{5}$
b $15.7\,\text{cm}$ c $5.025\,\text{cm}^2$

10 a $2\sqrt{3}\,\text{cm}$ b $2\pi\,\text{cm}^2$
c Perimeter $= 2\sqrt{3} + 2\sqrt{3} + 2\sqrt{3} \times \frac{\pi}{3} = \frac{2\sqrt{3}}{3}(\pi + 6)$

11 a $70^2 = 44^2 + 44^2 - 2 \times 44 \times 44\cos C$
$\cos C = -\frac{257}{968}$
$C = \cos^{-1}\left(-\frac{257}{968}\right) = 1.84$
b i $80.9\,\text{m}$ ii $26.7\,\text{m}$ iii $847\,\text{m}^2$

12 a Arc $AB = 6 \times 2\theta = 12\theta$
Length DC = Chord AB
Chord $AB = 2 \times 6\sin\theta = 12\sin\theta$

Perimeter $ABCD = 12\theta + 4 + 12\sin\theta + 4 = 2(7 + \pi)$
$12\theta + 12\sin\theta + 8 = 2(7 + \pi)$
$6\theta + 6\sin\theta - 3 = \pi$
$2\theta + 2\sin\theta - 1 = \frac{\pi}{3}$

b $2 \times \frac{\pi}{6} + 2\sin\left(\frac{\pi}{6}\right) - 1 = \frac{\pi}{3} + 2 \times \frac{1}{2} - 1 = \frac{\pi}{3}$

c 20.7 cm^2

13 a $O_1A = O_2A = 12$, as they are radii of their respective circles.
$O_1O_2 = 12$, as O_2 is on the circumference of C_1 and hence is a radius (and vice versa).
Therefore,
O_1AO_2 is an equilateral triangle $\Rightarrow \angle AO_1O_2 = \frac{\pi}{3}$.
By symmetry, $\angle BO_1O_2$ is $\frac{\pi}{3} \Rightarrow \angle AO_1B = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

b 16π cm **c** 177 cm^2

14 a Student has used an angle measured in degrees – it needs to be measured in radians to use that formula.

b $\frac{5\pi}{4}\text{ cm}^2$

15 a $-\frac{1}{4}$ **b** $\theta + 1$

16 a $\frac{7 + 2\cos 2\theta}{\tan 2\theta + 3} \approx \frac{7 + 2\left(1 - \frac{(2\theta)^2}{2}\right)}{2\theta + 3}$
$= \frac{7 + 2\left(1 - \frac{4\theta^2}{2}\right)}{2\theta + 3} = \frac{9 - 4\theta^2}{2\theta + 3}$
$= \frac{(3 + 2\theta)(3 - 2\theta)}{2\theta + 3} = 3 - 2\theta$

b 3

17 a $32\cos 5\theta + 203\tan 10\theta = 182$
$32\left(1 - \frac{(5\theta)^2}{2}\right) + 203\,(10\theta) = 182$
$32 - 16(25\theta^2) + 2030\theta = 182$
$0 = 400\theta^2 - 2030\theta + 150$
$0 = 40\theta^2 - 203\theta + 15$

b $5, \frac{3}{40}$

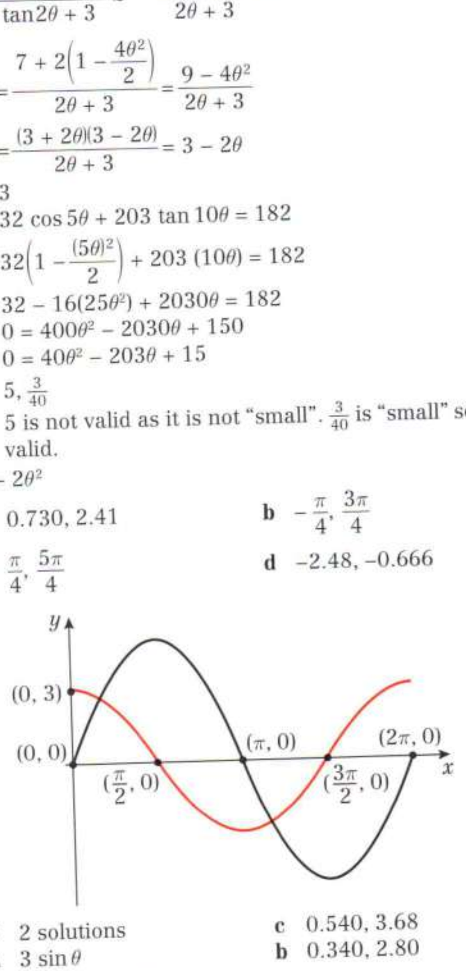
c 5 is not valid as it is not "small". $\frac{3}{40}$ is "small" so is valid.

18 $1 - 2\theta^2$

19 a 0.730, 2.41 **b** $-\frac{\pi}{4}, \frac{3\pi}{4}$

c $\frac{\pi}{4}, \frac{5\pi}{4}$ **d** −2.48, −0.666

20 a



b 2 solutions **c** 0.540, 3.68

21 a $3\sin\theta$ **b** 0.340, 2.80

22 $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

23 a Cosine can be negative so do not reject $-\frac{1}{\sqrt{2}}$. Cosine squared cannot be negative but the student has already square rooted it so no need to reject $-\frac{1}{\sqrt{2}}$.

b Rearranged incorrectly – square rooted incorrectly

c $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

24 a Not found all the solutions

b 0.595, 2.17, 3.74, 5.31

25 a $5\sin x = 1 + 2\cos^2 x \Rightarrow 5\sin x = 1 + 2(1 - \sin^2 x) \Rightarrow$
$2\sin^2 x + 5\sin x - 3 = 0$

b $\frac{\pi}{6}, \frac{5\pi}{6}$

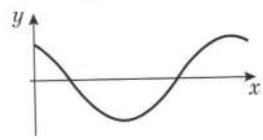
26 a $4\sin^2 x + 9\cos x - 6 = 0 \Rightarrow$
$4(1 - \cos^2 x) + 9\cos x - 6 = 0 \Rightarrow$
$4\cos^2 x - 9\cos x + 2 = 0$

b 1.3, 5.0, 7.6, 11.2

27 a $\tan 2x = 5\sin 2x \Rightarrow \frac{\sin 2x}{\cos 2x} = 5\sin 2x \Rightarrow$
$(1 - 5\cos 2x)\sin 2x = 0$

b $0, 0.7, \frac{\pi}{2}, 2.5, \pi$

28 a



b $\left(0, \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{3}, 0\right), \left(\frac{4\pi}{3}, 0\right)$ **c** 0.34, 4.90

29 $x = 0.54$, 1.90 or 2.64 (2 d.p)

Challenge

a $\theta = \frac{2}{9}$ or $\theta = -3$
$\theta = \frac{2}{9}$ is small, so this value is valid. $\theta = -3$ is not small so this value is not valid. Small in this context is "close to 0".

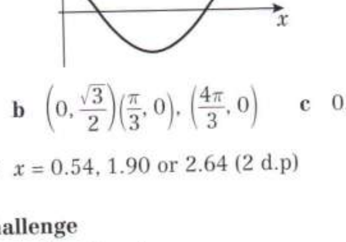
b $\theta = -\frac{1}{4}$ or $\theta = \frac{1}{5}$
Both θ could be considered "small" in this case so both are valid.

c No solutions

CHAPTER 6

Prior knowledge 6

1



a 53.1°, 126.9° (1 d.p.) **b** −23.6°, −156.4° (1 d.p.)

2 $\frac{1}{\sin x\cos x} - \frac{1}{\tan x} = \frac{1}{\sin x\cos x} - \frac{\cos x}{\sin x} = \frac{1 - \cos^2 x}{\sin x\cos x}$
$= \frac{\sin^2 x}{\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x$

3 0.308, 1.26, 1.88, 2.83, 3.45, 4.40, 5.02, 5.98 (3 s.f.)

Exercise 6A

1 a +ve b −ve c −ve d +ve
e −ve

2 a −5.76 b −1.02 c −1.02 d 5.67
e 0.577 f −1.36 g −3.24 h 1.04

3 a 1 b −1 c −1 d −2
e $-\frac{2\sqrt{3}}{3}$ f −1 g 2 h 2
i $-\sqrt{2}$ j $\frac{\sqrt{3}}{3}$ k $\frac{2\sqrt{3}}{3}$ l $-\sqrt{2}$

4 $\operatorname{cosec}(\pi - x) = \frac{1}{\sin(\pi - x)} = \frac{1}{\sin x} = \operatorname{cosec} x$

5 $\cot 30° \sec 30° = \frac{1}{\tan 30°} \times \frac{1}{\cos 30°} = \frac{\sqrt{3}}{1} \times \frac{2}{\sqrt{3}} = 2$

6 $\operatorname{cosec}\left(\frac{2\pi}{3}\right) + \sec\left(\frac{2\pi}{3}\right) = \frac{1}{\sin\left(\frac{2\pi}{3}\right)} + \frac{1}{\cos\left(\frac{2\pi}{3}\right)}$

$$= \frac{1}{\frac{\sqrt{3}}{2}} + \frac{1}{-\frac{1}{2}}$$

$$= -2 + \frac{2}{\sqrt{3}} = -2 + \frac{2}{3}\sqrt{3}$$

Challenge

a Using triangle OBP, $OB\cos\theta = 1$
$\Rightarrow OB = \frac{1}{\cos\theta} = \sec\theta$

b Using triangle OAP, $OA\sin\theta = 1$
$\Rightarrow OA = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$

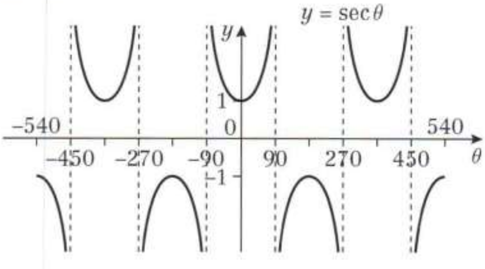
c Using Pythagoras' theorem, $AP^2 = OA^2 - OP^2$
So $AP^2 = \operatorname{cosec}^2\theta - 1 = \frac{1}{\sin^2\theta} - 1$

$= \frac{1 - \sin^2\theta}{\sin^2\theta} = \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$

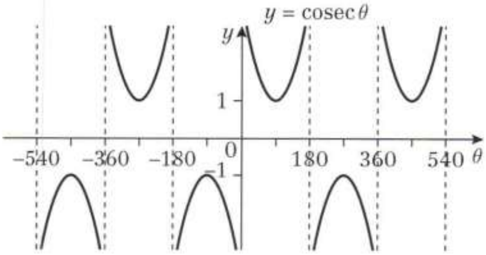
Therefore $AP = \cot\theta$.

Exercise 6B

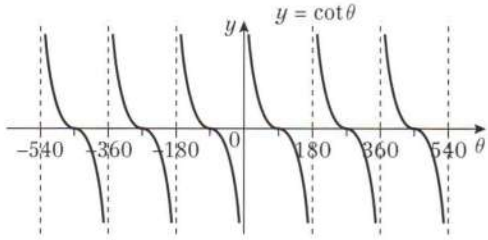
1 a



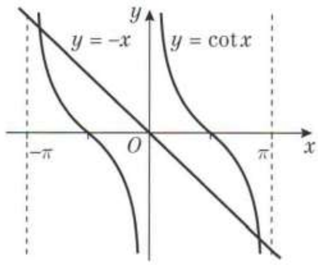
b



c

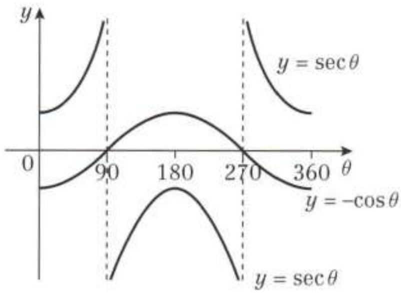


2 a



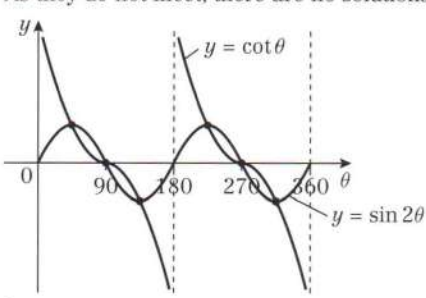
b 2 solutions

3 a



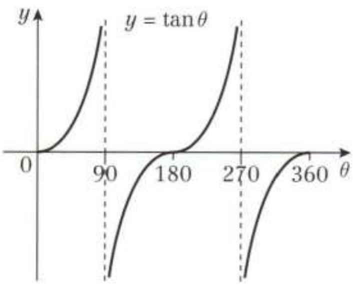
b The solutions of $\sec\theta = -\cos\theta$ are the θ values of the points of intersection of $y = \sec\theta$ and $y = -\cos\theta$. As they do not meet, there are no solutions.

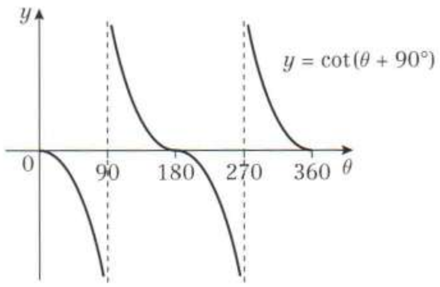
4 a



b 6

5 a





b $\cot(90° + \theta) = -\tan\theta$

6 **a** **i** The graph of $y = \tan\left(\theta + \frac{\pi}{2}\right)$ is the same as that of $y = \tan\theta$ translated by $\frac{\pi}{2}$ to the left.

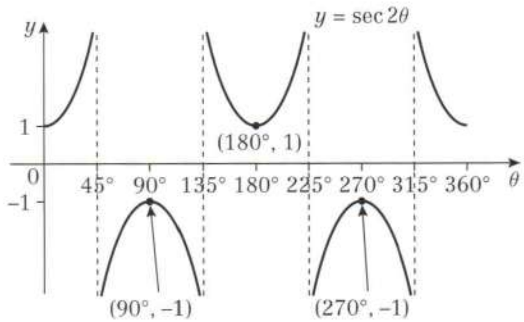
ii The graph of $y = \cot(-\theta)$ is the same as that of $y = \cot\theta$ reflected in the y-axis.

iii The graph of $y = \text{cosec}\left(\theta + \frac{\pi}{4}\right)$ is the same as that of $y = \text{cosec}\,\theta$ translated by $\frac{\pi}{4}$ to the left.

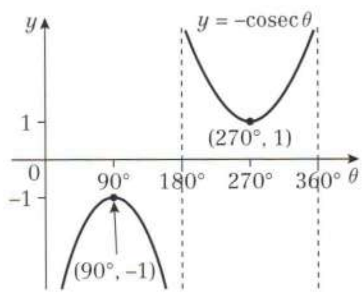
iv The graph of $y = \sec\left(\theta - \frac{\pi}{4}\right)$ is the same as that of $y = \sec\theta$ translated by $\frac{\pi}{4}$ to the right.

b $\tan\left(\theta + \frac{\pi}{2}\right) = \cot(-\theta)$; $\text{cosec}\left(\theta + \frac{\pi}{4}\right) = \sec\left(\theta - \frac{\pi}{4}\right)$

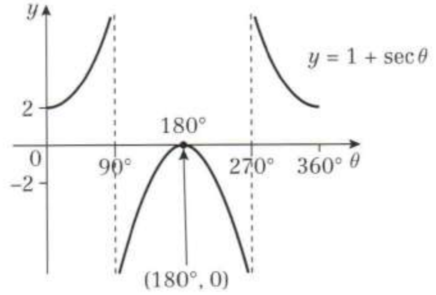
7 **a**



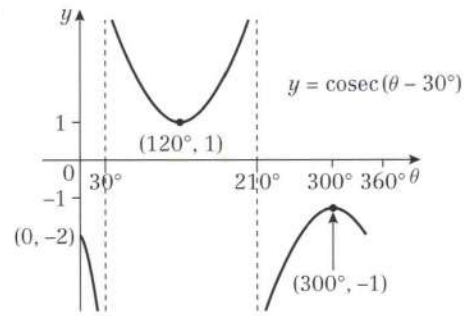
b



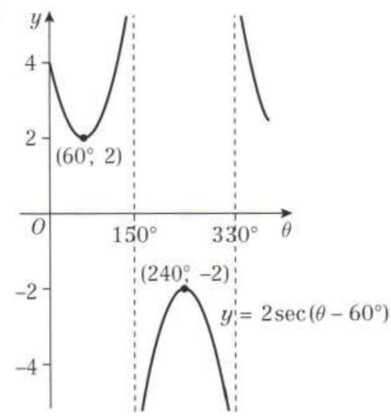
c



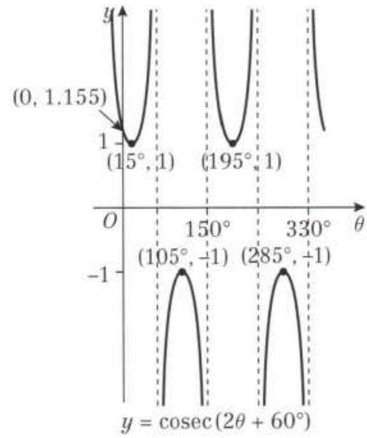
d



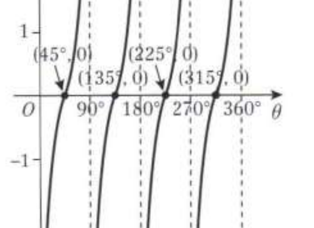
e



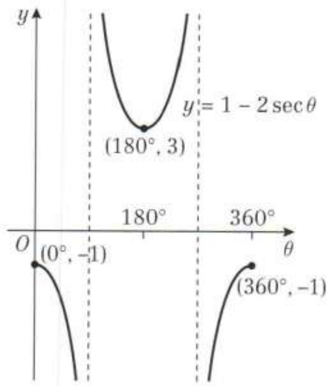
f



g

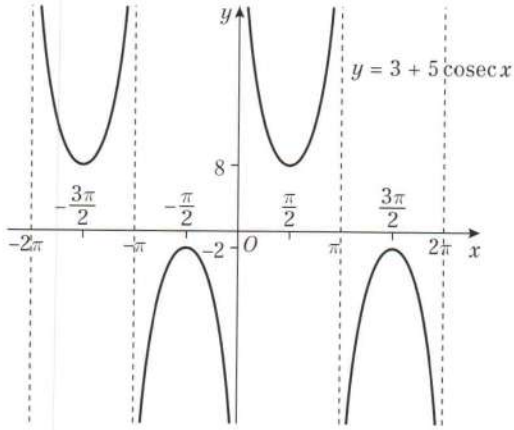


h



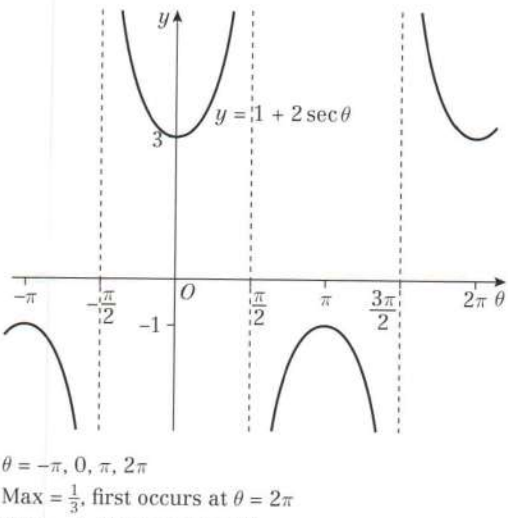
8 **a** $\frac{2\pi}{3}$ **b** 4π **c** π **d** 2π

9 **a**



b $-2 < k < 8$

10 **a**



b $\theta = -\pi, 0, \pi, 2\pi$

c Max $= \frac{1}{3}$, first occurs at $\theta = 2\pi$

Min $= -1$, first occurs at $\theta = \pi$

Exercise 6C

1 **a** $\operatorname{cosec}^3\theta$ **b** $4\cot^6\theta$ **c** $\frac{1}{2}\sec^2\theta$

d $\cot^2\theta$ **e** $\sec^5\theta$ **f** $\operatorname{cosec}^2\theta$

g $2\cot^{\frac{1}{2}}\theta$ **h** $\sec^3\theta$

2 **a** $\frac{5}{4}$ **b** $-\frac{1}{2}$ **c** $\pm\sqrt{3}$

3 **a** $\cos\theta$ **b** 1 **c** $\sec 2\theta$

d 1 **e** 1 **f** $\cos A$

g $\cos x$

4 **a** L.H.S. $= \cos\theta + \sin\theta\frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos\theta}$

$= \frac{1}{\cos\theta} = \sec\theta =$ R.H.S.

b L.H.S. $= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \equiv \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$

$\equiv \frac{1}{\sin\theta\cos\theta} = \frac{1}{\sin\theta} \times \frac{1}{\cos\theta}$

$\equiv \operatorname{cosec}\theta\sec\theta =$ R.H.S.

c L.H.S. $= \frac{1}{\sin\theta} - \sin\theta \equiv \frac{1 - \sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta}$

$\equiv \cos\theta \times \frac{\cos\theta}{\sin\theta} \equiv \cos\theta\cot\theta =$ R.H.S.

d L.H.S. $= (1 - \cos x)\left(1 + \frac{1}{\cos x}\right) \equiv 1 - \cos x + \frac{1}{\cos x} - 1$

$\equiv \frac{1}{\cos x} - \cos x \equiv \frac{1 - \cos^2 x}{\cos x} \equiv \frac{\sin^2 x}{\cos x}$

$\equiv \sin x \times \frac{\sin x}{\cos x} \equiv \sin x\tan x =$ R.H.S.

e L.H.S. $= \frac{\cos^2 x + (1 - \sin x)^2}{(1 - \sin x)\cos x}$

$\equiv \frac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{(1 - \sin x)\cos x}$

$\equiv \frac{2 - 2\sin x}{(1 - \sin x)\cos x} \equiv \frac{2(1 - \sin x)}{(1 - \sin x)\cos x}$

$\equiv 2\sec x =$ R.H.S.

f L.H.S. $= \frac{\cos\theta}{1 + \frac{1}{\tan\theta}} \equiv \frac{\cos\theta}{\left(\frac{\tan\theta + 1}{\tan\theta}\right)}$

$\equiv \frac{\cos\theta\tan\theta}{\tan\theta + 1} \equiv \frac{\sin\theta}{1 + \tan\theta} =$ R.H.S.

5 **a** 45°, 315° **b** 199°, 341°

c 112°, 292° **d** 30°, 150°

e 30°, 150°, 210°, 330° **f** 36.9°, 90°, 143°, 270°

g 26.6°, 207° **h** 45°, 135°, 225°, 315°

6 **a** 90° **b** ±109°

c −164°, 16.2° **d** 41.8°, 138°

e ±45°, ±135° **f** ±60°

g −173°, −97.2°, 7.24°, 82.8°

h −152°, −36.5°, 28.4°, 143°

7 **a** π **b** $\frac{5\pi}{6}, \frac{11\pi}{6}$ **c** $\frac{2\pi}{3}, \frac{4\pi}{3}$ **d** $\frac{\pi}{4}, \frac{3\pi}{4}$

8 **a** $\frac{AB}{AD} = \cos\theta \Rightarrow AD = 6\sec\theta$

$\frac{AC}{AB} = \cos\theta \Rightarrow AC = 6\cos\theta$

$CD = AD - AC \Rightarrow CD = 6\sec\theta - 6\cos\theta$

$= 6(\sec\theta - \cos\theta)$

b 2 cm

9 **a** $\frac{\operatorname{cosec} x - \cot x}{1 - \cos x} \equiv \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{1 - \cos x} \equiv \frac{1}{\sin x} \times \frac{1 - \cos x}{1 - \cos x}$

$\equiv \operatorname{cosec} x$

b $x = \frac{\pi}{6}, \frac{5\pi}{6}$

10 **a** $\frac{\sin x\tan x}{1 - \cos x} - 1 \equiv \frac{\sin^2 x}{\cos x(1 - \cos x)} - 1$

$\equiv \frac{\sin^2 x - \cos x + \cos^2 x}{\cos x(1 - \cos x)} \equiv \frac{1 - \cos x}{\cos x(1 - \cos x)}$

$\equiv \frac{1}{\cos x} \equiv \sec x$

b Would need to solve $\sec x = -\frac{1}{2}$, which is equivalent to $\cos x = -2$, which has no solutions.

11 $x = 11.3°, 191.3°$ (1 d.p.)

Exercise 6D

1 **a** $\sec^2\left(\frac{1}{2}\theta\right)$ **b** $\tan^2\theta$ **c** 1
d $\tan\theta$ **e** 1 **f** 3
g $\sin\theta$ **h** 1 **i** $\cos\theta$
j 1 **k** $4\operatorname{cosec}^4 2\theta$

2 $\pm\sqrt{k-1}$

3 **a** $\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$

4 **a** $-\frac{5}{4}$ **b** $-\frac{4}{5}$ **c** $-\frac{3}{5}$

5 **a** $-\frac{7}{24}$ **b** $-\frac{25}{7}$

6 **a** L.H.S. $\equiv (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$
$\equiv 1(\sec^2\theta + \tan^2\theta) =$ R.H.S.

b L.H.S. $\equiv (1 + \cot^2 x) - (1 - \cos^2 x)$
$\equiv \cot^2 x + \cos^2 x =$ R.H.S.

c L.H.S. $\equiv \dfrac{1}{\cos^2 A}\left(\dfrac{\cos^2 A}{\sin^2 A} - \cos^2 A\right) \equiv \dfrac{1}{\sin^2 A} - 1$
$\equiv \operatorname{cosec}^2 A - 1 = \cot^2 A =$ R.H.S.

d R.H.S. $\equiv \tan^2\theta \times \cos^2\theta \equiv \dfrac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta \equiv \sin^2\theta$
$\equiv 1 - \cos^2\theta =$ L.H.S.

e L.H.S. $= \dfrac{1 - \tan^2 A}{\sec^2 A} \equiv \cos^2 A\left(1 - \dfrac{\sin^2 A}{\cos^2 A}\right)$
$\equiv \cos^2 A - \sin^2 A \equiv (1 - \sin^2 A) - \sin^2 A$
$\equiv 1 - 2\sin^2 A =$ R.H.S.

f L.H.S. $= \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta} \equiv \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$
$\equiv \dfrac{1}{\cos^2\theta\sin^2\theta} \equiv \sec^2\theta\operatorname{cosec}^2\theta =$ R.H.S.

g L.H.S. $= \operatorname{cosec} A(1 + \tan^2 A) \equiv \operatorname{cosec} A\left(1 + \dfrac{\sin^2 A}{\cos^2 A}\right)$
$\equiv \operatorname{cosec} A + \dfrac{1}{\sin A}\cdot\dfrac{\sin^2 A}{\cos^2 A} \equiv \operatorname{cosec} A + \dfrac{\sin A}{\cos A}\cdot\dfrac{1}{\cos A}$
$\equiv \operatorname{cosec} A + \tan A\sec A =$ R.H.S.

h L.H.S. $= \sec^2\theta - \sin^2\theta \equiv (1 + \tan^2\theta) - (1 - \cos^2\theta)$
$\equiv \tan^2\theta + \cos^2\theta \equiv$ R.H.S.

7 $\frac{\sqrt{2}}{4}$

8 **a** $20.9°, 69.1°, 201°, 249°$ **b** $\pm\frac{\pi}{3}$
c $-153°, -135°, 26.6°, 45°$ **d** $\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$
e $120°$ **f** $0, \frac{\pi}{4}, \pi$
g $0°, 180°$ **h** $\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{4}, \frac{4\pi}{3}$

9 **a** $1 + \sqrt{2}$
b $\cos k = \dfrac{1}{1+\sqrt{2}} = \dfrac{\sqrt{2}-1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \sqrt{2} - 1$
c $65.5°, 294.5°$

10 **a** $b = \frac{4}{a}$
b $c^2 = \cot^2 x = \dfrac{\cos^2 x}{\sin^2 x} = \dfrac{b^2}{1 - b^2} = \dfrac{\left(\frac{4}{a}\right)^2}{1 - \left(\frac{4}{a}\right)^2}$
$= \dfrac{16}{a^2} \times \dfrac{a^2}{(a^2 - 16)} = \dfrac{16}{a^2 - 16}$

11 **a** $\dfrac{1}{x} = \dfrac{1}{\sec\theta + \tan\theta} = \dfrac{\sec\theta - \tan\theta}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}$
$= \dfrac{\sec\theta - \tan\theta}{(\sec^2\theta - \tan^2\theta)} = \dfrac{\sec\theta - \tan\theta}{1}$

b $x^2 + \dfrac{1}{x^2} + 2 = \left(x + \dfrac{1}{x}\right)^2 = (2\sec\theta)^2 = 4\sec^2\theta$

12 $p = 2(1 + \tan^2\theta) - \tan^2\theta = 2 + \tan^2\theta$
$\Rightarrow \tan^2\theta = p - 2 \Rightarrow \cot^2\theta = \dfrac{1}{p-2}$
$\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \dfrac{1}{p-2} = \dfrac{(p-2)+1}{p-2} = \dfrac{p-1}{p-2}$

Exercise 6E

1 **a** $\frac{\pi}{2}$ **b** $\frac{\pi}{2}$ **c** $-\frac{\pi}{4}$ **d** $-\frac{\pi}{6}$
e $\frac{3\pi}{4}$ **f** $-\frac{\pi}{6}$ **g** $\frac{\pi}{3}$ **h** $\frac{\pi}{3}$

2 **a** 0 **b** $-\frac{\pi}{3}$ **c** $\frac{\pi}{2}$

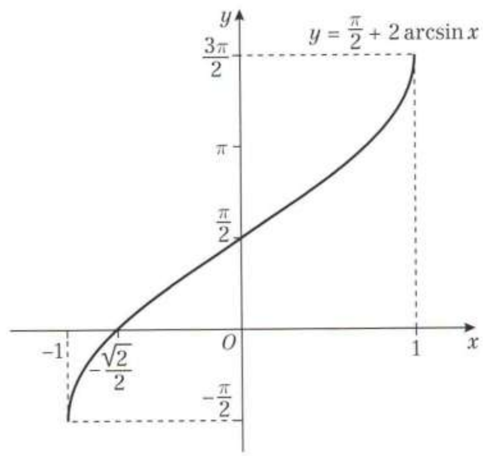
3 **a** $\frac{1}{2}$ **b** $-\frac{1}{2}$ **c** -1 **d** 0

4 **a** $\frac{\sqrt{3}}{2}$ **b** $\frac{\sqrt{3}}{2}$ **c** -1 **d** 2
e -1 **f** 1

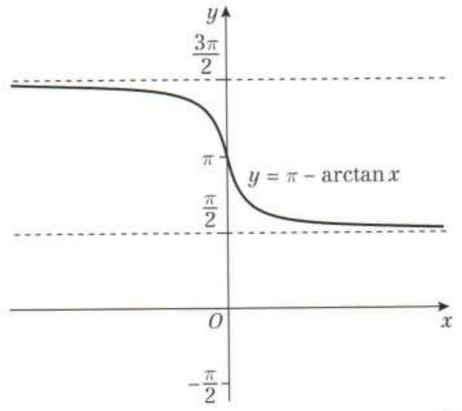
5 $\alpha, \pi - \alpha$

6 **a** $0 < x < 1$
b **i** $\sqrt{1 - x^2}$ **ii** $\dfrac{x}{\sqrt{1 - x^2}}$
c **i** no change **ii** no change

7 **a**

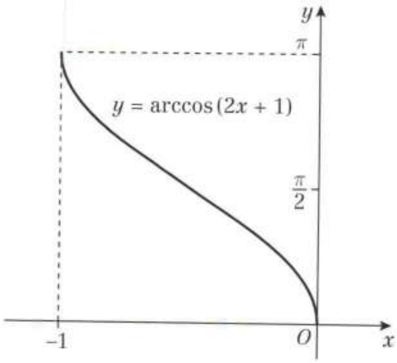


b

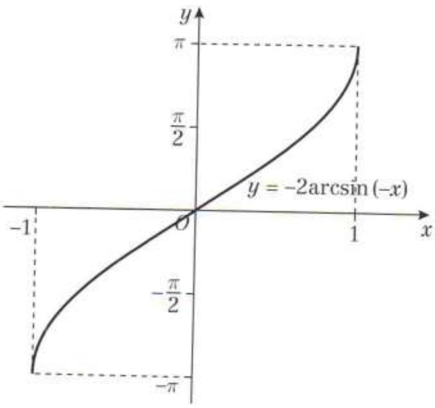


Online Full worked solutions are available in SolutionBank.

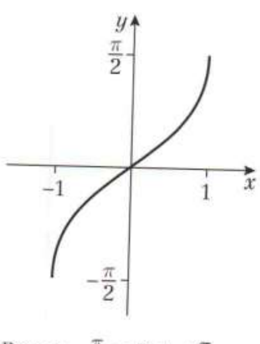
c



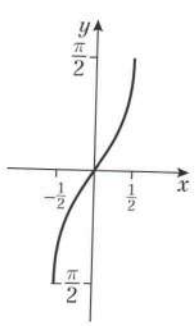
d



8 a



b



Range: $-\frac{\pi}{2} \leqslant f(x) \leqslant \frac{\pi}{2}$

c $g: x \to \arcsin 2x, -\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$

d $g^{-1}: x \to \frac{1}{2}\sin x, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

a Let $y = \arccos x$. $x \in [0,1] \Rightarrow y \in \left[0, \frac{\pi}{2}\right]$

$\cos y = x$, so $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$

(Note, $\sin y \neq -\sqrt{1-x^2}$ since $y \in \left[0, \frac{\pi}{2}\right]$, so $\sin y \geqslant 0$)

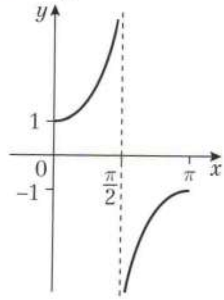
$y = \arcsin\sqrt{1-x^2}$

Therefore, $\arccos x = \arcsin\sqrt{1-x^2}$ for $x \in [0,1]$.

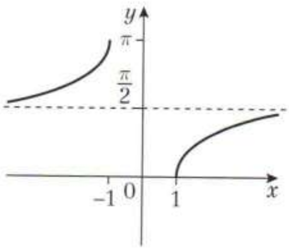
b For $x \in [-1,0]$, $\arccos x \in \left(\frac{\pi}{2}, \pi\right)$, but arcsin only has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Challenge

a



b



Range: $0 \leqslant \text{arcsec}\, x \leqslant \pi$, $\text{arcsec}\, x \neq \frac{\pi}{2}$

Mixed exercise 6

1 $-125.3°, \pm 54.7°$

2 $p = \frac{8}{q}$

3 $p^2q^2 = \sin^2\theta \times 4^2\cot^2\theta = 16\sin^2\theta \times \frac{\cos^2\theta}{\sin^2\theta}$

$= 16\cos^2\theta = 16(1-\sin^2\theta) = 16(1-p^2)$

4 a i $60°$

ii $30°, 41.8°, 138.2°, 150°$

b i $30°, 165°, 210°, 345°$

ii $45°, 116.6°, 225°, 296.6°$

c i $\frac{71\pi}{60}, \frac{101\pi}{60}$

ii $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

5 $-\frac{8}{5}$

6 a $\text{L.H.S.} \equiv \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)(\sin\theta + \cos\theta)$

$\equiv \frac{(\sin^2\theta + \cos^2\theta)}{\cos\theta\sin\theta}(\sin\theta + \cos\theta)$

$\equiv \frac{\sin\theta}{\sin\theta\cos\theta} + \frac{\cos\theta}{\cos\theta\sin\theta}$

$\equiv \sec\theta + \text{cosec}\,\theta \equiv \text{R.H.S.}$

b $\text{L.H.S.} \equiv \frac{\frac{1}{\sin x}}{\frac{1}{\sin x} - \sin x}$

$\equiv \frac{\frac{1}{\sin x}}{\frac{1-\sin^2 x}{\sin x}} \equiv \frac{1}{\sin x} \times \frac{\sin x}{\cos^2 x} \equiv \frac{1}{\cos^2 x} \equiv \sec^2 x \equiv \text{R.H.S.}$

c $\text{L.H.S.} \equiv 1 - \sin x + \text{cosec}\, x - 1 \equiv \frac{1}{\sin x} - \sin x$

$\equiv \frac{1-\sin^2 x}{\sin x} \equiv \frac{\cos^2 x}{\sin x} \equiv \cos x \frac{\cos x}{\sin x} \equiv \cos x \cot x$

$\equiv \text{R.H.S.}$

d $\text{L.H.S.} \equiv \frac{\cot x(1+\sin x) - \cos x(\text{cosec}\, x - 1)}{(\text{cosec}\, x - 1)(1 + \sin x)}$

$\equiv \frac{\cot x + \cos x - \cot x + \cos x}{\text{cosec}\, x - 1 + 1 - \sin x} \equiv \frac{2\cos x}{\text{cosec}\, x - \sin x}$

$\equiv \frac{2\cos x}{\frac{1}{\sin x} - \sin x} \equiv \frac{2\cos x}{\left(\frac{1-\sin^2 x}{\sin x}\right)} \equiv \frac{2\cos x \sin x}{\cos^2 x}$

$\equiv 2\tan x \equiv \text{R.H.S.}$

e L.H.S. $\equiv \dfrac{\operatorname{cosec}\theta + 1 + \operatorname{cosec}\theta - 1}{(\operatorname{cosec}^2\theta - 1)} \equiv \dfrac{2\operatorname{cosec}\theta}{\cot^2\theta}$

$\equiv \dfrac{2}{\sin\theta} \cdot \dfrac{\sin^2\theta}{\cos^2\theta} \equiv \dfrac{2\sin\theta}{\cos^2\theta} \equiv \dfrac{2}{\cos\theta} \cdot \dfrac{\sin\theta}{\cos\theta}$

$\equiv 2\sec\theta\tan\theta \equiv$ R.H.S.

f L.H.S. $\equiv \dfrac{\sec^2\theta - \tan^2\theta}{\sec^2\theta} \equiv \dfrac{1}{\sec^2\theta} \equiv \cos^2\theta \equiv$ R.H.S.

7 a L.H.S. $\equiv \dfrac{\sin^2 x + (1 + \cos x)^2}{(1 + \cos x)\sin x}$

$\equiv \dfrac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{(1 + \cos x)\sin x} \equiv \dfrac{2 + 2\cos x}{(1 + \cos x)\sin x}$

$\equiv \dfrac{2(1 + \cos x)}{(1 + \cos x)\sin x} \equiv \dfrac{2}{\sin x} \equiv 2\operatorname{cosec} x$

b $-\frac{\pi}{3}, -\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

8 R.H.S. $\equiv \left(\dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^2 \equiv \dfrac{(1 + \cos\theta)^2}{\sin^2\theta} \equiv \dfrac{(1 + \cos\theta)^2}{1 - \cos^2\theta}$

$\equiv \dfrac{(1 + \cos\theta)^2}{(1 - \cos\theta)(1 + \cos\theta)} \equiv \dfrac{1 + \cos\theta}{1 - \cos\theta} \equiv$ L.H.S.

9 a $-2\sqrt{2}$

b $\operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{1}{8} = \frac{9}{8}$

$\Rightarrow \operatorname{cosec} A = \pm\dfrac{3}{2\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{4}$

As A is obtuse, $\operatorname{cosec} A$ is +ve, $\Rightarrow \operatorname{cosec} A = \dfrac{3\sqrt{2}}{4}$

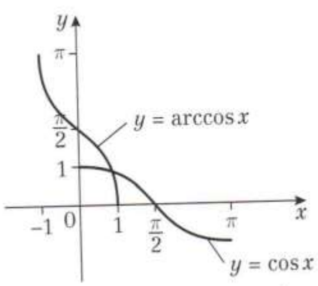
10 a $\dfrac{1}{k}$ **b** $k^2 - 1$ **c** $-\dfrac{1}{\sqrt{k^2 - 1}}$ **d** $-\dfrac{k}{\sqrt{k^2 - 1}}$

11 $\frac{\pi}{12}, \frac{17\pi}{12}$ **12** $\frac{\pi}{3}$ **13** $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$

14 a $(\sec x - 1)(\operatorname{cosec} x - 2)$ **b** $30°, 150°$

15 $2 - \sqrt{3}$

16



17 a $-\frac{1}{3}$ **b i** $-\frac{5}{3}$, **ii** $-\frac{4}{3}$ **c** $126.9°$

18 $pq = (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = \sec^2\theta - \tan^2\theta$

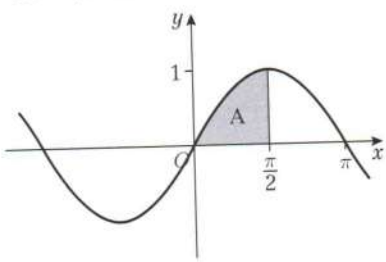
$= 1 \Rightarrow p = \dfrac{1}{q}$

19 a L.H.S. $= (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$

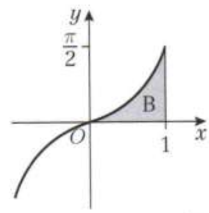
$= 1 \times (\sec^2\theta + \tan^2\theta) = \sec^2\theta + \tan^2\theta =$ R.H.S.

b $-153.4°, -135°, 26.6°, 45°$

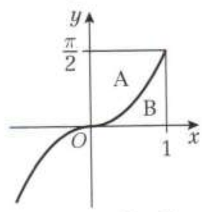
20 a



b



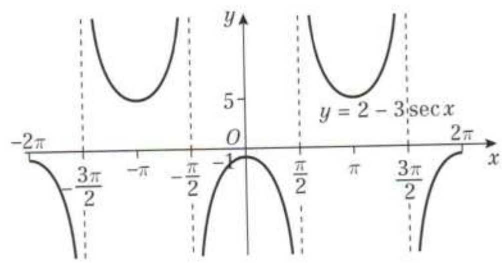
c The regions A and B fit together to make a rectangle.



Area $= 1 \times \frac{\pi}{2} = \frac{\pi}{2}$

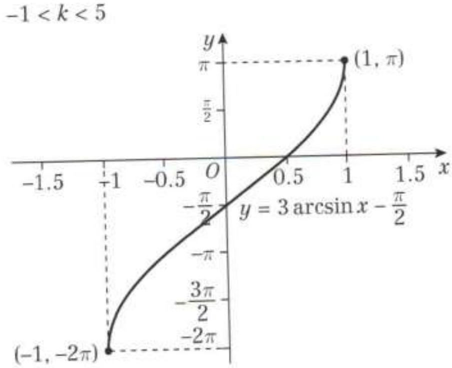
21 $\cot 60° \sec 60° = \dfrac{1}{\tan 60°} \times \dfrac{1}{\cos 60°} = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

22 a



b $-1 < k < 5$

23 a



b $\left(\frac{1}{2}, 0\right)$

24 a Let $y = \arccos x$. So $\cos y = x$, $\sin y = \sqrt{1 - x^2}$.

Thus $\tan y = \dfrac{\sqrt{1 - x^2}}{x}$, which is valid for $x \in (0, 1]$.

Therefore $\arccos x = \arctan\dfrac{\sqrt{1 - x^2}}{x}$ for $0 < x \le 1$.

b Letting $y = \arccos x$, $x \in [-1, 0) \Rightarrow y \in \left(\frac{\pi}{2}, \pi\right]$

$\tan y = \dfrac{\sin y}{\cos y} = \dfrac{\sqrt{1 - x^2}}{x}$

$\arctan\dfrac{\sqrt{1 - x^2}}{x}$ gives values in the range $\left(-\frac{\pi}{2}, 0\right]$,

so for $y \in \left(\frac{\pi}{2}, \pi\right]$ you need to add π:

$y = \pi + \arctan\dfrac{\sqrt{1 - x^2}}{x}$

Therefore $\arccos x = \pi + \arctan\dfrac{\sqrt{1 - x^2}}{x}$

CHAPTER 7

Prior knowledge 7

1 **a** $\frac{1}{\sqrt{2}}$ **b** $\frac{\sqrt{3}}{2}$ **c** $\sqrt{3}$

2 **a** 194.2°, 245.8° **b** 45°, 165°, 225°, 345° **c** 270°

3 **a** $\text{LHS} \equiv \cos x + \sin x \tan x \equiv \cos x + \sin x\left(\frac{\sin x}{\cos x}\right)$

$\equiv \frac{\cos^2 x + \sin^2 x}{\cos x} \equiv \frac{1}{\cos x} \equiv \sec x \equiv \text{RHS}$

b $\text{LHS} \equiv \cot x \sec x \sin x \equiv \left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{1}\right)$

$\equiv 1 \equiv \text{RHS}$

c $\text{LHS} \equiv \frac{\cos^2 x + \sin^2 x}{1 + \cot^2 x} \equiv \frac{1}{\operatorname{cosec}^2 x} \equiv \sin^2 x \equiv \text{RHS}$

Exercise 7A

1 **a** **i** $(\alpha - \beta) + \beta = \alpha$. So $\angle FAB = \alpha$.

ii $\angle FAB = \angle ABD$ (alternate angles)
$\angle CBE = 90 - \alpha$, so $\angle BCE = 90 - (90 - \alpha) = \alpha$.

iii $\cos\beta = \frac{AB}{1} \Rightarrow AB = \cos\beta$

iv $\sin\beta = \frac{BC}{1} \Rightarrow BC = \sin\beta$

b **i** $\sin\alpha = \frac{AD}{\cos\beta} \Rightarrow AD = \sin\alpha\cos\beta$

ii $\cos\alpha = \frac{BD}{\cos\beta} \Rightarrow BD = \cos\alpha\cos\beta$

c **i** $\cos\alpha = \frac{CE}{\sin\beta} \Rightarrow CE = \cos\alpha\sin\beta$

ii $\sin\alpha = \frac{BE}{\sin\beta} \Rightarrow BE = \sin\alpha\sin\beta$

d **i** $\sin(\alpha - \beta) = \frac{FC}{1} \Rightarrow FC = \sin(\alpha - \beta)$

ii $\cos(\alpha - \beta) = \frac{FA}{1} \Rightarrow FA = \cos(\alpha - \beta)$

e **i** $FC + CE = AD$, so $FC = AD - CE$
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

ii $AF = DB + BE$
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

2 $\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$

$= \dfrac{\frac{\sin A\cos B}{\cos A\cos B} - \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} + \frac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

$\sin(A + B) = \sin A\cos B + \cos A\sin B$
$\sin(P + (-Q)) = \sin P\cos(-Q) + \cos P\sin(-Q)$
$\sin(P - Q) = \sin P\cos Q - \cos P\sin Q$

Example: with $A = 60°$, $B = 30°$,

$\sin(A + B) = \sin 90° = 1$; $\sin A + \sin B = \frac{\sqrt{3}}{2} + \frac{1}{2} \neq 1$

[You can find examples of A and B for which the statement is true, e.g. $A = 30°$, $B = -30°$, but one counter-example shows that it is not an identity.]

$\cos(\theta - \theta) \equiv \cos\theta\cos\theta + \sin\theta\sin\theta$
$\Rightarrow \sin^2\theta + \cos^2\theta \equiv 1$ as $\cos 0 = 1$

a $\sin\left(\frac{\pi}{2} - \theta\right) \equiv \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta$

$\equiv (1)\cos\theta - (0)\sin\theta = \cos\theta$

b $\cos\left(\frac{\pi}{2} - \theta\right) \equiv \cos\frac{\pi}{2}\cos\theta - \sin\frac{\pi}{2}\sin\theta$

$\equiv (0)\cos\theta - (1)\sin\theta = \sin\theta$

7 $\sin\left(x + \frac{\pi}{6}\right) = \sin x\cos\frac{\pi}{6} + \cos x\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$

8 $\cos\left(x + \frac{\pi}{3}\right) = \cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

9 **a** $\sin 35°$ **b** $\sin 35°$ **c** $\cos 210°$ **d** $\tan 31°$
e $\cos\theta$ **f** $\cos 7\theta$ **g** $\sin 3\theta$ **h** $\tan 5\theta$
i $\sin A$ **j** $\cos 3x$

10 **a** $\sin\left(x + \frac{\pi}{4}\right)$ or $\cos\left(x - \frac{\pi}{4}\right)$ **b** $\cos\left(x + \frac{\pi}{4}\right)$

c $\sin\left(x + \frac{\pi}{3}\right)$ or $\cos\left(x - \frac{\pi}{6}\right)$ **d** $\sin\left(x - \frac{\pi}{4}\right)$

11 $\cos y = \sin x\cos y + \sin y\cos x$
Divide by $\cos x\cos y \Rightarrow \sec x = \tan x + \tan y$,
so $\tan y = \sec x - \tan x$

12 $\frac{\tan x - 3}{3\tan x + 1}$ **13** 2

14 **a** $\frac{5}{3}$ **b** $\sqrt{3}$ **c** $-\left(\frac{8 + 5\sqrt{3}}{11}\right)$

15 $\frac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x} = \frac{1}{2} \Rightarrow (2 + \sqrt{3})\tan x = 1 - 2\sqrt{3}$, so

$\tan x = \frac{1 - 2\sqrt{3}}{2 + \sqrt{3}} = \frac{(1 - 2\sqrt{3})(2 - \sqrt{3})}{1} = 8 - 5\sqrt{3}$

16 Write θ as $\left(\theta + \frac{2\pi}{3}\right) - \frac{2\pi}{3}$ and $\theta + \frac{4\pi}{3}$ as $\left(\theta + \frac{2\pi}{3}\right) + \frac{2\pi}{3}$.
Use the addition formulae for cos and simplify.

Challenge

a **i** $\text{Area} = \frac{1}{2}ab\sin\theta = \frac{1}{2}x(y\cos B)(\sin A) = \frac{1}{2}xy\sin A\cos B$

ii $\text{Area} = \frac{1}{2}ab\sin\theta = \frac{1}{2}y(x\cos A)(\sin B) = \frac{1}{2}xy\cos A\sin B$

iii $\text{Area} = \frac{1}{2}ab\sin\theta = \frac{1}{2}xy\sin(A + B)$

b Area of large triangle = area T_1 + area T_2

$\frac{1}{2}xy\sin(A + B) = \frac{1}{2}xy\sin A\cos B + \frac{1}{2}xy\cos A\sin B$

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

Exercise 7B

1 **a** $\frac{\sqrt{2}(\sqrt{3} + 1)}{4}$ **b** $\frac{\sqrt{2}(\sqrt{3} + 1)}{4}$ **c** $\frac{\sqrt{2}(\sqrt{3} - 1)}{4}$ **d** $\sqrt{3} - 2$

2 **a** 1 **b** 0 **c** $\frac{\sqrt{3}}{2}$ **d** $\frac{\sqrt{2}}{2}$ **e** $\frac{\sqrt{2}}{2}$
f $-\frac{1}{2}$ **g** $\sqrt{3}$ **h** $\frac{\sqrt{3}}{3}$ **i** 1 **j** $\sqrt{2}$

3 **a** $\tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45°\tan 30°}$

b $\tan 75° = \dfrac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{(3 + \sqrt{3})(3 + \sqrt{3})}{(3 - \sqrt{3})(3 + \sqrt{3})}$

$= \frac{12 + 6\sqrt{3}}{9 - 3} = 2 + \sqrt{3}$

4 $-\frac{6}{7}$

5 a $\cos 105° = \cos(45° + 60°)$
$= \cos 45° \cos 60° - \sin 45° \sin 60°$
$= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

b $a = 2, b = 3$

6 a $\frac{3+4\sqrt{3}}{10}$ b $\frac{4+3\sqrt{3}}{10}$ c $\frac{10(3\sqrt{3}-4)}{11}$

7 a $\frac{3}{5}$ b $\frac{4}{5}$ c $\frac{3-4\sqrt{3}}{10}$ d $\frac{1}{7}$

8 a $-\frac{77}{85}$ b $-\frac{36}{85}$ c $\frac{36}{77}$

9 a $-\frac{36}{325}$ b $\frac{204}{253}$ c $-\frac{325}{36}$

10 a $45°$ b $225°$

Exercise 7C

1 $\sin 2A = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A$

2 a $\cos 2A = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$

b i $\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A)$
$= 2\cos^2 A - 1$

ii $\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A$

3 $\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2\tan A}{1 - \tan^2 A}$

4 a $\sin 20°$ b $\cos 50°$ c $\cos 80°$
d $\tan 10°$ e $\operatorname{cosec} 49°$ f $3\cos 60°$
g $\frac{1}{2}\sin 16°$ h $\cos\left(\frac{\pi}{8}\right)$

5 a $\frac{\sqrt{2}}{2}$ b $\frac{\sqrt{3}}{2}$ c $\frac{1}{2}$ d 1

6 a $(\sin A + \cos A)^2 = \sin^2 A + 2\sin A \cos A + \cos^2 A$
$= 1 + \sin 2A$

b $\left(\sin\frac{\pi}{8} + \cos\frac{\pi}{8}\right)^2 = 1 + \sin\frac{\pi}{4} = 1 + \frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{2}$

7 a $\cos 6\theta$ b $3\sin 4\theta$ c $\tan\theta$
d $2\cos\theta$ e $\sqrt{2}\cos\theta$ f $\frac{1}{4}\sin^2 2\theta$
g $\sin 4\theta$ h $-\frac{1}{2}\tan 2\theta$ i $\cos^2 2\theta$

8 $q = \frac{p^2}{2} - 1$

9 a $y = 2(1-x)$ b $2xy = 1 - x^2$
c $y^2 = 4x^2(1-x^2)$ d $y^2 = \frac{2(4-x)}{3}$

10 $-\frac{7}{8}$ 11 $\pm\frac{1}{5}$

12 a i $\frac{24}{7}$ ii $\frac{24}{25}$ iii $\frac{7}{25}$ b $\frac{336}{625}$

13 a i $-\frac{7}{9}$ ii $\frac{2\sqrt{2}}{3}$ iii $-\frac{9\sqrt{2}}{8}$

b $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{4\sqrt{2}}{9} \times -\frac{9}{7} = \frac{4\sqrt{2}}{7}$

14 -3 15 mn

16 a $\cos 2\theta = \frac{3^2 + 6^2 - 5^2}{2 \times 3 \times 6} = \frac{20}{36} = \frac{5}{9}$ b $\frac{\sqrt{2}}{3}$

17 a $\frac{3}{4}$ b $m = \tan 2\theta = \frac{2\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$

18 a $\cos 2A = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$
$= \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$

b $4\cos 2x = 6\cos^2 x - 3\sin 2x$
$\cos 2x + 3\cos 2x - 6\cos^2 x + 3\sin 2x = 0$
$\cos 2x + 3(2\cos^2 x - 1) - 6\cos^2 x + 3\sin 2x = 0$
$\cos 2x - 3 + 3\sin 2x = 0$
$\cos 2x + 3\sin 2x - 3 = 0$

19 $\tan 2A \equiv \frac{\sin 2A}{\cos 2A} \equiv \frac{2\sin A \cos A}{\cos^2 A - \sin^2 A}$

$\equiv \frac{\frac{2\sin A\cos A}{\cos^2 A}}{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}} \equiv \frac{2\tan A}{1 - \tan^2 A}$

Exercise 7D

1 a $51.7°, 231.7°$ b $170.1°, 350.1°$
c $56.5°, 303.5°$ d $150°, 330°$

2 a $\sin\left(\theta + \frac{\pi}{4}\right) \equiv \sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}$

$\equiv \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta \equiv \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$

b $0, \frac{\pi}{2}, 2\pi$ c $0, \frac{\pi}{2}, 2\pi$

3 a $30°, 270°$ b $30°, 270°$

4 a $3(\sin x \cos y - \cos x \sin y)$
$- (\sin x \cos y + \cos x \sin y) = 0$
$\Rightarrow 2\sin x \cos y - 4\cos x \sin y = 0$
Divide throughout by $2\cos x \cos y$
$\Rightarrow \tan x - 2\tan y = 0$, so $\tan x = 2\tan y$

b Using **a** $\tan x = 2\tan y = 2\tan 45° = 2$
so $x = 63.4°, 243.4°$

5 a $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ b $\pm 38.7°$
c $30°, 150°, 210°, 330°$ d $\frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$
e $60°, 300°, 443.6°, 636.4°$ f $\frac{\pi}{8}, \frac{5\pi}{8}$
g $\frac{\pi}{4}, \frac{5\pi}{4}$
h $0°, 30°, 150°, 180°, 210°, 330°$ i $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$
j $-104.0°, 0°, 76.0°$
k $0°, 35.3°, 144.7°, 180°, 215.3°, 324.7°, 360°$

6 $51.3°$

7 a $5\sin 2\theta = 10\sin\theta\cos\theta$, so equation becomes
$10\sin\theta\cos\theta + 4\sin\theta = 0$, or $2\sin\theta(5\cos\theta + 2) = 0$

b $0°, 180°, 113.6°, 246.4°$

8 a $2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta = 1$
$\Rightarrow 2\sin\theta\cos\theta - 2\sin^2\theta = 0$
$\Rightarrow 2\sin\theta(\cos\theta - \sin\theta) = 0$

b $0°, 180°, 45°, 225°$

9 a L.H.S. $= \cos^2 2\theta + \sin^2 2\theta - 2\sin 2\theta\cos 2\theta$
$= 1 - \sin 4\theta =$ R.H.S.

b $\frac{\pi}{24}, \frac{17\pi}{24}$

10 a i R.H.S. $= \frac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} = 2\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \times \frac{\cos^2\left(\frac{\theta}{2}\right)}{1}$

$= 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \sin\theta$

ii R.H.S. $= \frac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} = \frac{1 - \tan^2\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)}$

$= \cos^2\left(\frac{\theta}{2}\right)\left\{1 - \tan^2\left(\frac{\theta}{2}\right)\right\} = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$

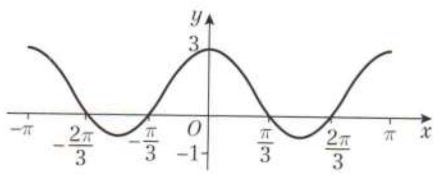
$= \cos\theta =$ L.H.S.

b i $90°, 323.1°$ ii $13.3°, 240.4°$

11 a L.H.S. $\equiv \dfrac{3(1+\cos 2x)}{2} - \dfrac{(1-\cos 2x)}{2}$

$\equiv 1 + 2\cos 2x$

b



Crosses y-axis at $(0, 3)$

Crosses x-axis at $\left(-\frac{2\pi}{3}, 0\right), \left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{2\pi}{3}, 0\right)$

12 a $2\cos^2\left(\frac{\theta}{2}\right) - 4\sin^2\left(\frac{\theta}{2}\right) = 2\left(\frac{1+\cos\theta}{2}\right) - 4\left(\frac{1-\cos\theta}{2}\right)$

$= 1 + \cos\theta - 2 + 2\cos\theta = 3\cos\theta - 1$

b 131.8°, 228.2°

13 a $(\sin^2 A + \cos^2 A)^2 \equiv \sin^4 A + \cos^4 A + 2\sin^2 A\cos^2 A$

So $\quad 1 \equiv \sin^4 A + \cos^4 A + \dfrac{(2\sin A\cos A)^2}{2}$

$\Rightarrow \quad 2 \equiv 2(\sin^4 A + \cos^4 A) + \sin^2 2A$

$\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$

b Using **a**: $\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$

$\equiv \frac{1}{2}\left\{2 - \dfrac{(1-\cos 4A)}{2}\right\} \equiv \dfrac{(4 - 1 + \cos 4A)}{4} \equiv \dfrac{3 + \cos 4A}{4}$

c $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

14 a $\cos 3\theta \equiv \cos(2\theta + \theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3(1 - \cos^2\theta)\cos\theta$

$\equiv 4\cos^3\theta - 3\cos\theta$

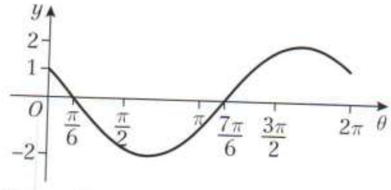
b $\frac{\pi}{9}, \frac{5\pi}{9}$ and $\frac{7\pi}{9}$

Exercise 7E

1 $R = 13$; $\tan\alpha = \frac{12}{5}$ **2** 35.3° **3** 41.8°

4 a $\cos\theta - \sqrt{3}\sin\theta \equiv R\cos(\theta + \alpha)$ gives $R = 2$, $\alpha = \frac{\pi}{3}$

b $y = 2\cos\left(\theta + \frac{\pi}{3}\right)$



5 a $25\cos(\theta + 73.7°)$ **b** $(0, 7)$

c $25, -25$ **d i** 2 **ii** 0 **iii** 1

6 a $R = \sqrt{10}$, $\alpha = 71.6°$ **b** $\theta = 69.2°, 327.7°$

7 a $\sqrt{5}\cos(2\theta + 1.107)$ **b** $\theta = 0.60, 1.44$

8 a 6.9°, 66.9° **b** 16.6°, 65.9°

c 8.0°, 115.9° **d** −165.2°, 74.8°

9 a $5\sin(3\theta - 53.1°)$

b Minimum value is −5, when $3\theta - 53.1° = 270° \Rightarrow \theta = 107.7°$

c 21.6°, 73.9°, 141.6°

10 a $5\left(\frac{1-\cos 2\theta}{2}\right) - 3\left(\frac{1+\cos 2\theta}{2}\right) + 3\sin 2\theta$

$\equiv 1 + 3\sin 2\theta - 4\cos 2\theta$, so $a = 3$, $b = -4$, $c = 1$

b Maximum = 6, minimum = −4 **c** 14.8°, 128.4°

11 a $R = \sqrt{10}$, $\alpha = 18.4°$, $\theta = 69.2°, 327.7°$

b $9\cos^2\theta = 4 - 4\sin\theta + \sin^2\theta$

$\Rightarrow 9(1 - \sin^2\theta) = 4 - 4\sin\theta + \sin^2\theta$

So $10\sin^2\theta - 4\sin\theta - 5 = 0$

c 69.2°, 110.8°, 212.3°, 327.7°

d When you square you are also solving $3\cos\theta = -(2 - \sin\theta)$. The other two solutions are for this equation.

12 a $\dfrac{\cos\theta}{\sin\theta} \times \sin\theta + 2\sin\theta = \dfrac{1}{\sin\theta} \times \sin\theta \Rightarrow$

$\cos\theta + 2\sin\theta = 1$

b $\theta = 126.9°$ (1 d.p.)

13 a $\sqrt{2}\cos\theta\cos\frac{\pi}{4} + \sqrt{2}\sin\theta\sin\frac{\pi}{4} + \sqrt{3}\sin\theta - \sin\theta = 2$

$\Rightarrow \cos\theta + \sin\theta - \sin\theta + \sqrt{3}\sin\theta = 2$

$\Rightarrow \cos\theta + \sqrt{3}\sin\theta = 2$

b $\frac{\pi}{3}$

14 a $R = 41$, $\alpha = 77.320°$ **b i** $\frac{18}{91}$ **ii** 77.320°

15 a $R = 13$, $\alpha = 22.6°$ **b** $\theta = 48.7°, 108.7°$

c $a = 12$, $b = -5$, $c = 12$ **d** minimum value = −1

Exercise 7F

1 a L.H.S. $= \dfrac{\cos^2 A - \sin^2 A}{\cos A + \sin A} = \dfrac{(\cos A + \sin A)(\cos A - \sin A)}{\cos A + \sin A}$

$= \cos A - \sin A =$ R.H.S.

b R.H.S. $= \dfrac{2}{2\sin A\cos A}\{\sin B\cos A - \cos B\sin A\}$

$= \dfrac{\sin B}{\sin A} - \dfrac{\cos B}{\cos A} =$ L.H.S.

c L.H.S. $= \dfrac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta} = \dfrac{2\sin^2\theta}{2\sin\theta\cos\theta} = \tan\theta =$ R.H.S.

d L.H.S. $= \dfrac{1 + \tan^2\theta}{1 - \tan^2\theta} = \dfrac{1 + \frac{\sin^2\theta}{\cos^2\theta}}{1 - \frac{\sin^2\theta}{\cos^2\theta}}$

$= \dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta} = \dfrac{1}{\cos 2\theta} = \sec 2\theta =$ R.H.S.

e L.H.S. $= 2\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$

$= 2\sin\theta\cos\theta = \sin 2\theta =$ R.H.S.

f L.H.S. $= \dfrac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta} = \dfrac{\sin(3\theta - \theta)}{\sin\theta\cos\theta}$

$= \dfrac{\sin 2\theta}{\sin\theta\cos\theta} = \dfrac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 =$ R.H.S.

g L.H.S. $= \dfrac{1}{\sin\theta} - \dfrac{2\cos 2\theta\cos\theta}{\sin 2\theta} = \dfrac{1}{\sin\theta} - \dfrac{2\cos 2\theta\cos\theta}{2\sin\theta\cos\theta}$

$= \dfrac{1 - \cos 2\theta}{\sin\theta} = \dfrac{1 - (1 - 2\sin^2\theta)}{\sin\theta} = 2\sin\theta =$ R.H.S.

h L.H.S. $= \dfrac{\frac{1}{\cos\theta} - 1}{\frac{1}{\cos\theta} + 1} = \dfrac{1 - \cos\theta}{1 + \cos\theta} = \dfrac{1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)}{1 + \left(2\cos^2\frac{\theta}{2} - 1\right)}$

$= \dfrac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2} =$ R.H.S.

i L.H.S. $= \dfrac{1 - \tan x}{1 + \tan x} = \dfrac{\cos x - \sin x}{\cos x + \sin x}$

$= \dfrac{(\cos x - \sin x)(\cos x - \sin x)}{\cos^2 x - \sin^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x - 2\sin x \cos x}{\cos^2 x - \sin^2 x} = \dfrac{1 - \sin 2x}{\cos 2x} =$ R.H.S.

2 a L.H.S. $= \sin(A + 60°) + \sin(A - 60°) = \sin A \cos 60°$
$+ \cos A \sin 60° + \sin A \cos 60° - \cos A \sin 60°$
$= 2\sin A \cos 60° \equiv \sin A =$ R.H.S.

b L.H.S. $= \dfrac{\cos A}{\sin B} - \dfrac{\sin A}{\cos B} = \dfrac{\cos A \cos B - \sin A \sin B}{\sin B \cos B}$

$\equiv \dfrac{\cos(A + B)}{\sin B \cos B} =$ R.H.S.

c L.H.S. $= \dfrac{\sin(x + y)}{\cos x \cos y} = \dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}$

$= \dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y} \equiv \tan x + \tan y =$ R.H.S.

d L.H.S. $= \dfrac{\cos(x + y)}{\sin x \sin y} + 1 = \dfrac{\cos x \cos y - \sin x \sin y}{\sin x \sin y} + 1$

$= \dfrac{\cos x \cos y}{\sin x \sin y} - \dfrac{\sin x \sin y}{\sin x \sin y} + 1 = \dfrac{\cos x \cos y}{\sin x \sin y}$

$\equiv \cot x \cot y =$ R.H.S.

e L.H.S. $= \cos\left(\theta + \dfrac{\pi}{3}\right) + \sqrt{3}\sin\theta = \cos\theta\cos\dfrac{\pi}{3}$

$- \sin\theta\sin\dfrac{\pi}{3} + \sqrt{3}\sin\theta = \dfrac{1}{2}\cos\theta - \dfrac{\sqrt{3}}{2}\sin\theta + \sqrt{3}\sin\theta$

$= \dfrac{1}{2}\cos\theta + \dfrac{\sqrt{3}}{2}\sin\theta \equiv \sin\left(\theta + \dfrac{\pi}{6}\right) =$ R.H.S.

f L.H.S. $= \cot(A + B) = \dfrac{\cos(A + B)}{\sin(A + B)}$

$= \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

$= \dfrac{\dfrac{\cos A \cos B}{\sin A \sin B} - \dfrac{\sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}} \equiv \dfrac{\cot A \cot B - 1}{\cot A + \cot B} =$ R.H.S.

g L.H.S. $= \sin^2(45° + \theta) + \sin^2(45° - \theta) = (\sin(45° + \theta))^2$
$+ (\sin(45° - \theta))^2 = (\sin 45° \cos\theta + \cos 45° \sin\theta)^2$
$+ (\sin 45° \cos\theta - \cos 45° \sin\theta)^2$

$= \left(\dfrac{\sqrt{2}}{2}\cos\theta + \dfrac{\sqrt{2}}{2}\sin\theta\right)^2 + \left(\dfrac{\sqrt{2}}{2}\cos\theta - \dfrac{\sqrt{2}}{2}\sin\theta\right)^2$

$= \dfrac{1}{2}\cos^2\theta + \cos\theta\sin\theta + \dfrac{1}{2}\sin^2\theta + \dfrac{1}{2}\cos^2\theta$

$- \cos\theta\sin\theta + \dfrac{1}{2}\sin^2\theta = \cos^2\theta + \sin^2\theta \equiv 1 =$ R.H.S.

h L.H.S. $= \cos(A + B)\cos(A - B)$
$= (\cos A \cos B - \sin A \sin B) \times (\cos A \cos B + \sin A \sin B)$
$= (\cos^2 A \cos^2 B) - (\sin^2 A \sin^2 B) = (\cos^2 A(1 - \sin^2 B))$
$- ((1 - \cos^2 A)\sin^2 B) = \cos^2 A - \cos^2 A \sin^2 B$
$- \sin^2 B + \cos^2 A \sin^2 B \equiv \cos^2 A - \sin^2 B =$ R.H.S.

3 a L.H.S. $= \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$

$= \dfrac{1}{\left(\frac{1}{2}\right)\sin 2\theta} = 2\operatorname{cosec} 2\theta =$ R.H.S.

b 4

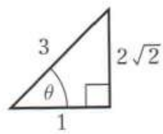
4 a Use $\sin 3\theta \equiv \sin(2\theta + \theta)$ and substitute $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$.

b Use $\cos 3\theta \equiv \cos(2\theta + \theta)$ and substitute $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$.

c $\tan 3\theta \equiv \dfrac{\sin 3\theta}{\cos 3\theta} = \dfrac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta}$

$= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

d



$\tan\theta = 2\sqrt{2}$

so $\tan 3\theta = \dfrac{6\sqrt{2} - 16\sqrt{2}}{1 - 24} = \dfrac{-10\sqrt{2}}{-23} = \dfrac{10\sqrt{2}}{23}$

5 a i $\cos x \equiv 2\cos^2\dfrac{x}{2} - 1$

$\Rightarrow 2\cos^2\dfrac{x}{2} \equiv 1 + \cos x \Rightarrow \cos^2\dfrac{x}{2} \equiv \dfrac{1 + \cos x}{2}$

ii $\cos x \equiv 1 - 2\sin^2\dfrac{x}{2}$

$\Rightarrow 2\sin^2\dfrac{x}{2} \equiv 1 - \cos x \Rightarrow \sin^2\dfrac{x}{2} \equiv \dfrac{1 - \cos x}{2}$

b i $\dfrac{2\sqrt{5}}{5}$ ii $\dfrac{\sqrt{5}}{5}$ iii $\frac{1}{2}$

c $\cos^4\dfrac{A}{2} \equiv \left(\dfrac{1 + \cos A}{2}\right)^2 \equiv \dfrac{1 + 2\cos A + \cos^2 A}{4}$

$\equiv \dfrac{1 + 2\cos A + \left(\dfrac{1 + \cos 2A}{2}\right)}{4}$

$\equiv \dfrac{2 + 4\cos A + 1 + \cos 2A}{8} \equiv \dfrac{3 + 4\cos A + \cos 2A}{8}$

6 L.H.S. $\equiv \cos^4\theta \equiv (\cos^2\theta)^2 \equiv \left(\dfrac{1 + \cos 2\theta}{2}\right)^2$

$\equiv \dfrac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta) \equiv \dfrac{1}{4} + \dfrac{1}{2}\cos 2\theta$

$+ \dfrac{1}{4}\left(\dfrac{1 + \cos 4\theta}{2}\right) \equiv \dfrac{1}{4} + \dfrac{1}{2}\cos 2\theta + \dfrac{1}{8} + \dfrac{1}{8}\cos 4\theta$

$\equiv \dfrac{3}{8} + \dfrac{1}{2}\cos 2\theta + \dfrac{1}{8}\cos 4\theta \equiv$ R.H.S.

7 $[\sin(x + y) + \sin(x - y)][\sin(x + y) - \sin(x - y)]$
$\equiv [2\sin x \cos y][2\cos x \sin y]$
$\equiv [2\sin x \cos x][2\cos y \sin y]$
$\equiv \sin 2x \sin 2y$

8 $2\cos\left(2\theta + \dfrac{\pi}{3}\right) \equiv 2\left(\cos 2\theta\cos\dfrac{\pi}{3} - \sin 2\theta\sin\dfrac{\pi}{3}\right)$

$\equiv 2\left(\cos 2\theta\dfrac{1}{2} - \sin 2\theta\dfrac{\sqrt{3}}{2}\right) \equiv \cos 2\theta - \sqrt{3}\sin 2\theta$

9 $4\cos\left(2\theta - \dfrac{\pi}{6}\right) \equiv 4\cos 2\theta\cos\dfrac{\pi}{6} + 4\sin 2\theta\sin\dfrac{\pi}{6}$

$\equiv 2\sqrt{3}\cos 2\theta + 2\sin 2\theta \equiv 2\sqrt{3}(1 - 2\sin^2\theta) + 4\sin\theta\cos\theta$

$\equiv 2\sqrt{3} - 4\sqrt{3}\sin^2\theta + 4\sin\theta\cos\theta$

10 a R.H.S. $= \sqrt{2}\left\{\sin\theta\cos\dfrac{\pi}{4} + \cos\theta\sin\dfrac{\pi}{4}\right\}$

$= \sqrt{2}\left\{\sin\theta\dfrac{1}{\sqrt{2}} + \cos\theta\dfrac{1}{\sqrt{2}}\right\} = \sin\theta + \cos\theta =$ L.H.S.

b R.H.S. $= 2\left\{\sin 2\theta\cos\dfrac{\pi}{6} - \cos 2\theta\sin\dfrac{\pi}{6}\right\}$

$= 2\left\{\sin 2\theta\dfrac{\sqrt{3}}{2} - \cos 2\theta\dfrac{1}{2}\right\} = \sqrt{3}\sin 2\theta - \cos 2\theta =$ L.H.S.

Challenge

1 a $\cos(A+B) - \cos(A-B)$
$\equiv \cos A\cos B - \sin A\sin B - (\cos A\cos B + \sin A\sin B)$
$\equiv -2\sin A\sin B$

b Let $A + B = P$ and $A - B = Q$. Solve to get $A = \dfrac{P+Q}{2}$ and $B = \dfrac{P-Q}{2}$. Then use result from part **a** to get
$\cos P - \cos Q = -2\sin\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$

c $-\frac{3}{2}(\cos 8x - \cos 6x)$

2 a $\sin(A+B) + \sin(A-B)$
$= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$
$= 2\sin A\cos B$
Let $A + B = P$ and $A - B = Q$
$\therefore A = \dfrac{P+Q}{2}$ and $B = \dfrac{P-Q}{2}$
$\therefore \sin P + \sin Q = 2\sin\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$

b $\dfrac{11\pi}{24} = \dfrac{P+Q}{2}, \dfrac{5\pi}{24} = \dfrac{P-Q}{2}$
$\dfrac{22\pi}{24} = P+Q, \dfrac{10\pi}{24} = P-Q$
$\dfrac{32\pi}{24} = 2P \Rightarrow P = \dfrac{2\pi}{3}, Q = \dfrac{\pi}{4},$
$\sin\left(\dfrac{2\pi}{3}\right) + \sin\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{3}+\sqrt{2}}{2}$

Exercise 7G

1 a 0.25 m b 0.013 minutes, 0.8 seconds
c 0.2 minutes or 12 seconds

2 a 0.03 radians b 0.0085 radians
c 0.251 seconds
d 0.0492, 0.2021, 0.3006, 0.4534 seconds

3 a £17.12, £17.08
b £19.40, 6.53 hours or 6 h 32 min
c After 4.37 hours (4 h 22 min after market opens)

4 a 224.7°C
b 2 m 17 s, 5 m 26 s, 8 m 34 s
c 17.6 seconds.

5 a $R = 0.5, \alpha = 53.13°$
b i 0.5 ii $\theta = 143.1°$
c Minimum value is 22.5, occurs at 17.95 minutes
d 3, 13, 23, 33, 43, 53 minutes

6 a $R = 68.0074, \alpha = 0.2985$ b 138.0 m
c 31.4 minutes d 11.1 minutes

7 a $R = 250, \alpha = 0.6435$
b i 1950 V/m ii $x = 4.41$ cm, $x = 16.91$ cm
c $2.10 \leqslant x \leqslant 6.71, 14.60 \leqslant x \leqslant 19.21$

Challenge

a $0\text{ cm} \leqslant x < 0.39\text{ cm}, 8.42\text{ cm} < x < 12.89\text{ cm}, 20.92\text{ cm} < x < 25\text{ cm}$

b Identifying the exact point of maximum field strength; microwave oven would not work exactly the same every time it is used.

Mixed exercise 7

a $\frac{1}{2}$ b $\frac{1}{2}$ c $\dfrac{\sqrt{3}}{3}$

2 $\sin x = \dfrac{1}{\sqrt{5}}$, so $\cos x = \dfrac{2}{\sqrt{5}}$
$\cos(x-y) = \sin y \Rightarrow \dfrac{2}{\sqrt{5}}\cos y + \dfrac{1}{\sqrt{5}}\sin y = \sin y$
$\Rightarrow (\sqrt{5}-1)\sin y = 2\cos y \Rightarrow \tan y = \dfrac{2}{\sqrt{5}-1} = \dfrac{\sqrt{5}+1}{2}$

3 a $\tan A = 2, \tan B = \frac{1}{3}$ b 45°

4 Use the sine rule and addition formulae to get
$\dfrac{1}{20}\sin\theta \times \dfrac{\sqrt{3}}{2} = \dfrac{9}{20}\cos\theta \times \dfrac{1}{2}$
Then rearrage to get $\tan\theta = 3\sqrt{3}$.

5 75°

6 a i $\frac{56}{65}$ ii $\frac{120}{119}$
b Use $\cos\{180° - (A+B)\} \equiv -\cos(A+B)$ and expand. You can work out all the required trig. ratios (A and B are acute).

7 a Use $\cos 2x \equiv 1 - 2\sin^2 x$ b $\frac{4}{5}$
c i Use $\tan x = 2, \tan y = \frac{1}{3}$ in the expansion of $\tan(x+y)$.
ii Find $\tan(x-y) = 1$ and note that $x - y$ has to be acute.

8 a Show that both sides are equal to $\frac{5}{6}$.
b $\dfrac{3k}{2}$ c $\dfrac{12k}{4-9k^2}$

9 a $\sqrt{3}\sin 2\theta = 1 - 2\sin^2\theta = \cos 2\theta$
$\Rightarrow \sqrt{3}\tan 2\theta = 1 \Rightarrow \tan 2\theta = \dfrac{1}{\sqrt{3}}$
b $\dfrac{\pi}{12}, \dfrac{7\pi}{12}$

10 a $a = 2, b = 5, c = -1$ b 0.187, 2.95

11 a $\cos(x - 60°) = \cos x\cos 60° + \sin x\sin 60°$
$= \dfrac{1}{2}\cos x + \dfrac{\sqrt{3}}{2}\sin x$
So $\left(2 - \dfrac{\sqrt{3}}{2}\right)\sin x = \dfrac{1}{2}\cos x \Rightarrow \tan x = \dfrac{\frac{1}{2}}{2 - \frac{\sqrt{3}}{2}} = \dfrac{1}{4-\sqrt{3}}$
b 23.8°, 203.8°

12 a Using addition formulae:
$\cos x\cos 20° - \sin x\sin 20°$
$= \sin 70°\cos x - \cos 70°\sin x$
Rearrange to get: $\sin x(5\cos 70°) + \cos x(3\sin 70°) = 0$
$\Rightarrow \tan x = \dfrac{\sin x}{\cos x} = -\dfrac{3\sin 70°}{5\cos 70°} = -\dfrac{3}{5}\tan 70°$
b 121.2°

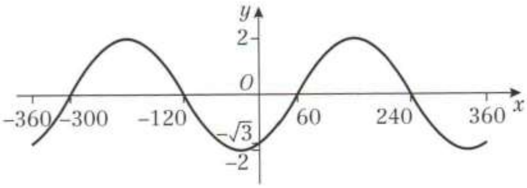
13 a Find $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$ and insert in expansions on L.H.S. Result follows.
b 0.6, 0.8

14 a Example: $A = 60°, B = 0°$; $\sec(A+B) = 2$,
$\sec A + \sec B = 2 + 1 = 3$
b L.H.S. $= \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} \equiv \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$
$\equiv \dfrac{1}{\frac{1}{2}\sin 2\theta} \equiv 2\operatorname{cosec} 2\theta =$ R.H.S.

15 a Setting $\theta = \frac{\pi}{8}$ gives resulting quadratic equation in t, $t^2 + 2t - 1 = 0$, where $t = \tan\left(\frac{\pi}{8}\right)$.
Solving this and taking +ve value for t gives result.
b Expanding $\tan\left(\dfrac{\pi}{4} + \dfrac{\pi}{8}\right)$ gives answer: $\sqrt{2} + 1$

16 a $2\sin(x - 60)°$

b



Graph crosses y-axis at $(0, -\sqrt{3})$
Graph crosses x-axis at $(-300°, -0)$, $(-120°, 0)$, $(60°, 0)$, $(240°, 0)$

17 a $R = 25, \alpha = 1.29$ **b** 32 **c** $\theta = 0.12, 1.17$

18 a $2.5\sin(2x + 0.927)$ **b** $\frac{3}{2}\sin 2x + 2\cos 2x + 2$ **c** 4.5

19 a $\alpha = 14.0°$ **b** $0°, 151.9°, 360°$

20 a $R = \sqrt{13}$, $\alpha = 56.3°$ **b** $\theta = 17.6°, 229.8°$

21 a L.H.S. $= \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta} \equiv \frac{1}{\frac{1}{2}\sin 2\theta} \equiv 2\operatorname{cosec} 2\theta =$ R.H.S.

b L.H.S. $= \frac{1 + \tan x}{1 - \tan x} - \frac{1 - \tan x}{1 + \tan x}$

$\equiv \frac{(1 + \tan x)^2 - (1 - \tan x)^2}{(1 + \tan x)(1 - \tan x)}$

$\equiv \frac{(1 + 2\tan x + \tan^2 x) - (1 - 2\tan x + \tan^2 x)}{1 - \tan^2 x}$

$\equiv \frac{4\tan x}{1 - \tan^2 x} = \frac{2(2\tan x)}{1 - \tan^2 x} = 2\tan 2x =$ R.H.S.

c L.H.S. $= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)$
$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$
$= (1 - \cos^2 x)\cos^2 y - \cos^2 x(1 - \sin^2 y) =$ R.H.S.

d L.H.S. $= 2\cos 2\theta + 1 + (2\cos^2 2\theta - 1)$
$\equiv 2\cos 2\theta(1 + \cos 2\theta) \equiv 2\cos 2\theta(2\cos^2\theta)$
$\equiv 4\cos^2\theta\cos 2\theta \equiv$ R.H.S.

22 a $\frac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)} \equiv \frac{2\sin^2 x}{2\cos^2 x}$

$\equiv \tan^2 x$

b $\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$

23 a L.H.S. $= \cos^4 2\theta - \sin^4 2\theta$
$\equiv (\cos^2 2\theta - \sin^2 2\theta)(\cos^2 2\theta + \sin^2 2\theta)$
$\equiv (\cos^2 2\theta - \sin^2 2\theta)(1)$
$\equiv \cos 4\theta =$ R.H.S.

b $15°, 75°, 105°, 165°$

24 a Use $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.
b $\sin 360° = 0$, $2 - 2\cos(360°) = 2 - 2 = 0$
c $26.6°, 206.6°$

25 a $R = 3, \alpha = 0.841$ **b** $x = 1.07, 3.53$

26 a $R = 5.772, \alpha = 75.964°$ **b** 5.772 when $\theta = 166.0°$
c 6.228 hours **d** 350.8 days

27 a $13\sin(x + 22.6°)$ **b** 3.8 m/s
c 168.5 minutes

Challenge

1 a $\frac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} \equiv \frac{2\cos 3\theta\cos\theta}{-2\cos 3\theta\sin\theta} \equiv -\cot\theta$

b $\cos 5x + \cos x + 2\cos 3x$
$\equiv 2\cos 3x\cos 2x + 2\cos 3x$
$\equiv 2\cos 3x(\cos 2x + 1)$
$\equiv 2\cos 3x(2\cos^2 x)$
$\equiv 4\cos^2 x\cos 3x$

2 a $\theta = \angle OAB = \angle OBA \Rightarrow \angle AOB = \pi - 2\theta$, so $\angle BOD = 2\theta$
$OB = 1$, $OD = \cos 2\theta$
$BD = \sin 2\theta$, $AB = 2\cos\theta$
$\sin\theta = \frac{BD}{AB} = \frac{BD}{2\cos\theta}$
So $BD = 2\sin\theta\cos\theta$
But $BD = \sin 2\theta$
So $\sin 2\theta \equiv 2\sin\theta\cos\theta$

b $AB = 2\cos\theta$
$AD = (2\cos\theta)\cos\theta = 2\cos^2\theta$
$OD = 2\cos^2\theta - 1$
From part **a**, $OD = \cos 2\theta$, so $\cos 2\theta = 2\cos^2\theta - 1$.

CHAPTER 8

Prior knowledge 8

1 a $t = \frac{x}{4 - k}$ **b** $t = \pm\sqrt{\frac{y}{3}}$
c $t = e^{\frac{2-y}{4}}$ **d** $t = -\frac{1}{3}\ln\left(\frac{x-1}{2}\right)$

2 a $7 - 3\cos^2 x$ **b** $2\cos x\sqrt{1 - \cos^2 x}$
c $\frac{\cos x}{\sqrt{1 - \cos^2 x}}$ **d** $2\cos x + 2\cos^2 x - 1$

3 a $y > 0$ **b** $0 < y < 2$
c $-6 \leq y < 3$ **d** $0 < y < 1$

4 (4,7) and (−4.8,2.6)

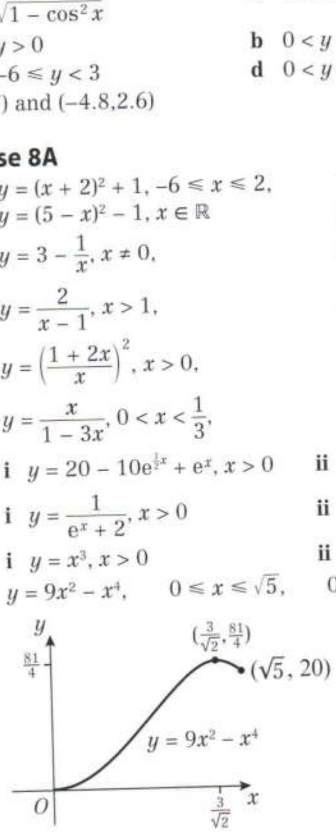
Exercise 8A

1 a $y = (x + 2)^2 + 1, -6 \leq x \leq 2$, $1 \leq y \leq 17$
b $y = (5 - x)^2 - 1, x \in \mathbb{R}$ $y \geq -1$
c $y = 3 - \frac{1}{x}, x \neq 0$, $y \neq 3$
d $y = \frac{2}{x - 1}, x > 1$, $y > 0$
e $y = \left(\frac{1 + 2x}{x}\right)^2, x > 0$, $y > 4$
f $y = \frac{x}{1 - 3x}, 0 < x < \frac{1}{3}$, $y > 0$

2 a i $y = 20 - 10e^{\frac{1}{2}x} + e^x, x > 0$ **ii** $y \geq -5$
b i $y = \frac{1}{e^x + 2}, x > 0$ **ii** $0 < y < \frac{1}{3}$
c i $y = x^3, x > 0$ **ii** $y > 0$

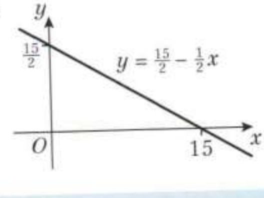
3 a $y = 9x^2 - x^4$, $0 \leq x \leq \sqrt{5}$, $0 \leq y \leq \frac{81}{4}$

b



4 a i $y = \frac{15}{2} - \frac{1}{2}x$ **ii** $x > -3, y < 9$

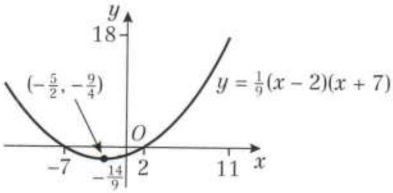
iii



b **i** $y = \frac{1}{9}(x - 2)(x + 7)$

ii $-13 < x < 11, -\frac{9}{4} < y < 18$

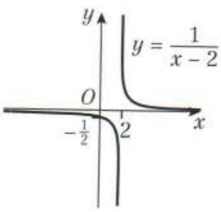
iii



c **i** $y = \dfrac{1}{x - 2}$

ii $x \in \mathbb{R}, x \neq 2, y \in \mathbb{R}, y \neq 0$

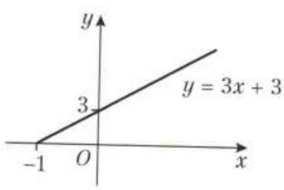
iii



d **i** $y = 3x + 3$

ii $x > -1, y > 0$

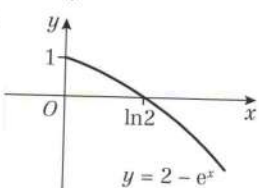
iii



e **i** $y = 2 - e^x$

ii $x > 0, y < 1$

iii



5 **a** C_1: $x = 1 + 2t, t = \dfrac{x - 1}{2}$

Sub t into $y = 2 + 3t$:

$$y = 2 + 3\left(\frac{x - 1}{2}\right) = 2 + \frac{3}{2}x - \frac{3}{2} = \frac{3}{2}x + \frac{1}{2}$$

$$C_2: x = \frac{1}{2t - 3}, t = \frac{1 + 3x}{2x}$$

Sub into $y = \dfrac{t}{2t - 3}$.

$$y = \frac{\frac{1 + 3x}{2x}}{2\left(\frac{1 + 3x}{2x}\right) - 3} = \frac{\frac{1 + 3x}{2x}}{\frac{1}{x}} = \frac{1 + 3x}{2} = \frac{3}{2}x + \frac{1}{2}$$

Therefore $C_1$and C_2 represent a segment of the same straight line.

b Length of $C_1 = 3\sqrt{13}$, length of $C_2 = \dfrac{\sqrt{13}}{3}$

a $x \neq 2; y \leqslant -2, y \neq -3$

b $x = \dfrac{3}{t} + 2, t = \dfrac{3}{x - 2}$

Sub into $y = 2t - 3 - t^2$

$$y = 2\left(\frac{3}{x - 2}\right) - 3 - \left(\frac{3}{x - 2}\right)^2 = \frac{6}{x - 2} - 3 - \frac{9}{(x - 2)^2}$$

$$= \frac{6(x - 2) - 3(x - 2)^2 - 9}{(x - 2)^2}$$

$$= \frac{6x - 12 - 3x^2 + 12x - 12 - 9}{(x - 2)^2} = \frac{-3x^2 + 18x - 33}{(x - 2)^2}$$

$$= \frac{-3(x^2 - 6x + 11)}{(x - 2)^2} \text{ so } A = -3, b = -6, c = 11$$

7 **a** $x = \ln(t + 3) \qquad t = e^x - 3 \qquad$ Sub into $y = \dfrac{1}{t + 5}$

$$y = \frac{1}{e^x - 3 + 5} = \frac{1}{e^x + 2}, \quad x > 0$$

b $0 < y < \dfrac{1}{3}$

8 **a** $y = \dfrac{x^6}{729} - \dfrac{2x^2}{9}, 0 \leqslant x \leqslant 3\sqrt{2}$

b $y = t^3 - 2t, \dfrac{dy}{dt} = 3t^2 - 2$

$0 = 3t^2 - 2 \qquad t^2 = \dfrac{2}{3} \qquad t = \sqrt{\dfrac{2}{3}}$

c $-\dfrac{4\sqrt{6}}{9} \leqslant f(x) \leqslant 4$

9 **a** $y = 4 - t^2 \Rightarrow t = \sqrt{4 - y}$

Sub into $x = t^3 - t = t(t^2 - 1)$

$x = \sqrt{4 - y}(4 - y - 1) = \sqrt{4 - y}(3 - y)$

$x^2 = (4 - y)(3 - y)^2$

$a = 4, b = 3$

b Max y is 4

Challenge

a $x^2 = \dfrac{(1 - t^2)^2}{(1 + t^2)^2}, y^2 = \dfrac{4t^2}{(1 + t^2)^2}$

$$x^2 + y^2 = \frac{(1 - t^2)^2}{(1 + t^2)^2} + \frac{4t^2}{(1 + t^2)^2} = \frac{1 - 2t^2 + t^4}{(1 + t^2)^2} + \frac{4t^2}{(1 + t^2)^2}$$

$$= \frac{1 + 2t^2 + t^4}{(1 + t^2)^2} = \frac{(1 + t^2)^2}{(1 + t^2)^2} = 1$$

So $x^2 + y^2 = 1$

b Circle, centre (0,0), radius 1.

Exercise 8B

1 **a** $25(x + 1)^2 + 4(y - 4)^2 = 100$

b $y^2 = 4x^2(1 - x^2)$

c $y = 4x^2 - 2$

d $y = \dfrac{2x\sqrt{1 - x^2}}{1 - 2x^2}$

e $y = \dfrac{4}{x - 2}$

f $y^2 = 1 + \left(\dfrac{x}{3}\right)^2$

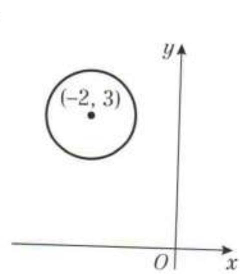
2 **a** $(x + 5)^2 + (y - 2)^2 = 1$

b Centre (−5, 2), radius 1

c $0 \leqslant t < 2\pi$

3 Centre (3,−1), radius 4

4 $(x + 2)^2 + (y - 3)^2 = 1$



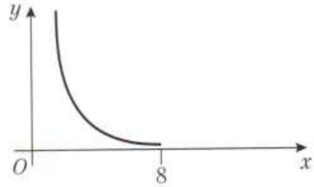
5 **a** $y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2(1 - x^2)}}{2}, -1 < x < 1$

b $y = \frac{\sqrt{3}}{3}x - \frac{\sqrt{9 - x^2}}{3}, \frac{3}{2} < x < 3$

c $y = -3x, -1 \leqslant x \leqslant 1$

6 **a** $y = \frac{16}{x^2}, 0 < x \leqslant 8$

b



7 $y = \frac{9}{3 + x}$ Domain: $x > 0$

8 **a** $y = 9x(1 - 12x^2) \Rightarrow a = 9, b = 12$

b Domain: $0 < x < \frac{1}{3}$, Range: $-1 < y \leqslant 1$

9 $y = \sin t \cos\left(\frac{\pi}{6}\right) - \cos t \sin\left(\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t = \frac{\sqrt{3\left(1 - \frac{x^2}{4}\right)}}{2} - \frac{1}{4}x$

$= \frac{1}{4}\left(2\sqrt{3 - \frac{3}{4}x^2} - x\right) = \frac{1}{4}(\sqrt{12 - 3x^2} - x)$

$t = 0 \Rightarrow x = 2, t = \pi \Rightarrow x = -2$, so $-2 < x < 2$.

10 **a** $y^2 = 25\left(1 - \frac{1}{x - 4}\right)$ **b** $x > 5, 0 < y < 5$

11 $x = -\frac{y}{\sqrt{9 - y^2}}, x > 0$

Challenge

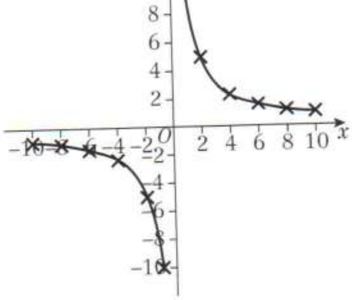
$(4y^2 - 2 + 2x)^2 + 12x^2 - 3 = 0$

Exercise 8C

1

t	−5	−4	−3	−2	−1	−0.5
$x = 2t$	−10	−8	−6	−4	−2	−1
$y = \frac{5}{t}$	−1	−1.25	−1.67	−2.5	−5	−10

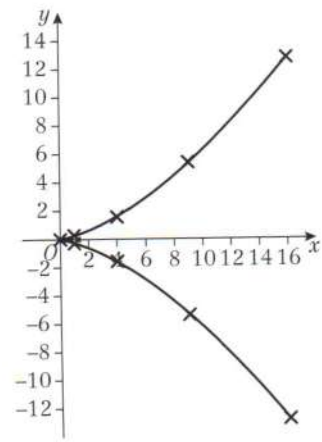
t	0.5	1	2	3	4	5
$x = 2t$	1	2	4	6	8	10
$y = \frac{5}{t}$	10	5	2.5	1.67	1.25	1



2

t	−4	−3	−2	−1	0
$x = t^2$	16	9	4	1	0
$y = \frac{t^3}{5}$	−12.8	−5.4	−1.6	−0.2	0

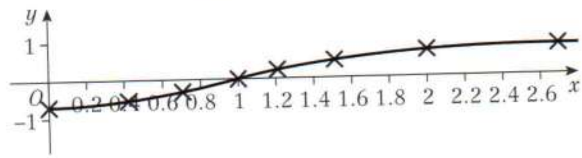
t	1	2	3	4
$x = t^2$	1	4	9	16
$y = \frac{t^3}{5}$	0.2	1.6	5.4	12.8



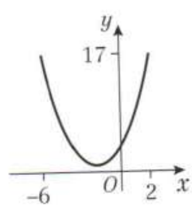
3

t	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0
$x = \tan t + 1$	0	0.423	0.732	1
$y = \sin t$	−0.707	−0.5	−0.259	0

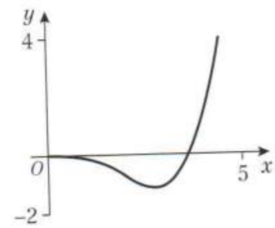
t	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$x = \tan t + 1$	1.268	1.577	2	2.732
$y = \sin t$	0.259	0.5	0.707	0.866



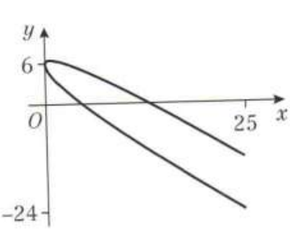
4 **a**



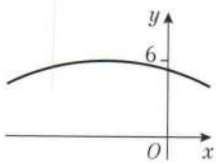
b



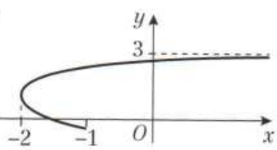
c



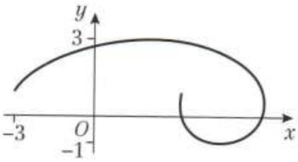
d



e

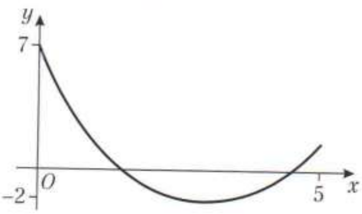


f



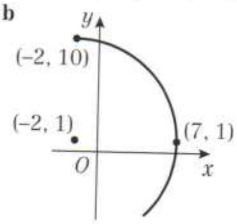
5 **a** $y = (3 - x)^2 - 2$

b



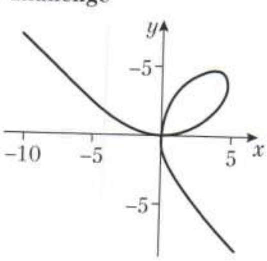
6 **a** $(x + 2)^2 + (y - 1)^2 = 81$

b



c 6π

Challenge



As t approaches −1 from the positive direction, the curve heads off to infinity in the 2nd quadrant.
As t approaches −1 from the negative direction, the curve heads off to infinity in the 4th quadrant.

Exercise 8D

1 **a** (11, 0) **b** (7, 0) **c** (1, 0), (9, 0)
d (1, 0), (2, 0) **e** $\left(\frac{9}{5}, 0\right)$

2 **a** (0, −5) **b** $\left(0, \frac{9}{16}\right)$ **c** (0, 0), (0, 12)
d $\left(0, \frac{1}{2}\right)$ **e** (0, 1)

3 4 **4** 4 **5** $\left(\frac{1}{2}, \frac{3}{2}\right)$

6 $t = \frac{5}{2}, t = -\frac{3}{2}$; $\left(\frac{25}{4}, 5\right), \left(\frac{9}{4}, -3\right)$

7 (1, 2), (1, −2), (4, 4), (4, −4)

8 **a** $\left(\frac{\pi^2}{4} - 1, 0\right)$, (0, cos 1)

b $\left(\frac{\sqrt{3}}{2}, 0\right)$, (0, 1)

c (1, 0)

9 **a** (e + 5, 0) **b** (ln 8, 0), (0, −63) **c** $\left(\frac{5}{4}, 0\right)$

10 $t = \frac{2}{3}, t = -1, \left(\frac{4}{9}, \frac{2}{3}\right), (1, -1)$

11 $t = \frac{14}{5}, \left(\ln\frac{9}{5}, \ln\frac{3}{5}\right)$

12 **a** $\left(6\cos\left(\frac{\pi}{12}\right), 0\right), \left(6\cos\left(\frac{5\pi}{12}\right), 0\right)$

b $4\sin 2t + 2 = 4 \Rightarrow 4\sin 2t = 2 \Rightarrow \sin 2t = 0.5$
$2t = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow t = \frac{\pi}{12}, \frac{5\pi}{12}$

c $\left(6\cos\left(\frac{\pi}{12}\right), 4\right), \left(6\cos\left(\frac{5\pi}{12}\right), 4\right)$

13 $y = 2x - 5 \Rightarrow 4t(t - 1) = 2(2t) - 5 \Rightarrow 4t^2 - 8t + 5 = 0$
Discriminant $= 8^2 - 4 \times 4 \times 5 = 64 - 80 = -16 < 0$
Since the discriminant is less than 0, the quadratic has no solutions, therefore the two equations do not intersect.

14 **a** $\cos 2t + 1 = k$
max of $\cos 2t = 1$, so $k = 1 + 1 = 2$
min of $\cos 2t = -1$, so $k = -1 + 1 = 0$
Therefore, $0 \leqslant k \leqslant 2$

b $y = 1 - 2\sin^2 t + 1 = 2 - 2\sin^2 t = 2 - 2x^2$
$k = 2 - 2x^2 \Rightarrow 2x^2 + k - 2 = 0$
Tangent when discriminant = 0
Discriminant $= 0^2 - 4 \times 2 \times (k - 2) = 0$
$-8(k - 2) = 0 \Rightarrow k - 2 = 0 \Rightarrow k = 2$
Therefore, $y = k$ is a tangent to the curve when $k = 2$.

15 **a** $A(4, 1), B(9, 2)$ **b** Gradient of $l = \frac{2 - 1}{9 - 4} = \frac{1}{5}$

c $x - 5y + 1 = 0$

16 $y + \sqrt{3}x - \sqrt{3} = 0$

17 **a** A(0, −3), $\mathrm{B}\left(\frac{3}{4}, 0\right)$

b Gradient of $l_1 = 4$
Equation of l_2 and l_3: $y = 4x + c$
$t - 4 = \frac{4(t - 1)}{t} + c \Rightarrow t^2 - 4t = 4t - 4 + ct$
$\Rightarrow t^2 - (8 + c)t + 4 = 0$
Tangent when discriminant = 0
$(-(8+c))^2 - 4 \times 1 \times 4 = 0$
$64 + 16c + c^2 - 16 = 0$
$c^2 + 16c + 48 = 0$
$(c + 4)(c + 12) = 0 \Rightarrow c = -4$ or $c = -12$
So, two possible equations for l_2 and l_3 are
$y = 4x - 4$ and $y = 4x - 12$

c $\left(\frac{1}{2}, -2\right), \left(\frac{3}{2}, -6\right)$

Challenge
(1, 1), (e, 2)

Exercise 8E

1 **a** 83.3 seconds **b** 267 m

c $t = \frac{x}{0.9} \Rightarrow y = -3.2\frac{x}{0.9} \Rightarrow y = -\frac{32}{9}x$
which is in the form, $y = mx + c$ and is therefore a straight line.

d $3.32\,\mathrm{ms}^{-1}$

2 **a** 3000 m

b Initial point is when $t = 0$. For $t \geqslant 330$, y is negative ie. the plane is underground or below sea level.

c 26 400 m (3 s.f.)

3 **a** 35.3 m

b Between 1.75 and 1.88 seconds (3 s.f.)

c 30.3 (3 s.f.).

4 **a** $\frac{100}{49}$ seconds **b** $\frac{200}{49}$ m

c $t = \frac{x}{2} \Rightarrow y = -4.9\left(\frac{x}{2}\right)^2 + 10\left(\frac{x}{2}\right) = -\frac{49}{40}x^2 + 5x$

Therefore, the dolphin's path is a quadratic curve

d $\frac{250}{49}$ m

5 **a** $\sin t = \frac{x}{12}$, $\cos t = \frac{y-12}{-12}$

$\left(\frac{x}{12}\right)^2 + \left(\frac{y-12}{-12}\right)^2 = 1 \Rightarrow x^2 + (y-12)^2 = 144$

Therefore, motion is a circle with centre (0,12) and radius $\sqrt{144} = 12$.

b 24 m **c** 2π minutes, 12 m/min

6 **a** 4.86 (3 s.f.) **b** Depth = 2

7 **a** $\frac{\sqrt{13}}{2}$ **b** (0, 2), (0, 4)

c $2t = 2\left(\frac{t^2 - 3t + 2}{t}\right) + 10$

$2t^2 = 2t^2 - 6t + 4 + 10t \Rightarrow 0 = 4t + 4 \Rightarrow t = -1$

Since, $t > 0$, the paths do not intersect.

8 **a** 10 m

b $k = 1.89$ (3 s.f.). Therefore, time taken is 1.89 seconds.

c 34.1 m (3 s.f.)

d $t = \frac{x}{18}$

$y = -4.9\left(\frac{x}{18}\right)^2 + 4\left(\frac{x}{18}\right) + 10 = -\frac{49}{3240}x^2 + \frac{2}{9}x + 10$

Therefore, the ski jumper's path is a quadratic equation. Maximum height = 10.8 m (3 s.f.)

9 **a** $t = \frac{\pi}{4}$ **b** (50, 20)

c (77.87, 18.19)

$\frac{\pi}{4} < 1 < \frac{\pi}{2}$, which is when $\sin 2t$ is decreasing, hence when y is decreasing, hence the cyclist is descending.

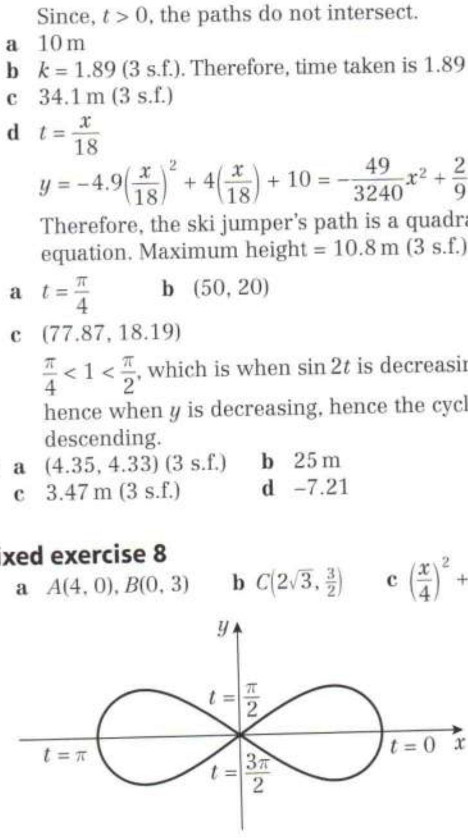
10 **a** (4.35, 4.33) (3 s.f.) **b** 25 m

c 3.47 m (3 s.f.) **d** −7.21

Mixed exercise 8

1 **a** $A(4, 0), B(0, 3)$ **b** $C\left(2\sqrt{3}, \frac{3}{2}\right)$ **c** $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

2



3 **a** $y = \ln(2\sqrt{x-1}) - \frac{1}{2}, x > e^3 + 1$

b $y > 1 + \ln 2$

4 $y = -\ln(4) - 2\ln(x), 0 < x < \frac{1}{2}, y > 0$

5 **a** $y = 1 - 2x^2$ **b** $\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

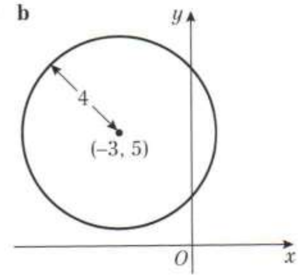
6 $t = \frac{1}{x} - 1$

Sub into $y = \frac{1}{(1+t)(1-t)}$

$y = \frac{1}{\left(1 + \frac{1}{x} - 1\right)\left(1 - \frac{1}{x} + 1\right)} = \frac{1}{\left(\frac{1}{x}\right)\left(2 - \frac{1}{x}\right)}$

$y = \frac{x^2}{x^2\left(\frac{1}{x}\right)\left(2 - \frac{1}{x}\right)} = \frac{x^2}{2x - 1}$

7 **a** $(x+3)^2 + (y-5)^2 = 16$ **b**



c $(0, 5 + \sqrt{7}), (0, 5 - \sqrt{7})$

8 **a** $x(1+t) = 2 - 3t \Rightarrow xt + 3t = 2 - x \Rightarrow t(x+3) = 2 - x$

$\Rightarrow t = \frac{2-x}{x+3}$

Sub into $y = \frac{3+2t}{1+t}$

$y = \frac{3 + 2\left(\frac{2-x}{x+3}\right)}{1 + \left(\frac{2-x}{x+3}\right)} = \frac{3(x+3) + 2(2-x)}{x + 3 + 2 - x} = \frac{3x + 9 + 4 - 2x}{5}$

$= \frac{x+13}{5} \Rightarrow y = \frac{1}{5}x + \frac{13}{5}$

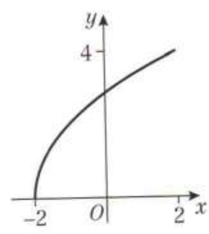
This is in the form $y = mx + c$, therefore the curve C is a straight line.

b $\frac{4\sqrt{26}}{5}$

9 **a** $y = 2\sqrt{x+2}$

b Domain: $-2 \leqslant x \leqslant 2$, Range: $0 \leqslant y \leqslant 4$

c

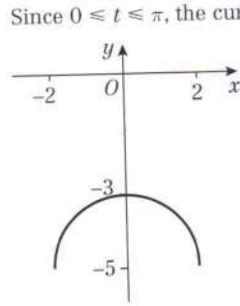


10 **a** $\cos t = \frac{x}{2}$, $\sin t = \frac{y+5}{2}$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y+5}{2}\right)^2 = 1 \Rightarrow x^2 + (y+5)^2 = 4$

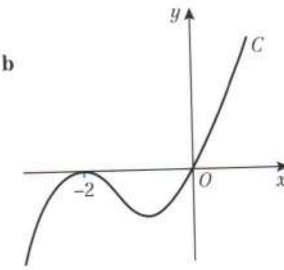
Since $0 \leqslant t \leqslant \pi$, the curve C forms half of a circle.

b



c 2π

11 **a** $y = x^3 + 4x^2 + 4x$ **b**



12 $4 - t^2 = 4(t - 3) + 20 \Rightarrow 0 = t^2 + 4t + 4$
Discriminant $= 4^2 - 4 \times 1 \times 4 = 16 - 16 = 0$
So, the line and the curve only intersect once.
Therefore, $y = 4x + 20$ is a tangent to the curve.

13 a $(5, e^5 - 1)$ b $k > -1$

14 a $A(0, -\frac{1}{2}), B(1, 0)$ b $x - 2y - 1 = 0$

15 $x + y\ln 2 - \ln 2 = 0$

16 a $t = \frac{x}{80}$, sub into $y = 3000 - 30t$

$y = 3000 - 30\left(\frac{x}{80}\right) \Rightarrow y = 3000 - \frac{3}{8}x$

This is in the form $y = mx + c$, therefore the plane's descent is a straight line.

b $k = 99$ c 8458.56 m

17 a 1022 m

b $1000 = 50\sqrt{2}\,t \Rightarrow t = 10\sqrt{2}$
Sub into $y = 1.5 - 4.9t^2 + 50\sqrt{2}\,t$
$y = 1.5 - 4.9(10\sqrt{2})^2 + 50\sqrt{2}(10\sqrt{2})$
$y = 21.5$ m
$21.5 > 10$, therefore, the arrow will be too high

c 11.8 m (3 s.f.)

18 a 976 m, 2 hours b 600 m

19 a 10 m b 80 m

20 a 10 m b 1 second c 0.9 m

Challenge

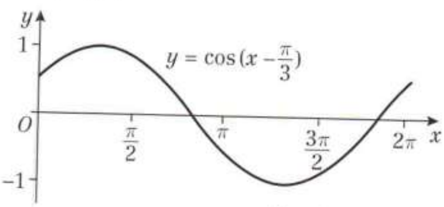
a $k = \frac{3}{2}$ b $\left(4, \frac{5}{2}\right)$

Review exercise 2

1 x-axis: $\left(-\frac{7\pi}{4}, 0\right), \left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right)$

y-axis: $\left(0, \frac{1}{\sqrt{2}}\right)$

2 a



b y-axis at (0, 0.5). x-axis at $\left(\frac{5\pi}{6}, 0\right)$ and $\left(\frac{11\pi}{6}, 0\right)$

c $x = 2.89, x = 5.49$

3 a 1.287 radians b 6.44 cm

4 $12 + 2\pi$ cm

5 a $\frac{1}{2}(r + 10)^2\theta - \frac{1}{2}r^2\theta = 40 \Rightarrow 20r\theta + 100\theta = 80$

$\Rightarrow r\theta + 5\theta = 4 \Rightarrow r = \frac{4}{\theta} - 5$

b 28 cm

6 a 6 cm b 6.7 cm²

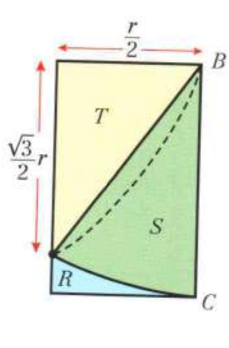
7 a 119.7 cm² b 40.3 cm

8 Split each half of the rectangle as shown.

Area $S = \frac{\pi}{12}r^2$

Area $T = \frac{\sqrt{3}}{8}r^2$

$\Rightarrow$ Area $R = \left(\frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right)r^2$



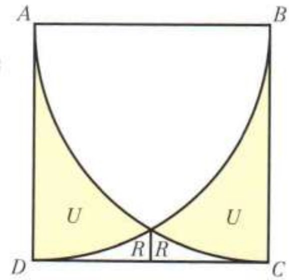
$U = \left(r^2 - \frac{\pi}{4}r^2\right) - 2R$

$= \left(1 - \frac{\pi}{4} - 1 + \frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)r^2$

$= r^2\left(\frac{\sqrt{3}}{4} - \frac{\pi}{12}\right)$

$\therefore$ Shaded area

$= \frac{r^2}{6}(3\sqrt{3} - \pi)$



9 a $3\sin^2 x + 7\cos x + 3 = 3(1 - \cos^2 x) + 7\cos x + 3$
$= -3\cos^2 x + 7\cos x + 6 = 0$
Therefore $0 = 3\cos^2 x - 7\cos x - 6$

b $x = 2.30, 3.98$

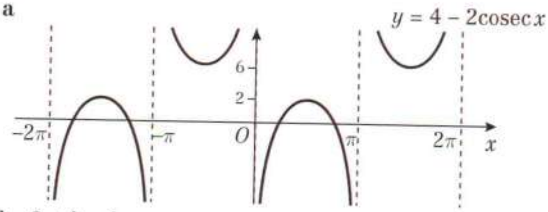
10 a For small values of θ:
$\sin 4\theta \approx 4\theta$, $\cos 4\theta \approx 1 - \frac{1}{2}(4\theta)^2$ $\tan 3\theta \approx 3\theta$

$\sin 4\theta - \cos 4\theta + \tan 3\theta \approx 4\theta - \left(1 - \frac{(4\theta)^2}{2}\right) + 3\theta$

$= 8\theta^2 + 7\theta - 1$

b -1

11 a



b $2 < k < 6$

12 a $\frac{\pi}{3}$ b $k = 2$ c $-\frac{11\pi}{12}, -\frac{5\pi}{12}$

13 a $\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} = \frac{\cos^2 x + (1 - \sin x)^2}{\cos x(1 - \sin x)}$

$= \frac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{\cos x(1 - \sin x)} = \frac{2 - 2\sin x}{\cos x(1 - \sin x)}$

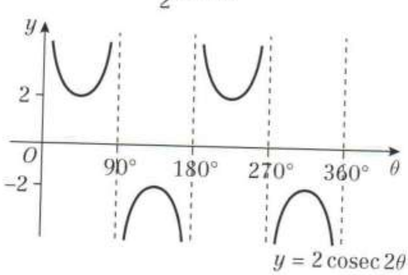
$= \frac{2}{\cos x} = 2\sec x$

b $x = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$

14 a $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{1}{\frac{1}{2}\sin 2\theta} = \frac{2}{\sin 2\theta} = 2\,\mathrm{cosec}\,2\theta$

b



c 20.9°, 69.1°, 200.9°, 249.1°

15 a Note the angle $BDC = \theta$

$\cos\theta = \frac{BC}{10} \Rightarrow BC = 10\cos\theta$

$\sin\theta = \frac{BC}{BD} \Rightarrow BD = 10\cot\theta$

b $10\cot\theta = \frac{10}{\sqrt{3}} \Rightarrow \cot\theta = \frac{1}{\sqrt{3}}, \theta = \frac{\pi}{3}$

$DC = 10\cos\theta\cot\theta = 10\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{5}{\sqrt{3}}$

16 a $\sin^2\theta + \cos^2\theta = 1$

$\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} \Rightarrow \tan^2\theta + 1 = \sec^2\theta$

b $0.0°, 131.8°, 228.2°$

17 a $ab = 2, a = \dfrac{2}{b}$

b $\dfrac{4 - b^2}{a^2 - 1} = \dfrac{4 - b^2}{\dfrac{4}{b^2} - 1} = \dfrac{4 - b^2}{\dfrac{4 - b^2}{b^2}} = b^2$

18 a $\dfrac{\pi}{2} - y = \arccos x$ **b** $\dfrac{\pi}{2}$

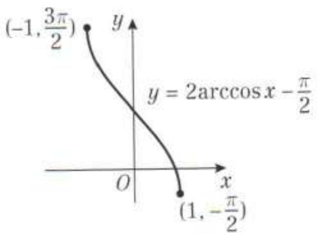
19 a $\arccos\dfrac{1}{x} = p \Rightarrow \cos p = \dfrac{1}{x}$

Use Pythagorean Theorem to show that opposite side of right triangle is $\sqrt{x^2 - 1}$

$\sin p = \dfrac{\sqrt{x^2 - 1}}{x} \Rightarrow p = \arcsin\dfrac{\sqrt{x^2 - 1}}{x}$

b Possible answer: cannot take the square root of a negative number and for $0 \leqslant x < 1$, $x^2 - 1$ is negative.

20 a



b $\left(\dfrac{1}{\sqrt{2}}, 0\right)$

21 $\tan\left(x + \dfrac{\pi}{6}\right) = \dfrac{1}{6} \Rightarrow \dfrac{\tan x + \dfrac{\sqrt{3}}{3}}{1 - \dfrac{\sqrt{3}}{3}\tan x} = \dfrac{1}{6}$

$6\tan x + 2\sqrt{3} = 1 - \dfrac{\sqrt{3}}{3}\tan x$

$\left(\dfrac{18 + \sqrt{3}}{3}\right)\tan x = 1 - 2\sqrt{3}$

$\tan x = \dfrac{3 - 6\sqrt{3}}{18 + \sqrt{3}} \times \dfrac{18 - \sqrt{3}}{18 - \sqrt{3}} = \dfrac{72 - 111\sqrt{3}}{321}$

22 a $\sin(x + 30°) = 2\sin(x - 60°)$

$\sin x\cos 30° + \cos x\sin 30°$
$= 2(\sin x\cos 60° - \cos x\sin 60°)$

$\dfrac{\sqrt{3}}{2}\sin x + \dfrac{1}{2}\cos x = 2\left(\dfrac{1}{2}\sin x - \dfrac{\sqrt{3}}{2}\cos x\right)$

$\sqrt{3}\sin x + \cos x = 2\sin x - 2\sqrt{3}\cos x$

$(-2 + \sqrt{3})\sin x = (-1 - 2\sqrt{3})\cos x$

$\dfrac{\sin x}{\cos x} = \dfrac{-1 - 2\sqrt{3}}{-2 + \sqrt{3}} = \dfrac{-1 - 2\sqrt{3}}{-2 + \sqrt{3}} \times \dfrac{-2 - \sqrt{3}}{-2 - \sqrt{3}}$

$= \dfrac{2 + 4\sqrt{3} + \sqrt{3} + 6}{4 + 2\sqrt{3} - 2\sqrt{3} - 3} = 8 + 5\sqrt{3}$

b $8 - 5\sqrt{3}$

23 a $\sin 165° = \sin(120° + 45°)$
$= \sin 120°\cos 45° + \cos 120°\sin 45°$

$= \dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}} + \dfrac{-1}{2} \times \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$

b $\operatorname{cosec} 165° = \dfrac{1}{\sin 165°}$

$= \dfrac{4}{(\sqrt{6} - \sqrt{2})} \times \dfrac{(\sqrt{6} + \sqrt{2})}{(\sqrt{6} + \sqrt{2})} = \dfrac{4(\sqrt{6} + \sqrt{2})}{6 - 2} = \sqrt{6} + \sqrt{2}$

24 a $\cos A = \dfrac{3}{4} \Rightarrow \sin A = \dfrac{-\sqrt{7}}{4}$

$\sin 2A = 2\sin A\cos A = 2\left(\dfrac{-\sqrt{7}}{4}\right)\left(\dfrac{3}{4}\right) = \dfrac{-3\sqrt{7}}{8}$

b $\cos 2A = 2\cos^2 A - 1 = \dfrac{1}{8}$

$\tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{\left(\dfrac{-3\sqrt{7}}{8}\right)}{\left(\dfrac{1}{8}\right)} = -3\sqrt{7}$

25 a $-180°, 0°, 30°, 150°, 180°$

b $-148.3°, -58.3°, 31.7°, 121.7°$ (1 d.p.)

26 a $3\sin x + 2\cos x = \sqrt{13}\sin(x + 0.588\ldots)$

b 169

c $\Rightarrow x = 2.273, 5.976$ (3 d.p.)

27 a $\cot\theta - \tan\theta = \dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = \dfrac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$

$= \dfrac{\cos 2\theta}{\dfrac{1}{2}\sin 2\theta} = \dfrac{2\cos 2\theta}{\sin 2\theta} = 2\cot 2\theta$

b $\theta = -2.95, -1.38, 0.190, 1.76$ (3 s.f.)

28 a $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$
$= (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$
$= \cos^3\theta - 3\sin^2\theta\cos\theta$
$= \cos^3\theta - 3(1 - \cos^2\theta)\cos\theta$
$= 4\cos^3\theta - 3\cos\theta$

b $\sec 3\theta = \dfrac{-27}{19\sqrt{2}} = \dfrac{-27\sqrt{2}}{38}$

29 $\sin^4\theta = (\sin^2\theta)(\sin^2\theta)$

$\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$

$\sin^4\theta = \left(\dfrac{1 - \cos 2\theta}{2}\right)\left(\dfrac{1 - \cos 2\theta}{2}\right)$

$\sin^4\theta = \dfrac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta)$

$\sin^4\theta = \dfrac{1}{4}\left(1 - 2\cos 2\theta + \dfrac{1 + \cos 4\theta}{2}\right)$

$\sin^4\theta = \dfrac{3}{8} - \dfrac{1}{2}\cos 2\theta + \dfrac{1}{8}\cos 4\theta$

30 a $\sqrt{40}\sin(\theta + 0.32)$

b i $\sqrt{40}$ **ii** $\theta = 1.25$

c Minimum of 2.68°C, occurs 16.77 hours after 9 am ≈ 1:46 am

d $t = 2.25$, $t = 7.29$. So 11:15 am and 4:17 pm

31 a $x \neq 1, y \geqslant -1.25$

b $t = \dfrac{-4}{x - 1} = \dfrac{4}{1 - x}$

$$y = \left(\frac{4}{1-x}\right)^2 - 3\left(\frac{4}{1-x}\right) + 1$$

$$y = \frac{16}{(1-x)^2} - \frac{12(1-x)}{(1-x)^2} + \frac{(1-x)^2}{(1-x)^2}$$

$$y = \frac{16 - 12 + 12x + 1 - 2x + x^2}{(1-x)^2}$$

$$y = \frac{x^2 + 10x + 5}{(1-x)^2} \Rightarrow a = 1, b = 10, c = 5$$

32 a $t = e^x - 2$

$$y = \frac{3t}{t+3} = \frac{3e^x - 6}{e^x + 1}$$

$t > 4 \Rightarrow e^x - 2 > 4 \Rightarrow e^x > 6 \Rightarrow x > \ln 6$

b $t = 4 \Rightarrow y = \frac{12}{7}, x \to \infty, y \to 3, \frac{12}{7} < y < 3$

33 $x = \frac{1}{1+t} \Rightarrow t = \frac{1-x}{x}$

$$y = \frac{1}{1 - \frac{1-x}{x}} = \frac{x}{x - (1-x)} = \frac{x}{2x-1}$$

34 a $y = \cos 3t = \cos(2t + t) = \cos 2t \cos t - \sin 2t \sin t$

$= (2\cos^2 t - 1)\cos t - 2\sin^2 t \cos t$

$= 2\cos^3 t - \cos t - 2(1 - \cos^2 t)\cos t$

$= 4\cos^3 t - 3\cos t$

$x = 2\cos t \Rightarrow \cos t = \frac{x}{2}$

$$y = 4\left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right) = \frac{x}{2}(x^2 - 3)$$

b $0 \leqslant x \leqslant 2, -1 \leqslant y \leqslant 1$

35 a $y = \sin\left(t + \frac{\pi}{6}\right) = \sin t \cos\frac{\pi}{6} + \cos t \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\cos t$

$= \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\sqrt{1 - \sin^2 t}$

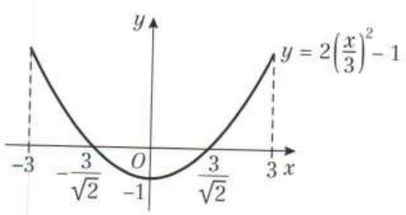
$= \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1 - x^2}$

$-1 \leqslant \sin t \leqslant 1 \Rightarrow -1 \leqslant x \leqslant 1$

b $A = (-0.5, 0), B = (0, 0.5)$

36 a $y = 2\left(\frac{x}{3}\right)^2 - 1, -3 \leqslant x \leqslant 3$

b



37 $y = 3x + c \Rightarrow 8t(2t - 1) = 3(4t) + c \Rightarrow 16t^2 - 20t - c = 0$

$(-20)^2 - 4(16)(-c) < 0$ so $64c < -400 \Rightarrow c < -\frac{25}{4}$

38 a $\left(-\frac{3\sqrt{3}}{2}, 0\right)$ and $\left(\frac{3\sqrt{3}}{2}, 0\right)$

b $3\sin 2t = 1.5 \Rightarrow \sin 2t = \frac{1}{2}$

$$2t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots \Rightarrow t = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \ldots$$

$$t = \frac{13\pi}{12}, \frac{17\pi}{12}$$

39 a $-4.9t^2 + 25t + 50 = 0$

$$t = \frac{-25 \pm \sqrt{25^2 - 4(-4.9)(50)}}{2(-4.9)}$$

$t \neq -1.54, t = 6.64s \Rightarrow k = 6.64$

b $t = \frac{x}{25\sqrt{3}}$

$$y = 25\left(\frac{x}{25\sqrt{3}}\right) - 4.9\left(\frac{x}{25\sqrt{3}}\right)^2 + 50$$

$$= \frac{x}{\sqrt{3}} - \frac{49}{18\,750}x^2 + 50$$

$t = 6.64$ $x = 25\sqrt{3}t = 25\sqrt{3}(6.64) = 287.5$

Domain of f(x) is $0 \leqslant x \leqslant 287.5$

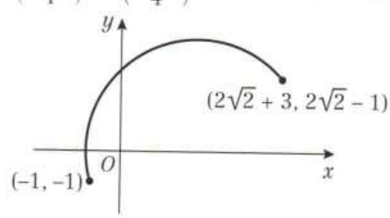
Challenge

1 $\frac{\pi - 2}{2 + 3\pi} : 1$

2 **a** $\sin x$ **b** $\cos x$ **c** $\operatorname{cosec} x$
d $\cot x$ **e** $\tan x$ **f** $\sec x$

3 **a** $\sin^2 t + \cos^2 t = 1$

$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y+1}{4}\right)^2 = 1 \Rightarrow (x-3)^2 + (y+1)^2 = 16$$



b $\frac{3}{8}(2\pi \times 4) = 3\pi$

CHAPTER 9

Prior knowledge 9

1 **a** $6x - 5$ **b** $-\frac{2}{x^2} - \frac{1}{2\sqrt{x}}$ **c** $8x - 16x^3$

2 $y = -6x + 17$ **3** $(0, 2), \left(0, \frac{179}{27}\right), (11.1, 0)$

4 0.588, 3.73

Exercise 9A

1 a $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$

$$= \lim_{h \to 0}\left(\frac{\cos(x+h) - \cos x}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\cos x(\cos h - 1) - \sin x \sin h}{h}\right)$$

$$= \lim_{h \to 0}\left(\left(\frac{\cos h - 1}{h}\right)\cos x - \left(\frac{\sin h}{h}\right)\sin x\right)$$

b As $h \to 0$, $\cos h \to 1$, so $\left(\frac{\cos h - 1}{h}\right) \to 0$

and $\left(\frac{\sin h}{h}\right) \to 1$

So $f'(x) = \lim_{h \to 0}\left(\left(\frac{\cos h - 1}{h}\right)\cos x - \left(\frac{\sin h}{h}\right)\sin x\right)$

$= 0\cos x - \sin x = -\sin x$

2 **a** $-2\sin x$ **b** $\cos\left(\frac{1}{2}x\right)$
c $8\cos 8x$ **d** $4\cos\left(\frac{2}{3}x\right)$

3 **a** $-2\sin x$ **b** $-5\sin\left(\frac{5}{6}x\right)$
c $-2\sin\left(\frac{1}{2}x\right)$ **d** $-6\sin 2x$

4 a $2\cos 2x - 3\sin 3x$ b $-8\sin 4x + 4\sin x - 14\sin 7x$
c $2x - 12\sin 3x$ d $-\frac{1}{x^2} + 10\cos 5x$

5 (0.41, −0.532), (1.68, 2.63), (2.50, 1.56)

6 8 7 (0.554, 2.24), (2.12, −2.24)

8 $y = -5x + 5\pi - 1$

9 $\frac{dy}{dx} = 4x - \cos x$

At $x = \pi$, $y = 2\pi^2$, $\frac{dy}{dx} = 4\pi - \cos\pi = 4\pi + 1$

Gradient of normal $= -\frac{1}{4\pi + 1}$

Equation of normal:

$y - 2\pi^2 = -\frac{1}{4\pi+1}(x - \pi)$

$(4\pi + 1)y - 2\pi^2(4\pi + 1) = -x + \pi$

$x + (4\pi + 1)y - 8\pi^3 - 2\pi^2 - \pi = 0$

$x + (4\pi + 1)y - \pi(8\pi^2 + 2\pi + 1) = 0$

10 Let $f(x) = \sin x$

$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$

$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$

$= \lim_{h\to 0}\left[\left(\frac{\cos h - 1}{h}\right)\sin x + \left(\frac{\sin h}{h}\right)\cos x\right]$

Since $\frac{\cos h - 1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$ the expression inside the limit $\to (0 \times \sin x + 1 \times \cos x)$

So $\lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \cos x$

Hence the derivative of $\sin x$ is $\cos x$.

Challenge

Let $f(x) = \sin(kx)$

$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{\sin(kx + kh) - \sin kx}{h}\right)$

$= \lim_{h\to 0}\left(\frac{\sin kx\cos kh + \cos kx\sin kh - \sin kx}{h}\right)$

$= \lim_{h\to 0}\left(\left(\frac{\cos kh - 1}{h}\right)\sin kx + \left(\frac{\sin kh}{h}\right)\cos kx\right)$

As $h \to 0$, $\left(\frac{\sin kh}{h}\right) \to k$ and $\left(\frac{\cos kh - 1}{h}\right) \to 0$ as given,

So $f'(x) = 0\sin kx + k\cos kx = k\cos kx$

Exercise 9B

1 a $28e^{7x}$ b $3^x\ln 3$ c $\left(\frac{1}{2}\right)^x\ln\frac{1}{2}$ d $\frac{1}{x}$
e $4\left(\frac{1}{3}\right)^x\ln\frac{1}{3}$ f $\frac{3}{x}$ g $3e^{3x} + 3e^{-3x}$ h $-e^{-x} + e^x$

2 a $3^{4x}4\ln 3$ b $\left(\frac{3}{2}\right)^{2x}2\ln\frac{3}{2}$
c $2^{4x}8\ln 2$ d $2^{3x}3\ln 2 - 2^{-x}\ln 2$

3 323.95 (2 d.p.) 4 $4y = 15\ln 2(x-2) + 17$

5 $\frac{dy}{dx} = 2e^{2x} - \frac{1}{x}$ At $x = 1$, $y = e^2$, $\frac{dy}{dx} = 2e^2 - 1$

Equation of tangent: $y - e^2 = (2e^2 - 1)(x - 1)$

$\Rightarrow y = (2e^2 - 1)x - 2e^2 + 1 + e^2 \Rightarrow y = (2e^2 - 1)x - e^2 + 1$

6 −9.07 millicuries/day

7 a $P_0 = 37\,000$, $k = 1.01$ (2 d.p.) b 1178
c The rate of change of the population in the year 2000

8 The student has treated "ln kx" as if it is "e^{kx}" – they have applied the incorrect standard differential.
Correct differential is: $\frac{1}{x}$

9 Let $y = a^{kx} \Rightarrow y = e^{\ln a^{kx}} = e^{kx\ln a}$

$\frac{dy}{dx} = k\ln a\, e^{kx\ln a} = k\ln a\, e^{\ln a^{kx}} = a^{kx}k\ln a$

10 a $2e^{2x} - \frac{2}{x}$
b $2e^{2a} - \frac{2}{a} = 2 \Rightarrow 2ae^{2a} - 2 = 2a \Rightarrow a(e^{2a} - 1) = 1$

11 a $5\sin(3\times 0) + 2\cos(3\times 0) = 0 + 2 = 2 = y$
When $x = 0$, $y = 2$, therefore (0, 2) lies on C.
b $y = -\frac{1}{15}x + 2$

12 $y = -\frac{1}{648\ln 3}x + \frac{1}{648\ln 3} + 162$

Challenge

$y = 3x - 2\ln 2 + 2$

Exercise 9C

1 a $8(1+2x)^3$ b $20x(3-2x^2)^{-6}$
c $2(3 + 4x)^{-\frac{1}{2}}$ d $7(6 + 2x)(6x + x^2)^6$
e $-\frac{2}{(3 + 2x)^2}$ f $-\frac{1}{2\sqrt{7 - x}}$
g $128(2 + 8x)^3$ h $18(8 - x)^{-7}$

2 a $-\sin x\, e^{\cos x}$ b $-2\sin(2x - 1)$ c $\frac{1}{2x\sqrt{\ln x}}$
d $5(\cos x - \sin x)(\sin x + \cos x)^4$
e $(6x - 2)\cos(3x^2 - 2x + 1)$
f $\cot x$ g $-8\sin 4x\, e^{\cos 4x}$ h $-2e^{2x}\sin(e^{2x} + 3)$

3 −1 4 $y = -54x + 81$ 5 $12e^{-3}$

6 a $\frac{1}{2y + 1}$ b $\frac{1}{e^y + 4}$ c $\frac{1}{2}\sec 2y$ d $\frac{4y}{1 + 3y^3}$

7 $\frac{1}{10}$ 8 $\frac{16}{3}$

9 a $e^y = \frac{dx}{dy}$
b $y = \ln x$, $e^y = x$
Differentiate with respect to y using part **a**

$e^y = \frac{dx}{dy} \Rightarrow \frac{1}{e^y} = \frac{dy}{dx}$

Since $x = e^y$, $\frac{dy}{dx} = \frac{1}{x}$

10 a $4\cos 2\left(\frac{\pi}{6}\right) = 4\left(\frac{1}{2}\right) = 2$

When $y = \frac{\pi}{6}$, $x = 2$, therefore $\left(2, \frac{\pi}{6}\right)$ lies on C.

b $\frac{dx}{dy} = -8\sin 2y$

At $Q\left(2, \frac{\pi}{6}\right)$: $\frac{dx}{dy} = -8\sin 2\left(\frac{\pi}{6}\right) = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$

So, $\frac{dy}{dx} = -\frac{1}{4\sqrt{3}}$

c $4\sqrt{3}\,x - y - 8\sqrt{3} + \frac{\pi}{6} = 0$

11 a $6\sin 3x\cos 3x$ b $2(x + 1)e^{(x+1)^2}$ c $-2\tan x$
d $\frac{2\sin 2x}{(3 + \cos 2x)^2}$ e $-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

12 $3125x - 100y - 9371 = 0$ 13 $9\ln 3$

Challenge

a $\frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$ b $9e^{\sin^3(3x+4)}\cos(3x + 4)\sin^2(3x + 4)$

Exercise 9D

1 a $(3x+1)^4(18x+1)$ b $2(3x^2+1)^2(21x^2+1)$
c $16x^2(x+3)^3(7x+9)$ d $3x(5x-2)(5x-1)^{-2}$

2 a $-4(x-3)(2x-1)^4\,e^{-2x}$
b $2\cos 2x\cos 3x - 3\sin 2x\sin 3x$
c $e^x(\sin x+\cos x)$ d $5\cos 5x\ln(\cos x) - \tan x\sin 5x$

3 a 52 b 13 c $\frac{3}{25}$

4 $(2, 0), \left(-\frac{1}{3}, \frac{343}{27}\right)$ 5 $\frac{5\pi^4}{256}$

6 $\sqrt{2\pi}\,(\pi-4)x + 8y - \pi\sqrt{2}\left(\frac{\pi-2}{2}\right) = 0$

7 $6x(5x-3)^3 + 3x^2[3(5x-3)^2(5)] = 6x(5x-3)^3 + 45x^2(5x-3)^2$
$= 3x(5x-3)^2(2(5x-3)+15x) = 3x(5x-3)^2(10x-6+15x)$
$= 3x(5x-3)^2(25x-6) \Rightarrow n=2, A=3, B=25, C=-6$

8 a $(x+3)(3x+11)e^{3x}$ b $85e^6$

9 a $(3\sin x + 2\cos x)\ln(3x) + \frac{2\sin x - 3\cos x}{x}$
b $x^3(7x+4)e^{7x-3}$

10 21.25

Challenge

a $-e^x\sin x\,(\sin^2 x - \cos x\sin x - 2\cos^2 x)$
b $-(4x-3)^5(4x-1)^8(256x^2-148x+3)$

Exercise 9E

1 a $\frac{5}{(x+1)^2}$ b $-\frac{4}{(3x-2)^2}$ c $-\frac{5}{(2x+1)^2}$
d $-\frac{6x}{(2x-1)^3}$ e $\frac{15x+18}{(5x+3)^{\frac{3}{2}}}$

2 a $\frac{e^{4x}(\sin x+4\cos x)}{\cos^2 x}$ b $\frac{1}{x(x+1)} - \frac{\ln x}{(x+1)^2}$
c $\frac{e^{-2x}(2x(e^{4x}-1)\ln x - e^{4x}-1)}{x(\ln x)^2}$
d $\frac{(e^x+3)^2((e^x+3)\sin x + 3e^x\cos x)}{\cos^2 x}$
e $\frac{2\sin x\cos x}{\ln x} - \frac{\sin^2 x}{x(\ln x)^2}$

3 $\frac{1}{16}$ 4 $\frac{2}{25}$ 5 $(0.5, 2e^4)$

6 $y = \frac{1}{3}e$ 7 $\frac{6\sqrt{3} - 2\pi\ln\left(\frac{\pi}{9}\right)}{\pi}$

8 a $\left(\frac{1}{3}, 0\right)$ b $y = -\frac{1}{9}x + \frac{1}{27}$

9 $\frac{x^3(3x\sin 3x + 4\cos 3x)}{\cos^2 3x}$

10 a $\frac{(x-2)^2(2e^{2x}) - e^{2x}[2(x-2)]}{(x-2)^4} = \frac{2(x-2)^2e^{2x} - 2e^{2x}(x-2)}{(x-2)^4}$
$= \frac{2(x-2)e^{2x} - 2e^{2x}}{(x-2)^3} = \frac{2e^{2x}(x-2-1)}{(x-2)^3} = \frac{2e^{2x}(x-3)}{(x-2)^3}$
$A=2, B=1, C=3$
b $y = 4e^2x - 3e^2$

11 a $\frac{2x}{x+5} + \frac{6x}{(x+5)(x+2)} = \frac{2x(x+2)}{(x+5)(x+2)} + \frac{6x}{(x+5)(x+2)}$
$= \frac{2x(x+2+3)}{(x+5)(x+2)} = \frac{2x(x+5)}{(x+5)(x+2)} = \frac{2x}{(x+2)}$
b $\frac{4}{25}$

12 a $f'(x) = -2e^{x-2}(2\sin 2x - \cos 2x) = 0$
$2\sin 2x - \cos 2x = 0 \Rightarrow \tan 2x = \frac{1}{2}$
b $-1.47 < f(x) < 6.26$

Exercise 9F

1 a $3\sec^2 3x$ b $12\tan^2 x\sec^2 x$ c $\sec^2(x-1)$
d $\frac{1}{2}x^2\sec^2\frac{1}{2}x + 2x\tan\frac{1}{2}x + \sec^2\left(x-\frac{1}{2}\right)$

2 a $-4\operatorname{cosec}^2 4x$ b $5\sec 5x\tan 5x$
c $-4\operatorname{cosec} 4x\cot 4x$ d $6\sec^2 3x\tan 3x$
e $\cot 3x - 3x\operatorname{cosec}^2 3x$ f $\frac{\sec^2 x(2x\tan x - 1)}{x^2}$
g $-6\operatorname{cosec}^3 2x\cot 2x$
h $-4\cot(2x-1)\operatorname{cosec}^2(2x-1)$

3 a $\frac{1}{2}(\sec x)^{\frac{1}{2}}\tan x$ b $-\frac{1}{2}(\cot x)^{-\frac{1}{2}}\operatorname{cosec}^2 x$
c $-2\operatorname{cosec}^2 x\cot x$ d $2\tan x\sec^2 x$
e $3\sec^3 x\tan x$ f $-3\cot^2 x\operatorname{cosec}^2 x$

4 a $2x\sec 3x + 3x^2\sec 3x\tan 3x$
b $\frac{2x\sec^2 2x - \tan 2x}{x^2}$ c $\frac{2x\tan x - x^2\sec^2 x}{\tan^2 x}$
d $e^x\sec 3x(1+3\tan 3x)$ e $\frac{\tan x - x\sec^2 x\ln x}{x\tan^2 x}$
f $e^{\tan x}\sec x(\tan x + \sec^2 x)$

5 a $\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}$ b 2
c $24x - 9y + 12\sqrt{3} - 8\pi = 0$

6 $y = \frac{1}{\cos x}, \frac{dy}{dx} = \frac{\cos x \times 0 - 1 \times -\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$
$= \sec x\tan x$

7 $y = \frac{1}{\tan x}$
$\frac{dy}{dx} = \frac{\tan x \times 0 - 1 \times \sec^2 x}{\tan^2 x} = -\frac{\sec^2 x}{\tan^2 x} = -\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} = -\operatorname{cosec}^2 x$

8 a Let $y = \arccos x \Rightarrow \cos y = x \Rightarrow \frac{dx}{dy} = -\sin y$
$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$
b Let $y = \arctan x$
Then, $\tan y = x$
$\frac{dx}{dy} = \sec^2 y$
$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

9 **a** $\frac{-1}{5\cot 5y \operatorname{cosec} 5y}$ **b** $-\frac{1}{5x\sqrt{x^2-1}}$

Exercise 9G

1 **a** $\frac{2t-3}{2}$ **b** $\frac{6t^2}{6t}=t$ **c** $\frac{4}{1+6t}$ **d** $\frac{15t^3}{2}$
e $-3t^3$ **f** $t(1-t)$ **g** $\frac{2t}{t^2-1}$ **h** $\frac{2}{(t^2+2t)e^t}$
i $-\frac{3}{4}\tan 3t$ **j** $4\tan t$ **k** $\operatorname{cosec} t$ **l** $\frac{2\sin 2t}{2-2\cos 2t}$
m $\frac{1}{te^t}$ **n** $2t^2$ **o** $\frac{1}{e^t}$

2 **a** $y=-\frac{1}{2}x+\frac{3}{2}-\pi$ **b** $2y+5x=57$

3 **a** $x=1$ **b** $y+\sqrt{3}x=\sqrt{3}$

4 (0, 0) and (−2, −4)

5 **a** $y=\frac{1}{4}x$
b $\frac{dy}{dx}=\frac{1}{2}e^{-t}=0 \Rightarrow e^{-t}=0$
No solution, therefore no stationary points.

6 $y=x+7$

7 **a** $-\frac{1}{2}\sec t \operatorname{cosec}^3 t$ **b** $8x+\sqrt{3}y-10=0$

8 **a** $\frac{\pi}{3}$
b $\frac{dy}{dt}=-4\cot 2t \operatorname{cosec} 2t$, $\frac{dx}{dt}=4\cos t$
$\frac{dy}{dx}=\frac{-4\cot 2t \operatorname{cosec} 2t}{4\cos t}=\frac{-\cot 2t \operatorname{cosec} 2t}{\cos t}$
At $t=\frac{\pi}{3}$, $\frac{dy}{dx}=\frac{4}{3}$
Gradient of normal: $-\frac{3}{4}$
Equation of normal:
$y-\frac{4\sqrt{3}}{3}=-\frac{3}{4}(x-2\sqrt{3}) \Rightarrow 9x+12y-34\sqrt{3}=0$

9 **a** (30, 101) **b** $y=2x+41$
c $t^2-10t+5=2(t^2+t)+41$
$t^2-10t+5=2t^2+2t+41$
$0=t^2+12t+36$
Discriminant $=12^2-4\times 1\times 36=144-144=0$
Therefore the curve and the line only intersects once.
Therefore it does not intersect the curve again.

10 **a** $-2\sqrt{2}\sin t$ **b** $x-\sqrt{6}y-2\sqrt{3}=0$
c $2\sin t-\sqrt{12}\cos 2t-2\sqrt{3}=0$
$\sin t-\sqrt{3}\cos 2t-\sqrt{3}=0$
$2\sqrt{3}\sin^2 t+\sin t-2\sqrt{3}=0$
$(2\sin t-\sqrt{3})(\sqrt{3}\sin t+2)=0$
$\sin t=\frac{\sqrt{3}}{2}\left(\sin t\neq-\frac{2}{\sqrt{3}}\right) \Rightarrow t=\frac{\pi}{3}$ or $\frac{2\pi}{3}$
B is when $t=\frac{2\pi}{3}$: $\left(2\sin\frac{2\pi}{3}, \sqrt{2}\cos\frac{4\pi}{3}\right)=\left(\sqrt{3}, -\frac{1}{\sqrt{2}}\right)$
Same point as A, so l only intersects C once.

11 **a** $-\frac{\cos 2t}{\sin t}$ **b** $y=-x+\frac{3\sqrt{3}}{4}$
c $y=-x$ and $y=-x-\frac{3\sqrt{3}}{4}$

Exercise 9H

1 Letting $u=y^n$, $\frac{du}{dy}=ny^{n-1}$
$\frac{d}{dx}(y^n)=\frac{du}{dx}=\frac{du}{dy}\times\frac{dy}{dx}=ny^{n-1}\frac{dy}{dx}$

2 $\frac{d}{dx}(xy)=x\frac{d}{dx}(y)+\frac{d}{dx}(x)y=x\frac{dy}{dx}+1\times y=x\frac{dy}{dx}+y$

3 **a** $-\frac{2x}{3y^2}$ **b** $-\frac{x}{5y}$ **c** $\frac{-3-x}{5y-4}$
d $\frac{4-6xy}{3x^2+3y^2}$ **e** $\frac{3x^2-2y}{6y-2+2x}$ **f** $\frac{3x^2-y}{2+x}$
g $\frac{4(x-y)^3-1}{1+4(x-y)^3}$ **h** $\frac{e^x y-e^y}{xe^y-e^x}$ **i** $\frac{-2\sqrt{xy}-y}{4y\sqrt{xy}+x}$

4 $y=-\frac{7}{9}x+\frac{23}{9}$ **5** $y=2x-2$

6 (3, 1) and (3, 3) **7** $3x+2y+1=0$

8 $2-3\ln 3$ **9** $\frac{1}{4}(4+3\ln 3)$

10 **a** $\frac{\cos x}{\sin y}$ **b** $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{\pi}{2}, -\frac{2\pi}{3}\right)$

11 **a** $\frac{3+3ye^{-3x}}{e^{-3x}-2y}$
b At O, $\frac{dy}{dx}=\frac{3+0}{e^0-0}=3$
So the tangent is $y-0=3(x-0)$, or $y=3x$.

Challenge

a $6+2y\frac{dy}{dx}+2y+2x\frac{dy}{dx}=2x \Rightarrow \frac{dy}{dx}=\frac{x-y-3}{y+x}$
So $\frac{dy}{dx}=0 \Leftrightarrow x-y=3$
Substitute: $6x+(x-3)^2+2x(x-3)=x^2$
So $2x^2-6x+9=0$
Discriminant $=-36$, so no real solutions to quadratic.
Therefore no points on C s.t. $\frac{dy}{dx}=0$.

b (0, 0) and (3, −3)

Exercise 9I

1 **a** **i** $[1, \infty)$ **ii** $(-\infty, 1]$
b **i** $(-\infty, 0]\cup\left[\frac{3}{2}\ \infty\right)$ **ii** $\left[0, \frac{3}{2}\right]$
c **i** $[\pi, 2\pi)$ **ii** $(0, \pi]$
d **i** nowhere **ii** $(-\infty, \infty)$
e **i** $[\ln 2, \infty)$ **ii** $(-\infty, \ln 2]$
f **i** nowhere **ii** $(0, \infty)$

2 **a** Let $y = \text{f}(x)$. Then $x = \sin y$.

$\frac{dx}{dy} = \cos y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$

so $\text{f}'(x) = \frac{1}{\sqrt{1 - x^2}}$

b $\text{f}'(x) = \frac{1}{\sqrt{1 - x^2}}$, $\text{f}''(x) = \frac{x}{(1 - x^2)^{\frac{3}{2}}}$

$\text{f}''(x) \leqslant 0 \Rightarrow x \leqslant 0$, so f(x) concave for $x \in (-1, 0)$

c $\text{f}''(x) \geqslant 0 \Rightarrow x \geqslant 0$, so f(x) convex for $x \in (0, 1)$

d (0, 0)

3 **a** $\left(\frac{\pi}{6}, -\frac{1}{4}\right), \left(\frac{5\pi}{6}, -\frac{1}{4}\right)$

b (1, −1) **c** (0, 0)

4 $\text{f}'(x) = 2x + 4x\ln x = 2x(1 + 2\ln x)$, $\text{f}''(x) = 6 + 4\ln x$

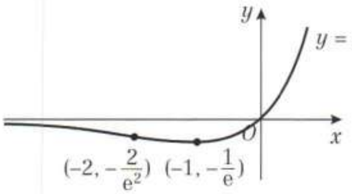
$\text{f}''(x) = 0 \Rightarrow 4\ln x = -6 \Rightarrow \ln x = -\frac{3}{2} \Rightarrow x = e^{-\frac{3}{2}}$

There is one point of inflection where $x = e^{-\frac{3}{2}}$

5 **a** (0, 2), point of inflection **b** $\left(-2, \frac{10}{e^2}\right)$

6 **a** $\left(-1, -\frac{1}{e}\right)$, minimum **b** $\left(-2, -\frac{2}{e^2}\right)$

c



7 A **i** negative **ii** positive
B **i** zero **ii** positive
C **i** positive **ii** negative
D **i** zero **ii** zero

8 $\text{f}'(x) = \sec^2 x$, $\text{f}''(x) = 2\sin x \sec^3 x$

$\text{f}''(x) = 0 \Leftrightarrow \sin x = 0$ (as $\sec x \neq 0$) $\Leftrightarrow x = 0$

So there is one point of inflection at (0, tan 0) = (0, 0).

9 **a** $\frac{dy}{dx} = 15x(3x - 1)^4 + (3x - 1)^5$

$\frac{d^2y}{dx^2} = 30(3x - 1)^4 + 180x(3x - 1)^3$

b $\left(\frac{1}{9}, -\frac{32}{2187}\right), \left(\frac{1}{3}, 0\right)$

10 **a** Although $\frac{d^2y}{dx^2} = 0$, the sign does not change, so there is not a point of inflection when $x = 5$.

b (5, 0); minimum

11 $\frac{dy}{dx} = \frac{2}{3}x\ln x + \frac{1}{3}x - 2$, $\frac{d^2y}{dx^2} = \frac{2}{3}\ln x + 1$

$\frac{d^2y}{dx^2} \geqslant 0 \Leftrightarrow \frac{2}{3}\ln x \geqslant -1 \Leftrightarrow x \geqslant e^{-\frac{3}{2}}$

Challenge

1 A general cubic can be written as $\text{f}(x) = ax^3 + bx^2 + cx + d$.

$\text{f}''(x) = 6ax + 2b$. $\text{f}''(x) = 0 \Leftrightarrow x = -\frac{b}{3a}$

Let $\varepsilon > \mathbb{R}$, $\varepsilon > 0$:

$\text{f}''\left(-\frac{b}{3a} + \varepsilon\right) = 6a\varepsilon > 0$, $\text{f}''\left(-\frac{b}{3a} - \varepsilon\right) = -6a\varepsilon < 0$

So the sign of f″(x) changes either side of $x = -\frac{b}{3a}$, and this is a point of inflection.

2 **a** $\text{f}''(x) = 12ax^2 + 6bx + 2c$ is quadratic, so there are at most two values of x at which $\text{f}''(x) = 0$.

b $\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 2c$

Discriminant $= 36b^2 - 96ac < 0 \Leftrightarrow 3b^2 < 8ac$

So when $3b^2 < 8ac$, $\frac{d^2y}{dx^2} = 0$ has no solutions.

Therefore C has no points of inflection.

Exercise 9J

1 6π **2** $15e^2$ **3** $-\frac{9}{2}$ **4** $\frac{8}{9\pi}$ **5** $\frac{dP}{dt} = kP$

6 $\frac{dy}{dx} = kxy$; at (4, 2) $\frac{dy}{dx} = \frac{1}{2}$, so $8k = \frac{1}{2}$, $k = \frac{1}{16}$

Therefore $\frac{dy}{dx} = \frac{xy}{16}$

7 $\frac{dV}{dt}$ = rate in − rate out $= 30 - \frac{2}{15}V \Rightarrow 15\frac{dV}{dt} = 450 - 2V$

So $-15\frac{dV}{dt} = 2V - 450$

8 $\frac{dQ}{dt} = -kQ$ **9** $\frac{dx}{dt} = \frac{k}{x^2}$

10 **a** Circumference, C, $= 2\pi r$, so $\frac{dC}{dt} = 2\pi \times 0.4$
$= 0.8\pi \text{ cm s}^{-1}$

Rate of increase of circumference with respect to time.

b $8\pi \text{ cm}^2 \text{ s}^{-1}$ **c** $\frac{25}{\pi}$ cm

11 **a** 0.070 cm per second **b** 20.5 cm³

12 $\frac{dV}{dt} \propto \sqrt{V} \Rightarrow \frac{dV}{dt} = -k_1\sqrt{V}$, $V \propto h \Rightarrow h = k_2 V$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = k_2 \times (-k_1\sqrt{V}) = -k_1 k_2 \sqrt{\frac{h}{k_2}}$

$= \frac{-k_1 k_2}{\sqrt{k_2}}\sqrt{h} = -k\sqrt{h}$

13 **a** $V = \left(\frac{A}{6}\right)^{\frac{3}{2}}$ **b** $\frac{1}{4}\left(\frac{A}{6}\right)^{\frac{1}{2}}$

c $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt} = \frac{1}{4}\left(\frac{A}{6}\right)^{\frac{1}{2}} \times 2 = \frac{1}{2}\left(V^{\frac{2}{3}}\right)^{\frac{1}{2}} = \frac{1}{2}V^{\frac{1}{3}}$

14 $V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}(h\tan 30°)^2 h = \frac{\pi}{9}h^3$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dh}{dV}} \times \frac{dV}{dt} = \frac{1}{\frac{\pi}{3}h^2} \times (-6) = -\frac{18}{\pi h^2}$

So $\frac{dh}{dt} \propto \frac{1}{h^2}$

Mixed exercise 9

1 **a** $\frac{2}{x}$ **b** $2x\sin 3x + 3x^2\cos 3x$

2 **a** $2\frac{dy}{dx} = 1 - \sin x\frac{d}{dx}(\cos x) - \frac{d}{dx}(\sin x)\cos x$
$= 1 + \sin^2 x - \cos^2 x = 2\sin^2 x$

So $\frac{dy}{dx} = \sin^2 x$

b $\left(\frac{\pi}{2}, \frac{\pi}{4}\right), \left(\pi, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{4}\right)$

3 **a** $\frac{x\cos x - \sin x}{x^2}$ **b** $-\frac{2x}{x^2 + 9}$

4 **a** $k = \sqrt{2}$ **b** $(0, 0), \left(\pm\sqrt{6}, \pm\frac{\sqrt{3}}{4\sqrt{2}}\right)$

5 **a** $x > 0$ **b** $\left(\sqrt[3]{256}, 32\ln 2 + 16\right)$

6 $\left(\frac{\pi}{6}, \frac{5}{4}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{5}{4}\right), \left(\frac{3\pi}{2}, -1\right)$

7 Maximum is when $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \sqrt{\sin x} + x\left(\cos x \times \frac{1}{2\sqrt{\sin x}}\right) = \frac{2\sin x + x\cos x}{2\sqrt{\sin x}} = 0$

So $2\sin x + x\cos x = 0 \Rightarrow 2\sin x = -x\cos x \Rightarrow 2\tan x = -x$

$\therefore 2\tan x + x = 0$

8 **a** $\text{f}'(x) = 0.5e^{0.5x} - 2x$

b $\text{f}'(6) = -1.957... < 0$, $\text{f}'(7) = 2.557... > 0$

So there exists $p \in (6, 7)$ such that $\text{f}'(p) = 0$.

$\therefore$ there is a stationary point for some $x = p \in (6, 7)$.

9 **a** $\left(\frac{3\pi}{8}, \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}\right), \left(\frac{7\pi}{8}, -\frac{e^{\frac{7\pi}{4}}}{\sqrt{2}}\right)$

b $f''(x) = 2e^{2x}(-2\sin 2x + 2\cos 2x) + 4e^{2x}(\cos 2x + \sin 2x)$
$= 4e^{2x}(-\sin 2x + \cos 2x + \cos 2x + \sin 2x)$
$= 8e^{2x}\cos 2x$

c $\left(\frac{3\pi}{8}, \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}\right)$ is a maximum; $\left(\frac{7\pi}{8}, -\frac{e^{\frac{7\pi}{4}}}{\sqrt{2}}\right)$ is a minimum.

d $\left(\frac{\pi}{4}, e^{\frac{\pi}{2}}\right), \left(\frac{3\pi}{4}, -e^{\frac{3\pi}{2}}\right)$

10 $x + 2y - 8 = 0$

11 **a** $x = \frac{1}{3}$ **b** $y = -\frac{1}{2}x + 1\frac{1}{2}$

12 **a** $f'(x) = e^{2x}(2\cos x - \sin x)$
$2\cos x - \sin x = 0 \Rightarrow \tan x = 2$
b $y = 2x + 1$

13 **a** $y + 2y\ln y$ **b** $\frac{1}{3e}$

14 **a** $e^{-x}(-x^3 + 3x^2 + 2x - 2)$

b $f'(0) = -2 \Rightarrow$ gradient of normal $= \frac{1}{2}$
Equation of normal is $y = \frac{1}{2}x$
$(x^3 - 2x)e^{-x} = \frac{1}{2}x \Rightarrow 2x^3 - 4x = xe^x \Rightarrow 2x^2 = e^x + 4$

15 **a** $1 + x + (1 + 2x)\ln x$

b $1 + x + (1 + 2x)\ln x = 0 \Rightarrow x = e^{-\frac{1+x}{1+2x}}$

16 **a** $\frac{dy}{dx} = -\frac{4}{t^3}$ **b** $y = 2x - 8$

17 $3y + x = 33$ **18** $y = \frac{2}{3}x + \frac{1}{3}$

19 **a** $\frac{dx}{dt} = -2\sin t + 2\cos 2t$; $\frac{dy}{dt} = -\sin t - 4\cos 2t$

b $\frac{1}{2}$ **c** $y + 2x = \frac{5\sqrt{2}}{2}$

20 **a** $\frac{dy}{dt} = 3t^2 - 4$, $\frac{dx}{dt} = 2$, $\frac{dy}{dx} = \frac{3t^2 - 4}{2}$
At $t = -1$, $\frac{dy}{dx} = -\frac{1}{2}$, $x = 1$, $y = 3$.
Equation of l is $2y + x = 7$.

b 2

21 $\frac{dV}{dt} = -kV$ **22** $\frac{dM}{dt} = -kM$ **23** $\frac{dP}{dt} = kP - Q$

24 $\frac{dr}{dt} = \frac{k}{r}$ **25** $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

26 **a** $\frac{\pi}{6}$ **b** $-\frac{3}{16}\operatorname{cosec} t$

c $\frac{dy}{dx} = -\frac{3}{8} \Rightarrow$ gradient of normal $= \frac{8}{3}$
$y - \frac{3}{2} = \frac{8}{3}(x - 2) \Rightarrow 6y - 16x + 23 = 0$

d $-\frac{123}{64}$

27 **a** $-\frac{1}{2}\sec t$ **b** $4y + 4x = 5a$

c Tangent crosses the x-axis at $x = \frac{5}{4}a$, and crosses the y-axis at $y = \frac{5}{4}a$.
So area $AOB = \frac{1}{2}\left(\frac{5}{4}a\right)^2 = \frac{25}{32}a^2$, $k = \frac{25}{32}$

28 $y + x = 16$ **29** $\frac{1}{7}$

30 $\frac{y - 2e^{2x}}{2e^{2y} - x}$ **31** $(1, 1)$ and $(-\sqrt[3]{-3}, \sqrt[3]{-3})$.

32 **a** $\frac{2x - 2 - y}{1 + x - 2y}$ **b** $\frac{4}{3}, -\frac{1}{3}$

c $\left(\frac{5 + 2\sqrt{13}}{3}, \frac{4 + \sqrt{13}}{3}\right)$ and $\left(\frac{5 - 2\sqrt{13}}{3}, \frac{4 - \sqrt{13}}{3}\right)$

33 $14x + 48y + 48x\frac{dy}{dx} - 14y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-7x - 24y}{24x - 7y}$
So $\frac{-7x - 24y}{24x - 7y} = \frac{2}{11}$
$\Rightarrow -77x - 264y = 48x - 14y \Rightarrow x + 2y = 0$

34 $\ln y = x\ln x \Rightarrow \frac{1}{y} \times \frac{dy}{dx} = x\frac{d}{dx}(\ln x) + \frac{d}{dx}(x)\ln x = 1 + \ln x$
So $\frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x)$

35 **a** $\ln a^x = \ln e^{kx} \Rightarrow x\ln a = kx\ln e = kx \Rightarrow k = \ln a$

b $y = e^{(\ln 2)x} \Rightarrow \frac{dy}{dx} = \ln 2\, e^{(\ln 2)x} = 2^x \ln 2$

c $\frac{dy}{dx} = 2^2\ln 2 = 4\ln 2 = \ln 2^4 = \ln 16$

36 **a** $\frac{\ln P - \ln P_0}{\ln 1.09}$ **b** 8.04 years **c** $0.172P_0$

37 **a** $\left(\frac{\pi}{2}, 0\right)$

b $\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$. $\operatorname{cosec}^2 x > 0$ for all x,
hence $-\operatorname{cosec}^2 x < 0$, so $\frac{d^2y}{dx^2} < 0$ for all x.
Thus C is concave for all values of x.

38 **a** $40e^{-0.183} = 33.31...$ **b** $-9.76e^{-0.244t}$

c The mass is decreasing

39 **a** $f'(x) = -\frac{2\sin 2x + \cos 2x}{e^x}$
$f'(x) = 0 \Leftrightarrow 2\sin 2x + \cos 2x = 0 \Leftrightarrow \tan 2x = -0.5$
$A(1.34, -0.234)$, $B(2.91, 0.0487)$

b Maximum (6.91, 2.19); minimum (5.34, 1.06) to 3 s.f.

c $0 < x \leqslant 0.322$, $1.89 \leqslant x < \pi$ (decimals to 3 s.f.)

Challenge

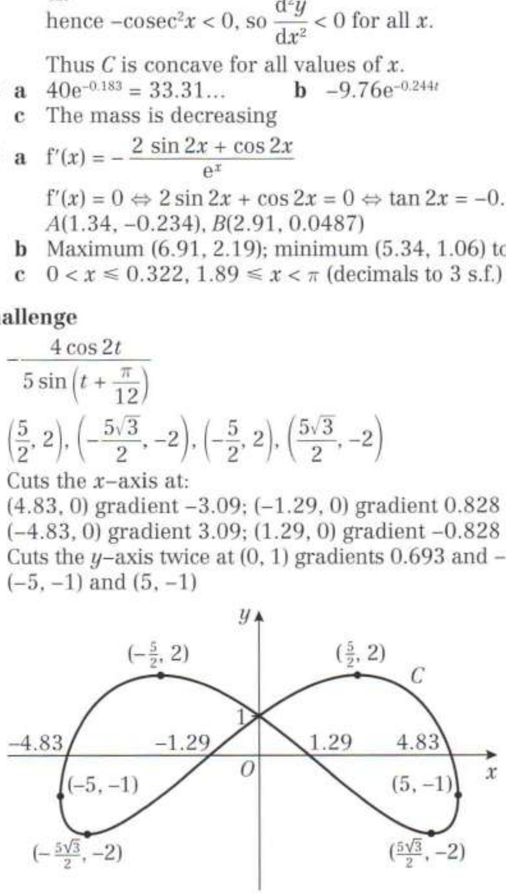
a $-\frac{4\cos 2t}{5\sin\left(t + \frac{\pi}{12}\right)}$

b $\left(\frac{5}{2}, 2\right), \left(-\frac{5\sqrt{3}}{2}, -2\right), \left(-\frac{5}{2}, 2\right), \left(\frac{5\sqrt{3}}{2}, -2\right)$

c Cuts the x-axis at:
(4.83, 0) gradient −3.09; (−1.29, 0) gradient 0.828
(−4.83, 0) gradient 3.09; (1.29, 0) gradient −0.828
Cuts the y-axis twice at (0, 1) gradients 0.693 and −0.693

d (−5, −1) and (5, −1)

e



Online Full worked solutions are available in SolutionBank.

CHAPTER 10

Prior knowledge 10

1 a 3.25 b 11.24

2 a $f'(x) = \frac{3}{2\sqrt{x}} + 8x + \frac{15}{x^4}$ b $f'(x) = \frac{5}{x+2} - 7e^{-x}$

c $f'(x) = x^2 \cos x + 2x \sin x + 4 \sin x$

3 $u_1 = 2, u_2 = 2.5, u_3 = 2.9$

Exercise 10A

1 a $f(-2) = -1 < 0$, $f(-1) = 5 > 0$. Sign change implies root.

b $f(3) = -2.732 < 0$, $f(4) = 4 > 0$. Sign change implies root.

c $f(-0.5) = -0.125 < 0$, $f(-0.2) = 2.992 > 0$. Sign change implies root.

d $f(1.65) = -0.294 < 0$, $f(1.75) = 0.195 > 0$. Sign change implies root.

2 a $f(1.8) = 0.408 > 0$, $f(1.9) = -0.249$. Sign change implies root.

b $f(1.8635) = 0.00138... > 0$, $f(1.8645) = -0.00531... < 0$. Sign change implies root.

3 a $h(1.4) = -0.0512... < 0$, $h(1.5) = 0.0739.... > 0$. Sign change implies root.

b $h(1.4405) = -0.00055... < 0$, $h(1.4415) = 0.00069... > 0$. Sign change implies root.

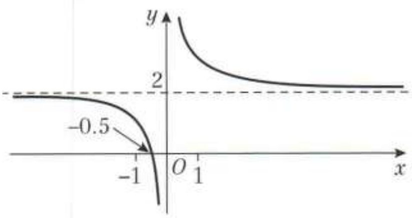
4 a $f(2.2) = 0.020 > 0$, $f(2.3) = -0.087$. Sign change implies root.

b $f(2.2185) = 0.00064... > 0$, $f(2.2195) = -0.00041... < 0$. There is a sign change in the interval $2.2185 < x < 2.2195$, so $\alpha = 2.219$ correct to 3 decimal places.

5 a $f(1.5) = 16.10... > 0$, $f(1.6) = -32.2... < 0$. Sign change implies root.

b There is an asymptote in the graph of $y = f(x)$ at $x = \frac{\pi}{2} \approx 1.57$. So there is not a root in this interval.

6



Alternatively: $\frac{1}{x} + 2 = 0 \Rightarrow \frac{1}{x} = -2 \Rightarrow x = -\frac{1}{2}$

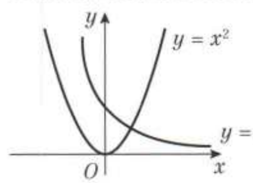
7 a $f(0.2) = -0.4421...$, $f(0.8) = -0.1471...$

b There are either no roots or an even numbers of roots in the interval $0.2 < x < 0.8$.

c $f(0.3) = 0.01238... > 0$, $f(0.4) = -0.1114... < 0$, $f(0.5) = -0.2026... < 0$, $f(0.6) = 0$, $f(0.7) = 0.2710... > 0$

d There exists at least one root in the interval $0.2 < x < 0.3$, $0.3 < x < 0.4$ and $0.7 < x < 0.8$. Additionally $x = 0.6$ is a root. Therefore there are at least four roots in the interval $0.2 < x < 0.8$.

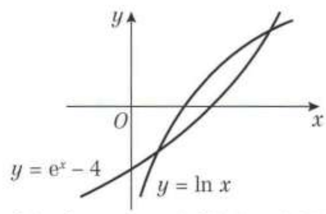
8 a



b One point of intersection, so one root.

c $f(0.7) = 0.0065... > 0$, $f(0.71) = -0.0124... < 0$. Sign change implies root.

9 a



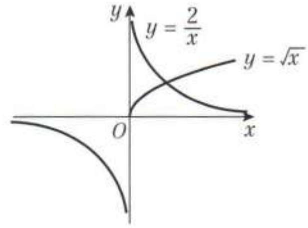
b 2

c $f(x) = \ln x - e^x + 4$. $f(1.4) = 0.2812... < 0$, $f(1.5) = -0.0762... < 0$. Sign change implies root.

10 a $h'(x) = 2\cos 2x + 4e^{4x}$. $h'(-0.9) = -0.3451... < 0$. $h'(-0.8) = 0.1046... > 0$. Sign change implies slope changes from decreasing to increasing over interval, which implies turning point.

b $h'(-0.8235) = -0.003839.... < 0$, $h'(-0.8225) = 0.00074... > 0$. Sign change implies α lies in the range $-0.8235 < \alpha < -0.8225$, so $\alpha = -0.823$ correct to 3 decimal places.

11 a



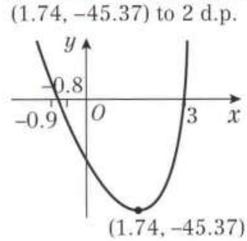
b 1 point of intersection $\Rightarrow$ 1 root

c $f(1) = -1$, $f(2) = 0.414...$ d $p = 3$, $q = 4$ e $4^{\frac{1}{3}}$

12 a $f(-0.9) = 1.5561 > 0$, $f(-0.8) = -0.7904 < 0$. There is a change of sign in the interval $[-0.9, -0.8]$, so there is at least one root in this interval.

b $(1.74, -45.37)$ to 2 d.p. c $a = 3$, $b = 9$ and $c = 6$

d



Exercise 10B

1 a i $x^2 - 6x + 2 = 0 \Rightarrow 6x = x^2 + 2 \Rightarrow x = \frac{x^2 + 2}{6}$

ii $x^2 - 6x + 2 = 0 \Rightarrow x^2 = 6x - 2 \Rightarrow x = \sqrt{6x - 2}$

iii $x^2 - 6x + 2 = 0 \Rightarrow x - 6 + \frac{2}{x} = 0 \Rightarrow x = 6 - \frac{2}{x}$

b i $x = 0.354$ ii $x = 5.646$ iii $x = 5.646$

c $a = 3$, $b = 7$

2 a i $x^2 - 5x - 3 = 0 \Rightarrow x^2 = 5x + 3 \Rightarrow x = \sqrt{5x + 3}$

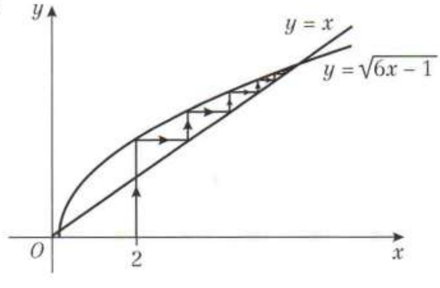
ii $x^2 - 5x - 3 = 0 \Rightarrow x^2 - 3 = 5x \Rightarrow x = \frac{x^2 - 3}{5}$

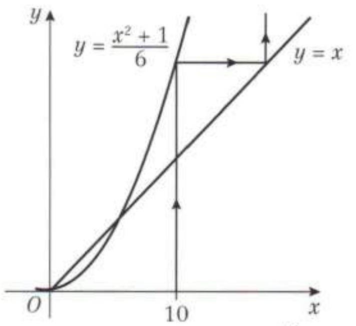
b i 5.5 (1 d.p.) ii −0.5 (1 d.p.)

3 a $x^2 - 6x + 1 = 0 \Rightarrow x^2 = 6x - 1 \Rightarrow x = \sqrt{6x - 1}$

c The graph shows there are two roots of $f(x) = 0$

b, d



e 

4 a $xe^{-x} - x + 2 = 0 \Rightarrow e^{-x} = \frac{x-2}{x} \Rightarrow e^x = \frac{x}{x-2}$
$\Rightarrow x = \ln\left|\frac{x}{x-2}\right|$

b $x_1 = -1.10$, $x_2 = -1.04$, $x_3 = -1.07$

5 a i $x^3 + 5x^2 - 2 = 0 \Rightarrow x^3 = 2 - 5x^2 \Rightarrow x = \sqrt[3]{2-5x^2}$

ii $x^3 + 5x^2 - 2 = 0 \Rightarrow x + 5 - \frac{2}{x^2} = 0 \Rightarrow x = \frac{2}{x^2} - 5$

iii $x^3 + 5x^2 - 2 = 0 \Rightarrow 5x^2 = 2 - x^3 \Rightarrow x^2 = \frac{2-x^3}{5}$
$\Rightarrow x = \sqrt{\frac{2-x^3}{5}}$

b $x = -4.917$ c $x = 0.598$

d It is not possible to take the square root of a negative number over ℝ.

6 a $x^4 - 3x^3 - 6 = 0 \Rightarrow \frac{1}{3}x^4 - x^3 - 2 = 0$
$\Rightarrow \frac{1}{3}x^4 - 2 = x^3 \Rightarrow x = \sqrt[3]{\frac{1}{3}x^4 - 2} \Rightarrow p = \frac{1}{3}, q = -2$

b $x_1 = -1.260$, $x_2 = -1.051$, $x_3 = -1.168$

c f(−1.1315) = −0.014… < 0, f(−1.1325) = 0.0024… > 0
There is a sign change in this interval, which implies $\alpha = -1.132$ correct to 3 decimal places.

7 a $3\cos(x^2) + x - 2 = 0 \Rightarrow \cos(x^2) = \frac{2-x}{3}$
$\Rightarrow x^2 = \arccos\left(\frac{2-x}{3}\right) \Rightarrow x = \left[\arccos\left(\frac{2-x}{3}\right)\right]^{\frac{1}{2}}$

b $x_1 = 1.109$, $x_2 = 1.127$, $x_3 = 1.129$

c f(1.12975) = 0.000423… > 0,
f(1.12985) = −0.0001256… < 0. There is a sign change in this interval, which implies $\alpha = 1.1298$ correct to 4 decimal places.

8 a f(0.8) = 0.484…, f(0.9) = −1.025…. There is a change of sign in the interval, so there must exist a root in the interval, since f is continuous over the interval.

b $\frac{4\cos x}{\sin x} - 8x + 3 = 0 \Rightarrow 8x = \frac{4\cos x}{\sin x} + 3$
$\Rightarrow x = \frac{\cos x}{2\sin x} + \frac{3}{8}$

c $x_1 = 0.8142$, $x_2 = 0.8470$, $x_3 = 0.8169$

d f(0.8305) = 0.0105… > 0, f(0.8315) = −0.0047… < 0.
There is a change of sign in the interval, so there must exist a root in the interval.

9 a $e^{x-1} + 2x - 15 = 0 \Rightarrow e^{x-1} = 15 - 2x$
$\Rightarrow x - 1 = \ln(15 - 2x)$
$\Rightarrow x = \ln(15 - 2x) + 1$ where $x < \frac{15}{2}$

b $x_1 = 3.1972$, $x_2 = 3.1524$, $x_3 = 3.1628$

c f(3.155) = −0.062… < 0, f(3.165) = 0.044… > 0.
There is a sign change in this interval, which implies $\alpha = 3.16$ correct to 2 decimal places.

10 a $A(0, 0)$ and $B(\ln 4, 0)$

b $f'(x) = xe^x + e^x - 4 = e^x(x + 1) - 4$

c f′(0.7) = −0.5766… < 0, f′(0.8) = 0.0059… > 0.
There is a sign change in this interval, which implies f′(x) = 0 in this range. f′(x) = 0 at a turning point.

d $e^x(x+1) - 4 = 0 \Rightarrow e^x = \frac{4}{x+1} \Rightarrow x = \ln\left(\frac{4}{x+1}\right)$

e $x_1 = 1.386$, $x_2 = 0.517$, $x_3 = 0.970$, $x_4 = 0.708$

Exercise 10C

1 a f(1) = −2, f(2) = 3. There is a sign change in the interval $1 < \alpha < 2$, so there is a root in this interval.

b $x_1 = 1.632$

2 a $f'(x) = 2x + \frac{4}{x^2} + 6$ b −0.326

3 a It's a turning point, so f′(x) = 0, and you cannot divide by zero in the Newton–Raphson formula.

b 1.247

4 a f(1.4) = −0.020…, f(1.5) = 0.12817… As there is a change of sign in the interval, there must be a root α in this interval.

b $x_1 = 1.413$

c f(1.4125) = −0.00076…, f(1.4135) = 0.0008112….

5 a f(1.3) = −0.085…, f(1.4) = 0.429… As there is a change of sign in the interval, there must be a root α in this interval.

b $f'(x) = 2x + \frac{6}{x^3}$ c 1.316

6 a f(0.6) = 0.0032… > 0, f(0.7) = −0.0843… < 0.
Sign change implies root in the interval.
f(1.2) = −0.0578… < 0, f(1.3) = 0.0284… > 0.
Sign change implies root in the interval.
f(2.4) = 0.0906… > 0, f(2.5) = −0.2595… < 0.
Sign change implies root in the interval.

b It's a turning point, so f′(x) = 0, and you cannot divide by zero in the Newton–Raphson formula.

c 2.430

7 a f(3.4) = 0.2645… > 0, f(3.5) = −0.3781…. < 0.
Sign change implies root in the interval.

b $f'(x) = \frac{3}{3x-4} - 2x$ c 3.442

Challenge

a From the graph, f(x) > 0 for all values of x > 0. Note also that $xe^{-x^2} > 0$ when x > 0. So the same must be true for $x > \frac{1}{\sqrt{2}}$.
$f'(x) = e^{-x^2}(1 - 2x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$
So f′(x) < 0 for $x > \frac{1}{\sqrt{2}}$.
$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$ is an increasing sequence as f(x) > 0 and f′(x) < 0, for $x > \frac{1}{\sqrt{2}}$. Therefore the Newton–Raphson method will fail to converge.

b −0.209

Exercise 10D

1 a $\frac{\pi}{6} = E - 0.1\sin E$, if E is a root then f(E) = 0
$E - 0.1\sin E - k = 0 \Rightarrow E - 0.1\sin E = k \Rightarrow \frac{\pi}{6} = k$

b 0.5782…

c f(0.5775) = −0.00069…< 0, f(0.5785) = 0.00022 > 0.
Change of sign implies root in interval [0.5775, 0.5785], so root is 0.578 to 3 d.p.

2 a $A(0, 0)$ and $B(19, 0)$

b $f'(t) = \frac{10}{t+1} - \left(\frac{\ln(t+1)}{2} + \frac{1}{2}\right)$

c $f'(5.8) = \dfrac{10}{5.8+1} - \left(\dfrac{\ln(5.8+1)}{2} + \dfrac{1}{2}\right) = 0.0121... > 0$

$f'(5.9) = \dfrac{10}{5.9+1} - \left(\dfrac{\ln(5.9+1)}{2} + \dfrac{1}{2}\right) = -0.0164... < 0$

The sign change implies that the speed changes from increasing to decreasing, so the greatest speed of the skier lies between 5.8 and 5.9.

d $f'(t) = \dfrac{10}{t+1} - \left(\dfrac{\ln(t+1)}{2} + \dfrac{1}{2}\right) = 0$

$\dfrac{\ln(t+1)+1}{2} = \dfrac{10}{t+1}$

$(t+1)(\ln(t+1)+1) = 20$

$t + 1 = \dfrac{20}{\ln(t+1)+1}$

$t = \dfrac{20}{\ln(t+1)+1} - 1$

e $t_1 = 6.164$, $t_2 = 5.736$, $t_3 = 5.879$

3 a $d(x) = 0 \Rightarrow x^2 - 3x = 0$

$x(x-3) = 0 \Rightarrow x = 0, x = 3$

The river bed is 3 m wide so the function is only valid for $0 \leqslant x \leqslant 3$.

b $d'(x) = 2xe^{-0.6x} - \dfrac{3}{5}x^2e^{-0.6x} - 3e^{-0.6x} + \dfrac{9}{5}xe^{-0.6x}$

$d'(x) = e^{-0.6x}\left(-\dfrac{3}{5}x^2 + \dfrac{19}{5}x - \dfrac{15}{5}\right)$

$d'(x) = -\dfrac{1}{5}e^{-0.6x}(3x^2 - 19x + 15)$

So $a = 3$, $b = -19$ and $c = 15$

c $-\frac{1}{5}e^{-0.6x} \neq 0$ so $d'(x) = 0 \Rightarrow 3x^2 - 19x + 15 = 0$

i $3x^2 - 19x + 15 = 0 \Rightarrow 3x^2 = 19x - 15$

$\Rightarrow x^2 = \dfrac{19x-15}{3} \Rightarrow x = \sqrt{\dfrac{19x-15}{3}}$

ii $3x^2 - 19x + 15 = 0 \Rightarrow 3x^2 + 15 = 19x$

$\Rightarrow x = \dfrac{3x^2+15}{19}$

iii $3x^2 - 19x + 15 = 0 \Rightarrow 3x^2 = 19x - 15$

$\Rightarrow x = \dfrac{19x-15}{3x}$

d Part **i** and **iii** tend to 5.408… which is outside the required range. Part **ii** tends to $x = 0.924$.

e 1.10 m.

4 a $h(t) = 0$

$40\sin\left(\dfrac{t}{10}\right) - 9\cos\left(\dfrac{t}{10}\right) - 0.5t^2 + 9 = 0$

$40\sin\left(\dfrac{t}{10}\right) - 9\cos\left(\dfrac{t}{10}\right) + 9 = 0.5t^2$

$80\sin\left(\dfrac{t}{10}\right) - 18\cos\left(\dfrac{t}{10}\right) + 18 = t^2$

$\Rightarrow t = \sqrt{18 + 80\sin\left(\dfrac{t}{10}\right) - 18\cos\left(\dfrac{t}{10}\right)}$

b $t_1 = 7.928$, $t_2 = 7.896$, $t_3 = 7.882$, $t_4 = 7.876$

c $h'(t) = 4\cos\left(\dfrac{t}{10}\right) + 0.9\sin\left(\dfrac{t}{10}\right) - t$

d 7.874 (3 d.p.)

e Restrict the range of validity to $0 \leqslant t \leqslant A$

5 a $c'(x) = -5e^{-x} + 2\cos\left(\dfrac{x}{2}\right) + \dfrac{1}{2}$

b i $-5e^{-x} + 2\cos\left(\dfrac{x}{2}\right) + \dfrac{1}{2} = 0$

$\Rightarrow \cos\left(\dfrac{x}{2}\right) = \dfrac{5}{2}e^{-x} - \dfrac{1}{4} \Rightarrow x = 2\arccos\left[\dfrac{5}{2}e^{-x} - \dfrac{1}{4}\right]$

ii $-5e^{-x} + 2\cos\left(\dfrac{x}{2}\right) + \dfrac{1}{2} = 0 \Rightarrow e^{-x} = \dfrac{4\cos\left(\frac{x}{2}\right)+1}{10}$

$\Rightarrow e^{x} = \dfrac{10}{4\cos\left(\frac{x}{2}\right)+1} \Rightarrow x = \ln\left(\dfrac{10}{4\cos\left(\frac{x}{2}\right)+1}\right)$

c $x_1 = 3.393$, $x_2 = 3.475$, $x_3 = 3.489$, $x_4 = 3.491$

d $x_1 = 0.796$, $x_2 = 0.758$, $x_3 = 0.752$, $x_4 = 0.751$

e The model does support the assumption that the crime rate was increasing. The model shows that there is a minimum point $\frac{3}{4}$ of the way through 2000 and a maximum point mid-way through 2003. So, the crime rate is increasing in the interval between October 2000 and June 2003.

Mixed exercise 10

1 a $x^3 - 6x - 2 = 0 \Rightarrow x^3 = 6x + 2$

$\Rightarrow x^2 = 6 + \dfrac{2}{x} \Rightarrow x = \pm\sqrt{6 + \dfrac{2}{x}}$; $a = 6$, $b = 2$

b $x_1 = 2.6458$, $x_2 = 2.5992$, $x_3 = 2.6018$, $x_4 = 2.6017$

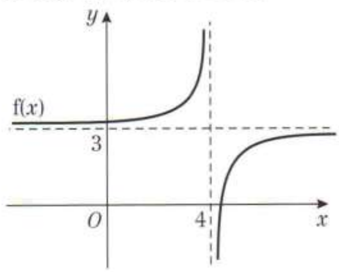
c $f(2.6015) = (2.6015)^3 - 6(2.6015) - 2 = -0.0025... < 0$

$f(2.6025) = (2.6025)^3 - 6(2.6025) - 2 = 0.0117 > 0$

There is a sign change in the interval $2.6015 < x < 2.6025$, so this implies there is a root in the interval.

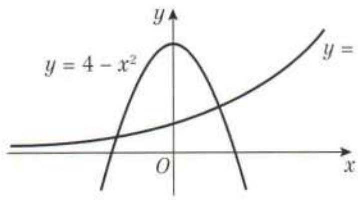
2 a $f(3.9) = 13$, $f(4.1) = -7$

b There is an asymptote at $x = 4$ which causes the change of sign, not a root.



c $\alpha = \frac{13}{3}$

3 a



b 2 roots – 1 positive and 1 negative

c $x^2 + e^x - 4 = 0 \Rightarrow x^2 = 4 - e^x \Rightarrow x = \pm(4 - e^x)^{1/2}$

d $x_1 = -1.9659$, $x_2 = -1.9647$, $x_3 = -1.9646$, $x_4 = -1.9646$

e You would need to take the square root of a negative number.

4 a $g(1) = -10 < 0$, $g(2) = 16 > 0$. The sign change implies there is a root in this interval.

b $g(x) = 0 \Rightarrow x^5 - 5x - 6 = 0$

$\Rightarrow x^5 = 5x + 6 \Rightarrow x = (5x+6)^{\frac{1}{5}}$

$p = 5$, $q = 6$, $r = 5$

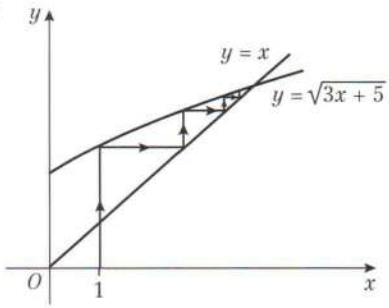
c $x_1 = 1.6154$, $x_2 = 1.6971$, $x_3 = 1.7068$

d $g(1.7075) = -0.0229... < 0$, $g(1.7085) = 0.0146... > 0$. The sign change implies there is a root in this interval.

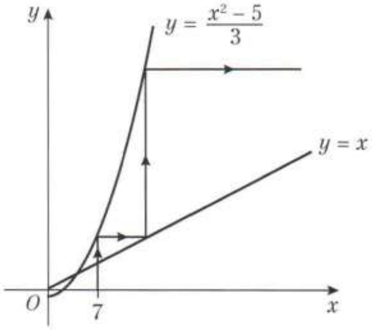
5 a $g(x) = 0 \Rightarrow x^2 - 3x - 5 = 0$

$\Rightarrow x^2 = 3x + 5 \Rightarrow x = \sqrt{3x+5}$

b, c



d



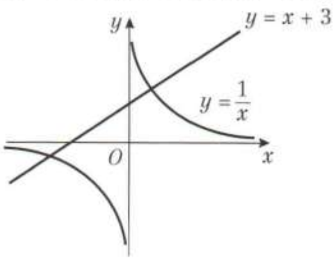
6 a $f(1.1) = -0.0648... < 0$, $f(1.15) = 0.0989... > 0$. The sign change implies there is a root in this interval.

b $5x - 4\sin x - 2 = 0 \Rightarrow 5x = 4\sin x + 2$

$\Rightarrow x = \frac{4}{5}\sin x + \frac{2}{5} \Rightarrow p = \frac{4}{5}, q = \frac{2}{5}$

c $x_1 = 1.113, x_2 = 1.118, x_3 = 1.119, x_4 = 1.120$

7 a



b 2

c $\frac{1}{x} = x + 3 \Rightarrow 0 = x + 3 - \frac{1}{x}$, let $f(x) = x + 3 - \frac{1}{x}$

$f(0.30) = -0.0333...< 0$, $f(0.31) = 0.0841... > 0$.
Sign change implies root.

d $\frac{1}{x} = x + 3 \Rightarrow 1 = x^2 + 3x \Rightarrow 0 = x^2 + 3x - 1$

e 0.303

8 a $g'(x) = 3x^2 - 14x + 2$ **b** 6.606

c $(x - 1)(x^2 - 6x - 4) \Rightarrow x^2 - 6x - 4 = 0 \Rightarrow x = 3 \pm \sqrt{13}$

d 0.007%

9 a $f(0.4) = -0.0285... < 0$, $f(0.5) = 0.2789... > 0$.
Sign change implies root.

b 0.410

c $f(-1.1905) = 0.0069... > 0$,
$f(-1.1895) = -0.0044... < 0$.
Sign change implies root.

10 a $e^{0.8x} - \frac{1}{3 - 2x} = 0 \Rightarrow (3 - 2x)e^{0.8x} - 1 = 0$

$\Rightarrow (3 - 2x)e^{0.8x} = 1 \Rightarrow 3 - 2x = e^{-0.8x}$

$\Rightarrow 3 - e^{-0.8x} = 2x \Rightarrow x = 1.5 - 0.5e^{-0.8x}$

b $x_1 = 1.32327..., x_2 = 1.32653..., x_3 = 1.32698...,$
root = 1.327 (3 d.p.)

c $e^{0.8x} - \frac{1}{3 - 2x} = 0 \Rightarrow e^{0.8x} = \frac{1}{3 - 2x} \Rightarrow 3 - 2x = e^{-0.8x}$

$\Rightarrow -0.8x = \ln(3 - 2x) \Rightarrow x = -1.25\ln(3 - 2x)$
$p = -1.25$

d $x_1 = -2.6302, x_2 = -2.6393, x_3 = -2.6421,$
root = −2.64 (2 d.p.)

11 a $\ln y = x\ln x \Rightarrow \frac{1}{y}\frac{dy}{dx} = (1)(\ln x) + (x)\left(\frac{1}{x}\right)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x + 1 \Rightarrow \frac{dy}{dx} = y(\ln x + 1) = x^x(1 + \ln x)$

b $f(1.4) = -0.3983... < 0$, $f(1.6) = 0.1212... > 0$.
Sign change implies root in the interval.

c $x_1 = 1.5631$ (4 d.p.)

d $f(1.55955) = -0.00017... < 0$,
$f(1.55965) = 0.00011... > 0$. Sign change implies root in the interval.

12 a $f(1.3) = -0.18148...$, $f(1.4) = 0.07556.....$ There is a sign change in the interval [1.3, 1.4], so there is a root in this interval.

b (0.817, −1.401)

c $x_1 = 0.3424, x_2 = 0.3497, x_3 = 0.3488, x_4 = 0.3489$

d $x_1 = 1.708$

e $f(1.7075) = 0.000435....$, $f(1.7085) = -0.002151....$
There is a sign change in the interval [1.7075, 1.7085], so there is a root in this interval.

Challenge

a $f(x) = x^6 + x^3 - 7x^2 - x + 3$
$f'(x) = 6x^5 + 3x^2 - 14x - 1$
$f''(x) = 30x^4 + 6x - 14$
$f''(x) = 0 \Rightarrow 15x^4 + 3x - 7 = 0$

i $15x^4 + 3x - 7 = 0 \Rightarrow 3x = 7 - 15x^4 \Rightarrow x = \frac{7 - 15x^4}{3}$

ii $15x^4 + 3x - 7 = 0 \Rightarrow 15x^4 + 3x = 7$

$\Rightarrow x(15x^3 + 3) = 7 \Rightarrow x = \frac{7}{15x^3 + 3}$

iii $15x^4 + 3x - 7 = 0 \Rightarrow 15x^4 = 7 - 3x$

$\Rightarrow x^4 = \frac{7 - 3x}{15} \Rightarrow x = \sqrt[4]{\frac{7 - 3x}{15}}$

b Using formula **iii**, root = 0.750 (3 d.p.)

c Formula **iii** gives the positive fourth root, so cannot be used to find a negative root.

d −0.897 (3 d.p.)

CHAPTER 11

Prior knowledge 11

1 a $12(2x - 7)^5$ **b** $5\cos 5x$ **c** $\frac{1}{3}e^{\frac{x}{3}}$

2 a $y = \frac{16}{3}x^{\frac{3}{2}} - 12x^{\frac{1}{2}}$ **b** $\frac{268}{3}$

3 $\frac{7}{4x - 1} - \frac{1}{x + 3}$ **4** 6 units²

Exercise 11A

1 a $3\tan x + 5\ln|x| - \frac{2}{x} + c$ **b** $5e^x + 4\cos x + \frac{x^4}{2} + c$

c $-2\cos x - 2\sin x + x^2 + c$ **d** $3\sec x - 2\ln|x| + c$

e $5e^x + 4\sin x + \frac{2}{x} + c$ **f** $\frac{1}{2}\ln|x| - 2\cot x + c$

g $\ln|x| - \frac{1}{x} - \frac{1}{2x^2} + c$ **h** $e^x - \cos x + \sin x + c$

i $-2\operatorname{cosec} x - \tan x + c$ **j** $e^x + \ln|x| + \cot x + c$

2 a $\tan x - \frac{1}{x} + c$ **b** $\sec x + 2e^x + c$

c $-\cot x - \operatorname{cosec} x - \frac{1}{x} + \ln|x| + c$

d $-\cot x + \ln|x| + c$ **e** $-\cos x + \sec x + c$

f $\sin x - \operatorname{cosec} x + c$ **g** $-\cot x + \tan x + c$

h $\tan x + \cot x + c$ i $\tan x + e^x + c$

j $\tan x + \sec x + \sin x + c$

3 a $2e^7 - 2e^3$ b $\frac{95}{72}$ c -5 d $2 - \sqrt{2}$

4 $a = 2$ 5 $a = 7$ 6 $b = 2$

7 a $x = 4$ b $\frac{1}{20}x^{\frac{5}{2}} - 4\ln|x| + c$

c $\frac{31}{20} - 4\ln 4$

Exercise 11B

1 a $-\frac{1}{2}\cos(2x + 1) + c$ b $\frac{3}{2}e^{2x} + c$

c $4e^{x+5} + c$ d $-\frac{1}{2}\sin(1 - 2x) + c$

e $-\frac{1}{3}\cot 3x + c$ f $\frac{1}{4}\sec 4x + c$

g $-6\cos\left(\frac{1}{2}x + 1\right) + c$ h $-\tan(2 - x) + c$

i $-\frac{1}{2}\operatorname{cosec} 2x + c$ j $\frac{1}{3}(\sin 3x + \cos 3x) + c$

2 a $\frac{1}{2}e^{2x} + \frac{1}{4}\cos(2x - 1) + c$ b $\frac{1}{2}e^{2x} + 2e^x + x + c$

c $\frac{1}{2}\tan 2x + \frac{1}{2}\sec 2x + c$

d $-6\cot\left(\frac{1}{2}x\right) + 4\operatorname{cosec}\left(\frac{1}{2}x\right) + c$

e $-e^{3-x} + \cos(3 - x) - \sin(3 - x) + c$

3 a $\frac{1}{2}\ln|2x + 1| + c$ b $-\dfrac{1}{2(2x + 1)} + c$

c $\dfrac{(2x + 1)^3}{6} + c$ d $\frac{3}{4}\ln|4x - 1| + c$

e $-\frac{3}{4}\ln|1 - 4x| + c$ f $\dfrac{3}{4(1 - 4x)} + c$

g $\dfrac{(3x + 2)^6}{18} + c$ h $\dfrac{3}{4(1 - 2x)^2} + c$

4 a $-\frac{3}{2}\cos(2x + 1) + 2\ln|2x + 1| + c$

b $\frac{1}{5}e^{5x} - \dfrac{(1 - x)^6}{6} + c$

c $-\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1 + 2x| - \dfrac{1}{2(1 + 2x)} + c$

d $\dfrac{(3x + 2)^3}{9} - \dfrac{1}{3(3x + 2)} + c$

5 a 1 b $\frac{7}{4}$ c $\dfrac{2\sqrt{3}}{9}$ d $\frac{5}{2}\ln 3$

6 $b = 6$ 7 $k = 24$ 8 $k = \frac{1}{12}$

Challenge

$a = 4, b = -3$ or $a = 8, b = -6$

Exercise 11C

1 a $-\cot x - x + c$ b $\frac{1}{2}x + \frac{1}{4}\sin 2x + c$

c $-\frac{1}{8}\cos 4x + c$ d $\frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + c$

e $\frac{1}{3}\tan 3x - x + c$ f $-2\cot x - x + 2\operatorname{cosec} x + c$

g $x - \frac{1}{2}\cos 2x + c$ h $\frac{1}{8}x - \frac{1}{32}\sin 4x + c$

i $-2\cot 2x + c$ j $\frac{3}{2}x + \frac{1}{8}\sin 4x - \sin 2x + c$

2 a $\tan x - \sec x + c$ b $-\cot x - \operatorname{cosec} x + c$

c $2x - \tan x + c$ d $-\cot x - x + c$

e $-2\cot x - x - 2\operatorname{cosec} x + c$

f $-\cot x - 4x + \tan x + c$ g $x + \frac{1}{2}\cos 2x + c$

h $-\frac{3}{2}x + \frac{1}{4}\sin 2x + \tan x + c$ i $-\frac{1}{2}\operatorname{cosec} 2x + c$

3 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{8} + \frac{1}{4} = \frac{2 + \pi}{8}$

4 a $\dfrac{4\sqrt{3}}{3}$ b $\dfrac{9\sqrt{3} - 10 - \pi}{8}$ c $2\sqrt{2} - \dfrac{\pi}{4}$ d $\dfrac{\sqrt{2} - 1}{2}$

5 a $\sin(3x + 2x) = \sin 3x\cos 2x + \cos 3x\sin 2x$
$\sin(3x - 2x) = \sin 3x\cos 2x - \cos 3x\sin 2x$
Adding gives $\sin 5x + \sin x = 2\sin 3x\cos 2x$

b So $\int \sin 3x\cos 2x\,dx = \int \frac{1}{2}(\sin 5x + \sin x)\,dx$

$= \frac{1}{2}(-\frac{1}{5}\cos 5x - \cos x) + c = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$

6 a $5\sin^2 x + 7\cos^2 x = 5 + 2\cos^2 x = 6 + (2\cos^2 x - 1)$
$= \cos 2x + 6$

b $\frac{1}{2}(1 + 3\pi)$

7 a $\cos^4 x = (\cos^2 x)^2 = \left(\dfrac{1 + \cos 2x}{2}\right)^2 = \dfrac{1}{4} + \dfrac{1}{2}\cos 2x$

$+ \dfrac{1}{4}\cos^2 2x = \dfrac{1}{4} + \dfrac{1}{2}\cos 2x + \dfrac{1}{4}\left(\dfrac{1 + \cos 4x}{2}\right)$

$= \dfrac{3}{8} + \dfrac{1}{2}\cos 2x + \dfrac{1}{8}\cos 4x$

b $\dfrac{1}{32}\sin 4x + \dfrac{1}{4}\sin 2x + \dfrac{3}{8}x + c$

Exercise 11D

1 a $\frac{1}{2}\ln|x^2 + 4| + c$ b $\frac{1}{2}\ln|e^{2x} + 1| + c$

c $-\frac{1}{4}(x^2 + 4)^{-2} + c$ d $-\frac{1}{4}(e^{2x} + 1)^{-2} + c$

e $\frac{1}{2}\ln|3 + \sin 2x| + c$ f $\frac{1}{4}(3 + \cos 2x)^{-2} + c$

g $\frac{1}{2}e^{x^2} + c$ h $\frac{1}{10}(1 + \sin 2x)^5 + c$

i $\frac{1}{3}\tan^3 x + c$ j $\tan x + \frac{1}{3}\tan^3 x + c$

2 a $\frac{1}{10}(x^2 + 2x + 3)^5 + c$ b $-\frac{1}{4}\cot^2 2x + c$

c $\frac{1}{18}\sin^6 3x + c$ d $e^{\sin x} + c$

e $\frac{1}{2}\ln|e^{2x} + 3| + c$ f $\frac{1}{5}(x^2 + 1)^{\frac{5}{2}} + c$

g $\frac{2}{3}(x^2 + x + 5)^{\frac{3}{2}} + c$ h $2(x^2 + x + 5)^{\frac{1}{2}} + c$

i $-\frac{1}{2}(\cos 2x + 3)^{\frac{1}{2}} + c$ j $-\frac{1}{4}\ln|\cos 2x + 3| + c$

3 a 468 b $2\ln 3$ c $\frac{1}{2}\ln\left(\frac{16}{5}\right)$ d $\frac{1}{4}(e^4 - 1)$

4 $k = 2$ 5 $\theta = \dfrac{\pi}{2}$

6 a $\ln|\sin x| + c$

b $\int \tan x\,dx = -\ln|\cos x| + c = \ln\left|\dfrac{1}{\cos x}\right| + c$

$= \ln|\sec x| + c$

Exercise 11E

1 a $\frac{2}{5}(1 + x)^{\frac{5}{2}} - \frac{2}{3}(1 + x)^{\frac{3}{2}} + c$ b $-\ln|1 - \sin x| + c$

c $\dfrac{\cos^3 x}{3} - \cos x + c$ d $\ln\left|\dfrac{\sqrt{x} - 2}{\sqrt{x} + 2}\right| + c$

e $\frac{2}{5}(1 + \tan x)^{\frac{5}{2}} - \frac{2}{3}(1 + \tan x)^{\frac{3}{2}} + c$

f $\tan x + \frac{1}{3}\tan^3 x + c$

2 a $\frac{506}{15}$ b $\frac{392}{5}$ c $\frac{14}{9}$ d $\frac{16}{3} - 2\sqrt{3}$ e $\frac{1}{2}\ln\frac{9}{5}$

3 a $\dfrac{(3 + 2x)^7}{28} - \dfrac{(3 + 2x)^6}{8} + c$ b $\frac{2}{3}(1 + x)^{\frac{3}{2}} - 2\sqrt{1 + x} + c$

c $\sqrt{x^2 + 4} + \ln\left|\dfrac{\sqrt{x^2 + 4} - 2}{\sqrt{x^2 + 4} + 2}\right| + c$

4 a $\frac{886}{15}$ b $2 + 2\ln\frac{2}{3}$ c $2 - 2\ln 2$

5 $\frac{592}{3}$

6 $\displaystyle\int_{\ln 3}^{\ln 4} \frac{e^{4x}}{e^x - 2}\,dx = \int_1^{\sqrt{2}} \frac{2(u^2 + 2)^3}{u}\,du$

$= \displaystyle\int_1^{\sqrt{2}} \left(2u^5 + 12u^3 + 24u + \frac{16}{u}\right) du$

$= \left[\dfrac{1}{3}u^6 + 3u^4 + 12u^2 + 16\ln u\right]_1^{\sqrt{2}}$

$= \left(\frac{116}{3} + 16\ln\sqrt{2}\right) - \left(\frac{46}{3} + 16\ln 1\right)$

$= \frac{70}{3} + 16\ln\sqrt{2} = \frac{70}{3} + 8\ln 2 \Rightarrow$

$a = 70, b = 3, c = 8, d = 2$

7 $x = \cos\theta, \frac{dx}{d\theta} = -\sin\theta$

$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \int -\frac{1}{\sin\theta}(-\sin\theta)\,d\theta$

$= \int 1\,d\theta = \theta + c = \arccos x + c$

8 $\int_0^{\frac{\pi}{3}} \sin^3 x\cos^2 x\,dx = \int_1^{\frac{1}{2}}(u^2-1)u^2\,du = \int_1^{\frac{1}{2}}(u^4 - u^2)\,du$

$= \left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right]_1^{\frac{1}{2}} = \frac{47}{480}$

9 $\frac{2\pi + 3\sqrt{3}}{96}$

Challenge

$x = 3\sin u, \frac{dx}{du} = 3\cos u \Rightarrow dx = 3\cos u\,du$

$(3\sin u)^2 + (3\cos u)^2 = 9$

$\Rightarrow x^2 + (3\cos u)^2 = 9 \Rightarrow \cos u = \frac{\sqrt{9-x^2}}{3}$

$\int \frac{1}{x^2\sqrt{9-x^2}}\,dx = \int \frac{1}{9\sin^2 u\cos u}(3\cos u)\,du$

$= \frac{1}{9}\int \operatorname{cosec}^2 u\,du = -\frac{1}{9}\cot u + c = -\frac{\cos u}{9\sin u} + c$

$= -\frac{\frac{\sqrt{9-x^2}}{3}}{3x} + c = -\frac{\sqrt{9-x^2}}{9x} + c$

Exercise 11F

1 **a** $-x\cos x + \sin x + c$ **b** $xe^x - e^x + c$
c $x\tan x - \ln|\sec x| + c$ **d** $x\sec x - \ln|\sec x + \tan x| + c$
e $-x\cot x + \ln|\sin x| + c$

2 **a** $3x\ln x - 3x + c$ **b** $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$
c $-\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c$ **d** $x(\ln x)^2 - 2x\ln x + 2x + c$
e $\frac{x^3}{3}\ln x - \frac{x^3}{9} + x\ln x - x + c$

3 **a** $-e^{-x}x^2 - 2xe^{-x} - 2e^{-x} + c$
b $x^2\sin x + 2x\cos x - 2\sin x + c$
c $x^2(3+2x)^6 - \frac{x(3+2x)^7}{7} + \frac{(3+2x)^8}{112} + c$
d $-x^2\cos 2x + x\sin 2x + \frac{1}{2}\cos 2x + c$
e $x^2\sec^2 x - 2x\tan x + 2\ln|\sec x| + c$

4 **a** $2\ln 2 - \frac{3}{4}$ **b** 1 **c** $\frac{\pi}{2} - 1$
d $\frac{1}{2}(1 - \ln 2)$ **e** 9.8 **f** $2\sqrt{2}\pi + 8\sqrt{2} - 16$
g $\frac{1}{2}(1 - \ln 2)$

5 **a** $\frac{1}{16}(4x\sin 4x + \cos 4x) + c$
b $\frac{1}{32}((1 - 8x^2)\cos 4x + 4x\sin 4x)) + c$

6 **a** $-\frac{2}{3}(8-x)^{\frac{3}{2}} + c$
b $u = x - 2 \Rightarrow \frac{du}{dx} = 1;\ \frac{dv}{dx} = \sqrt{8-x} \Rightarrow v = -\frac{2}{3}(8-x)^{\frac{3}{2}}$

$I = (x-2)\left(-\frac{2}{3}(8-x)^{\frac{3}{2}}\right) - \int -\frac{2}{3}(8-x)^{\frac{3}{2}}\,dx$

$= -\frac{2}{3}(x-2)(8-x)^{\frac{3}{2}} + \frac{2}{3}\int(8-x)^{\frac{3}{2}}\,dx$

$= -\frac{2}{3}(x-2)(8-x)^{\frac{3}{2}} - \frac{4}{15}(8-x)^{\frac{5}{2}} + c$

$= -\frac{2}{3}(x-2)(8-x)^{\frac{3}{2}} - \frac{4}{15}(8-x)(8-x)^{\frac{3}{2}} + c$

$= (8-x)^{\frac{3}{2}}\left(-\frac{2}{3}(x-2) - \frac{4}{15}(8-x)\right) + c$

$= (8-x)^{\frac{3}{2}}\left(-\frac{2x}{5} - \frac{4}{5}\right) + c = -\frac{2}{5}(8-x)^{\frac{3}{2}}(x+2) + c$

c 15.6

7 **a** $\frac{1}{3}\tan 3x + c$ **b** $\frac{1}{3}x\tan 3x - \frac{1}{9}\ln|\sec 3x| + c$

c $\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} x\sec^2 3x = \left[\frac{1}{3}x\tan 3x - \frac{1}{9}\ln|\sec 3x|\right]_{\frac{\pi}{18}}^{\frac{\pi}{9}}$

$= \left(\frac{\sqrt{3}\pi}{27} - \frac{1}{9}\ln 2\right) - \left(\frac{\sqrt{3}\pi}{162} - \frac{1}{9}\ln\frac{2}{\sqrt{3}}\right)$

$= \frac{5\sqrt{3}\pi}{162} - \frac{1}{9}\ln 2 + \frac{1}{9}\ln 2 - \frac{1}{9}\ln\sqrt{3}$

$= \frac{5\sqrt{3}\pi}{162} - \frac{1}{18}\ln 3 \Rightarrow p = \frac{5\sqrt{3}}{162}$ and $q = \frac{1}{18}$

Exercise 11G

1 **a** $\ln|(x+1)^2(x+2)| + c$ **b** $\ln|(x-2)\sqrt{2x+1}| + c$
c $\ln\left|\frac{(x+3)^3}{x-1}\right| + c$ **d** $\ln\left|\frac{2+x}{1-x}\right| + c$

2 **a** $x + \ln|(x+1)^2\sqrt{2x-1}| + c$ **b** $\frac{x^2}{2} + x + \ln\left|\frac{x^2}{(x+1)^3}\right| + c$
c $x + \ln\left|\frac{x-2}{x+2}\right| + c$ **d** $-x + \ln\left|\frac{(3+x)^2}{1-x}\right| + c$

3 **a** $A = 2, B = 2$ **b** $\ln\left|\frac{2x+1}{1-2x}\right| + c$ **c** $\ln\frac{5}{9}$, so $k = \frac{5}{9}$

4 **a** $f(x) = \frac{2}{3+2x} + \frac{1}{2-x} + \frac{1}{(2-x)^2}$ **b** $\frac{1}{2} + \ln\frac{10}{3}$

5 **a** $A = 1, B = 2, C = -2$ **b** $a = \frac{2}{3}, b = -\frac{4}{3}, c = 3$

6 **a** $f(x) = \frac{3}{x^2} - \frac{1}{x+2}$ **b** $a = \frac{3}{4}, b = \frac{2}{3}$

7 **a** $f(x) = 2 - \frac{3}{4x+1} + \frac{3}{4x-1}$, so $A = 2, B = -3$ and $C = 3$
b $k = \frac{3}{4}, m = \frac{35}{27}$

Exercise 11H

1 **a** $2\ln 2$ **b** $\ln(2+\sqrt{3})$ **c** $2\ln 2 - 1$
d $\sqrt{2} - 1$ **e** $\frac{8}{3}$

2 **a** $\ln\frac{8}{5}$ **b** $\ln 3 - \frac{2}{3}$ **c** 1
d $\frac{2\sqrt{2}-1}{3}$ **e** $\frac{1}{2}(1 - \ln 2)$

3 $\ln 4$ **4** $2e^2 - 2e + \ln 2$

5 **a** $A(0, 0), B(\pi, 0)$ and $C(2\pi, 0)$
b Area $= \int_0^{\pi} x\sin x\,dx - \int_{\pi}^{2\pi} x\sin x\,dx$

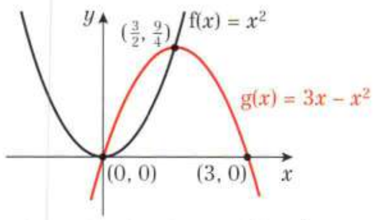
$= [-x\cos x + \sin x]_0^{\pi} - [-x\cos x + \sin x]_{\pi}^{2\pi}$

$= \pi + 3\pi = 4\pi$

6 **a** $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + c$ **b** $\frac{2}{3}(4\ln 2 - 1)$

7 **a** $A\left(-\frac{\pi}{2}, 0\right), B\left(\frac{\pi}{2}, 0\right), C\left(\frac{3\pi}{2}, 0\right)$ and $D(0, 3)$
b $2(\sin x + 1)^{\frac{3}{2}} + c$ **c** $a = 32$

8 **a**



b $\frac{9}{8}$

9 **a** $A\left(-\frac{\pi}{3}, 3\right)$, $B\left(\frac{\pi}{3}, 3\right)$ and $C\left(\frac{5\pi}{3}, 3\right)$

b $a = 4$, $b = -4$, $c = 3$ (or $a = 4$, $b = 4$, $c = -3$)

c $R_2 = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (-2\cos x + 4)\,dx - \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2\cos x + 2)\,dx$

$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (-4\cos x + 2)\,dx = \left[-4\sin x + 2x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$

$= \left(2\sqrt{3} + \frac{10\pi}{3}\right) - \left(-2\sqrt{3} + \frac{2\pi}{3}\right) = 4\sqrt{3} + \frac{8\pi}{3}$

$R_2 : R_1 \Rightarrow 4\sqrt{3} + \frac{8\pi}{3} : 4\sqrt{3} - \frac{4\pi}{3} \Rightarrow 3\sqrt{3} + 2\pi : 3\sqrt{3} - \pi$

10 $y = \sin\theta$: Area $= \int_0^{\pi} 2\sin\theta\,d\theta = [-2\cos\theta]_0^{\pi} = (2) - (-2) = 4$

$y = \sin 2\theta$: Area $= \int_0^{\frac{\pi}{2}} 4\sin 2\theta\,d\theta = \left[-2\cos 2\theta\right]_0^{\frac{\pi}{2}}$

$= (2) - (-2) = 4$

11 **a** $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

b **i** $\sqrt{2} - 1$ **ii** $2 - \sqrt{2}$ **iii** $\sqrt{2}$

c $R_1 : R_2 \Rightarrow \sqrt{2} - 1 : 2 - \sqrt{2}$

$\Rightarrow (\sqrt{2} - 1)(2 + \sqrt{2}) : (2 - \sqrt{2})(2 + \sqrt{2}) \Rightarrow \sqrt{2} : 2$

12 Area $= \int y \frac{dx}{dt}\,dt = \int_0^{\sqrt[3]{4}} t^2(3t^2)\,dt = \frac{3}{5}(\sqrt[3]{4})^5 = \frac{3}{5}2^{\frac{10}{3}}$

$= \frac{3}{5}(2^3)(2^{\frac{1}{3}}) = \frac{24}{5}\sqrt[3]{2}$

13 $\frac{2}{3}$

14 **a** $x + y = 16$ **b** 58.9

Challenge

Area of region $R = \frac{2 - \sqrt{2}}{2}$.

Exercise 11I

1 **a** 1.1260, 1.4142 **b** **i** 1.352 **ii** 1.341

2 **a** 0.7071, 0.7071 **b** 0.758

c The shape of the graph is concave, so the trapezium lines will underestimate the area.

d 0.8 **e** 5.25%

3 **a** 0.427 **b** 1.04

4 **a** 1, 1.4581 **b** **i** 2.549 **ii** 2.402

c Increasing the number of values decreases the interval. This leads to the approximation more closely following the curve.

d $\int x\ln x\,dx - \int 2\ln x\,dx + \int 1\,dx$

$= \left[\left(\frac{1}{2}x^2 - 2x\right)\ln x - \frac{1}{4}x^2 + 3x\right]_1^3$

$= \left(-\frac{3}{2}\ln 3 + \frac{27}{4}\right) - \left(\frac{11}{4}\right) = -\frac{3}{2}\ln 3 + 4$

5 **a** 1.0607 **b** 1.337 **c** $p = \frac{16}{15}$, $q = \frac{1}{2}$ **d** 11.4%

6 **a** $4x - 5 = 0$, $x = \frac{5}{4}$ **b** 0.3556

c 0.7313 **d** $\ln\left(\frac{49}{24}\right)$ **e** 2.5%

7 **a** 4.1133, 5.6522, 7.3891 **b** 23.25

c $t = \sqrt{2x + 1} \Rightarrow \frac{dt}{dx} = (2x + 1)^{-\frac{1}{2}}$

$\Rightarrow (2x + 1)^{\frac{1}{2}}\,dt = dx \Rightarrow t\,dt = dx$

$\int_0^3 e^{(2x+1)^{1/2}}\,dx = \int_1^{\sqrt{7}} te^t\,dt \Rightarrow a = 1, b = \sqrt{7}, k = 1$

d 23.20

Exercise 11J

1 **a** $y = Ae^{x - x^2} - 1$ **b** $y = k\sec x$

c $y = \frac{-1}{\tan x - x + c}$ **d** $y = \ln|2e^x + c|$

2 **a** $\frac{1}{24} - \frac{\cos^3 x}{3}$ **b** $\sin 2y + 2y = 4\tan x - 4$

c $\tan y = \frac{1}{2}\sin 2x + x + 1$ **d** $y = \arccos(e^{-\tan x})$

3 **a** $y = Axe^{-\frac{1}{x}}$ **b** $y = -e^3xe^{-\frac{1}{x}} = -xe^{\left(\frac{3x-1}{x}\right)}$

4 $y = \sqrt{\frac{x}{x + 1}}$ **5** $\ln|2 + e^y| = -xe^{-x} - e^{-x} + c$

6 $y = \frac{3}{1 - x}$ **7** $y = \frac{3(1 + x^2) + 1}{3(1 + x^2) - 1}$

8 $y = \ln\left|\frac{x^2 - 12}{2}\right|$ **9** $\tan y = x + \frac{1}{2}\sin 2x + \frac{2 - \pi}{4}$

10 $\ln|y| = -x\cos x + \sin x - 1$

11 **a** $3x + 4\ln|x| + c$ **b** $y = \left(\frac{3}{2}x + 2\ln|x| + \frac{5}{2}\right)^2$

12 **a** $\frac{5}{3x - 8} + \frac{1}{x - 2}$

b $\ln|y| = \frac{5}{3}\ln|3x - 8| + \ln|x - 2| + c$

c $y = 8(x - 2)(3x - 8)^{\frac{5}{3}}$

13 **a** $y = x^2 - 4x + c$

b Graphs of the form $y = x^2 - 4x + c$, where c is any real number

14 **a** $y = \frac{1}{x + 2} + c$

b Graphs of the form $y = \frac{1}{x + 2} + c$, where c is any real number

c $y = \frac{1}{x + 2} + 3$

15 **a** $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow \int y\,dy = \int -x\,dx$

$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + b \Rightarrow y^2 + x^2 = c$

b Circles with centre (0, 0) and radius $\sqrt{c}$, where c is any positive real number.

c $y^2 + x^2 = 49$

Exercise 11K

1 **a** $y = 200e^{kt}$ **b** 1 year

c The population could not increase in size in this way forever due to limitations such as available food or space.

2 **a** $M = \frac{e^t}{1 + e^t}$ **b** $\frac{2}{3}$ **c** M approaches 1

3 **a** $\frac{dx}{dt} = \frac{k}{x^2} \Rightarrow \frac{1}{3}x^3 = kt + c$

$t = 0, x = 1 \Rightarrow c = \frac{1}{3} \Rightarrow t = 20, x = 2 \Rightarrow k = \frac{7}{60}$

$\frac{1}{3}x^3 = \frac{7}{60}t + \frac{1}{3} \Rightarrow x = \sqrt[3]{\left(\frac{7}{20}t + 1\right)}$

b $x = 3$, $t = 74.3$ days. So it takes 54.3 days to increase from 2 cm to 3 cm.

4 **a** The difference in temperature is $T - 25$. The tea is cooling, so there should be a negative sign. k has to be positive or the tea would be warming.

b 46.2°C

5 **a** $\int A^{-\frac{1}{2}}\,\mathrm{d}A = \frac{1}{10}\int t^{-2}\mathrm{d}t \Rightarrow \frac{-2}{\sqrt{A}} = \frac{-1}{10t} + C \Rightarrow C = -\frac{19}{10}$

$\Rightarrow \frac{-2}{\sqrt{A}} = \frac{-1}{10t} - \frac{19}{10} \Rightarrow \sqrt{A} = \left(\frac{-20t}{-1-19t}\right)$

$\Rightarrow A = \left(\frac{20t}{1+19t}\right)^2$

b As t $\to \infty$, $A \to \left(\frac{20}{19}\right)^2 = \frac{400}{361}$ from below

6 **a** $V = 6000h \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}h} = 6000,\ \frac{\mathrm{d}V}{\mathrm{d}t} = 12000 - 500h,$

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{6000}(12000 - 500h)$

$60\frac{\mathrm{d}h}{\mathrm{d}t} = 120 - 5h$

b $t = 12\ln\left(\frac{9}{7}\right)$

7 **a** $\frac{\left(\frac{1}{10000}\right)}{P} + \frac{\left(\frac{1}{10000}\right)}{10000 - P}$

b $P = \frac{10000}{1 + 3\mathrm{e}^{-0.5t}}$ so $a = 10000$, $b = 1$ and $c = 3$.

c 10 000 deer

8 **a** $\frac{dV}{dt} = 40 - \frac{1}{4}V \Rightarrow -4\frac{dV}{dt} = V - 160$

b $V = 160 + 4840\mathrm{e}^{-\frac{1}{4}t}$, $a = 160$ and $b = 4840$

c $V \to 160$

9 **a** $\frac{\mathrm{d}R}{\mathrm{d}t} = -kR \Rightarrow \int\frac{1}{R}\mathrm{d}R = -k\int\mathrm{d}t$

$\Rightarrow \ln R = -kt + c \Rightarrow R = \mathrm{e}^{-kt+c}$

$\Rightarrow R = A\mathrm{e}^{-kt} \Rightarrow R_0 = A\mathrm{e}^0 \Rightarrow A = R_0 \Rightarrow R = R_0\mathrm{e}^{-kt}$

b $k = \frac{1}{5730}\ln 2$

c $0.1R_0 = R_0\mathrm{e}^{\frac{1}{5730}\ln\left(\frac{1}{2}\right)\times t}$

$\ln(0.1) = \frac{1}{5730}\ln\left(\frac{1}{2}\right)\times t \Rightarrow t \approx 19035$

Mixed exercise 11

1 **a** $\frac{1}{16}(2x-3)^8 + c$ **b** $\frac{1}{40}(4x-1)^{\frac{5}{2}} + \frac{1}{24}(4x-1)^{\frac{3}{2}} + c$

c $\frac{1}{3}\sin^3 x + c$ **d** $\frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$

e $-\frac{1}{4}\ln|\cos 2x| + c$ **f** $-\frac{1}{4}\ln|3 - 4x| + c$

2 **a** $\frac{995085}{4}$ **b** $\frac{1}{4}\pi - \frac{1}{2}\ln 2$ **c** $\frac{992}{5} - 2\ln 4$

d $\frac{\sqrt{3}-1}{4}$ **e** $\frac{1}{4}\ln\left(\frac{35}{19}\right)$ **f** $\ln\left(\frac{4}{3}\right)$

3 **a** $\int\frac{1}{x^2}\ln x\,\mathrm{d}x = (\ln x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\mathrm{d}x$

$= -\frac{\ln x}{x} + \int\frac{1}{x^2}\mathrm{d}x = -\frac{\ln x}{x} - \frac{1}{x} + c$

$\int_1^{\mathrm{e}}\frac{1}{x^2}\ln x\,\mathrm{d}x = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^{\mathrm{e}} = \left(-\frac{1}{\mathrm{e}} - \frac{1}{\mathrm{e}}\right) - (0 - 1) = 1 - \frac{2}{\mathrm{e}}$

b $\frac{1}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1} \Rightarrow A = -\frac{1}{3}, B = \frac{2}{3}$

$\int_1^p\frac{1}{(x+1)(2x-1)}\,\mathrm{d}x = \int_1^p\left(-\frac{1}{3(x+1)} + \frac{2}{3(2x-1)}\right)\mathrm{d}x$

$= \left[-\frac{1}{3}\ln(x+1) + \frac{1}{3}\ln(2x-1)\right]_1^p = \left[\frac{1}{3}\ln\left(\frac{2x-1}{x+1}\right)\right]_1^p$

$= \left(\frac{1}{3}\ln\left(\frac{2p-1}{p+1}\right)\right) - \left(\frac{1}{3}\ln\left(\frac{1}{2}\right)\right)$

$= \frac{1}{3}\ln\left(\frac{2(2p-1)}{p+1}\right) = \frac{1}{3}\ln\left(\frac{4p-2}{p+1}\right)$

4 $b = 2$ **5** $\theta = \frac{\pi}{3}$

6 **a** $\frac{2}{3}(x-2)\sqrt{x+1} + c$ **b** $\frac{8}{3}$

7 **a** $-\frac{1}{8}x\cos 8x + \frac{1}{64}\sin 8x + c$

b $\frac{1}{8}x^2\sin 8x + \frac{1}{32}x\cos 8x - \frac{1}{256}\sin 8x + c$

8 **a** $A = \frac{1}{2}, B = 2, C = -1$

b $\frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1} + c$

c $\int_4^9 \mathrm{f}(x)\,\mathrm{d}x = \left[\frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1}\right]_4^9$

$= \left(\frac{1}{2}\ln 9 + 2\ln 8 + \frac{1}{8}\right) - \left(\frac{1}{2}\ln 4 + 2\ln 3 + \frac{1}{3}\right)$

$= \left(\ln 3 + \ln 64 + \frac{1}{8}\right) - \left(\ln 2 + \ln 9 + \frac{1}{3}\right)$

$= \ln\left(\frac{3\times 64}{2\times 9}\right) - \frac{5}{24} = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$

9 **a** $x = 4, y = 20$

b $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{96}{x^3}$

when $x = 4$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{15}{8} > 0 \Rightarrow$ minimum

c $\frac{62}{5} + 48\ln 4$; $p = \frac{62}{5}$, $q = 48$, $r = 4$

10 **a** $\frac{1}{3}x^3\ln 2x - \frac{1}{9}x^3 + c$

b $\left[\frac{1}{3}x^3\ln 2x - \frac{1}{9}x^3\right]_{\frac{1}{2}}^3 = (9\ln 6 - 3) - \left(0 - \frac{1}{72}\right)$

$= 9\ln 6 - \frac{215}{72}$

11 **a** $(1 + \sin 2x)^2 \equiv 1 + 2\sin 2x + \sin^2 2x$

$\equiv 1 + 2\sin 2x + \frac{1 - \cos 4x}{2} \equiv \frac{3}{2} + 2\sin 2x - \frac{\cos 4x}{2}$

$\equiv \frac{1}{2}(3 + 4\sin 2x - \cos 4x)$

b $\frac{9\pi}{8} + 1$ **c** $\left(\frac{\pi}{4}, 4\right)$

12 **a** $-x\mathrm{e}^{-x} - \mathrm{e}^{-x} + c$ **b** $\cos 2y = 2\mathrm{e}^{-x}(x - \mathrm{e}^x + 1)$

13 **a** $-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

b $\tan y = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x - \frac{1}{4}$

14 **a** $-\frac{1}{y} = \frac{1}{2}x^2 + c$

b $x = 1: -\frac{1}{1} = \frac{1}{2} + c \Rightarrow c = -\frac{3}{2}$

$-\frac{1}{y} = \frac{1}{2}x^2 - \frac{3}{2} \Rightarrow \frac{1}{y} = \frac{1}{2}(3 - x^2) \Rightarrow y = \frac{2}{3 - x^2}$

c 1 **d** $y = x$; $(-2, -2)$

15 **a** $\ln|1 + 2x| + \frac{1}{1 + 2x} + c$

b $2y - \sin 2y = \ln|1 + 2x| + \frac{1}{1+2x} + \frac{\pi}{2} - 2$

16 **a** $A_1 = \frac{1}{4} - \frac{1}{2\mathrm{e}}, A_2 = \frac{1}{4}$

b $A_1 : A_2 \Rightarrow \frac{1}{4} - \frac{1}{2\mathrm{e}} : \frac{1}{4} \Rightarrow 1 - \frac{2}{\mathrm{e}} : 1 \Rightarrow (\mathrm{e} - 2) : \mathrm{e}$

17 **a** $-\mathrm{e}^{-x}(x^2 + 2x + 2) + c$

b $y = -\frac{1}{3}\ln|3\mathrm{e}^{-x}(x^2 + 2x + 2) - 5|$

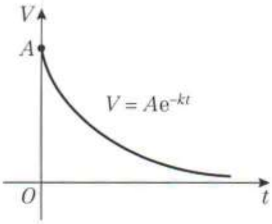
18 **a** $\frac{1}{3}\ln 7$

b $\int_0^{\frac{1}{3}\ln 7}(\mathrm{e}^{3x} + 1)\,\mathrm{d}x = \left[\frac{\mathrm{e}^{3x}}{3} + x\right]_0^{\frac{1}{3}\ln 7}$

$= \left(\frac{7}{3} + \frac{1}{3}\ln 7\right) - \left(\frac{1}{3} - 0\right) = 2 + \frac{1}{3}\ln 7$

19 a $A = 1, B = \frac{1}{2}, C = -\frac{1}{2}$
b $x + \frac{1}{2}\ln|x - 1| - \frac{1}{2}\ln|x + 1| = 2t - \frac{1}{2}\ln 3$

20 a 4.00932 **b** 2.6254
c The curve is convex, so it is an overestimate.
d $\frac{e^3 - 3e + 2}{2e}$, $P = -3, Q = 2$ **e** 2.5%

21 a $\frac{dV}{dt} = -kV \Rightarrow \int \frac{1}{V}\,dV = \int -k\,dt \Rightarrow \ln V = -kt + c$
$\Rightarrow V = Ae^{-kt}$
b


c $\frac{1}{2}A = Ae^{-kT} \Rightarrow \frac{1}{2} = e^{-kT} \Rightarrow 2 = e^{kT} \Rightarrow \ln 2 = kT$

22 a $\int (k - y)\,dy = \int x\,dx \Rightarrow ky - \frac{1}{2}y^2 = \frac{1}{2}x^2 + c$
$x^2 + (y - k)^2 = C$
b Concentric circles with centre (0, 2).

23 a 0.9775 **b** 3.074
c Use more values, use smaller intervals. The lines would then more closely follow the curve.
d $\int_1^4 \left(\frac{1}{5}x^2\right)\ln x - x + 2\,dx$
$= \left[\frac{1}{15}x^3\ln x - \frac{1}{45}x^3 - \frac{1}{2}x^2 + 2x\right]_1^4$
$= \left(\frac{64}{15}\ln 4 - \frac{64}{45}\right) - \left(-\frac{1}{45} - \frac{1}{2} + 2\right) = \frac{-29}{10} + \frac{64}{15}\ln 4$
e 2.0%

24 a $\frac{(1 + 2x^2)^6}{24} + c$ **b** $\tan 2y = \frac{1}{12}(1 + 2x^2)^6 + \frac{11}{12}$

25 $\arctan x + c$ **26** $y^2 = \frac{8x}{x + 2}$

27 a $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$
$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{2\pi r} \times k\sin\left(\frac{t}{3\pi}\right) = \frac{k}{2\pi r}\sin\left(\frac{t}{3\pi}\right)$
b $r^2 = -6\cos\left(\frac{t}{3\pi}\right) + 7$ **c** 6 days, 5 hours

28 a $\frac{\pi}{2}$ **b** $\frac{3\pi}{2}$

29 a $-\frac{4}{5}$ **b** $y - 2\sqrt{2} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$ **c** $10 - \frac{5\pi}{2}$

30 $\frac{41}{60}$

Challenge
a 15 **b** −3

CHAPTER 12

Prior knowledge 12

1 a $5\mathbf{i}$ **b** $-13\mathbf{i} + 11\mathbf{j}$
2 a $\sqrt{34}$ **b** $\frac{5}{\sqrt{34}}\mathbf{i} - \frac{3}{\sqrt{34}}\mathbf{j}$
3 a $-2\mathbf{i} - \frac{1}{2}\mathbf{j}$ **b** $3\mathbf{i} + \frac{3}{4}\mathbf{j}$

Exercise 12A

1 $2\sqrt{21}$ **2** $7\sqrt{3}$
3 a $\sqrt{14}$ **b** 15 **c** $5\sqrt{2}$ **d** $\sqrt{30}$
4 $k = 5$ or $k = 9$ **5** $k = 10$ or $k = -4$

Challenge
a (1, −3, 4), (1, −3, −2), (7, 3, 4), (7, 3, −2), (7, −3, −2)
b $6\sqrt{5}$

Exercise 12B

1 a i $\begin{pmatrix}-3\\5\\-9\end{pmatrix}$ **ii** $\begin{pmatrix}11\\-11\\19\end{pmatrix}$
b $\mathbf{a} - \mathbf{b}$ is parallel as $-2(\mathbf{a} - \mathbf{b}) = 6\mathbf{i} - 10\mathbf{j} + 18\mathbf{k}$
$-\mathbf{a} + 3\mathbf{b}$ is not parallel as it is not a multiple of $6\mathbf{i} - 10\mathbf{j} + 18\mathbf{k}$.

2 $3\mathbf{a} + 2\mathbf{b} = 3\begin{pmatrix}3\\2\\-1\end{pmatrix} + 2\begin{pmatrix}-3\\-2\\4\end{pmatrix} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} = \frac{1}{2}(6\mathbf{i} + 4\mathbf{j} + 10\mathbf{k})$

3 $p = 2, q = 1, r = 2$

4 a $\sqrt{35}$ **b** $2\sqrt{5}$ **c** $\sqrt{3}$ **d** $\sqrt{170}$ **e** $5\sqrt{3}$

5 a $\begin{pmatrix}7\\1\\-1\end{pmatrix}$ **b** $\begin{pmatrix}-5\\5\\-5\end{pmatrix}$ **c** $\begin{pmatrix}14\\-3\\1\end{pmatrix}$ **d** $\begin{pmatrix}8\\4\\4\end{pmatrix}$ **e** $\begin{pmatrix}8\\-6\\10\end{pmatrix}$

6 $7\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ **7** 6 or −6 **8** $\sqrt{3}$ or $-\sqrt{3}$

9 a i A: $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, B: $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, C: $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
ii $-3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
b i $\sqrt{14}$ **ii** 3

10 a $-4\mathbf{i} + 3\mathbf{j} - 12\mathbf{k}$ **b** 13 **c** $-\frac{4}{13}\mathbf{i} + \frac{3}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$

11 a $-6\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ **b** $\sqrt{61}$ **c** $-\frac{6}{\sqrt{61}}\mathbf{i} + \frac{4}{\sqrt{61}}\mathbf{j} + \frac{3}{\sqrt{61}}\mathbf{k}$

12 a $\frac{3}{\sqrt{29}}\mathbf{i} - \frac{4}{\sqrt{29}}\mathbf{j} - \frac{2}{\sqrt{29}}\mathbf{k}$ **b** $\frac{\sqrt{2}}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} - \frac{\sqrt{7}}{5}\mathbf{k}$
c $\frac{\sqrt{5}}{4}\mathbf{i} - \frac{2\sqrt{2}}{4}\mathbf{j} - \frac{\sqrt{3}}{4}\mathbf{k}$

13 a $\overrightarrow{AB} = 4\mathbf{j} - \mathbf{k}$, $\overrightarrow{AC} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\overrightarrow{BC} = 4\mathbf{i} - 3\mathbf{j}$
b $|\overrightarrow{AB}| = \sqrt{17}$, $|\overrightarrow{AC}| = 3\sqrt{2}$, $|\overrightarrow{BC}| = 5$
c scalene

14 a $\overrightarrow{AB} = -2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$, $\overrightarrow{AC} = 4\mathbf{i} - 9\mathbf{j} - \mathbf{k}$,
$\overrightarrow{BC} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$
b $|\overrightarrow{AB}| = 7$, $|\overrightarrow{AC}| = 7\sqrt{2}$, $|\overrightarrow{BC}| = 7$ **c** 45°

15 a i 98.0° **ii** 11.4° **iii** 82.0°
b i 69.6° **ii** 62.3° **iii** 35.5°
c i 56.3° **ii** 90° **iii** 146.3°

16 5.41

17 $|\overrightarrow{PQ}| = \sqrt{14}$, $|\overrightarrow{QR}| = \sqrt{29}$, $|\overrightarrow{PR}| = \sqrt{35}$
Let $\theta = \angle PQR$. $14 + 29 - 2\sqrt{406}\cos\theta = 35$
$\Rightarrow \cos\theta = 0.198... \Rightarrow \theta = 78.5°$ (1 d.p.)

Challenge
25.4°

Exercise 12C

1 a i $|\overrightarrow{OA}| = 9$; $|\overrightarrow{OB}| = 9 \Rightarrow |\overrightarrow{OA}| = |\overrightarrow{OB}|$
ii $\overrightarrow{AC} = \begin{pmatrix}9\\4\\22\end{pmatrix}$, $|\overrightarrow{AC}| = \sqrt{581}$; $\overrightarrow{BC} = \begin{pmatrix}6\\-4\\23\end{pmatrix}$, $|\overrightarrow{BC}| = \sqrt{581}$
Therefore $|\overrightarrow{AC}| = |\overrightarrow{BC}|$
b $OACB$ is a kite

2 a $\overrightarrow{AB} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \Rightarrow |\overrightarrow{AB}| = \sqrt{17}$
$\overrightarrow{AC} = 6\mathbf{j} \Rightarrow |\overrightarrow{AC}| = 6$
$\overrightarrow{BC} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \Rightarrow |\overrightarrow{BC}| = \sqrt{17}$
$|\overrightarrow{AB}| = |\overrightarrow{BC}|$, so ABC is isosceles.
b $6\sqrt{2}$ **c** (0, 4, 7)

3 **a** $\overrightarrow{AB} = 4\mathbf{i} - 10\mathbf{j} - 8\mathbf{k} = 2(2\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$

$\overrightarrow{CD} = -6\mathbf{i} + 15\mathbf{j} + 12\mathbf{k} = -3(2\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$

$\overrightarrow{CD} = -\frac{3}{2}\overrightarrow{AB}$, so AB is parallel to CD

$AB : CD = 2 : 3$

b $ABCD$ is a trapezium

4 $a = \frac{8}{3}, b = -1, c = \frac{3}{2}$

5 $(7, 14, -22), (-7, 14, -22)$ and $\left(\frac{1813}{20}, 14, -22\right)$

6 **a** 18.67 (2 d.p.) **b** 168.07 (2 d.p.)

7 Let H = point of intersection of OF and AG.

$\overrightarrow{OH} = r\overrightarrow{OF} = \overrightarrow{OA} + s\overrightarrow{AG}$

$\overrightarrow{OF} = \mathbf{a} + \mathbf{b} + \mathbf{c}, \overrightarrow{AG} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$

So $r(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a} + s(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$r = 1 - s = s \Rightarrow r = s = \frac{1}{2}$, so $\overrightarrow{OH} = \frac{1}{2}\overrightarrow{OF}$ and $\overrightarrow{AH} = \frac{1}{2}\overrightarrow{AG}$.

8 Show that $\overrightarrow{FP} = \frac{2}{3}\mathbf{a}$ (multiple methods possible)

Show that $\overrightarrow{PE} = \frac{1}{3}\mathbf{a}$ (multiple methods possible)

Therefore FP and PE are parallel, so P lies on FE

$FP:PE = 2:1$

Challenge

1 $p = \frac{24}{11}, q = \frac{32}{11}, r = -4$

2 $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \mathbf{b} + \mathbf{c}, \overrightarrow{BN} = \mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}, \overrightarrow{AF} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$

Let $\overrightarrow{OM}$ and $\overrightarrow{AF}$ intersect at X: $\overrightarrow{AX} = r\overrightarrow{AF} = r(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{OX} = s\overrightarrow{OM} = s\left(\frac{1}{2}\mathbf{a} + \mathbf{b} + \mathbf{c}\right)$ for scalars r and s

$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \mathbf{a} + r(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\Rightarrow s\left(\frac{1}{2}\mathbf{a} + \mathbf{b} + \mathbf{c}\right) = \mathbf{a} + r(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

Comparing coefficients in **a**, **b** and **c** gives $r = s = \frac{2}{3}$

Let $\overrightarrow{BN}$ and $\overrightarrow{AF}$ intersect at Y: $\overrightarrow{AY} = p\overrightarrow{AF} = p(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{BY} = q\overrightarrow{BN} = q\left(\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}\right)$ for scalars p and q

$\overrightarrow{BY} = \overrightarrow{BA} + \overrightarrow{AY} = \mathbf{a} - \mathbf{b} + p(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\Rightarrow q\left(\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}\right) = \mathbf{a} - \mathbf{b} + p(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

Comparing coefficients in **a**, **b** and **c** gives $p = \frac{1}{3}, q = \frac{2}{3}$

$\overrightarrow{AX} = \frac{2}{3}\overrightarrow{AF}, \overrightarrow{AY} = \frac{1}{3}\overrightarrow{AF}$

So the line segments OM and BN trisect the diagonal AF.

Exercise 12D

1 **a** $(5\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ N **b** $\sqrt{42}$ N

2 $2\sqrt{29}$ m

3 **a** $\left(\frac{1}{2}\mathbf{i} - \frac{5}{4}\mathbf{j} + \frac{3}{4}\mathbf{k}\right)$ m s^{-2} **b** 1.54 m s^{-2}

4 $(5\mathbf{i} - 3\mathbf{j} - 7\mathbf{k})$ N

5 **a** $a = -2, b = 4$ **b** $(\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})$ N

c $\left(\frac{1}{2}\mathbf{i} - \frac{3}{2}\mathbf{j} - 2\mathbf{k}\right)$ m s^{-2} **d** $\frac{1}{2}\sqrt{26}$ m s^{-2}

e 126°

6 **a** 1.96 m s^{-2} **b** Descending, 101.3°

Mixed exercise 12

1 $\sqrt{22}$ 2 $a = 5$ or $a = 6$

3 $|\overrightarrow{AB}| = 5\sqrt{2} \Rightarrow 9 + t^2 + 25 = 50 \Rightarrow t^2 = 16 \Rightarrow t = 4$

$6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k} = 6\mathbf{i} - 8\mathbf{j} - 10\mathbf{k} = -2(-3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) = -2\overrightarrow{AB}$

So $\overrightarrow{AB}$ is parallel to $6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k}$

4 **a** $\overrightarrow{PQ} = -3\mathbf{i} - 8\mathbf{j} + 3\mathbf{k}, \overrightarrow{PR} = -3\mathbf{i} - 9\mathbf{j} + 8\mathbf{k}, \overrightarrow{QR} = -\mathbf{j} + 5\mathbf{k}$

b 20.0

5 **a** $\overrightarrow{DE} = 4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \overrightarrow{EF} = -3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}, \overrightarrow{FD} = -\mathbf{i} + \mathbf{j} - 8\mathbf{k}$

b $|\overrightarrow{DE}| = \sqrt{41}, |\overrightarrow{EF}| = \sqrt{41}, |\overrightarrow{FD}| = \sqrt{66}$ **c** isosceles

6 **a** $\overrightarrow{PQ} = 9\mathbf{i} - 4\mathbf{j}, \overrightarrow{PR} = 7\mathbf{i} + \mathbf{j} - 3\mathbf{k}, \overrightarrow{QR} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

b $|\overrightarrow{PQ}| = \sqrt{97}, |\overrightarrow{PR}| = \sqrt{59}, |\overrightarrow{QR}| = \sqrt{38}$ **c** 51.3°

7 31.5°

8 184 (3 s.f.)

9 **a** $(2, -7, -2)$ **b** rhombus **c** 36.1

10 $\overrightarrow{PQ} = \frac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c}), \overrightarrow{RS} = \frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}), \overrightarrow{TU} = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Let $\overrightarrow{PQ}$, $\overrightarrow{RS}$ and $\overrightarrow{TU}$ intersect at X: $\overrightarrow{PX} = r\overrightarrow{PQ} = \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{RX} = s\overrightarrow{RS} = \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{TX} = t\overrightarrow{TU} = \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$ for scalars r, s and t

$\overrightarrow{RX} = \overrightarrow{RO} + \overrightarrow{OP} + \overrightarrow{PX} = \frac{1}{2}(-\mathbf{a} + \mathbf{c}) + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\Rightarrow \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = \frac{1}{2}(-\mathbf{a} + \mathbf{c}) + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

Comparing coefficients in **a**, **b** and **c** gives $r = s = \frac{1}{2}$

$\overrightarrow{TX} = \overrightarrow{TO} + \overrightarrow{OP} + \overrightarrow{PX} = \frac{1}{2}(-\mathbf{b} + \mathbf{c}) + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\Rightarrow \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c}) = \frac{1}{4}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Comparing coefficients in **a**, **b** and **c** gives $t = \frac{1}{2}$

So the line segments PQ, RS and TU meet at a point and bisect each other.

11 $b = 1$ or $\frac{17}{3}$

12 **a** Air resistance acts in opposition to the motion of the BASE jumper. The motion downwards will be greater than the motion in the other directions.

b $(16\mathbf{i} + 13\mathbf{j} - 40\mathbf{k})$ N **c** 20 seconds

Challenge

If $\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{b} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{c} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, then $\mathbf{a} + \mathbf{b} + \mathbf{c} = 2\mathbf{a} + 2\mathbf{b} + 0\mathbf{c}$.

Review exercise 3

1 $\frac{dy}{dx} = x - 4\sin x$

$x = \frac{\pi}{2}, \frac{dy}{dx} = \frac{\pi}{2} - 4, y = \frac{\pi^2}{8}, m_n = -\frac{1}{\frac{\pi}{2} - 4}$

$y - \frac{\pi^2}{8} = -\frac{1}{\frac{\pi}{2} - 4}\left(x - \frac{\pi}{2}\right)$

$\Rightarrow 8y(8 - \pi) - 16x + \pi(\pi^2 - 8\pi + 8) = 0$

2 $\frac{dy}{dx} = 3\,e^{3x} - \frac{2}{x}, x = 2, y = e^6 - \ln 4, \frac{dy}{dx} = 3e^6 - 1$

$y - e^6 + \ln 4 = (3\,e^6 - 1)(x - 2)$

$\Rightarrow y - (3\,e^6 - 1)x - 2 + \ln 4 + 5e^6 = 0$

3 $8x + 36y + 19 = 0$

4 **a** $\frac{dy}{dx} = 4(2x - 3)(e^{2x}) + 2(2x - 3)^2(e^{2x})$

$= 2(e^{2x})(2x - 3)(2x - 1)$

b $\left(\frac{3}{2}, 0\right)$ and $\left(\frac{1}{2}, 4e\right)$

5 **a** $\frac{dy}{dx} = \frac{(x - 1)(2\sin x + \cos x - x\cos x)}{\sin^2 x}$

b $x = \frac{\pi}{2}, y = \left(\frac{\pi}{2} - 1\right)^2, \frac{dy}{dx} = 2\left(\frac{\pi}{2} - 1\right)$

$y - \left(\frac{\pi}{2} - 1\right)^2 = (\pi - 2)\left(x - \frac{\pi}{2}\right)$

$\Rightarrow y = (\pi - 2)x + \left(1 - \frac{\pi^2}{4}\right)$

6 **a** $y = \operatorname{cosec} x = \dfrac{1}{\sin x}$

$\dfrac{dy}{dx} = -\dfrac{\cos x}{\sin^2 x} = -\dfrac{1}{\sin x} \times \dfrac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$

b $\dfrac{dy}{dx} = -\dfrac{1}{6x\sqrt{x^2 - 1}}$

7 $y = \arcsin x \Rightarrow x = \sin y$

$\dfrac{dx}{dy} = \cos y \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y}$

$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$

8 **a** $-2\sin^3 t \cos t$ **b** $y = -\frac{1}{2}x + 2$

c $y = \dfrac{8}{4 + x^2}$

$x \geqslant 0$ is the domain of the function.

9 **a** $y = -9x + 8$ **b** $y = \dfrac{x}{2x - 1}$

10 $7x + 2y - 2 = 0$

11 **a** $\dfrac{dy}{dx} = \dfrac{\cos x}{\sin y}$

b Stationary points at $\left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$ and $\left(\dfrac{\pi}{2}, \dfrac{-2\pi}{3}\right)$ only in the given range.

12 $\dfrac{d^2y}{dx^2} = e^{-x}(x^2 - 4x + 2)$ can show that roots of $x^2 - 4x + 2$ are $x = 2 \pm \sqrt{2}$ which means that $f''(x) \geqslant 0$ for all $x < 0$.

13 **a** $\dfrac{dV}{dr} = 4\pi r^2$ **b** $\dfrac{dr}{dt} = \dfrac{250}{\pi(2t + 1)^2 r^2}$

14 **a** $g(1.4) = -0.216 < 0$, $g(1.5) = 0.125 > 0$. Sign change implies root.

b $g(1.4655) = -0.00025\ldots < 0$, $g(1.4665) = 0.00326\ldots > 0$. Sign change implies root.

15 **a** $p(1.7) = 0.0538\ldots > 0$, $p(1.8) = -0.0619\ldots < 0$. Sign change implies root.

b $p(1.7455) = 0.00074\ldots > 0$, $p(1.7465) = -0.00041\ldots < 0$. Sign change implies root.

16 **a** $e^{x-2} - 3x + 5 = 0 \Rightarrow e^{x-2} = 3x - 5$

$\Rightarrow x - 2 = \ln(3x - 5) \Rightarrow x = \ln(3x - 5) + 2,\ x > \frac{5}{3}$

b $x_0 = 4$, $x_1 = 3.9459$, $x_2 = 3.9225$, $x_3 = 3.9121$

17 **a** $f(0.2) = -0.01146\ldots < 0$, $f(0.3) = 0.1564\ldots > 0$. Sign change implies root.

b $\dfrac{1}{(x - 2)^3} + 4x^2 = 0 \Rightarrow \dfrac{1}{(x - 2)^3} = -4x^2$

$\Rightarrow \dfrac{-1}{4x^2} = (x - 2)^3 \Rightarrow \sqrt[3]{\dfrac{-1}{4x^2}} + 2 = x$

c $x_0 = 1$, $x_1 = 1.3700, 75$, $x_2 = 1.4893$, $x_3 = 1.5170$, $x_4 = 1.5228$

d $f(1.5235) = 0.0412\ldots > 0$, $f(1.5245) = -0.0050\ldots < 0$

18 **a** It's a turning point, so the gradient is zero, which means dividing by zero in the Newton Raphson formula.

b $x_{n+1} = 2.9 - \dfrac{-0.5155\ldots}{23.825\ldots} = 2.922$

19 **a** **i** There is a sign change between f(0.2) and f(0.3). Sign change implies root.

ii There is a sign change between f(2.6) and f(2.7). Sign change implies root.

b $\dfrac{3}{10}x^3 - x^{\frac{2}{3}} + \dfrac{1}{x} - 4 = 0 \Rightarrow \dfrac{3}{10}x^3 = x^{\frac{2}{3}} - \dfrac{1}{x} + 4$

$\Rightarrow x^3 = \dfrac{10}{3}\left(4 + x^{\frac{2}{3}} - \dfrac{1}{x}\right) \Rightarrow x = \sqrt[3]{\left(\dfrac{10}{3}\left(4 + x^{\frac{2}{3}} - \dfrac{1}{x}\right)\right)}$

c $x_0 = 2.5$, $x_1 = 2.6275$, $x_2 = 2.6406$, $x_3 = 2.6419$, $x_4 = 2.6420$

d $0.3 - \dfrac{-1.10670714}{-12.02597883} = 0.208$

20 **a** $R = 0.37$, $\alpha = 1.2405$

b $v'(x) = -0.148\sin\left(\dfrac{2x}{5} + 1.2405\right)$

c $v'(4.7) = -0.00312\ldots < 0$, $v'(4.8) = 0.002798\ldots > 0$. Sign change implies maximum or minimum.

d 12.607

e $v'(12.60665) = 0.0000037\ldots > 0$, $v'(12.60675) = -0.0000021\ldots < 0$. Sign change implies maximum or minimum.

21 $a = 1$

22 **a** $\cos 7x + \cos 3x = \cos(5x + 2x) + \cos(5x - 2x)$
$= \cos 5x \cos 2x - \sin 5x \sin 2x + \cos 5x \cos 2x + \sin 5x \sin 2x = 2\cos 5x \cos 2x$

b $\frac{3}{7}\sin 7x + \sin 3x + c$

23 $m = 3$ **24** 16

25 $\dfrac{2}{3} - \dfrac{3\sqrt{3}}{8}$ **26** $\frac{1}{9}(2e^3 + 10)$

27 **a** $\dfrac{5x + 3}{(2x - 3)(x - 2)} \equiv \dfrac{3}{2x - 3} + \dfrac{1}{x + 2}$ **b** $\ln 54$

28 **a** $2\cos t = 1 \Rightarrow \cos t = \dfrac{1}{2} \Rightarrow t = \dfrac{\pi}{3}$ or $t = \frac{5\pi}{3}$

b $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} y \frac{dx}{dt}\,dt = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)(1 - 2\cos t)\,dt$

$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2\,dt$

c $4\pi + 3\sqrt{3}$

29 **a** $\frac{\sqrt{3}}{16}a^2$ **b** 6.796 (4 s.f.)

30 **a** $\frac{1}{4}e^2 + \frac{1}{4}$

b

x	0	0.2	0.4	0.6	0.8	1
y	0	0.29836	0.89022	1.99207	3.96243	7.38906

c 2.168 (4 s.f.) **d** 3.37%

31 **a** $\dfrac{2x - 1}{(x - 1)(2x - 3)} \equiv \dfrac{-1}{x - 1} + \dfrac{4}{2x - 3}$

b $y = \dfrac{A(2x - 3)^2}{(x - 1)}$ **c** $y = \dfrac{10(2x - 3)^2}{(x - 1)}$

32 **a** $\dfrac{3k}{16\pi^2 r^5}$ **b** $r = \left[\dfrac{9k}{8\pi^2}t + A'\right]^{\frac{1}{6}}$

33 **a** Rate in = 20, rate out = $-kV$. So $\dfrac{dV}{dt} = 20 - kV$

b $A = \dfrac{20}{k}$ and $B = -\dfrac{20}{k}$ **c** 108 cm^3 (3 s.f.)

34 **a** $\dfrac{dC}{dt} = -kC$, because k is the constant of proportionality. The negative sign and $k > 0$ indicates rate of decrease.

b $C = Ae^{-kt}$ **c** $k = \frac{1}{4}\ln 10$

35 $k = -4$, $k = 16$ **36** 130.3°

37 **a** $10\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$ **b** $\dfrac{10}{\sqrt{129}}\mathbf{i} - \dfrac{5}{\sqrt{129}}\mathbf{j} - \dfrac{2}{\sqrt{129}}\mathbf{k}$

c 100.1° **d** Not parallel: $\overrightarrow{PQ} \neq m\overrightarrow{AB}$.

38 $k = 2$ **39** $p = -2$, $q = -8$, $r = -4$

Challenge

1 **a** (0, 0) and $\left(\dfrac{-2a}{5}, \dfrac{a}{5}\right)$

b $\dfrac{dx}{dy} = 0 \Rightarrow y = 2x + \dfrac{a}{2} \Rightarrow 5x^2 + 2ax + \dfrac{a^2}{4} = 0$

$b^2 - 4ac = 4a^2 - 5a^2 = -a^2 < 0$

2 $3\sqrt{3} - 1$

Exam-style practice: Paper 1

1 $\frac{dy}{dx} = \frac{2\sec^2 t}{2\sin t\cos t} = \frac{1}{\sin t\cos^3 t} = \operatorname{cosec} t\sec^3 t$

2 **a** $x > -\frac{3}{2}$ **b** $x < -4, x > -1$ **c** $x > -1$

3 **a** $2x + y - 3 = 0 \rightarrow y = 3 - 2x$
$x^2 + kx + y^2 + 4y = 4$
$x^2 + kx + (3 - 2x)^2 + 4(3 - 2x) = 4$
$5x^2 + kx - 20x + 17 = 0$
$5x^2 + (k - 20)x + 17 > 0$ for no intersections.
b $20 - 2\sqrt{85} < k < 20 + 2\sqrt{85}$

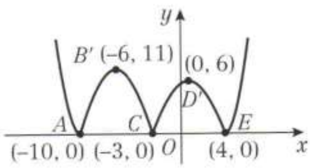
4 Let $f(\theta) = \cos\theta$
$$f'(\theta) = \lim_{h\to 0}\frac{f(\theta + h) - f(\theta)}{h} = \lim_{h\to 0}\frac{\cos(\theta + h) - \cos\theta}{h}$$
$$= \lim_{h\to 0}\frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}$$
$$= \lim_{h\to 0}\left[\left(\frac{\cos h - 1}{h}\right)\cos\theta - \left(\frac{\sin h}{h}\right)\sin\theta\right] = -\sin\theta$$

5 **a** $p = 4, p = -4$ **b** Use $p = -4$, $-18\,432$

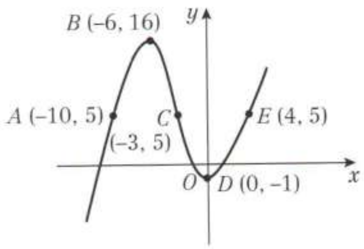
6 $\left(-\frac{49}{8}, \frac{705}{64}\right)$

7 **a** $u_1 = a, u_2 = 96 = ar, S_\infty = 600 = \frac{a}{1 - r}$
So $\frac{\frac{96}{r}}{1 - r} = 600 \Rightarrow 96 = 600r(1 - r) \Rightarrow 96 = 600r - 600r^2$ and therefore $25r^2 - 25r + 4 = 0$
b $r = 0.2, 0.8$ **c** $a = 120$ **d** $n = 39$

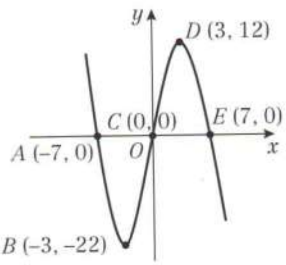
8 **a**



b



c



9 $x = 0.77, x = 5.51$

10 **a** $\frac{dV}{dt} = -kV \Rightarrow \ln V = -kt + c \Rightarrow V = V_0\, e^{-kt}$
b $k = \frac{1}{3}\ln\left(\frac{5}{3}\right)$, $V_0 = £35\,100$ **c** $t = 11.45$ years

11 **a** 14.9 miles
b It is unlikely that a road could be built in a straight line, so the actual length of a road will be greater than 14.9 miles.

12 **a** $y = 2.79 - 0.01(x - 11)^2$, A = 2.79, B = 0.01, C = −11
b 11 m from goal, height of 2.79 m.
c 27.7 m (or 27.70 m)
d $x = 0, y = 1.58$. The ball will enter the goal.

13 **a** Surface area of box $= 2x^2 + 2(2xh + xh) = 2x^2 + 6xh$
Surface area of lid $= 2x^2 + 2(6x + 3x) = 2x^2 + 18x$
Total surface area $= 4x^2 + 6xh + 18x = 5356$
So $h = \frac{5356 - 18x - 4x^2}{6x} = \frac{2678 - 9x - 2x^2}{3x}$
$V = 2x^2h = \frac{2}{3}(2678x - 9x^2 - 2x^3)$
b $6x^2 + 18x - 2678 = 0, x = 19.68$
c $\frac{d^2V}{dx^2} < 0 \Rightarrow$ maximum **d** $22\,648.7\text{ cm}^3$ **e** 21.1%

Exam-style practice: Paper 2

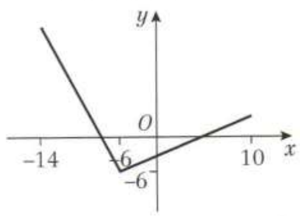
1 $a = -1, b = -4, c = 4$

2 **a** $y = -4x + 28$ **b** $y = \frac{1}{4}x + \frac{5}{2}$
c $R(-10, 0)$ **d** 204 units²

3 $y = \frac{1}{3}\ln(x + 1)$, $x > -1$

4 **a** Student did not apply the laws of logarithms correctly in moving from the first line to the second line: $4^{A+B} \neq 4^A + 4^B$
b $x = -2$. Note $x \neq -5$

5 **a**



b $-6 \leqslant y \leqslant 18$ **c** $a = -7, a = 0$

6 **a** $k = 4$ **b** $x = -2, x = 3 + \sqrt{7}, x = 3 - \sqrt{7}$

7 **a** Area $= \frac{1}{2}(x - 10)(x - 3)\sin 30° = \frac{1}{4}(x^2 - 13x + 30) = 11$
So $x^2 - 13x + 30 = 44$, and $x^2 - 13x - 14 = 0$
b $x = 14$ ($x \neq -1$, as $x - 10$ and $x - 3$ must be positive.)

8 **a** $h = -5, k = 2, c = 36$ **b** $\frac{13\pi}{2}$

9 **a** A = 4, B = 5, C = −6 **b** $-\frac{7}{8}x - \frac{23}{32}x^2$

10 $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$
$\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{BA} = \frac{1}{2}\mathbf{b} + \frac{1}{2}(-\mathbf{b} + \mathbf{a}) = \frac{1}{2}\mathbf{a}$
Therefore $\overrightarrow{OA}$ and $\overrightarrow{MN}$ are parallel and $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{OA}$ as required.

11 **a** 2.46740 **b** 2.922 **c** 3.023 **d** 3.3%

12 **a** £3550 **b** £40 950 **c** £45 599.17

13 **a** $R = 0.41, \alpha = 1.3495$
b 40 cm at time $t = 2.70$ seconds
c 0.38 seconds and 5.02 seconds.

14 **a** $h'(t) = 3e^{-0.3(t-6.4)} - 8e^{0.8(t-6.4)}$
b $\frac{3}{8}e^{-0.3(t-6.4)} = e^{0.8(t-6.4)} \Rightarrow \ln\left(\frac{3}{8}e^{-0.3(t-6.4)}\right) = 0.8(t - 6.4)$
$\Rightarrow \frac{5}{4}\ln\left(\frac{3}{8}e^{-0.3(t-6.4)}\right) = t - 6.4 \Rightarrow t = \frac{5}{4}\ln\left(\frac{3}{8}e^{-0.3(t-6.4)}\right) + 6.4$
c $t_0 = 5, t_1 = 5.6990, t_2 = 5.4369, t_3 = 5.5351, t_4 = 5.4983$
d $h'(5.5075) = 0.00360\ldots > 0$, $h'(5.5085) = -0.000702\ldots < 0$. Sign change implies slope change, which implies a turning point.

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