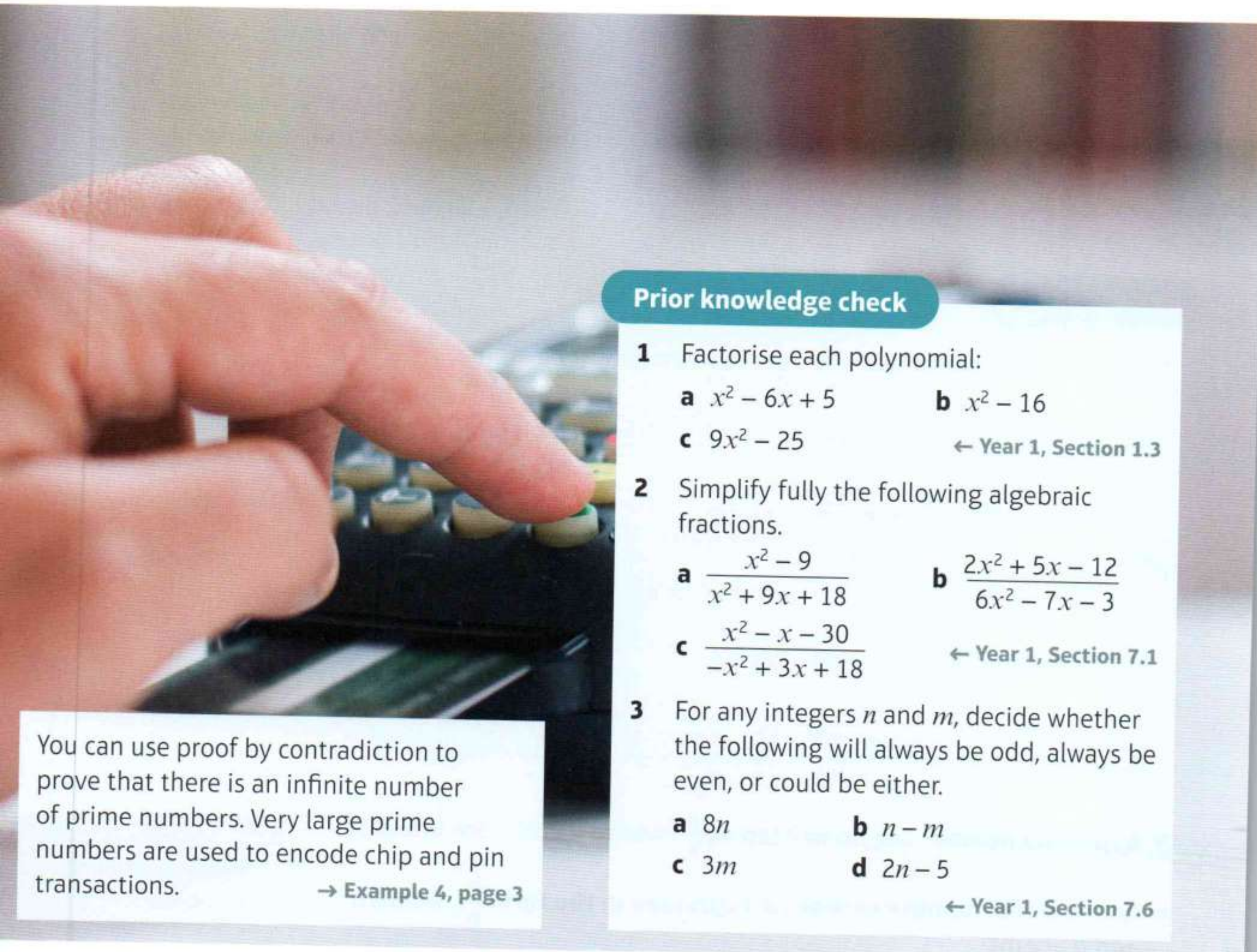
Algebraic methods

1

Objectives

After completing this chapter you should be able to:

- Use proof by contradiction to prove true statements → pages 2–5
- Multiply and divide two or more algebraic fractions → pages 5–7
- Add or subtract two or more algebraic fractions → pages 7–8
- Convert an expression with linear factors in the denominator into partial fractions → pages 9–11
- Convert an expression with repeated linear factors in the denominator into partial fractions → pages 12–13
- Divide algebraic expressions → pages 14–17
- Convert an improper fraction into partial fraction form → pages 17–18



You can use proof by contradiction to prove that there is an infinite number of prime numbers. Very large prime numbers are used to encode chip and pin transactions. → Example 4, page 3

Prior knowledge check

1 Factorise each polynomial:

a $x^2 - 6x + 5$ **b** $x^2 - 16$

c $9x^2 - 25$ ← Year 1, Section 1.3

2 Simplify fully the following algebraic fractions.

a $\dfrac{x^2 - 9}{x^2 + 9x + 18}$ **b** $\dfrac{2x^2 + 5x - 12}{6x^2 - 7x - 3}$

c $\dfrac{x^2 - x - 30}{-x^2 + 3x + 18}$ ← Year 1, Section 7.1

3 For any integers n and m, decide whether the following will always be odd, always be even, or could be either.

a $8n$ **b** $n - m$

c $3m$ **d** $2n - 5$

← Year 1, Section 7.6

1.1 Proof by contradiction

A **contradiction** is a disagreement between two statements, which means that both cannot be true. Proof by contradiction is a powerful technique.

- **To prove a statement by contradiction you start by assuming it is not true. You then use logical steps to show that this assumption leads to something impossible (either a contradiction of the assumption, or a contradiction of a fact you know to be true). You can conclude that your assumption was incorrect, and the original statement was true.**

Notation A statement that asserts the falsehood of another statement is called the negation of that statement.

Example 1

Prove by contradiction that there is no greatest odd integer.

Assumption: there is a greatest odd integer, n.

$n + 2$ is also an integer and $n + 2 > n$

$n + 2 = \text{odd} + \text{even} = \text{odd}$

So there exists an odd integer greater than n.

This contradicts the assumption that the greatest odd integer is n.

Therefore, there is no greatest odd integer.

Begin by assuming the original statement is false. This is the negation of the original statement.

You need to use logical steps to reach a contradiction. Show all of your working.

The existence of an odd integer greater than n contradicts your initial assumption.

Finish your proof by concluding that the original statement must be true.

Example 2

Prove by contradiction that if n^2 is even, then n must be even.

Assumption: there exists a number n such that n^2 is even but n is odd.

n is odd so write $n = 2k + 1$

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

So n^2 is odd.

This contradicts the assumption that n^2 is even.

Therefore, if n^2 is even then n must be even.

This is the negation of the original statement.

You can write any odd number in the form $2k + 1$ where k is an integer.

All multiples of 2 are even numbers, so 1 more than a multiple of 2 is an odd number.

Finish your proof by concluding that the original statement must be true.

- **A rational number can be written as $\frac{a}{b}$, where a and b are integers.**
- **An irrational number cannot be expressed in the form $\frac{a}{b}$, where a and b are integers.**

Notation $\mathbb{Q}$ is the set of all rational numbers.

Example 3

Prove by contradiction that $\sqrt{2}$ is an irrational number.

Assumption: $\sqrt{2}$ is a rational number. — Begin by assuming the original statement is false.

Then $\sqrt{2} = \frac{a}{b}$ for some integers, a and b. — This is the definition of a rational number.

Also assume that this fraction cannot be reduced further: there are no common factors between a and b. — If a and b did have a common factor you could just cancel until this fraction was in its simplest form.

So $2 = \frac{a^2}{b^2}$ or $a^2 = 2b^2$ — Square both sides and make a^2 the subject.

This means that a^2 must be even, so a is also even. — We proved this result in Example 2.

If a is even, then it can be expressed in the form $a = 2n$, where n is an integer.

So $a^2 = 2b^2$ becomes $(2n)^2 = 2b^2$ which means $4n^2 = 2b^2$ or $2n^2 = b^2$.

This means that b^2 must be even, so b is also even. — Again using the result from Example 2.

If a and b are both even, they will have a common factor of 2. — All even numbers are divisible by 2.

This contradicts the statement that a and b have no common factors.

Therefore $\sqrt{2}$ is an irrational number. — Finish your proof by concluding that the original statement must be true.

Example 4

Prove by contradiction that there are infinitely many prime numbers.

Assumption: there is a finite number of prime numbers. — Begin by assuming the original statement is false.

List all the prime numbers that exist:

$p_1, p_2, p_3, \ldots, p_n$ — This is a list of all possible prime numbers.

Consider the number

$N = p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1$ — This new number is one more than the product of the existing prime numbers.

When you divide N by any of the prime numbers $p_1, p_2, p_3, \ldots, p_n$ you get a remainder of 1. So none of the prime numbers $p_1, p_2, p_3, \ldots, p_n$ is a factor of N.

So N must either be prime or have a prime factor which is not in the list of all possible prime numbers.

This is a contradiction. — This contradicts the assumption that the list $p_1, p_2, p_3, \ldots, p_n$ contains all the prime numbers.

Therefore, there is an infinite number of prime numbers. — Conclude your proof by stating that the original statement must be true.

Exercise 1A

(P) **1** Select the statement that is the negation of 'All multiples of three are even'.

A All multiples of three are odd.

B At least one multiple of three is odd.

C No multiples of three are even.

(P) **2** Write down the negation of each statement.

a All rich people are happy.

b There are no prime numbers between 10 million and 11 million.

c If p and q are prime numbers then $(pq + 1)$ is a prime number.

d All numbers of the form $2^n - 1$ are either prime numbers or multiples of 3.

e At least one of the above four statements is true.

(P) **3** Statement: If n^2 is odd then n is odd.

a Write down the negation of this statement.

b Prove the original statement by contradiction.

(P) **4** Prove the following statements by contradiction.

a There is no greatest even integer.

b If n^3 is even then n is even.

c If pq is even then at least one of p and q is even.

d If $p + q$ is odd then at least one of p and q is odd.

(E/P) **5** **a** Prove that if ab is an irrational number then at least one of a and b is an irrational number. **(3 marks)**

b Prove that if $a + b$ is an irrational number then at least one of a and b is an irrational number. **(3 marks)**

c A student makes the following statement:

If $a + b$ is a rational number then at least one of a and b is a rational number.

Show by means of a counterexample that this statement is not true. **(1 mark)**

(P) **6** Use proof by contradiction to show that there exist no integers a and b for which $21a + 14b = 1$.

Hint Assume the opposite is true, and then divide both sides by the highest common factor of 21 and 14.

(E/P) **7** **a** Prove by contradiction that if n^2 is a multiple of 3, n is a multiple of 3. **(3 marks)**

b Hence prove by contradiction that $\sqrt{3}$ is an irrational number. **(3 marks)**

Hint Consider numbers in the form $3k + 1$ and $3k + 2$.

(P) **8** Use proof by contradiction to prove the statement: 'There are no integer solutions to the equation $x^2 - y^2 = 2$'

Hint You can assume that x and y are positive, since $(-x)^2 = x^2$.

(E/P) **9** Prove by contradiction that $\sqrt[3]{2}$ is irrational. **(5 marks)**

(E/P) **10** This student has attempted to use proof by contradiction to show that there is no least positive rational number:

> **Assumption:** There is a least positive rational number.
> Let this least positive rational number be n.
> As n is rational, $n = \frac{a}{b}$ where a and b are integers.
> $n - 1 = \frac{a}{b} - 1 = \frac{a-b}{b}$
> Since a and b are integers, $\frac{a-b}{b}$ is a rational number that is less than n.
> This contradicts the statement that n is the least positive rational number.
> Therefore, there is no least positive rational number.

Problem-solving

You might have to analyse student working like this in your exam. The question says, 'the error', so there should only be one error in the proof.

a Identify the error in the student's proof. **(1 mark)**

b Prove by contradiction that there is no least positive rational number. **(5 marks)**

1.2 Algebraic fractions

Algebraic fractions work in the same way as numeric fractions. You can simplify them by cancelling common factors and finding common denominators.

- **To multiply fractions, cancel any common factors, then multiply the numerators and multiply the denominators.**

Example 5

Simplify the following products:

a $\frac{3}{5} \times \frac{5}{9}$ **b** $\frac{a}{b} \times \frac{c}{a}$ **c** $\frac{x+1}{2} \times \frac{3}{x^2-1}$

a $\frac{{}^{1}\not{3}}{{}_{1}\not{5}} \times \frac{\not{5}^{1}}{\not{9}_{3}} = \frac{1 \times 1}{1 \times 3} = \frac{1}{3}$

Cancel any common factors and multiply numerators and denominators.

b $\frac{{}^{1}\not{a}}{b} \times \frac{c}{\not{a}_{1}} = \frac{1 \times c}{b \times 1} = \frac{c}{b}$

Cancel any common factors and multiply numerators and denominators.

c $\frac{x+1}{2} \times \frac{3}{x^2-1} = \frac{x+1}{2} \times \frac{3}{(x+1)(x-1)}$

Factorise $(x^2 - 1)$.

$= \frac{{}^{1}\cancel{x+1}}{2} \times \frac{3}{{}_{1}\cancel{(x+1)}(x-1)}$

Cancel any common factors and multiply numerators and denominators.

$= \frac{3}{2(x-1)}$

- **To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.**

Example 6

Simplify:

a $\frac{a}{b} \div \frac{a}{c}$ **b** $\frac{x+2}{x+4} \div \frac{3x+6}{x^2-16}$

a $\frac{a}{b} \div \frac{a}{c} = \frac{^1\cancel{a}}{b} \times \frac{c}{\cancel{a}_1}$ — Multiply the first fraction by the reciprocal of the second fraction. Cancel the common factor a.

$= \frac{1 \times c}{b \times 1}$ — Multiply numerators and denominators.

$= \frac{c}{b}$

b $\frac{x+2}{x+4} \div \frac{3x+6}{x^2-16}$

$= \frac{x+2}{x+4} \times \frac{x^2-16}{3x+6}$ — Multiply the first fraction by the reciprocal of the second fraction.

$= \frac{x+2}{x+4} \times \frac{(x+4)(x-4)}{3(x+2)}$ — Factorise as far as possible.

$= \frac{\cancel{x+2}^1}{\cancel{x+4}_1} \times \frac{\cancel{(x+4)}^1(x-4)}{3\cancel{(x+2)}_1}$ — Cancel any common factors and multiply numerators and denominators.

$= \frac{x-4}{3}$

Exercise 1B

1 Simplify:

a $\frac{a}{d} \times \frac{a}{c}$ **b** $\frac{a^2}{c} \times \frac{c}{a}$ **c** $\frac{2}{x} \times \frac{x}{4}$ **d** $\frac{3}{x} \div \frac{6}{x}$ **e** $\frac{4}{xy} \div \frac{x}{y}$ **f** $\frac{2r^2}{5} \div \frac{4}{r^3}$

2 Simplify:

a $(x+2) \times \frac{1}{x^2-4}$ **b** $\frac{1}{a^2+6a+9} \times \frac{a^2-9}{2}$ **c** $\frac{x^2-3x}{y^2+y} \times \frac{y+1}{x}$

d $\frac{y}{y+3} \div \frac{y^2}{y^2+4y+3}$ **e** $\frac{x^2}{3} \div \frac{2x^3-6x^2}{x^2-3x}$ **f** $\frac{4x^2-25}{4x-10} \div \frac{2x+5}{8}$

g $\frac{x+3}{x^2+10x+25} \times \frac{x^2+5x}{x^2+3x}$ **h** $\frac{3y^2+4y-4}{10} \div \frac{3y+6}{15}$ **i** $\frac{x^2+2xy+y^2}{2} \times \frac{4}{(x-y)^2}$

E/P **3** Show that $\frac{x^2-64}{x^2-36} \div \frac{64-x^2}{x^2-36} = -1$ **(4 marks)**

E/P **4** Show that $\frac{2x^2-11x-40}{x^2-4x-32} \times \frac{x^2+8x+16}{6x^2-3x-45} \div \frac{8x^2+20x-48}{10x^2-45x+45} = \frac{a}{b}$ and find the values of the constants a and b, where a and b are integers. **(4 marks)**

E/P **5** **a** Simplify fully $\frac{x^2 + 2x - 24}{2x^2 + 10x} \times \frac{x^2 - 3x}{x^2 + 3x - 18}$ **(3 marks)**

b Given that
$\ln((x^2 + 2x - 24)(x^2 - 3x)) = 2 + \ln((2x^2 + 10x)(x^2 + 3x - 18))$ find x in terms of e. **(4 marks)**

Hint Simplify and then solve the logarithmic equation.
← Year 1, Section 14.6

E/P **6** $f(x) = \frac{2x^2 - 3x - 2}{6x - 8} \div \frac{x - 2}{3x^2 + 14x - 24}$

a Show that $f(x) = \frac{2x^2 + 13x + 6}{2}$ **(4 marks)**

b Hence differentiate $f(x)$ and find $f'(4)$. **(3 marks)**

Hint Differentiate each term separately.
← Year 1, Section 12.5

- **To add or subtract two fractions, find a common denominator.**

Example 7

Simplify the following:

a $\frac{1}{3} + \frac{3}{4}$ **b** $\frac{a}{2x} + \frac{b}{3x}$ **c** $\frac{2}{x+3} - \frac{1}{x+1}$ **d** $\frac{3}{x+1} - \frac{4x}{x^2 - 1}$

a $\frac{1}{3} + \frac{3}{4}$ ($\times\frac{4}{4}$, $\times\frac{3}{3}$)

$= \frac{4}{12} + \frac{9}{12}$ — The lowest common multiple of 3 and 4 is 12.

$= \frac{13}{12}$

b $\frac{a}{2x} + \frac{b}{3x}$ — The lowest common multiple of $2x$ and $3x$ is $6x$.

$= \frac{3a}{6x} + \frac{2b}{6x}$ — Multiply the first fraction by $\frac{3}{3}$ and the second fraction by $\frac{2}{2}$

$= \frac{3a + 2b}{6x}$

c $\frac{2}{(x+3)} - \frac{1}{(x+1)}$

$= \frac{2(x+1)}{(x+3)(x+1)} - \frac{1(x+3)}{(x+3)(x+1)}$ — The lowest common multiple is $(x+3)(x+1)$, so change both fractions so that the denominators are $(x+3)(x+1)$.

$= \frac{2(x+1) - 1(x+3)}{(x+3)(x+1)}$ — Subtract the numerators.

$= \frac{2x + 2 - 1x - 3}{(x+3)(x+1)}$ — Expand the brackets.

$= \frac{x - 1}{(x+3)(x+1)}$ — Simplify the numerator.

d $\frac{3}{x+1} - \frac{4x}{x^2-1}$

$= \frac{3}{x+1} - \frac{4x}{(x+1)(x-1)}$ — Factorise $x^2 - 1$ to $(x + 1)(x - 1)$.

$= \frac{3(x-1)}{(x+1)(x-1)} - \frac{4x}{(x+1)(x-1)}$ — The LCM of $(x + 1)$ and $(x + 1)(x - 1)$ is $(x + 1)(x - 1)$.

$= \frac{3(x-1) - 4x}{(x+1)(x-1)}$

$= \frac{-x-3}{(x+1)(x-1)}$ — Simplify the numerator: $3x - 3 - 4x = -x - 3$.

Exercise 1C

1 Write as a single fraction:

a $\frac{1}{3} + \frac{1}{4}$ **b** $\frac{3}{4} - \frac{2}{5}$ **c** $\frac{1}{p} + \frac{1}{q}$ **d** $\frac{3}{4x} + \frac{1}{8x}$ **e** $\frac{3}{x^2} - \frac{1}{x}$ **f** $\frac{a}{5b} - \frac{3}{2b}$

2 Write as a single fraction:

a $\frac{3}{x} - \frac{2}{x+1}$ **b** $\frac{2}{x-1} - \frac{3}{x+2}$ **c** $\frac{4}{2x+1} + \frac{2}{x-1}$

d $\frac{1}{3}(x+2) - \frac{1}{2}(x+3)$ **e** $\frac{3x}{(x+4)^2} - \frac{1}{x+4}$ **f** $\frac{5}{2(x+3)} + \frac{4}{3(x-1)}$

3 Write as a single fraction:

a $\frac{2}{x^2+2x+1} + \frac{1}{x+1}$ **b** $\frac{7}{x^2-4} + \frac{3}{x+2}$ **c** $\frac{2}{x^2+6x+9} - \frac{3}{x^2+4x+3}$

d $\frac{2}{y^2-x^2} + \frac{3}{y-x}$ **e** $\frac{3}{x^2+3x+2} - \frac{1}{x^2+4x+4}$ **f** $\frac{x+2}{x^2-x-12} - \frac{x+1}{x^2+5x+6}$

E **4** Express $\frac{6x+1}{x^2+2x-15} - \frac{4}{x-3}$ as a single fraction in its simplest form. **(4 marks)**

5 Express each of the following as a fraction in its simplest form.

a $\frac{3}{x} + \frac{2}{x+1} + \frac{1}{x+2}$ **b** $\frac{4}{3x} - \frac{2}{x-2} + \frac{1}{2x+1}$ **c** $\frac{3}{x-1} + \frac{2}{x+1} + \frac{4}{x-3}$

E **6** Express $\frac{4(2x-1)}{36x^2-1} + \frac{7}{6x-1}$ as a single fraction in its simplest form. **(4 marks)**

E/P **7** $g(x) = x + \frac{6}{x+2} + \frac{36}{x^2-2x-8}, x \in \mathbb{R}, x \neq -2, x \neq 4$

a Show that $g(x) = \frac{x^3-2x^2-2x+12}{(x+2)(x-4)}$ **(4 marks)**

b Using algebraic long division, or otherwise, further show that $g(x) = \frac{x^2-4x+6}{x-4}$ **(4 marks)**

1.3 Partial fractions

- **A single fraction with two distinct linear factors in the denominator can be split into two separate fractions with linear denominators. This is called splitting it into partial fractions.**

$$\frac{5}{(x+1)(x-4)} \equiv \frac{A}{x+1} + \frac{B}{x-4}$$

A and B are constants to be found.

The denominator contains two linear factors: $(x + 1)$ and $(x - 4)$.

The expression is rewritten as the sum of two partial fractions.

Links Partial fractions are used for binomial expansions → **Chapter 4** and integration. → **Chapter 11**

There are two methods to find the constants A and B: by **substitution** and by **equating coefficients**.

Example 8

Split $\frac{6x-2}{(x-3)(x+1)}$ into partial fractions by: **a** substitution **b** equating coefficients.

a $\frac{6x-2}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$

Set $\frac{6x-2}{(x-3)(x+1)}$ identical to $\frac{A}{x-3} + \frac{B}{x+1}$

$\equiv \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}$

Add the two fractions.

$6x - 2 \equiv A(x + 1) + B(x - 3)$

$6 \times 3 - 2 = A(3 + 1) + B(3 - 3)$

$16 = 4A$

$A = 4$

To find A substitute $x = 3$. This value of x eliminates B from the equation.

$6 \times (-1) - 2 = A(-1 + 1) + B(-1 - 3)$

$-8 = -4B$

$B = 2$

To find B substitute $x = -1$. This value of x eliminates A from the equation.

$\therefore \frac{6x-2}{(x-3)(x+1)} \equiv \frac{4}{x-3} + \frac{2}{x+1}$

b $\frac{6x-2}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$

$\equiv \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}$

$6x - 2 \equiv A(x + 1) + B(x - 3)$

$\equiv Ax + A + Bx - 3B$

Expand the brackets.

$\equiv (A + B)x + (A - 3B)$

Collect like terms.

Equate coefficients of x:

$6 = A + B$ (1)

Equate constant terms:

$-2 = A - 3B$ (2)

You want $(A + B)x + A - 3B \equiv 6x - 2$. Hence coefficient of x is 6, and constant term is -2.

(1) − (2):

$8 = 4B$

$\Rightarrow B = 2$

Substitute $B = 2$ in (1) $\Rightarrow 6 = A + 2$

$A = 4$

Solve simultaneously.

- **The method of partial fractions can also be used when there are more than two distinct linear factors in the denominator.**

For example, the expression $\frac{7}{(x-2)(x+6)(x+3)}$

can be split into $\frac{A}{x-2}+\frac{B}{x+6}+\frac{C}{x+3}$

The constants A, B and C can again be found by either substitution or by equating coefficients.

Watch out This method cannot be used for a repeated linear factor in the denominator.

For example, the expression $\frac{9}{(x+4)(x-1)^2}$

cannot be rewritten as $\frac{A}{x+4}+\frac{B}{x-1}+\frac{C}{x-1}$

because $(x-1)$ is a repeated factor. There is more on this in the next section.

Example 9

Given that $\frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{A}{x}+\frac{B}{x-1}+\frac{C}{2x+1}$, find the values of the constants A, B and C.

Let $\frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{A}{x}+\frac{B}{x-1}+\frac{C}{2x+1}$

The denominators must be x, $(x-1)$ and $(2x+1)$.

$\equiv \frac{A(x-1)(2x+1)+Bx(2x+1)+Cx(x-1)}{x(x-1)(2x+1)}$

Add the fractions.

$\therefore 6x^2+5x-2 \equiv A(x-1)(2x+1) + Bx(2x+1) + Cx(x-1)$

The numerators are equal.

Let $x = 1$:

$6+5-2 = 0 + B \times 1 \times 3 + 0$

$9 = 3B$

$B = 3$

Let $x = 0$:

$0+0-2 = A \times (-1) \times 1 + 0 + 0$

$-2 = -A$

$A = 2$

Let $x = -\frac{1}{2}$:

$\frac{6}{4} - \frac{5}{2} - 2 = 0 + 0 + C \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right)$

$-3 = \frac{3}{4}C$

$C = -4$

Proceed by substitution OR by equating coefficients.
Here we used the method of substitution.

So $\frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{2}{x}+\frac{3}{x-1}-\frac{4}{2x+1}$

So $A = 2$, $B = 3$ and $C = -4$.

Finish the question by listing the coefficients.

Exercise 1D

1 Express the following as partial fractions:

a $\dfrac{6x-2}{(x-2)(x+3)}$ **b** $\dfrac{2x+11}{(x+1)(x+4)}$ **c** $\dfrac{-7x-12}{2x(x-4)}$

d $\dfrac{2x-13}{(2x+1)(x-3)}$ **e** $\dfrac{6x+6}{x^2-9}$

Hint First factorise the denominator.

f $\dfrac{7-3x}{x^2-3x-4}$ **g** $\dfrac{8-x}{x^2+4x}$ **h** $\dfrac{2x-14}{x^2+2x-15}$

(E) **2** Show that $\dfrac{-2x-5}{(4+x)(2-x)}$ can be written in the form $\dfrac{A}{4+x}+\dfrac{B}{2-x}$ where A and B are constants to be found.

(3 marks)

(P) **3** The expression $\dfrac{A}{(x-4)(x+8)}$ can be written in partial fractions as $\dfrac{2}{x-4}+\dfrac{B}{x+8}$

Find the values of the constants A and B.

(E) **4** $h(x)=\dfrac{2x^2-12x-26}{(x+1)(x-2)(x+5)}, x>2$

Given that $h(x)$ can be expressed in the form $\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{x+5}$, find the values of A, B and C.

(4 marks)

(E) **5** Given that, for $x<-1$, $\dfrac{-10x^2-8x+2}{x(2x+1)(3x-2)}\equiv\dfrac{D}{x}+\dfrac{E}{2x+1}+\dfrac{F}{3x-2}$, where D, E and F are constants. Find the values of D, E and F.

(4 marks)

6 Express the following as partial fractions:

$\dfrac{-5x^2-19x-32}{(x+1)(x+2)(x-5)}$

(P) **7** Express the following as partial fractions:

a $\dfrac{6x^2+7x-3}{x^3-x}$ **b** $\dfrac{8x+9}{10x^2+3x-4}$

Hint First factorise the denominator.

Challenge

Express $\dfrac{5x^2-15x-8}{x^3-4x^2+x+6}$ as a sum of fractions with linear denominators.

1.4 Repeated factors

- **A single fraction with a repeated linear factor in the denominator can be split into two or more separate fractions.**

In this case, there is a special method for dealing with the repeated linear factor.

$$\frac{2x+9}{(x-5)(x+3)^2} \equiv \frac{A}{x-5} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

A and B and C are constants to be found.

The denominator contains three linear factors: $(x-5)$, $(x+3)$ and $(x+3)$. $(x+3)$ is a repeated linear factor.

The expression is rewritten as the sum of three partial fractions. Notice that $(x-5)$, $(x+3)$ and $(x+3)^2$ are the denominators.

Example 10

Show that $\frac{11x^2+14x+5}{(x+1)^2(2x+1)}$ can be written in the form $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x+1}$, where A, B and C are constants to be found.

Let

$$\frac{11x^2+14x+5}{(x+1)^2(2x+1)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(2x+1)}$$

You need denominators of $(x+1)$, $(x+1)^2$ and $(2x+1)$.

$$\equiv \frac{A(x+1)(2x+1) + B(2x+1) + C(x+1)^2}{(x+1)^2(2x+1)}$$

Add the three fractions.

Hence $11x^2+14x+5$

$\equiv A(x+1)(2x+1) + B(2x+1) + C(x+1)^2$ (1)

The numerators are equal.

Let $x = -1$:

$11 - 14 + 5 = A \times 0 + B \times -1 + C \times 0$

To find B substitute $x = -1$.

$2 = -1B$

$B = -2$

Let $x = -\frac{1}{2}$:

$\frac{11}{4} - 7 + 5 = A \times 0 + B \times 0 + C \times \frac{1}{4}$

To find C substitute $x = -\frac{1}{2}$

$\frac{3}{4} = \frac{1}{4}C$

$C = 3$

$11 = 2A + C$

Equate terms in x^2 in (1). Terms in x^2 are $A \times 2x^2 + C \times x^2$.

$11 = 2A + 3$

Substitute $C = 3$.

$2A = 8$

$A = 4$

Hence $\frac{11x^2+14x+5}{(x+1)^2(2x+1)}$

$$\equiv \frac{4}{(x+1)} - \frac{2}{(x+1)^2} + \frac{3}{(2x+1)}$$

So $A = 4$, $B = -2$ and $C = 3$.

Finish the question by listing the coefficients.

Online Check your answer using the simultaneous equations function on your calculator.

Exercise 1E

E 1 $f(x) = \frac{3x^2 + x + 1}{x^2(x+1)}, x \neq 0, x \neq -1$

Given that f(x) can be expressed in the form $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$, find the values of A, B and C.

(4 marks)

E 2 $g(x) = \frac{-x^2 - 10x - 5}{(x+1)^2(x-1)}, x \neq -1, x \neq 1$

Find the values of the constants D, E and F such that $g(x) = \frac{D}{x+1} + \frac{E}{(x+1)^2} + \frac{F}{x-1}$ **(4 marks)**

E 3 Given that, for $x < 0$, $\frac{2x^2 + 2x - 18}{x(x-3)^2} \equiv \frac{P}{x} + \frac{Q}{x-3} + \frac{R}{(x-3)^2}$, where P, Q and R are constants, find the values of P, Q and R.

(4 marks)

E 4 Show that $\frac{5x^2 - 2x - 1}{x^3 - x^2}$ can be written in the form $\frac{C}{x} + \frac{D}{x^2} + \frac{E}{x-1}$ where C, D and E are constants to be found.

(4 marks)

E 5 $p(x) = \frac{2x}{(x+2)^2}, x \neq -2.$

Find the values of the constants A and B such that $p(x) = \frac{A}{x+2} + \frac{B}{(x+2)^2}$ **(4 marks)**

E 6 $\frac{10x^2 - 10x + 17}{(2x+1)(x-3)^2} \equiv \frac{A}{2x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}, x > 3$

Find the values of the constants A, B and C.

(4 marks)

E 7 Show that $\frac{39x^2 + 2x + 59}{(x+5)(3x-1)^2}$ can be written in the form $\frac{A}{x+5} + \frac{B}{3x-1} + \frac{C}{(3x-1)^2}$ where A, B and C are constants to be found.

(4 marks)

P 8 Express the following as partial fractions:

a $\frac{4x + 1}{x^2 + 10x + 25}$ **b** $\frac{6x^2 - x + 2}{4x^3 - 4x^2 + x}$

1.5 Algebraic division

- **An improper algebraic fraction is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions.**

$\frac{x^2 + 5x + 8}{x - 2}$ and $\frac{x^3 + 5x - 9}{x^3 - 4x^2 + 7x - 3}$ are both improper fractions.

The degree of the numerator is greater than the degree of the denominator.

The degrees of the numerator and denominator are equal.

Notation The **degree of a polynomial** is the largest exponent in the expression. For example, $x^3 + 5x - 9$ has degree 3.

- **You can either use:**
  - **algebraic division**
  - **or the relationship F(x) = Q(x) × divisor + remainder**

  to convert an improper fraction into a mixed fraction.

Watch out The divisor and the remainder can be numbers or functions of x.

Method 1

Use algebraic long division to show that:

$$\frac{x^2 + 5x + 8}{x - 2} \equiv x + 7 + \frac{22}{x - 2}$$

F(x) → $x^2 + 5x + 8$; Q(x) → $x + 7$; remainder → 22; divisor → $x - 2$

Method 2

Multiply by $(x - 2)$ and compare coefficients to show that:

$$x^2 + 5x + 8 \equiv (x + 7)(x - 2) + 22$$

F(x) → $x^2 + 5x + 8$; Q(x) → $x + 7$; remainder → 22; divisor → $x - 2$

Example 11

Given that $\frac{x^3 + x^2 - 7}{x - 3} \equiv Ax^2 + Bx + C + \frac{D}{x - 3}$, find the values of A, B, C and D.

Using algebraic long division:

$$\begin{array}{r|l} & x^2 + 4x + 12 \\ \hline x - 3 & x^3 + x^2 + 0x - 7 \\ & x^3 - 3x^2 \\ \hline & \quad 4x^2 + 0x \\ & \quad 4x^2 - 12x \\ \hline & \qquad 12x - 7 \\ & \qquad 12x - 36 \\ \hline & \qquad\qquad 29 \end{array}$$

Problem-solving

Solving this problem using algebraic long division will give you an answer in the form asked for in the question.

So $\frac{x^3 + x^2 - 7}{x - 3} = x^2 + 4x + 12$

with a remainder of 29.

$\frac{x^3 + x^2 - 7}{x - 3} = x^2 + 4x + 12 + \frac{29}{x - 3}$

So $A = 1$, $B = 4$, $C = 12$ and $D = 29$.

The divisor is $(x - 3)$ so you need to write the remainder as a fraction with denominator $(x - 3)$.

It's always a good idea to list the value of each unknown asked for in the question.

Example 12

Given that $x^3 + x^2 - 7 \equiv (Ax^2 + Bx + C)(x - 3) + D$, find the values of A, B, C and D.

Let $x = 3$:

$27 + 9 - 7 = (9A + 3B + C) \times 0 + D$

$D = 29$

Let $x = 0$:

$0 + 0 - 7 = (A \times 0 + B \times 0 + C) \times (0 - 3) + D$

$-7 = -3C + D$

$-7 = -3C + 29$

$3C = 36$

$C = 12$

Compare the coefficients of x^3 and x^2.

Compare coefficients in x^3: $1 = A$

Compare coefficients in x^2: $1 = -3A + B$

$1 = -3 + B$

Therefore $A = 1$, $B = 4$, $C = 12$ and $D = 29$ and we can write

$x^3 + x^2 - 7 \equiv (x^2 + 4x + 12)(x - 3) + 29$

This can also be written as:

$\frac{x^3 + x^2 - 7}{x - 3} \equiv x^2 + 4x + 12 + \frac{29}{x - 3}$

Problem-solving

The identity is given in the form $F(x) \equiv Q(x) \times$ divisor + remainder so solve the problem by equating coefficients.

Set $x = 3$ to find the value of D.

Set $x = 0$ and use your value of D to find the value of C.

You can find the remaining values by equating coefficients of x^3 and x^2.
Remember there are two x^2 terms when you expand the brackets on the RHS:
x^3 terms: LHS = x^3, RHS = Ax^3
x^2 terms: LHS = x^2, RHS = $(-3A + B)x^2$

Example 13

$f(x) = \frac{x^4 + x^3 + x - 10}{x^2 + 2x - 3}$

Show that f(x) can be written as $Ax^2 + Bx + C + \frac{Dx + E}{x^2 + 2x - 3}$ and find the values of A, B, C, D and E.

Using algebraic long division:

$$
\begin{array}{r}
x^2 - x + 5 \\
x^2 + 2x - 3 \overline{)\, x^4 + x^3 + 0x^2 + x - 10} \\
\underline{x^4 + 2x^3 - 3x^2} \\
- x^3 + 3x^2 + x \\
\underline{- x^3 - 2x^2 + 3x} \\
5x^2 - 2x - 10 \\
\underline{5x^2 + 10x - 15} \\
-12x + 5
\end{array}
$$

$$\frac{x^4 + x^3 + x - 10}{x^2 + 2x - 3} \equiv x^2 - x + 5 + \frac{-12x + 5}{x^2 + 2x - 3}$$

So $A = 1$, $B = -1$, $C = 5$, $D = -12$ and $E = 5$.

Write the remainder as a fraction over the whole divisor.

Watch out When you are dividing by a quadratic expression, the remainder can be a constant or a linear expression. The degree of $(-12x + 5)$ is smaller than the degree of $(x^2 + 2x - 3)$ so stop your division here. The remainder is $-12x + 5$.

Exercise 1F

E **1** $\dfrac{x^3 + 2x^2 + 3x - 4}{x + 1} \equiv Ax^2 + Bx + C + \dfrac{D}{x + 1}$

Find the values of the constants A, B, C and D. **(4 marks)**

E **2** Given that $\dfrac{2x^3 + 3x^2 - 4x + 5}{x + 3} \equiv ax^2 + bx + c + \dfrac{d}{x + 3}$ find the values of a, b, c and d. **(4 marks)**

E **3** $\mathrm{f}(x) = \dfrac{x^3 - 8}{x - 2}$

Show that $\mathrm{f}(x)$ can be written in the form $px^2 + qx + r$ and find the values of p, q and r. **(4 marks)**

E **4** Given that $\dfrac{2x^2 + 4x + 5}{x^2 - 1} \equiv m + \dfrac{nx + p}{x^2 - 1}$ find the values of m, n and p. **(4 marks)**

E **5** Find the values of the constants A, B, C and D in the following identity:

$8x^3 + 2x^2 + 5 \equiv (Ax + B)(2x^2 + 2) + Cx + D$ **(4 marks)**

E **6** $\dfrac{4x^3 - 5x^2 + 3x - 14}{x^2 + 2x - 1} \equiv Ax + B + \dfrac{Cx + D}{x^2 + 2x - 1}$

Find the values of the constants A, B, C and D. **(4 marks)**

E **7** $\mathrm{g}(x) = \dfrac{x^4 + 3x^2 - 4}{x^2 + 1}$. Show that $\mathrm{g}(x)$ can be written in the form $px^2 + qx + r + \dfrac{sx + t}{x^2 + 1}$ and find the values of p, q, r, s and t. **(4 marks)**

E **8** Given that $\dfrac{2x^4 + 3x^3 - 2x^2 + 4x - 6}{x^2 + x - 2} \equiv ax^2 + bx + c + \dfrac{dx + e}{x^2 + x - 2}$ find the values of a, b, c, d and e. **(5 marks)**

(E) **9** Find the values of the constants A, B, C, D and E in the following identity:

$$3x^4 - 4x^3 - 8x^2 + 16x - 2 \equiv (Ax^2 + Bx + C)(x^2 - 3) + Dx + E$$ **(5 marks)**

(E/P) **10** **a** Fully factorise the expression $x^4 - 1$. **(2 marks)**

b Hence, or otherwise, write the algebraic fraction $\frac{x^4 - 1}{x + 1}$ in the form $(ax + b)(cx^2 + dx + e)$ and find the values of a, b, c, d and e. **(4 marks)**

In order to express an improper algebraic fraction in partial fractions, it is first necessary to divide the numerator by the denominator. Remember an improper algebraic fraction is one where the degree of the numerator is greater than or equal to the degree of the denominator.

Example 14

Given that $\frac{3x^2 - 3x - 2}{(x - 1)(x - 2)} \equiv A + \frac{B}{x - 1} + \frac{C}{x - 2}$, find the values of A, B and C.

$\frac{3x^2 - 3x - 2}{(x - 1)(x - 2)} \equiv \frac{3x^2 - 3x - 2}{x^2 - 3x + 2}$ — Multiply out the denominator on the LHS.

$$\begin{array}{r} 3 \\ \equiv x^2 - 3x + 2 \overline{\big) 3x^2 - 3x - 2} \\ 3x^2 - 9x + 6 \\ \hline 6x - 8 \end{array}$$

Divide the denominator into the numerator. It goes in 3 times, with a remainder of $6x - 8$.

Therefore

$\frac{3x^2 - 3x - 2}{(x - 1)(x - 2)} \equiv 3 + \frac{6x - 8}{x^2 - 3x + 2}$ — Write $\frac{3x^2 - 3x - 2}{(x - 1)(x - 2)}$ as a mixed fraction.

$\equiv 3 + \frac{6x - 8}{(x - 1)(x - 2)}$ — Factorise $x^2 - 3x + 2$.

Let $\frac{6x - 8}{(x - 1)(x - 2)} \equiv \frac{B}{(x - 1)} + \frac{C}{(x - 2)}$ — The denominators must be $(x - 1)$ and $(x - 2)$.

$\equiv \frac{B(x - 2) + C(x - 1)}{(x - 1)(x - 2)}$ — Add the two fractions.

$6x - 8 \equiv B(x - 2) + C(x - 1)$ — The numerators are equal.

Let $x = 2$: $12 - 8 = B \times 0 + C \times 1$

$C = 4$ — Substitute $x = 2$ to find C.

Let $x = 1$: $6 - 8 = B \times -1 + C \times 0$

$B = 2$ — Substitute $x = 1$ to find B.

$\frac{3x^2 - 3x - 2}{(x - 1)(x - 2)} \equiv 3 + \frac{6x - 8}{(x - 1)(x - 2)}$

$\equiv 3 + \frac{2}{(x - 1)} + \frac{4}{(x - 2)}$ — Write out the full solution.

So $A = 3$, $B = 2$ and $C = 4$. — Finish the question by listing the coefficients.

Exercise 1G

(E) **1** $g(x) = \dfrac{x^2 + 3x - 2}{(x - 1)(x - 2)}$. Show that g(x) can we written in the form $A + \dfrac{B}{x - 1} + \dfrac{C}{x - 2}$ and find the values of the constants A, B and C. **(4 marks)**

(E) **2** Given that $\dfrac{x^2 - 10}{(x - 2)(x + 1)} \equiv A + \dfrac{B}{x - 2} + \dfrac{C}{x + 1}$, find the values of the constants A, B and C. **(4 marks)**

(E) **3** Find the values of the constants A, B, C and D in the following identity:

$$\frac{x^3 - x^2 - x - 3}{x(x - 1)} \equiv Ax + B + \frac{C}{x} + \frac{D}{x - 1}$$ **(5 marks)**

(E) **4** Show that $\dfrac{-3x^3 - 4x^2 + 19x + 8}{x^2 + 2x - 3}$ can be expressed in the form $A + Bx + \dfrac{C}{(x - 1)} + \dfrac{D}{(x + 3)}$, where A, B, C and D are constants to be found. **(5 marks)**

(E) **5** $p(x) \equiv \dfrac{4x^2 + 25}{4x^2 - 25}$

Show that p(x) can be written in the form $A + \dfrac{B}{2x - 5} + \dfrac{C}{2x + 5}$, where A, B and C are constants to be found. **(4 marks)**

(E) **6** Given that $\dfrac{2x^2 - 1}{x^2 + 2x + 1} \equiv A + \dfrac{B}{x + 1} + \dfrac{C}{(x + 1)^2}$, find the values of the constants A, B and C. **(4 marks)**

(P) **7** By factorising the denominator, express the following as partial fractions:

a $\dfrac{4x^2 + 17x - 11}{x^2 + 3x - 4}$ **b** $\dfrac{x^4 - 4x^3 + 9x^2 - 17x + 12}{x^3 - 4x^2 + 4x}$

(E) **8** Given that $\dfrac{6x^3 - 7x^2 + 3}{3x^2 + x - 10} \equiv Ax + B + \dfrac{C}{3x - 5} + \dfrac{D}{x + 2}$, find the values of the constants A, B, C and D. **(6 marks)**

(E) **9** $q(x) = \dfrac{8x^3 + 1}{4x^2 - 4x + 1}$

Show that q(x) can be written in the form $Ax + B + \dfrac{C}{2x - 1} + \dfrac{D}{(2x - 1)^2}$ and find the values of the constants A, B, C and D. **(6 marks)**

(E) **10** $h(x) = \dfrac{x^4 + 2x^2 - 3x + 8}{x^2 + x - 2}$

Show that h(x) can be written as $Ax^2 + Bx + C + \dfrac{D}{x + 2} + \dfrac{E}{x - 1}$ and find the values of A, B, C, D and E. **(5 marks)**

Mixed exercise 1

(E/P) **1** Prove by contradiction that $\sqrt{\frac{1}{2}}$ is an irrational number. **(5 marks)**

(P) **2** Prove that if q^2 is an irrational number then q is an irrational number.

3 Simplify:

a $\dfrac{x-4}{6} \times \dfrac{2x+8}{x^2-16}$ **b** $\dfrac{x^2-3x-10}{3x^2-21} \times \dfrac{6x^2+24}{x^2+6x+8}$ **c** $\dfrac{4x^2+12x+9}{x^2+6x} \div \dfrac{4x^2-9}{2x^2+9x-18}$

(E/P) **4** **a** Simplify fully $\dfrac{4x^2-8x}{x^2-3x-4} \times \dfrac{x^2+6x+5}{2x^2+10x}$ **(3 marks)**

b Given that $\ln((4x^2-8x)(x^2+6x+5)) = 6 + \ln((x^2-3x-4)(2x^2+10x))$ find x in terms of e. **(4 marks)**

(E/P) **5** $g(x) = \dfrac{4x^3-9x^2-9x}{32x+24} \div \dfrac{x^2-3x}{6x^2-13x-5}$

a Show that g(x) can be written in the form ax^2+bx+c, where a, b and c are constants to be found. **(4 marks)**

b Hence differentiate g(x) and find $g'(-2)$. **(3 marks)**

(E) **6** Express $\dfrac{6x+1}{x-5} + \dfrac{5x+3}{x^2-3x-10}$ as a single fraction in its simplest form. **(4 marks)**

(E) **7** $f(x) = x + \dfrac{3}{x-1} - \dfrac{12}{x^2+2x-3}, x \in \mathbb{R}, x > 1$

Show that $f(x) = \dfrac{x^2+3x+3}{x+3}$ **(4 marks)**

(E) **8** $f(x) = \dfrac{x-3}{x(x-1)}$

Show that f(x) can be written in the form $\dfrac{A}{x} + \dfrac{B}{x-1}$ where A and B are constants to be found. **(3 marks)**

(E) **9** $\dfrac{-15x+21}{(x-2)(x+1)(x-5)} \equiv \dfrac{P}{x-2} + \dfrac{Q}{x+1} + \dfrac{R}{x-5}$

Find the values of the constants P, Q and R. **(4 marks)**

(E) **10** Show that $\dfrac{16x-1}{(3x+2)(2x-1)}$ can be written in the form $\dfrac{D}{3x+2} + \dfrac{E}{2x-1}$ and find the values of the constants D and E. **(4 marks)**

(E) **11** $\dfrac{7x^2 + 2x - 2}{x^2(x + 1)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x + 1}$

Find the values of the constants A, B and C. **(4 marks)**

(E) **12** $h(x) = \dfrac{21x^2 - 13}{(x + 5)(3x - 1)^2}$

Show that h(x) can be written in the form $\dfrac{D}{x + 5} + \dfrac{E}{(3x - 1)} + \dfrac{F}{(3x - 1)^2}$ where D, E and F are constants to be found. **(5 marks)**

(E) **13** Find the values of the constants A, B, C and D in the following identity:

$$x^3 - 6x^2 + 11x + 2 \equiv (x - 2)(Ax^2 + Bx + C) + D$$ **(5 marks)**

(E) **14** Show that $\dfrac{4x^3 - 6x^2 + 8x - 5}{2x + 1}$ can be put in the form $Ax^2 + Bx + C + \dfrac{D}{2x + 1}$

Find the values of the constants A, B, C and D. **(5 marks)**

(E) **15** Show that $\dfrac{x^4 + 2}{x^2 - 1} \equiv Ax^2 + Bx + C + \dfrac{D}{x^2 - 1}$ where A, B, C and D are constants to be found. **(5 marks)**

(E) **16** $\dfrac{x^4}{x^2 - 2x + 1} \equiv Ax^2 + Bx + C + \dfrac{D}{x - 1} + \dfrac{E}{(x - 1)^2}$

Find the values of the constants A, B, C, D and E. **(5 marks)**

(E) **17** $h(x) = \dfrac{2x^2 + 2x - 3}{x^2 + 2x - 3}$

Show that h(x) can be written in the form $A + \dfrac{B}{x + 3} + \dfrac{C}{x - 1}$ where A, B and C are constants to be found. **(5 marks)**

(E) **18** Given that $\dfrac{x^2 + 1}{x(x - 2)} \equiv P + \dfrac{Q}{x} + \dfrac{R}{x - 2}$, find the values of the constants P, Q and R. **(5 marks)**

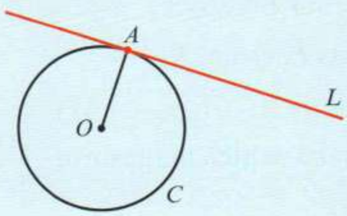
(P) **19** Given that $f(x) = 2x^3 + 9x^2 + 10x + 3$:

a show that -3 is a root of f(x)

b express $\dfrac{10}{f(x)}$ as partial fractions.

Challenge

The line L meets the circle C with centre O at exactly one point, A. Prove by contradiction that the line L is perpendicular to the radius OA.



Hint In a right-angled triangle, the side opposite the right-angle is always the longest side.

Summary of key points

1 To prove a statement by contradiction you start by assuming it is **not true**. You then use logical steps to show that this assumption leads to something impossible (either a contradiction of the assumption or a contradiction of a fact you know to be true). You can conclude that your assumption was incorrect, and the original statement **was true**.

2 A rational number can be written as $\frac{a}{b}$, where a and b are integers.

An irrational number cannot be expressed in the form $\frac{a}{b}$, where a and b are integers.

3 To multiply fractions, cancel any common factors, then multiply the numerators and multiply the denominators.

4 To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

5 To add or subtract two fractions, find a common denominator.

6 A single fraction with two distinct linear factors in the denominator can be split into two separate fractions with linear denominators. This is called splitting it into **partial fractions**:

$$\frac{5}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

7 The method of partial fractions can also be used when there are more than two distinct linear factors in the denominator:

$$\frac{7}{(x-2)(x+6)(x+3)} = \frac{A}{x-2} + \frac{B}{x+6} + \frac{C}{x+3}$$

8 A single fraction with a repeated linear factor in the denominator can be split into two or more separate fractions:

$$\frac{2x+9}{(x-5)(x+3)^2} = \frac{A}{x-5} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

9 An improper algebraic fraction is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions.

10 You can either use:

- algebraic division
- or the relationship $F(x) = Q(x) \times \text{divisor} + \text{remainder}$

to convert an improper fraction into a mixed fraction.