Trigonometric ratios 9

Objectives

After completing this unit you should be able to:

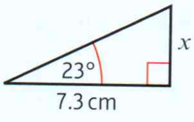
- Use the cosine rule to find a missing side or angle → pages 174–179
- Use the sine rule to find a missing side or angle → pages 179–185
- Find the area of a triangle using an appropriate formula → pages 185–187
- Solve problems involving triangles → pages 187–192
- Sketch the graphs of the sine, cosine and tangent functions → pages 192–194
- Sketch simple transformations of these graphs → pages 194–198

Trigonometry in both two and three dimensions is used by surveyors to work out distances and areas when planning building projects. You will also use trigonometry when working with vector quantities in mechanics.
→ Exercise 9B Q12 and Mixed exercise Q10, Q11

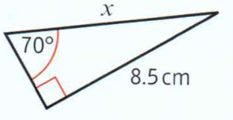
Prior knowledge check

1 Use trigonometry to find the lengths of the marked sides.

a



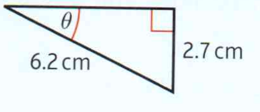
b



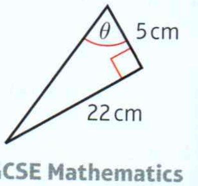
← GCSE Mathematics

2 Find the sizes of the angles marked.

a



b



← GCSE Mathematics

3 $f(x) = x^2 + 3x$. Sketch the graphs of

a $y = f(x)$ **b** $y = f(x + 2)$

c $y = f(x) - 3$ **d** $y = f(\frac{1}{2}x)$

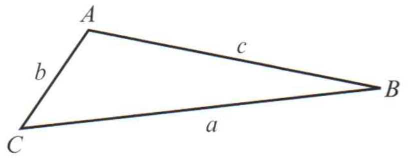
← Sections 4.5, 4.6

9.1 The cosine rule

The cosine rule can be used to work out missing sides or angles in triangles.

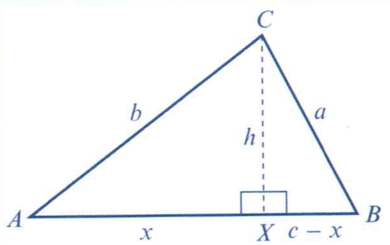
- **This version of the cosine rule is used to find a missing side if you know two sides and the angle between them:**

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Watch out You can exchange the letters depending on which side you want to find, as long as each side has the same letter as the **opposite** angle.

You can use the standard trigonometric ratios for right-angled triangles to prove the cosine rule:



$h^2 + x^2 = b^2$ — Use Pythagoras' theorem in $\triangle CAX$.

and $h^2 + (c - x)^2 = a^2$ — Use Pythagoras' theorem in $\triangle CBX$.

So $x^2 - (c - x)^2 = b^2 - a^2$ — Subtract the two equations.

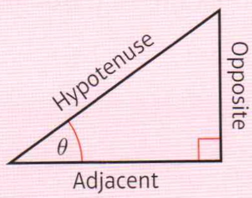
So $2cx - c^2 = b^2 - a^2$ — $(c - x)^2 = c^2 - 2cx + x^2$. So $x^2 - (c - x)^2 = x^2 - c^2 + 2cx - x^2$.

$a^2 = b^2 + c^2 - 2cx$ (1) — Rearrange.

but $x = b\cos A$ (2) — Use the cosine ratio $\cos A = \frac{x}{b}$ in $\triangle CAX$.

So $a^2 = b^2 + c^2 - 2bc\cos A$ — Combine (1) and (2). This is the cosine rule.

Hint For a right-angled triangle



$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

If you are given all three sides and asked to find an angle, the cosine rule can be rearranged.

$$a^2 + 2bc\cos A = b^2 + c^2$$

$$2bc\cos A = b^2 + c^2 - a^2$$

Hence $\qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$

You can exchange the letters depending on which angle you want to find.

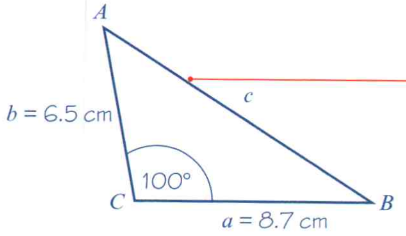
- **This version of the cosine rule is used to find an angle if you know all three sides:**

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Online Explore the cosine rule using technology.

Example 1

Calculate the length of the side AB of the triangle ABC in which $AC = 6.5$ cm, $BC = 8.7$ cm and $\angle ACB = 100°$.



$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 8.7^2 + 6.5^2 - 2 \times 8.7 \times 6.5 \times \cos 100°$

$= 75.69 + 42.25 - (-19.639\ldots)$

$= 137.57\ldots$

So $c = 11.729\ldots$

So $AB = 11.7$ cm (3 s.f.)

Label the sides of the triangle with small letters a, b and c opposite the angles marked.

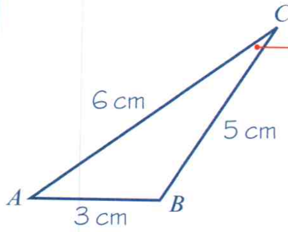
Write out the formula you are using as the first line of working, then substitute in the values given.

Don't round any values until the end of your working. You can write your final answer to 3 significant figures.

Find the square root.

Example 2

Find the size of the smallest angle in a triangle whose sides have lengths 3 cm, 5 cm and 6 cm.



$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \dfrac{5^2 + 6^2 - 3^2}{2 \times 5 \times 6}$

$= 0.8666\ldots$

$C = 29.9°$ (3 s.f.)

The size of the smallest angle is 29.9°.

Label the triangle ABC.
The smallest angle is opposite the smallest side so angle C is required.

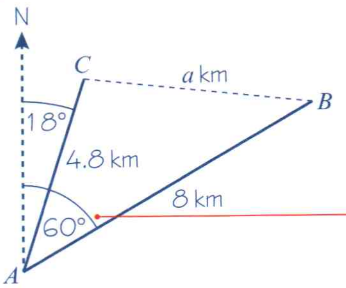
Use the cosine rule $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Online Use your calculator to work this out efficiently.

$C = \cos^{-1}(0.8666\ldots)$

Example 3

Coastguard station B is 8 km, on a bearing of 060°, from coastguard station A. A ship C is 4.8 km, on a bearing of 018°, away from A. Calculate how far C is from B.



$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 4.8^2 + 8^2 - 2 \times 4.8 \times 8 \times \cos 42°$

$= 29.966...$

$a = 5.47$ (3 s.f.)

C is 5.47 km from coastguard station B.

Problem-solving

If no diagram is given with a question you should draw one carefully. Double-check that the information given in the question matches your sketch.

In $\triangle ABC$, $\angle CAB = 60° - 18° = 42°$.

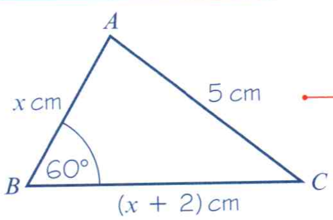
You now have $b = 4.8$ km, $c = 8$ km and $A = 42°$. Use the cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$.

If possible, work this out in one go using your calculator.

Take the square root of 29.966… and round your final answer to 3 significant figures.

Example 4

In $\triangle ABC$, $AB = x$ cm, $BC = (x + 2)$ cm, $AC = 5$ cm and $\angle ABC = 60°$.
Find the value of x.



$b^2 = a^2 + c^2 - 2ac\cos B$

$5^2 = (x + 2)^2 + x^2 - 2x(x + 2)\cos 60°$

So $25 = 2x^2 + 4x + 4 - x^2 - 2x$

So $x^2 + 2x - 21 = 0$

$x = \dfrac{-2 \pm \sqrt{88}}{2}$

$= 3.69$ (3 s.f.)

Use the information given in the question to draw a sketch.

Carefully expand and simplify the right-hand side. Note that $\cos 60° = \frac{1}{2}$

Rearrange to the form $ax^2 + bx + c = 0$.

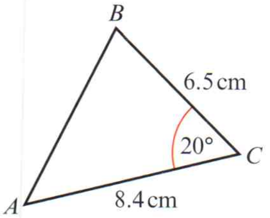
Solve the quadratic equation using the quadratic formula. ← Section 2.1

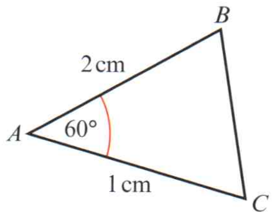
x represents a length so it cannot be negative.

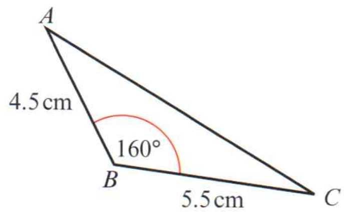
Exercise 9A

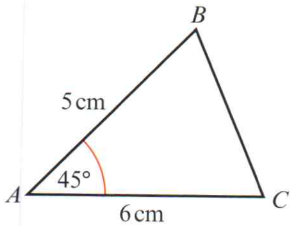
Give answers to 3 significant figures, where appropriate.

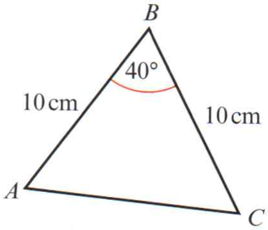
1 In each of the following triangles calculate the length of the missing side.

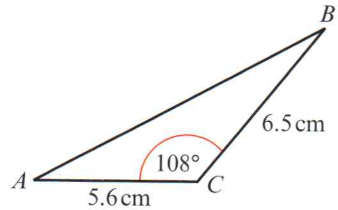
a


b


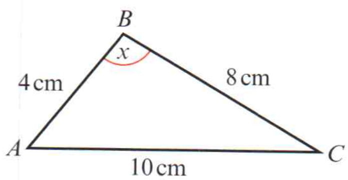
c


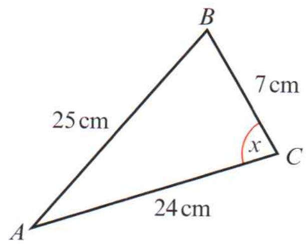
d


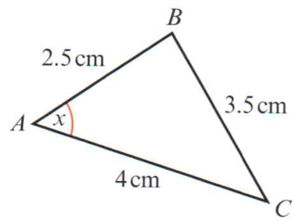
e


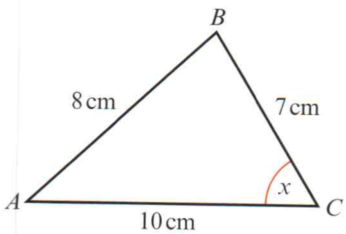
f


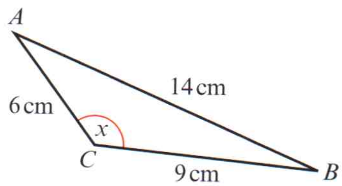
2 In the following triangles calculate the size of the angle marked x:

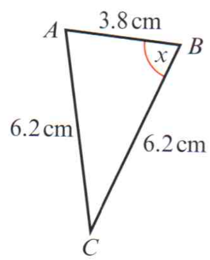
a


b


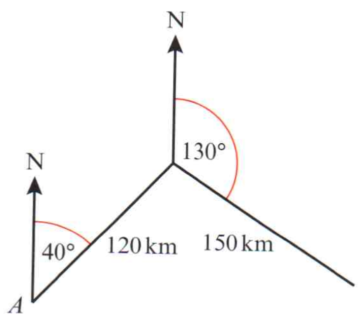
c


d


e


f


3 A plane flies from airport A on a bearing of 040° for 120 km and then on a bearing of 130° for 150 km. Calculate the distance of the plane from the airport.

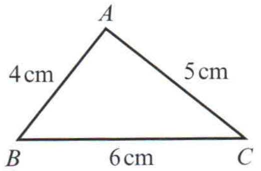


4 From a point A a boat sails due north for 7 km to B. The boat leaves B and moves on a bearing of 100° for 10 km until it reaches C. Calculate the distance of C from A.

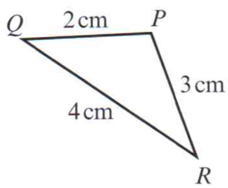
5 A helicopter flies on a bearing of 080° from A to B, where $AB = 50$ km.
It then flies for 60 km to a point C.
Given that C is 80 km from A, calculate the bearing of C from A.

6 The distance from the tee, T, to the flag, F, on a particular hole on a golf course is 494 yards.
A golfer's tee shot travels 220 yards and lands at the point S, where $\angle STF = 22°$.
Calculate how far the ball is from the flag.

(P) **7** Show that $\cos A = \frac{1}{8}$



(P) **8** Show that $\cos P = -\frac{1}{4}$



9 In $\triangle ABC$, $AB = 5$ cm, $BC = 6$ cm and $AC = 10$ cm.
Calculate the size of the smallest angle.

10 In $\triangle ABC$, $AB = 9.3$ cm, $BC = 6.2$ cm and $AC = 12.7$ cm.
Calculate the size of the largest angle.

(P) **11** The lengths of the sides of a triangle are in the ratio $2:3:4$.
Calculate the size of the largest angle.

12 In $\triangle ABC$, $AB = (x - 3)$ cm, $BC = (x + 3)$ cm, $AC = 8$ cm and $\angle BAC = 60°$.
Use the cosine rule to find the value of x.

(P) **13** In $\triangle ABC$, $AB = x$ cm, $BC = (x - 4)$ cm, $AC = 10$ cm and $\angle BAC = 60°$.
Calculate the value of x.

(P) **14** In $\triangle ABC$, $AB = (5 - x)$ cm, $BC = (4 + x)$ cm, $\angle ABC = 120°$ and $AC = y$ cm.

a Show that $y^2 = x^2 - x + 61$.

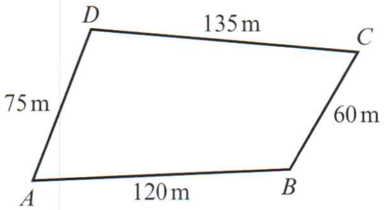
b Use the method of completing the square to find the minimum value of y^2, and give the value of x for which this occurs.

P **15** In $\triangle ABC$, $AB = x$ cm, $BC = 5$ cm, $AC = (10 - x)$ cm.

a Show that $\cos \angle ABC = \dfrac{4x - 15}{2x}$

b Given that $\cos \angle ABC = -\frac{1}{7}$, work out the value of x.

P **16** A farmer has a field in the shape of a quadrilateral as shown.



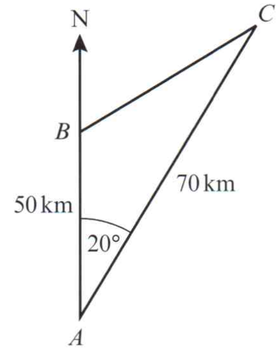
Problem-solving
You will have to use the cosine rule twice. Copy the diagram and write any angles or lengths you work out on your copy.

The angle between fences AB and AD is 74°. Find the angle between fences BC and CD.

/P **17** The diagram shows three cargo ships, A, B and C, which are in the same horizontal plane. Ship B is 50 km due north of ship A and ship C is 70 km from ship A. The bearing of C from A is 020°.

a Calculate the distance between ships B and C, in kilometres to 3 s.f. **(3 marks)**

b Calculate the bearing of ship C from ship B. **(4 marks)**

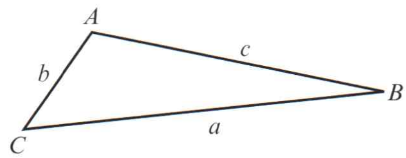


9.2 The sine rule

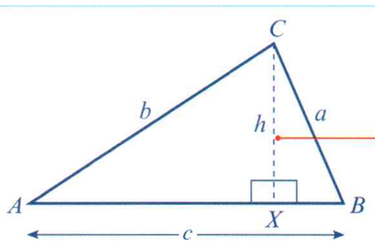
The sine rule can be used to work out missing sides or angles in triangles.

- **This version of the sine rule is used to find the length of a missing side:**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



You can use the standard trigonometric ratios for right-angled triangles to prove the sine rule:



$\sin B = \dfrac{h}{a} \Rightarrow h = a \sin B$

In a general triangle ABC, draw the perpendicular from C to AB. It meets AB at X.
The length of CX is h.

Use the sine ratio in triangle CBX.

Online Explore the sine rule using technology.

and $\sin A = \frac{h}{b} \Rightarrow h = b \sin A$ — Use the sine ratio in triangle CAX.

So $a \sin B = b \sin A$

So $\frac{a}{\sin A} = \frac{b}{\sin B}$ — Divide throughout by $\sin A \sin B$.

In a similar way, by drawing the perpendicular from B to the side AC, you can show that:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

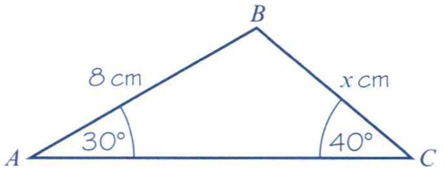
So $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ — This is the sine rule and is true for all triangles.

- **This version of the sine rule is used to find a missing angle:**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example 5

In $\triangle ABC$, $AB = 8$ cm, $\angle BAC = 30°$ and $\angle BCA = 40°$. Find BC.



Always draw a diagram and carefully add the data. Here $c = 8$ (cm), $C = 40°$, $A = 30°$, $a = x$ (cm).

In a triangle, the larger a side is, the larger the opposite angle is. Here, as $C > A$, then $c > a$, so you know that $8 > x$.

$\frac{a}{\sin A} = \frac{c}{\sin C}$ — Use this version of the sine rule to find a missing side. Write the formula you are going to use as the first line of working.

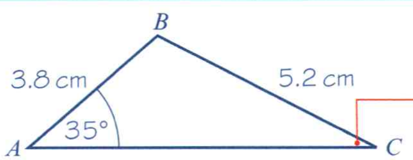
$$\frac{x}{\sin 30°} = \frac{8}{\sin 40°}$$

So $x = \frac{8 \sin 30°}{\sin 40°} = 6.2228...$ — Multiply throughout by sin 30°.

$= 6.22$ cm (3 s.f.) — Give your answer to 3 significant figures.

Example 6

In $\triangle ABC$, $AB = 3.8$ cm, $BC = 5.2$ cm and $\angle BAC = 35°$. Find $\angle ABC$.



Here $a = 5.2$ cm, $c = 3.8$ cm and $A = 35°$. You first need to find angle C.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

Use $\frac{\sin C}{c} = \frac{\sin A}{a}$

Write the formula you are going to use as the first line of working.

$$\frac{\sin C}{3.8} = \frac{\sin 35°}{5.2}$$

$$\text{So } \sin C = \frac{3.8 \sin 35°}{5.2}$$

$$C = 24.781\ldots$$

Use your calculator to find the value of C in a single step. Don't round your answer at this point.

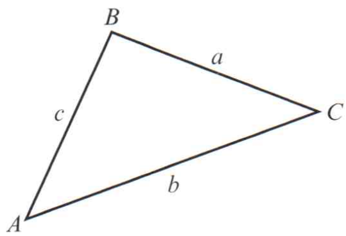
So $B = 120°$ (3 s.f.)

$B = 180° - (24.781\ldots° + 35°) = 120.21\ldots$ which rounds to 120° to 3 s.f.

Exercise 9B

Give answers to 3 significant figures, where appropriate.

1 In each of parts **a** to **d**, the given values refer to the general triangle.



a Given that $a = 8$ cm, $A = 30°$, $B = 72°$, find b.

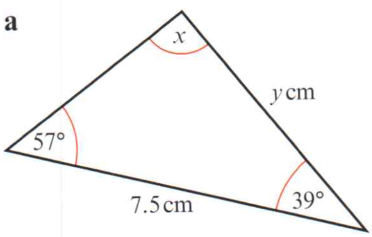
b Given that $a = 24$ cm, $A = 110°$, $C = 22°$, find c.

c Given that $b = 14.7$ cm, $A = 30°$, $C = 95°$, find a.

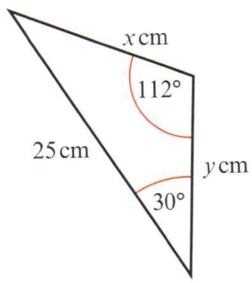
d Given that $c = 9.8$ cm, $B = 68.4°$, $C = 83.7°$, find a.

2 In each of the following triangles calculate the values of x and y.

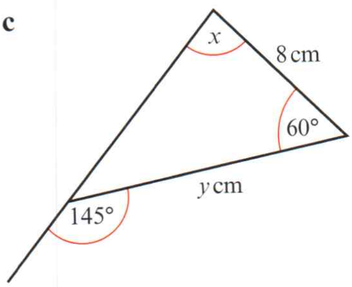
a



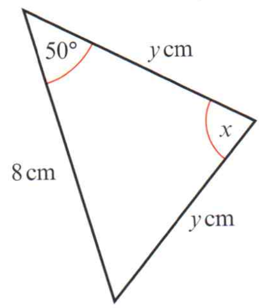
b



c

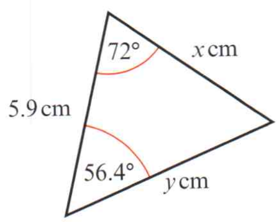


d

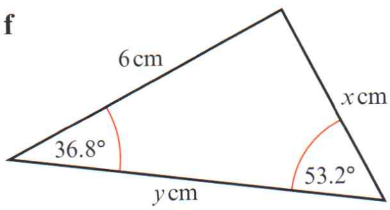


Hint In parts **c** and **d**, start by finding the size of the third angle.

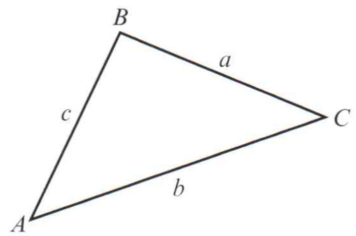
e



f



3 In each of the following sets of data for a triangle ABC, find the value of x.



a $AB = 6$ cm, $BC = 9$ cm, $\angle BAC = 117°$, $\angle ACB = x$

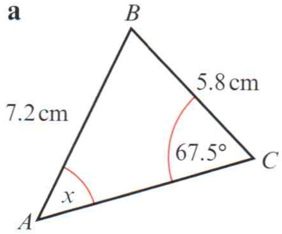
b $AC = 11$ cm, $BC = 10$ cm, $\angle ABC = 40°$, $\angle CAB = x$

c $AB = 6$ cm, $BC = 8$ cm, $\angle BAC = 60°$, $\angle ACB = x$

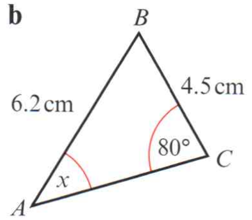
d $AB = 8.7$ cm, $AC = 10.8$ cm, $\angle ABC = 28°$, $\angle BAC = x$

4 In each of the diagrams shown below, work out the size of angle x.

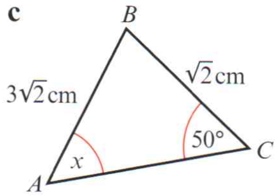
a



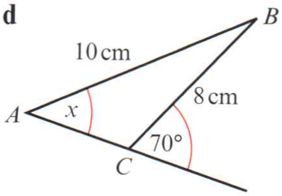
b



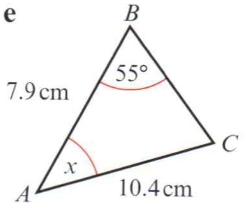
c



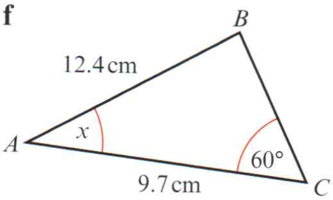
d



e



f

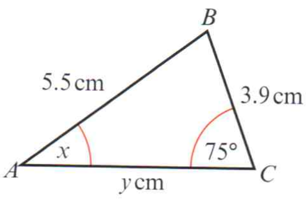


5 In $\triangle PQR$, $QR = \sqrt{3}$ cm, $\angle PQR = 45°$ and $\angle QPR = 60°$. Find **a** PR and **b** PQ.

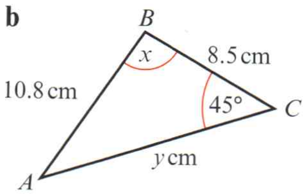
6 In $\triangle PQR$, $PQ = 15$ cm, $QR = 12$ cm and $\angle PRQ = 75°$. Find the two remaining angles.

7 In each of the following diagrams work out the values of x and y.

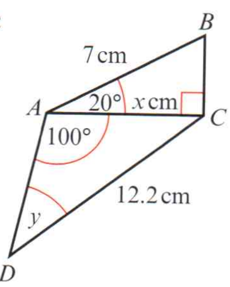
a



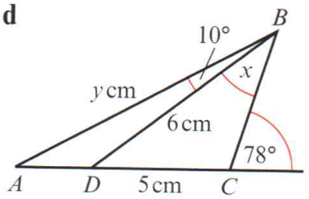
b



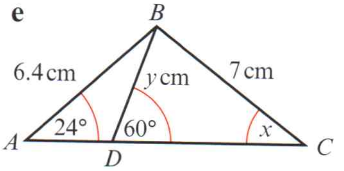
c



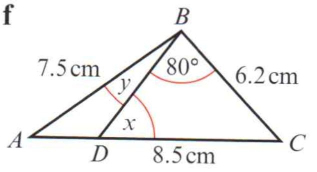
d



e



f



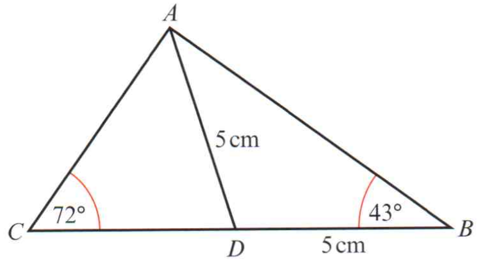
P **8** Town B is 6 km, on a bearing of 020°, from town A. Town C is located on a bearing of 055° from town A and on a bearing of 120° from town B. Work out the distance of town C from:

a town A **b** town B

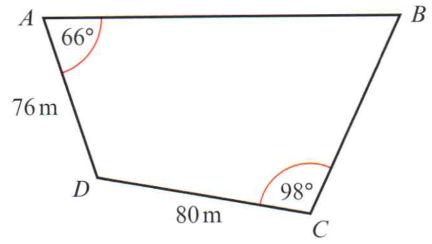
Problem-solving

Draw a sketch to show the information.

9 In the diagram $AD = DB = 5$ cm, $\angle ABC = 43°$ and $\angle ACB = 72°$.
Calculate:
a AB
b CD



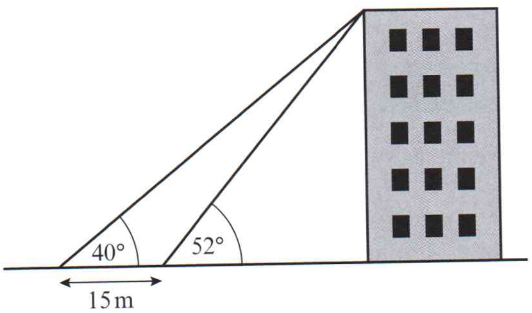
10 A zookeeper is building an enclosure for some llamas. The enclosure is in the shape of a quadrilateral as shown.
If the length of the diagonal BD is 136 m
a find the angle between the fences AB and BC
b find the length of fence AB



E/P 11 In $\triangle ABC$, $AB = x$ cm, $BC = (4 - x)$ cm, $\angle BAC = y$ and $\angle BCA = 30°$.
Given that $\sin y = \frac{1}{\sqrt{2}}$, show that
$x = 4(\sqrt{2} - 1)$ **(5 marks)**

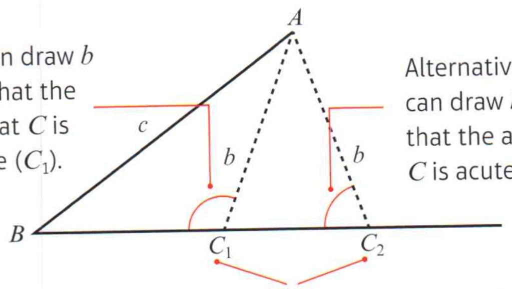
Problem-solving
You can use the value of sin y directly in your calculation. You don't need to work out the value of y.

E/P 12 A surveyor wants to determine the height of a building. She measures the angle of elevation of the top of the building at two points 15 m apart on the ground.
a Use this information to determine the height of the building. **(4 marks)**
b State one assumption made by the surveyor in using this mathematical model. **(1 mark)**



For given side lengths b and c and given angle B, you can draw the triangle in two different ways.

You can draw b such that the angle at C is obtuse (C_1).

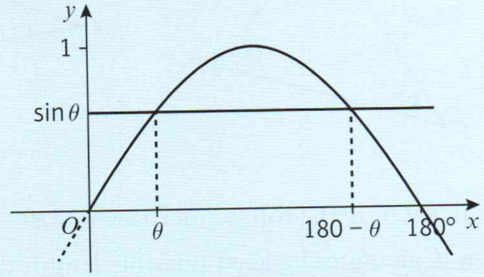


Alternatively you can draw b such that the angle at C is acute (C_2).

Since AC_1C_2 is an isosceles triangle, it follows that the angles AC_1B and AC_2B add together to make 180°.

- **The sine rule sometimes produces two possible solutions for a missing angle:**
  - **sin θ = sin (180° − θ)**

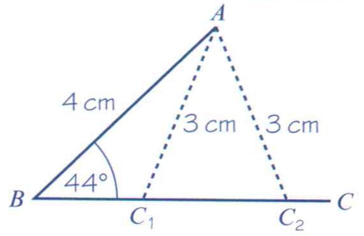
Links You can confirm this relationship by considering the graph of $y = \sin x$.



→ Section 9.5 and Chapter 10

Example 7

In $\triangle ABC$, $AB = 4$ cm, $AC = 3$ cm and $\angle ABC = 44°$. Work out the two possible values of $\angle ACB$.



$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{4} = \frac{\sin 44°}{3}$$

$$\sin C = \frac{4 \sin 44°}{3}$$

So $C = 67.851\ldots = 67.9°$ (3 s.f.)

or $C = 180 - 67.851\ldots = 112.14\ldots$

$= 112°$ (3 s.f.)

Problem-solving

Think about which lengths and angles are fixed, and which ones can vary. The length AC is fixed. If you drew a circle with radius 3 cm and centre A it would intersect the horizontal side of the triangle at two points, C_1 and C_2.

Use $\frac{\sin C}{c} = \frac{\sin B}{b}$, where $b = 3$, $c = 4$, $B = 44°$.

As $\sin (180 - \theta) = \sin \theta$.

Exercise 9C

Give answers to 3 significant figures, where appropriate.

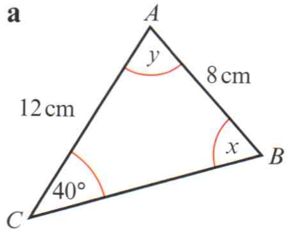
1 In $\triangle ABC$, $BC = 6$ cm, $AC = 4.5$ cm and $\angle ABC = 45°$.

a Calculate the two possible values of $\angle BAC$.

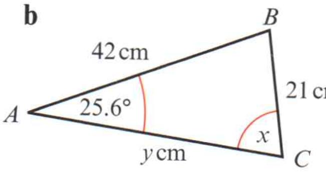
b Draw a diagram to illustrate your answers.

2 In each of the diagrams shown below, calculate the possible values of x and the corresponding values of y.

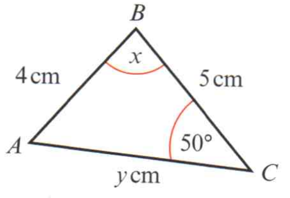
a



b



c



(P) **3** In each of the following cases $\triangle ABC$ has $\angle ABC = 30°$ and $AB = 10$ cm.

a Calculate the least possible length that AC could be.

b Given that $AC = 12$ cm, calculate $\angle ACB$.

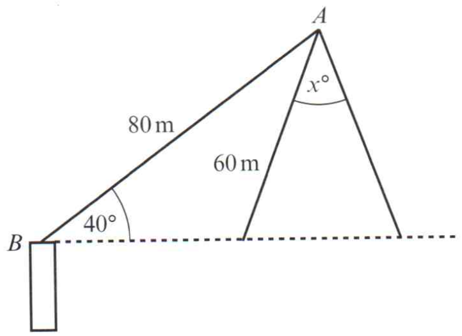
c Given instead that $AC = 7$ cm, calculate the two possible values of $\angle ACB$.

P **4** Triangle ABC is such that $AB = 4$ cm, $BC = 6$ cm and $\angle ACB = 36°$. Show that one of the possible values of $\angle ABC$ is 25.8° (to 3 s.f.). Using this value, calculate the length of AC.

P **5** Two triangles ABC are such that $AB = 4.5$ cm, $BC = 6.8$ cm and $\angle ACB = 30°$. Work out the value of the largest angle in each of the triangles.

E/P **6** **a** A crane arm AB of length 80 m is anchored at point B at an angle of 40° to the horizontal. A wrecking ball is suspended on a cable of length 60 m from A. Find the angle x through which the wrecking ball rotates as it passes the two points level with the base of the crane arm at B. **(6 marks)**

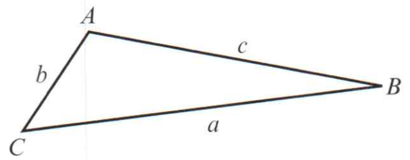
b Write down one modelling assumption you have made. **(1 mark)**



9.3 Areas of triangles

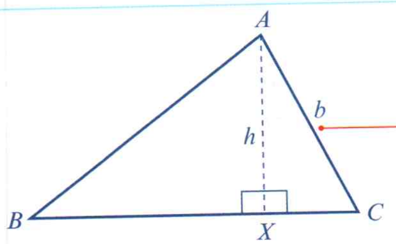
You need to be able to use the formula for finding the area of any triangle when you know two sides and the angle between them.

- **Area =** $\frac{1}{2}ab \sin C$



Hint As with the cosine rule, the letters are interchangeable. For example, if you know angle B and sides a and c, the formula becomes Area = $\frac{1}{2}ac \sin B$.

A proof of the formula:



The perpendicular from A to BC is drawn and it meets BC at X. The length of $AX = h$.

Area of $\triangle ABC = \frac{1}{2}ah$ (1)

But $h = b \sin C$ (2)

So Area $= \frac{1}{2}ab \sin C$

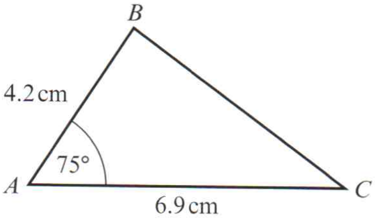
Area of triangle = $\frac{1}{2}$ base × height.

Use the sine ratio $\sin C = \frac{h}{b}$ in $\triangle AXC$.

Substitute (2) into (1).

Example 8

Work out the area of the triangle shown below.



Online Explore the area of a triangle using technology.

Area $= \frac{1}{2}bc \sin A$

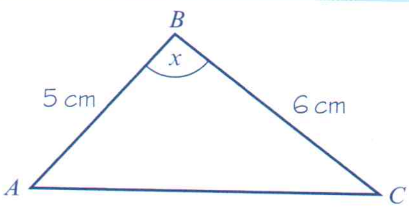
Area of $\triangle ABC = \frac{1}{2} \times 6.9 \times 4.2 \times \sin 75°\,\text{cm}^2$

$= 14.0\,\text{cm}^2$ (3 s.f.)

Here $b = 6.9$ cm, $c = 4.2$ cm and angle $A = 75°$, so use:
Area $= \frac{1}{2}bc \sin A$.

Example 9

In $\triangle ABC$, $AB = 5$ cm, $BC = 6$ cm and $\angle ABC = x$. Given that the area of $\triangle ABC$ is $12\,\text{cm}^2$ and that AC is the longest side, find the value of x.



Area $= \frac{1}{2}ac \sin B$

Area $\triangle ABC = \frac{1}{2} \times 6 \times 5 \times \sin x\,\text{cm}^2$

So $12 = \frac{1}{2} \times 6 \times 5 \times \sin x\,\text{cm}^2$

So $\sin x = 0.8$

$x = 126.86\ldots$

$= 127°$ (3 s.f.)

Here $a = 6$ cm, $c = 5$ cm and angle $B = x$, so use:
Area $= \frac{1}{2}ac \sin B$.

Area of $\triangle ABC$ is $12\,\text{cm}^2$.

$\sin x = \frac{12}{15}$

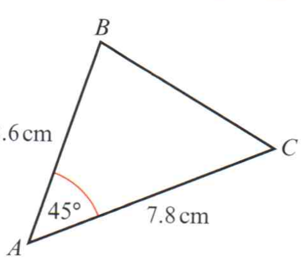
Problem-solving

There are two values of x for which $\sin x = 0.8$, $53.13\ldots°$ and $126.86\ldots°$, but here you know B is the largest angle because AC is the largest side.

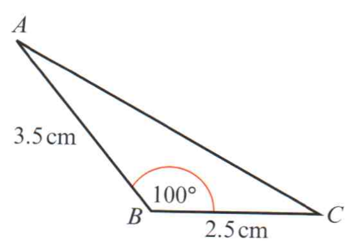
Exercise 9D

1 Calculate the area of each triangle.

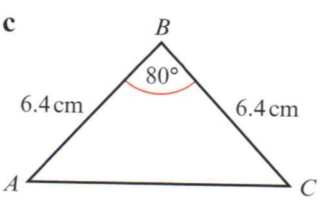
a



b

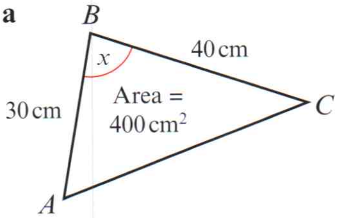


c

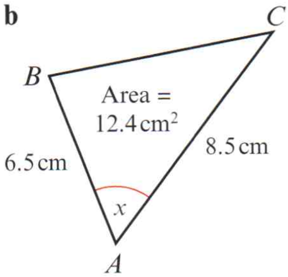


2 Work out the possible sizes of x in the following triangles.

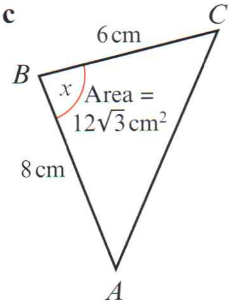
a



b

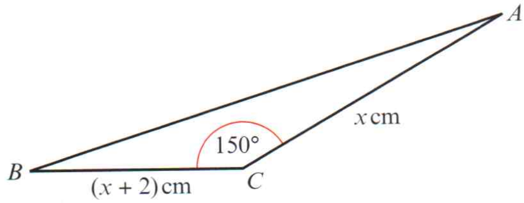


c



3 A fenced triangular plot of ground has area 1200 m^2. The fences along the two smaller sides are 60 m and 80 m respectively and the angle between them is θ. Show that $\theta = 150°$, and work out the total length of fencing.

P **4** In triangle ABC, $BC = (x + 2)$ cm, $AC = x$ cm and $\angle BCA = 150°$. Given that the area of the triangle is 5 cm^2, work out the value of x, giving your answer to 3 significant figures.



E/P **5** In $\triangle PQR$, $PQ = (x + 2)$ cm, $PR = (5 - x)$ cm and $\angle QPR = 30°$. The area of the triangle is $A\text{ cm}^2$.

a Show that $A = \frac{1}{4}(10 + 3x - x^2)$. **(3 marks)**

b Use the method of completing the square, or otherwise, to find the maximum value of A, and give the corresponding value of x. **(4 marks)**

E/P **6** In $\triangle ABC$, $AB = x$ cm, $AC = (5 + x)$ cm and $\angle BAC = 150°$. Given that the area of the triangle is $3\frac{3}{4}\text{ cm}^2$

Problem-solving x represents a length so it must be positive.

a Show that x satisfies the equation $x^2 + 5x - 15 = 0$. **(3 marks)**

b Calculate the value of x, giving your answer to 3 significant figures. **(3 marks)**

9.4 Solving triangle problems

You can solve problems involving triangles by using the sine and cosine rules along with Pythagoras' theorem and standard right-angled triangle trigonometry.

If some of the triangles are right-angled, try to use basic trigonometry and Pythagoras' theorem first to work out other information.

If you encounter a triangle which is not right-angled, you will need to decide whether to use the sine rule or the cosine rule. Generally, use the sine rule when you are considering two angles and two sides and the cosine rule when you are considering three sides and one angle.

Watch out The sine rule is often easier to use than the cosine rule. If you know one side and an opposite angle in a triangle, try to use the sine rule to find other missing sides and angles.

For questions involving area, check first whether you can use Area = $\frac{1}{2}$ × base × height, before using the formula involving sine.

- to find an unknown angle given two sides and one opposite angle, use the sine rule
- to find an unknown side given two angles and one opposite side, use the sine rule
- to find an unknown angle given all three sides, use the cosine rule
- to find an unknown side given two sides and the angle between them, use the cosine rule
- to find the area given two sides and the angle between them, use Area = $\frac{1}{2}ab \sin C$

Example 10

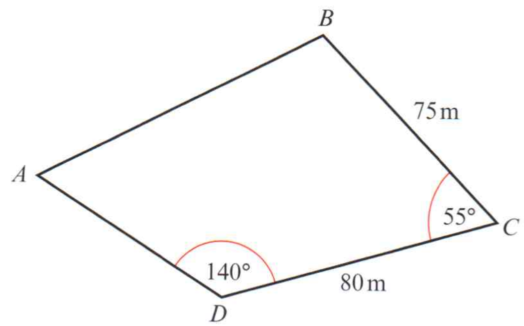
The diagram shows the locations of four mobile phone masts in a field. $BC = 75$ m, $CD = 80$ m, angle $BCD = 55°$ and angle $ADC = 140°$.

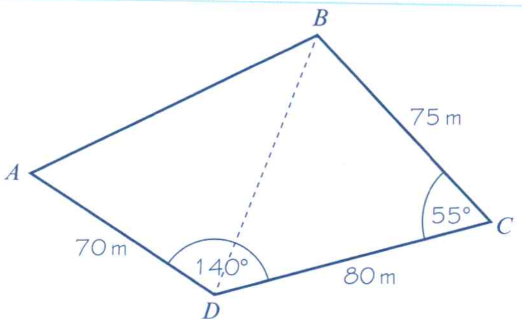
In order that the masts do not interfere with each other, they must be at least 70 m apart. Given that A is the minimum distance from D, find:

a the distance A is from B

b the angle BAD

c the area enclosed by the four masts





Problem-solving

Split the diagram into two triangles. Use the information in triangle BCD to work out the length BD. You are using three sides and one angle so use the **cosine rule**.

a $BD^2 = BC^2 + CD^2 - 2(BC)(CD)\cos(\angle BCD)$

Find BD first using the cosine rule.

$BD^2 = 75^2 + 80^2 - 2(75)(80)\cos 55°$

$BD^2 = 5142.08...$

$BD = \sqrt{5142.08...} = 71.708...$

Store this value in your calculator, or write down all the digits from your calculator display.

$$\frac{\sin(\angle BDC)}{BC} = \frac{\sin(\angle BCD)}{BD}$$

You know a side and its opposite angle (BD and $\angle BCD$), so use the sine rule to calculate angle BDC.

$$\sin(\angle BDC) = \frac{\sin(55°) \times 75}{71.708} = 0.85675...$$

$\angle BDC = 58.954...$

$\angle BDA = 140 - 58.954... = 81.045...$

Find BDA and store this value, or write down all the digits from your calculator display.

$AB^2 = AD^2 + BD^2 - 2(AD)(BD)\cos(\angle BDA)$

$AB^2 = 70^2 + 71.708...^2$
$\quad - 2(70)(71.708...)\cos(81.045...)$

$AB^2 = 8479.55...$

$AB = \sqrt{8479.55...} = 92.084...$
$\quad = 92.1\text{ m (3 s.f.)}$

> You can now use the cosine rule in triangle ABD to find AB.

> AB is a length, so you are only interested in the positive solution.

b $\dfrac{\sin(\angle BAD)}{BD} = \dfrac{\sin(\angle BDA)}{AB}$

$\sin(\angle BAD) = \dfrac{\sin(81.045...) \times 71.708...}{92.084...}$

$\quad = 0.769...$

$\angle BAD = 50.28... = 50.3°\text{ (3 s.f.)}$

> Use the sine rule to calculate angle BAD.

> Alternatively you could have used the cosine rule with sides AB, BD and AD.

c Area $ABCD$ = area BCD + area BDA

Area $ABCD = \frac{1}{2}(BC)(CD)\sin(\angle BCD)$
$\quad + \frac{1}{2}(AB)(AD)\sin(\angle BAD)$

Area $ABCD = \frac{1}{2}(75)(80)\sin(55°)$
$\quad + \frac{1}{2}(92.084...)(70)\sin(50.28...°)$

Area $ABCD = 2457.4... + 2479.2...$

Area $ABCD = 4936.6... = 4940\text{ m}^2\text{ (3 s.f.)}$

> Use the area formula twice.

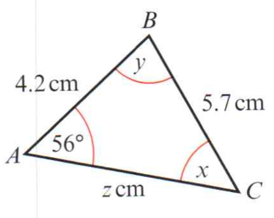
Online Explore the solution step-by-step using technology.

Exercise 9E

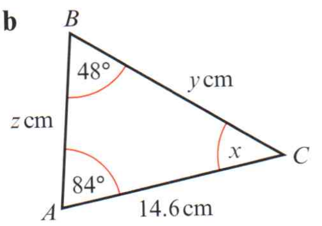
Try to use the most efficient method, and give answers to 3 significant figures.

1 In each triangle below find the values of x, y and z.

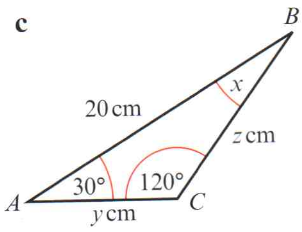
a



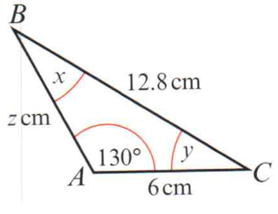
b



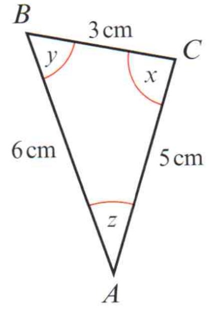
c



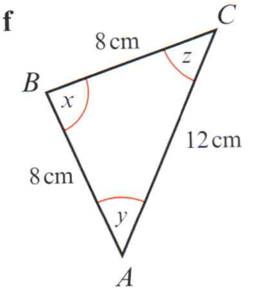
d



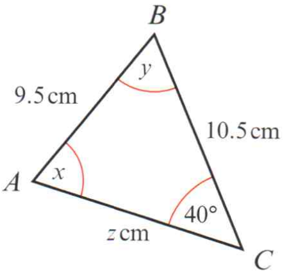
e



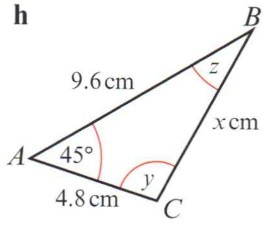
f



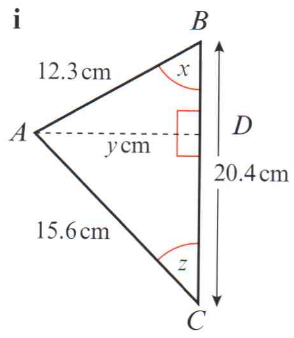
g



h



i



2 In $\triangle ABC$, calculate the size of the remaining angles, the lengths of the third side and the area of the triangle given that

a $\angle BAC = 40°$, $AB = 8.5$ cm and $BC = 10.2$ cm

b $\angle ACB = 110°$, $AC = 4.9$ cm and $BC = 6.8$ cm

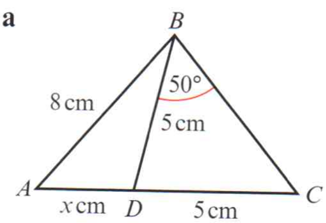
3 A hiker walks due north from A and after 8 km reaches B. She then walks a further 8 km on a bearing of 120° to C. Work out **a** the distance from A to C and **b** the bearing of C from A.

(P) **4** A helicopter flies on a bearing of 200° from A to B, where $AB = 70$ km. It then flies on a bearing of 150° from B to C, where C is due south of A. Work out the distance of C from A.

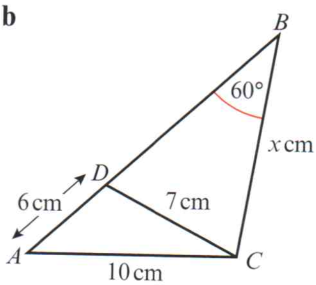
(P) **5** Two radar stations A and B are 16 km apart and A is due north of B. A ship is known to be on a bearing of 150° from A and 10 km from B. Show that this information gives two positions for the ship, and calculate the distance between these two positions.

(P) **6** Find x in each of the following diagrams:

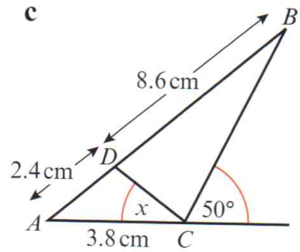
a



b



c



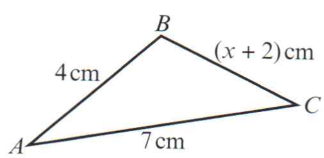
(P) **7** In $\triangle ABC$, $AB = 4$ cm, $BC = (x + 2)$ cm and $AC = 7$ cm.

a Explain how you know that $1 < x < 9$.

b Work out the value of x and the area of the triangle for the cases when

i $\angle ABC = 60°$ and

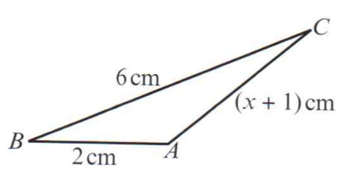
ii $\angle ABC = 45°$, giving your answers to 3 significant figures.



(P) **8** In the triangle, $\cos \angle ABC = \frac{5}{8}$

a Calculate the value of x.

b Find the area of triangle ABC.



(P) **9** In $\triangle ABC$, $AB = \sqrt{2}$ cm, $BC = \sqrt{3}$ cm and $\angle BAC = 60°$. Show that $\angle ACB = 45°$ and find AC.

(P) **10** In $\triangle ABC$, $AB = (2 - x)$ cm, $BC = (x + 1)$ cm and $\angle ABC = 120°$.

a Show that $AC^2 = x^2 - x + 7$.

b Find the value of x for which AC has a minimum value.

Problem-solving

Complete the square for the expression $x^2 - x + 7$ to find the minimum value of AC^2 and the value of x where it occurs.

11 Triangle ABC is such that $BC = 5\sqrt{2}$ cm, $\angle ABC = 30°$ and $\angle BAC = \theta$, where $\sin\theta = \frac{\sqrt{5}}{8}$

Work out the length of AC, giving your answer in the form $a\sqrt{b}$, where a and b are integers.

(P) **12** The perimeter of $\triangle ABC = 15$ cm. Given that $AB = 7$ cm and $\angle BAC = 60°$, find the lengths of AC and BC and the area of the triangle.

(E) **13** In the triangle ABC, $AB = 14$ cm, $BC = 12$ cm and $CA = 15$ cm.

a Find the size of angle C, giving your answer to 3 s.f. **(3 marks)**

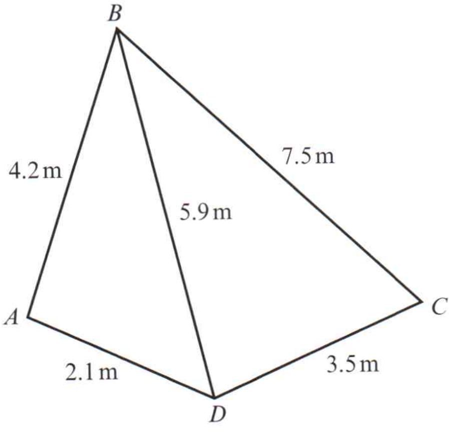
b Find the area of triangle ABC, giving your answer in cm² to 3 s.f. **(3 marks)**

(E/P) **14** A flower bed is in the shape of a quadrilateral as shown in the diagram.

a Find the sizes of angles DAB and BCD. **(4 marks)**

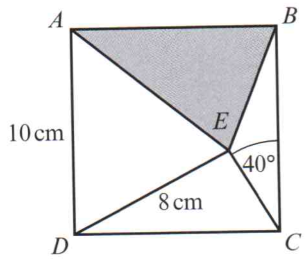
b Find the total area of the flower bed. **(3 marks)**

c Find the length of the diagonal AC. **(4 marks)**



(E/P) **15** $ABCD$ is a square. Angle CED is obtuse.

Find the area of the shaded triangle. **(7 marks)**

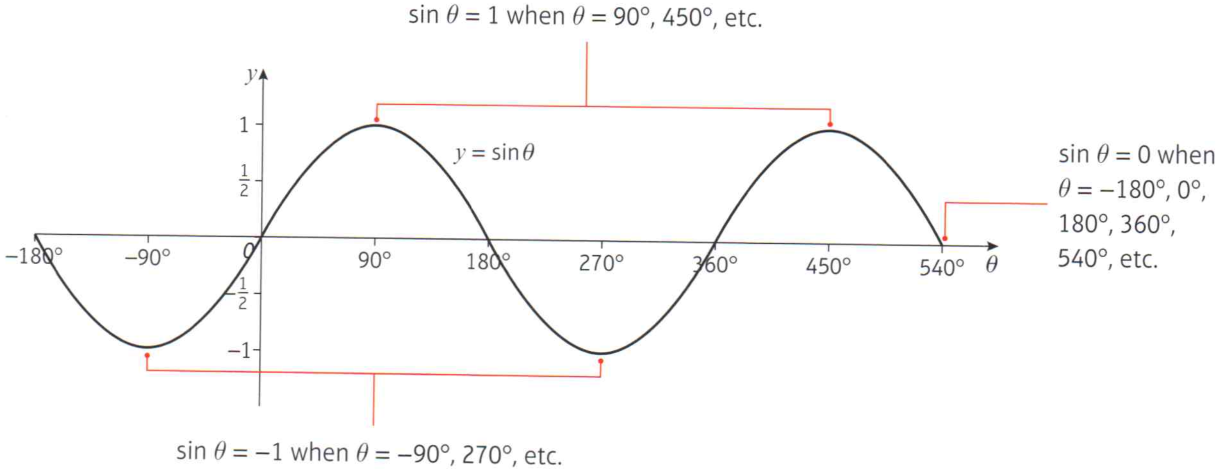


9.5 Graphs of sine, cosine and tangent

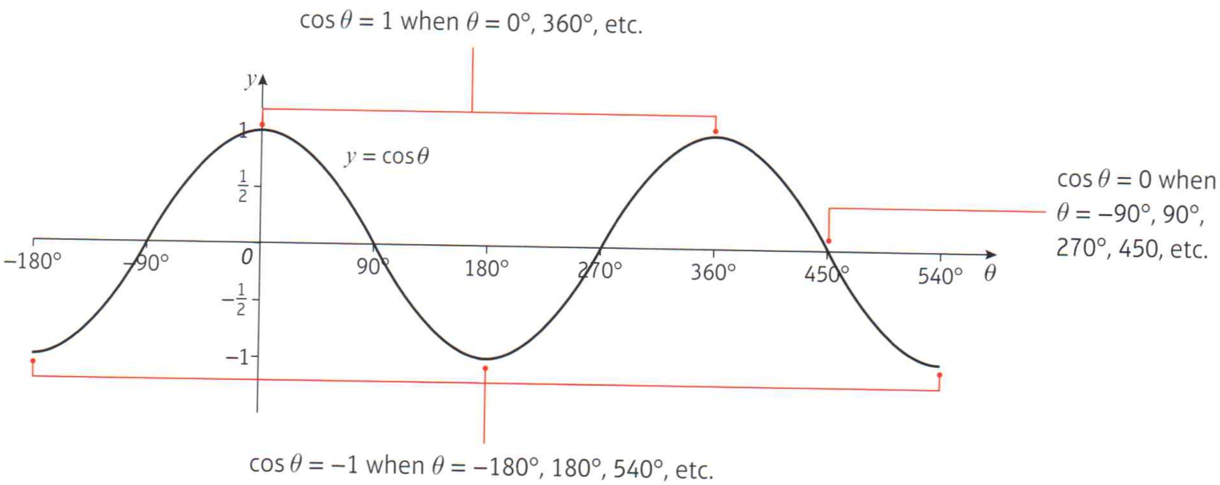
- **The graphs of sine, cosine and tangent are periodic. They repeat themselves after a certain interval.**

You need to be able to draw the graphs for a given range of angles.

- **The graph of $y = \sin\theta$:**
  - **repeats every 360° and crosses the x-axis at ..., −180°, 0, 180°, 360°, ...**
  - **has a maximum value of 1 and a minimum value of −1.**

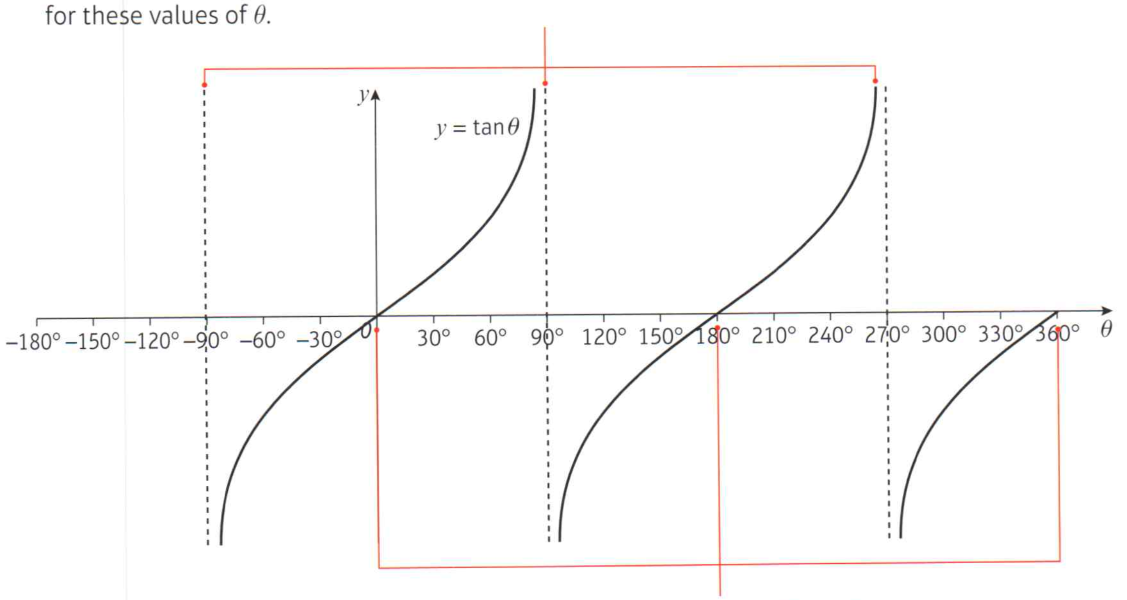


- **The graph of $y = \cos\theta$:**
  - **repeats every 360° and crosses the x-axis at ..., −90°, 90°, 270°, 450°, ...**
  - **has a maximum value of 1 and a minimum value of −1.**



- **The graph of $y = \tan \theta$:**
  - **repeats every 180° and crosses the x-axis at ... –180°, 0°, 180°, 360°, ...**
  - **has no maximum or minimum value**
  - **has vertical asymptotes at $x = -90°$, $x = 90°$, $x = 270°$, ...**

$\tan \theta$ does **not** have maximum and minimum points but approaches negative or positive infinity as the curve approaches the **asymptotes** at –90°, 90°, 270°, etc. $\tan \theta$ is **undefined** for these values of θ.



tan $\theta = 0$ when $\theta = 0°$, 180°, 360°, etc.

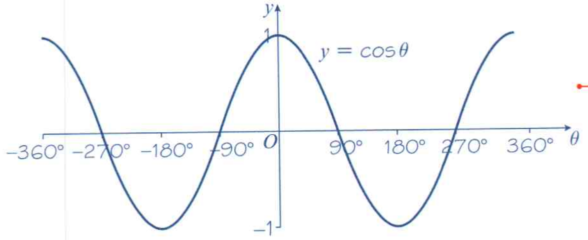
Example 11

a Sketch the graph of $y = \cos \theta$ in the interval $-360° \leqslant \theta \leqslant 360°$.

b i Sketch the graph of $y = \sin x$ in the interval $-180° \leqslant x \leqslant 270°$

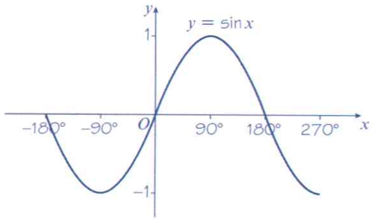
ii $\sin(-30°) = -0.5$. Use your graph to determine two further values of x for which $\sin x = -0.5$.

a



The axes are θ and y.
The curve meets the θ-axis at $\theta = \pm 270°$ and $\theta = \pm 90°$.
The curve crosses the y-axis at (0, 1).

b **i**



ii Using the symmetry of the graph:

$\sin(-150°) = -0.5$

$\sin 210° = -0.5$

$x = -150°$ or $210°$

The line $x = -90°$ is a line of symmetry.

The line $x = 90°$ is a line of symmetry.
You could also find this value by working out $\sin(180° - (-30°))$.

Exercise 9F

1 Sketch the graph of $y = \cos\theta$ in the interval $-180° \leqslant \theta \leqslant 180°$.

2 Sketch the graph of $y = \tan\theta$ in the interval $-180° \leqslant \theta \leqslant 180°$.

3 Sketch the graph of $y = \sin\theta$ in the interval $-180° \leqslant \theta \leqslant 180°$.

4 **a** $\cos 30° = \frac{\sqrt{3}}{2}$ Use your graph in question **1** to find another value of θ for which $\cos\theta = \frac{\sqrt{3}}{2}$

b $\tan 60° = \sqrt{3}$. Use your graph in question **2** to find other values of θ for which:

i $\tan\theta = \sqrt{3}$ **ii** $\tan\theta = -\sqrt{3}$

c $\sin 45° = \frac{1}{\sqrt{2}}$ Use your graph in question **3** to find other values of θ for which:

i $\sin\theta = \frac{1}{\sqrt{2}}$ **ii** $\sin\theta = -\frac{1}{\sqrt{2}}$

9.6 Transforming trigonometric graphs

You can use your knowledge of transforming graphs to transform the graphs of trigonometric functions.

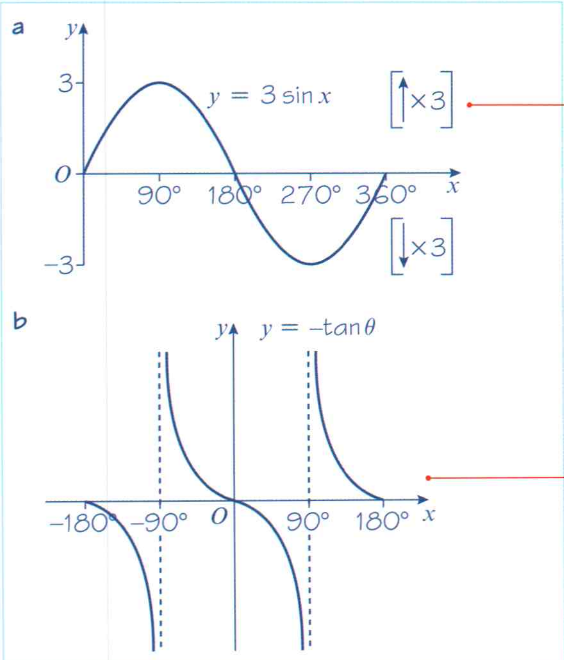
Links You need to be able to apply translations and stretches to graphs of trigonometric functions.
← Chapter 4

Example 12

Sketch on separate sets of axes the graphs of:

a $y = 3\sin x,\ 0 \leqslant x \leqslant 360°$

b $y = -\tan\theta,\ -180° \leqslant \theta \leqslant 180°$



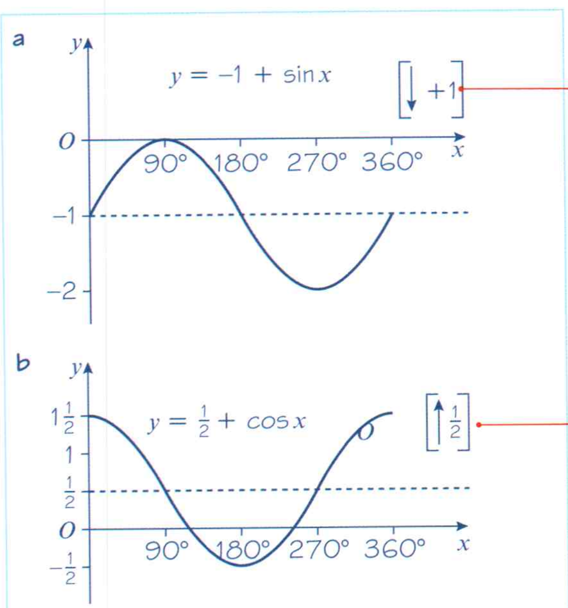
$y = 3f(x)$ is a vertical stretch of the graph $y = f(x)$ with scale factor 3. The intercepts on the x-axis remain unchanged, and the graph has a maximum point at (90°, 3) and a minimum point at (270°, −3).

$y = -f(x)$ is a reflection of the graph $y = f(x)$ in the x-axis. So this graph is a reflection of the graph $y = \tan x$ in the x-axis.

Example 13

Sketch on separate sets of axes the graphs of:

a $y = -1 + \sin x,\ 0 \leqslant x \leqslant 360°$ **b** $y = \frac{1}{2} + \cos x,\ 0 \leqslant x \leqslant 360°$



$y = f(x) - 1$ is a translation of the graph $y = f(x)$ by vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

The graph of $y = \sin x$ is translated by 1 unit in the negative y-direction.

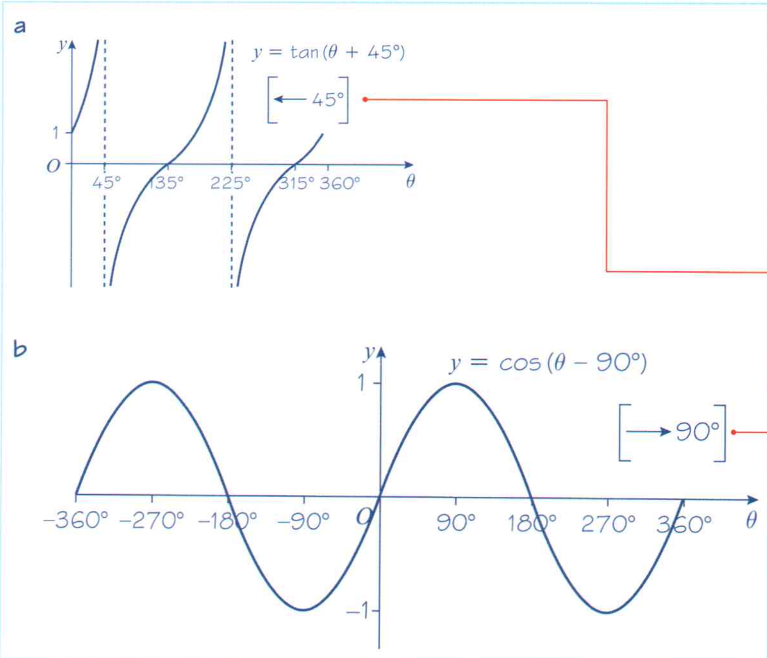
$y = f(x) + \frac{1}{2}$ is a translation of the graph $y = f(x)$ by vector $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$.

The graph of $y = \cos x$ is translated by $\frac{1}{2}$ unit in the positive y-direction.

Example 14

Sketch on separate sets of axes the graphs of:

a $y = \tan(\theta + 45°)$, $0 \leqslant \theta \leqslant 360°$ **b** $y = \cos(\theta - 90°)$, $-360° \leqslant \theta \leqslant 360°$



$y = f(\theta + 45°)$ is a translation of the graph $y = f(\theta)$ by vector $\begin{pmatrix}-45°\\0\end{pmatrix}$.
Remember to translate any asymptotes as well.

The graph of $y = \tan\theta$ is translated by 45° to the left. The asymptotes are now at $\theta = 45°$ and $\theta = 225°$. The curve meets the y-axis where $\theta = 0$ so $y = 1$.

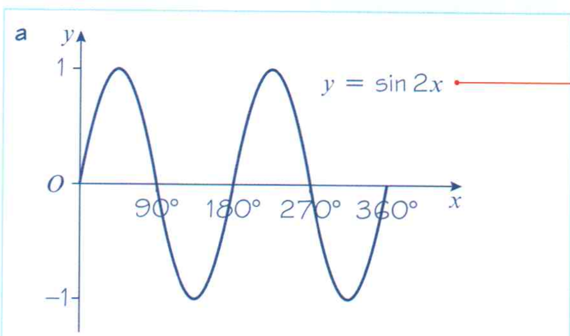
$y = f(\theta - 90°)$ is a translation of the graph $y = f(\theta)$ by vector $\begin{pmatrix}90°\\0\end{pmatrix}$.

The graph of $y = \cos\theta$ is translated by 90° to the right. Note that this is exactly the same curve as $y = \sin\theta$, so another property is that $\cos(\theta - 90°) = \sin\theta$.

Example 15

Sketch on separate sets of axes the graphs of:

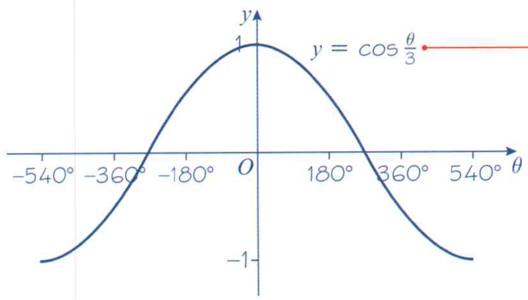
a $y = \sin 2x$, $0 \leqslant x \leqslant 360°$ **b** $y = \cos\frac{\theta}{3}$, $-540° \leqslant \theta \leqslant 540°$ **c** $y = \tan(-x)$, $-360° \leqslant x \leqslant 360°$



$y = f(2x)$ is a horizontal stretch of the graph $y = f(x)$ with scale factor $\frac{1}{2}$.

The graph of $y = \sin x$ is stretched horizontally with scale factor $\frac{1}{2}$
The period is now 180° and two complete 'waves' are seen in the interval $0 \leqslant x \leqslant 360°$.

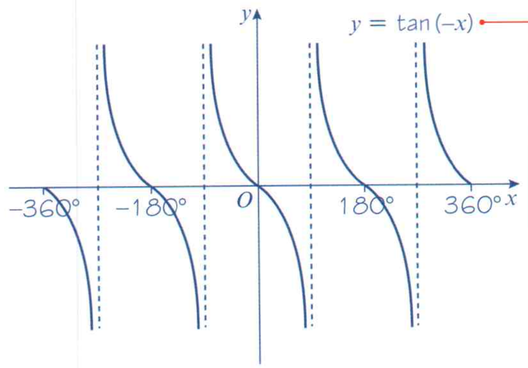
b



$y = f(\frac{1}{3}\theta)$ is a horizontal stretch of the graph $y = f(\theta)$ with scale factor 3.

The graph of $y = \cos\theta$ is stretched horizontally with scale factor 3. The period of $\cos\frac{\theta}{3}$ is 1080° and only one complete wave is seen while $-540 \leqslant \theta \leqslant 540°$. The curve crosses the θ-axis at $\theta = \pm 270°$.

c



$y = f(-x)$ is a reflection of the graph $y = f(x)$ in the y-axis.

The graph of $y = \tan(-x)$ is reflected in the y-axis. In this case the asymptotes are symmetric about the y-axis so they remain unchanged.

Online Plot transformations of trigonometric graphs using technology.

Exercise 9G

1 Write down **i** the maximum value, and **ii** the minimum value, of the following expressions, and in each case give the smallest positive (or zero) value of x for which it occurs.

a $\cos x$ **b** $4\sin x$ **c** $\cos(-x)$
d $3 + \sin x$ **e** $-\sin x$ **f** $\sin 3x$

2 Sketch, on the same set of axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of $\cos\theta$ and $\cos 3\theta$.

3 Sketch, on separate sets of axes, the graphs of the following, in the interval $0 \leqslant \theta \leqslant 360°$. Give the coordinates of points of intersection with the axes, and of maximum and minimum points where appropriate.

a $y = -\cos\theta$ **b** $y = \frac{1}{3}\sin\theta$ **c** $y = \sin\frac{1}{3}\theta$ **d** $y = \tan(\theta - 45°)$

4 Sketch, on separate sets of axes, the graphs of the following, in the interval $-180° \leqslant \theta \leqslant 180°$. Give the coordinates of points of intersection with the axes, and of maximum and minimum points where appropriate.

a $y = -2\sin\theta$ **b** $y = \tan(\theta + 180°)$ **c** $y = \cos 4\theta$ **d** $y = \sin(-\theta)$

5 Sketch, on separate sets of axes, the graphs of the following in the interval $-360° \leqslant \theta \leqslant 360°$. In each case give the periodicity of the function.

a $y = \sin\frac{1}{2}\theta$ **b** $y = -\frac{1}{2}\cos\theta$ **c** $y = \tan(\theta - 90°)$ **d** $y = \tan 2\theta$

(P) **6** **a** By considering the graphs of the functions, or otherwise, verify that:

i $\cos\theta = \cos(-\theta)$

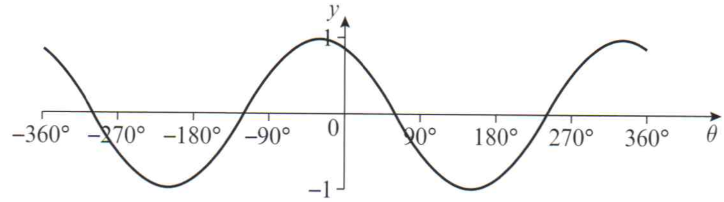
ii $\sin\theta = -\sin(-\theta)$

iii $\sin(\theta - 90°) = -\cos\theta$.

b Use the results in **a ii** and **iii** to show that $\sin(90° - \theta) = \cos\theta$.

c In Example 14 you saw that $\cos(\theta - 90°) = \sin\theta$.
Use this result with part **a i** to show that $\cos(90° - \theta) = \sin\theta$.

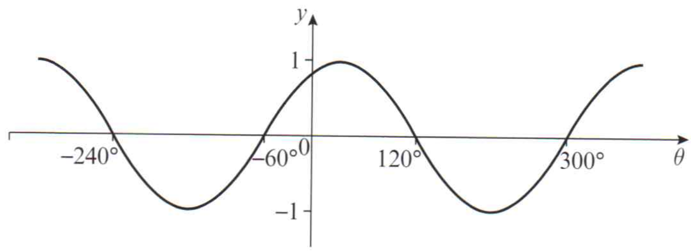
(E) **7** The graph shows the curve $y = \cos(x + 30°)$, $-360° \leqslant x \leqslant 360°$.



a Write down the coordinates of the points where the curve crosses the x-axis. **(2 marks)**

b Find the coordinates of the point where the curve crosses the y-axis. **(1 mark)**

(E/P) **8** The graph shows the curve with equation $y = \sin(x + k)$, $-360° \leqslant x \leqslant 360°$, where k is a constant.



a Find one possible value for k. **(2 marks)**

b Is there more than one possible answer to part **a**? Give a reason for your answer. **(2 marks)**

(E/P) **9** The variation in the depth of water in a rock pool can be modelled using the function $y = \sin(30t)°$, where t is the time in hours and $0 \leqslant t \leqslant 6$.

a Sketch the function for the given interval. **(2 marks)**

b If $t = 0$ represents midday, during what times will the rock pool be at least half full? **(3 marks)**

Mixed exercise 9

Give non-exact answers to 3 significant figures.

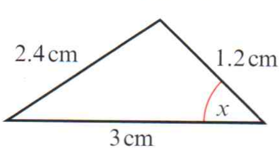
1 Triangle ABC has area 10 cm^2. $AB = 6$ cm, $BC = 8$ cm and $\angle ABC$ is obtuse. Find:

a the size of $\angle ABC$

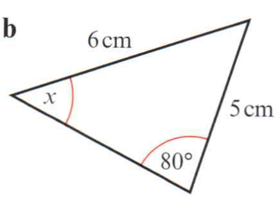
b the length of AC

2 In each triangle below, find the size of x and the area of the triangle.

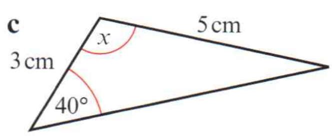
a



b



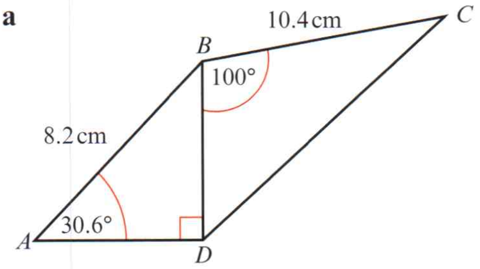
c



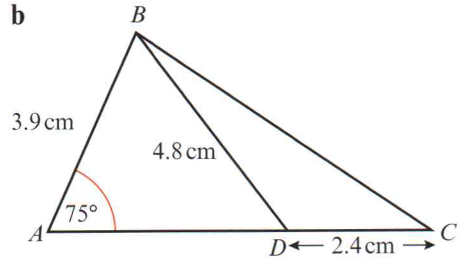
3 The sides of a triangle are 3 cm, 5 cm and 7 cm respectively. Show that the largest angle is 120°, and find the area of the triangle.

P **4** In each of the figures below calculate the total area.

a



b



5 In $\triangle ABC$, $AB = 10$ cm, $BC = a\sqrt{3}$ cm, $AC = 5\sqrt{13}$ cm and $\angle ABC = 150°$. Calculate:

a the value of a

b the exact area of $\triangle ABC$.

P **6** In a triangle, the largest side has length 2 cm and one of the other sides has length $\sqrt{2}$ cm. Given that the area of the triangle is 1 cm^2, show that the triangle is right-angled and isosceles.

E/P **7** The three points A, B and C, with coordinates $A(0, 1)$, $B(3, 4)$ and $C(1, 3)$ respectively, are joined to form a triangle.

a Show that $\cos \angle ACB = -\frac{4}{5}$ **(5 marks)**

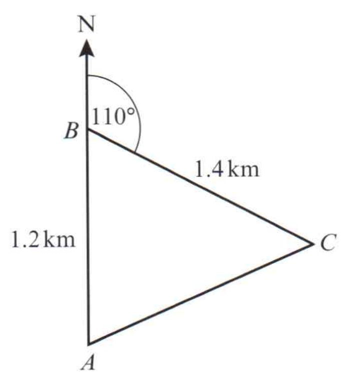
b Calculate the area of $\triangle ABC$. **(2 marks)**

E/P **8** The longest side of a triangle has length $(2x - 1)$ cm. The other sides have lengths $(x - 1)$ cm and $(x + 1)$ cm. Given that the largest angle is 120°, work out

a the value of x **(5 marks)**

b the area of the triangle. **(3 marks)**

E/P **9** A park keeper walks 1.2 km due north from his hut at point A to point B. He then walks 1.4 km on a bearing of 110° from point B to point C.



a Find how far he is from his hut when at point C. Give your answer in km to 3 s.f. **(3 marks)**

b Work out the bearing of the hut from point C. Give your answer to the nearest degree. **(3 marks)**

c Work out the area enclosed by his walk. **(3 marks)**

E/P **10** A windmill has four identical triangular sails made from wood. If each triangle has sides of length 12 m, 15 m and 20 m, work out the total area of wood needed. **(5 marks)**

E/P **11** Two points, A and B are on level ground. A church tower at point C has an angle of elevation from A of 15° and an angle of elevation from B of 32°. A and B are both on the same side of C, and A, B and C lie on the same straight line. The distance $AB = 75$ m.
Find the height of the church tower. **(4 marks)**

12 Describe geometrically the transformations which map:

a the graph of $y = \tan x$ onto the graph of $\tan \frac{1}{2}x$

b the graph of $y = \tan \frac{1}{2}x$ onto the graph of $3 + \tan \frac{1}{2}x$

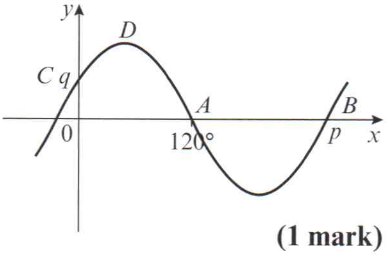
c the graph of $y = \cos x$ onto the graph of $-\cos x$

d the graph of $y = \sin(x - 10)$ onto the graph of $\sin(x + 10)$.

E/P **13** **a** Sketch on the same set of axes, in the interval $0 \leqslant x \leqslant 180°$, the graphs of $y = \tan(x - 45°)$ and $y = -2\cos x$, showing the coordinates of points of intersection with the axes. **(6 marks)**

b Deduce the number of solutions of the equation $\tan(x - 45°) + 2\cos x = 0$, in the interval $0 \leqslant x \leqslant 180°$. **(2 marks)**

E **14** The diagram shows part of the graph of $y = f(x)$.
It crosses the x-axis at $A(120°, 0)$ and $B(p, 0)$.
It crosses the y-axis at $C(0, q)$ and has a maximum value at D, as shown.



Given that $f(x) = \sin(x + k)$, where $k > 0$, write down

a the value of p **(1 mark)**

b the coordinates of D **(1 mark)**

c the smallest value of k **(1 mark)**

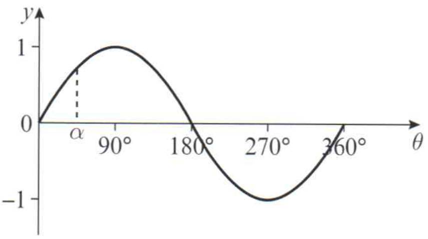
d the value of q. **(1 mark)**

E/P **15** Consider the function $f(x) = \sin px$, $p \in \mathbb{R}$, $0 \leqslant x \leqslant 360°$.
The closest point to the origin that the graph of $f(x)$ crosses the x-axis has x-coordinate 36°.

a Determine the value of p and sketch the graph of $y = f(x)$. **(5 marks)**

b Write down the period of $f(x)$. **(1 mark)**

16 The graph shows $y = \sin\theta$, $0 \leqslant \theta \leqslant 360°$, with one value of θ ($\theta = \alpha$) marked on the axis.



a Copy the graph and mark on the θ-axis the positions of $180° - \alpha$, $180° + \alpha$, and $360° - \alpha$.

b Verify that:
$\sin\alpha = \sin(180° - \alpha) = -\sin(180° + \alpha) = -\sin(360° - \alpha)$.

17 **a** Sketch on separate sets of axes the graphs of $y = \cos\theta$ ($0 \leqslant \theta \leqslant 360°$) and $y = \tan\theta$ ($0 \leqslant \theta \leqslant 360°$), and on each θ-axis mark the point $(\alpha, 0)$ as in question **16**.

b Verify that:

i $\cos\alpha = -\cos(180° - \alpha) = -\cos(180° + \alpha) = \cos(360° - \alpha)$

ii $\tan\alpha = -\tan(180° - \alpha) = \tan(180° + \alpha) = -\tan(360° - \alpha)$

E/P **18** A series of sand dunes has a cross-section which can be modelled using a sine curve of the form $y = \sin(60x)°$ where x is the length of the series of dunes in metres.

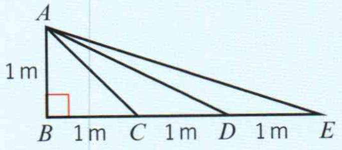
a Draw the graph of $y = \sin(60x)°$ for $0 \leqslant x \leqslant 24°$. **(3 marks)**

b Write down the number of sand dunes in this model. **(1 mark)**

c Give one reason why this may not be a realistic model. **(1 mark)**

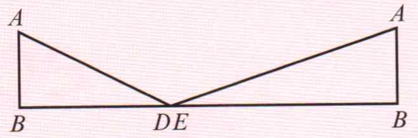
Challenge

In this diagram $AB = BC = CD = DE = 1$ m.



Prove that $\angle AEB + \angle ADB = \angle ACB$.

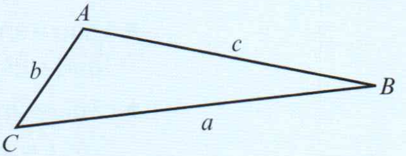
Hint Try drawing triangles ADB and AEB back to back.



Summary of key points

1 This version of the cosine rule is used to find a missing side if you know two sides and the angle between them:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



2 This version of the cosine rule is used to find an angle if you know all three sides:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

3 This version of the sine rule is used to find the length of a missing side:

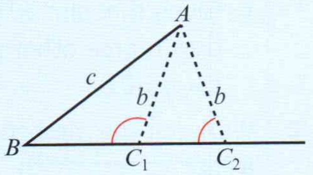
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

4 This version of the sine rule is used to find a missing angle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

5 The sine rule sometimes produces two possible solutions for a missing angle:

$$\sin\theta = \sin(180° - \theta)$$



6 Area of a triangle $= \frac{1}{2}ab\sin C$.

7 The graphs of sine, cosine and tangent are **periodic**. They repeat themselves after a certain interval.

- The graph of $y = \sin\theta$: repeats every 360° and crosses the x-axis at …, −180°, 0, 180°, 360°, … has a maximum value of 1 and a minimum value of −1.
- The graph of $y = \cos\theta$: repeats every 360° and crosses the x-axis at …, −90°, 90°, 270°, 450°, … has a maximum value of 1 and a minimum value of −1
- The graph of $y = \tan\theta$: repeats every 180° and crosses the x-axis at … −180°, 0°, 180°, 360°, … has no maximum or minimum value has vertical asymptotes at $x = -90°$, $x = 90°$, $x = 270°$, …