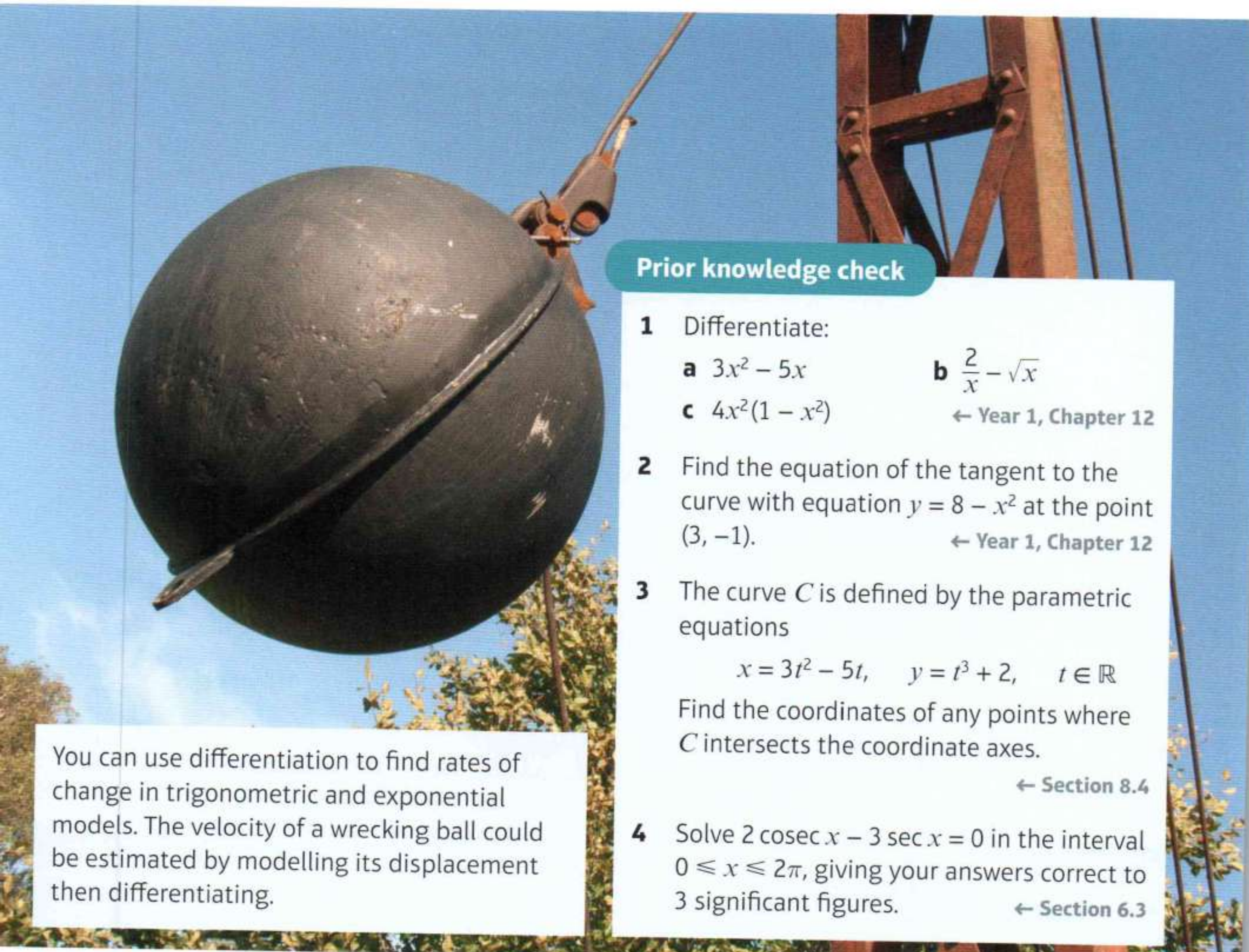
# Differentiation

# 9

## Objectives

After completing this chapter you should be able to:

- Differentiate trigonometric functions → pages 232–234, 246–251
- Differentiate exponentials and logarithms → pages 235–237
- Differentiate functions using the chain, product and quotient rules → pages 237–245
- Differentiate parametric equations → pages 251–254
- Differentiate functions which are defined implicitly → pages 254–257
- Use the second derivative to describe the behaviour of a function → pages 257–261
- Solve problems involving connected rates of change and construct simple differential equations → pages 261–264



You can use differentiation to find rates of change in trigonometric and exponential models. The velocity of a wrecking ball could be estimated by modelling its displacement then differentiating.

## Prior knowledge check

**1** Differentiate:

**a** $3x^2 - 5x$ **b** $\frac{2}{x} - \sqrt{x}$

**c** $4x^2(1 - x^2)$ ← **Year 1, Chapter 12**

**2** Find the equation of the tangent to the curve with equation $y = 8 - x^2$ at the point $(3, -1)$. ← **Year 1, Chapter 12**

**3** The curve $C$ is defined by the parametric equations

$$x = 3t^2 - 5t, \quad y = t^3 + 2, \quad t \in \mathbb{R}$$

Find the coordinates of any points where $C$ intersects the coordinate axes. ← **Section 8.4**

**4** Solve $2 \operatorname{cosec} x - 3 \sec x = 0$ in the interval $0 \leqslant x \leqslant 2\pi$, giving your answers correct to 3 significant figures. ← **Section 6.3**

## 9.1 Differentiating sin $x$ and cos $x$

You need to be able to differentiate sin $x$ and cos $x$ from first principles. You can use the following small angle approximations for sin and cos when the angle is measured in **radians**:

- **$\sin x \approx x$**
- **$\cos x \approx 1 - \frac{1}{2}x^2$**

**Watch out** You will always need to use radians when differentiating trigonometric functions.

This means that $\lim_{h \to 0} \frac{\sin h}{h} = \lim_{h \to 0} \frac{h}{h} = 1$, and

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} \frac{1 - \frac{1}{2}h^2 - 1}{h} = \lim_{h \to 0} \left(-\frac{1}{2}h\right) = 0$$

You will need to use these two limits when you differentiate sin and cos from first principles.

### Example 1

Prove, from first principles, that the derivative of $\sin x$ is $\cos x$.

You may assume that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$.

Let $f(x) = \sin x$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

Use the formula for $\sin(A + B)$ to expand $\sin(x + h)$, then write the resulting expression in terms of $\frac{\cos h - 1}{h}$ and $\frac{\sin h}{h}$ ← Section 7.1

$$= \lim_{h \to 0} \left( \left(\frac{\cos h - 1}{h}\right) \sin x + \left(\frac{\sin h}{h}\right) \cos x \right)$$

Since $\frac{\cos h - 1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$ the expression inside the limit tends to $(0 \times \sin x + 1 \times \cos x)$

Make sure you state where you are using the two limits given in the question.

So $\lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \cos x$

Hence the derivative of $\sin x$ is $\cos x$.

Write down what you have proved.

**Problem-solving**

Use the rule for differentiating from first principles. This is provided in the formula booklet. If you don't want to use limit notation, you could write an expression for the gradient of the chord joining $(x, \sin x)$ to $(x + h, \sin(x + h))$ and show that as $h \to 0$ the gradient of the chord tends to $\cos x$. ← Year 1, Section 12.2

- **If $y = \sin kx$, then $\frac{dy}{dx} = k \cos kx$**

You can use a similar technique to find the derivative of $\cos x$.

- **If $y = \cos kx$, then $\frac{dy}{dx} = -k \sin kx$**

**Online** Explore the relationship between sin and cos and their derivatives using technology.

## Example 2

Find $\frac{dy}{dx}$ given that:

**a** $y = \sin 2x$ **b** $y = \cos 5x$ **c** $y = 3\cos x + 2\sin 4x$

**a** $y = \sin 2x$

$\frac{dy}{dx} = 2\cos 2x$

Use the standard result for $\sin kx$ with $k = 2$.

**b** $y = \cos 5x$

$\frac{dy}{dx} = -5\sin 5x$

Use the standard result for $\cos kx$ with $k = 5$.

**c** $y = 3\cos x + 2\sin 4x$

$\frac{dy}{dx} = 3 \times (-\sin x) + 2 \times (4\cos 4x)$

$= -3\sin x + 8\cos 4x$

Differentiate each term separately.

## Example 3

A curve has equation $y = \frac{1}{2}x - \cos 2x$. Find the stationary points on the curve in the interval $0 \leqslant x \leqslant \pi$.

$\frac{dy}{dx} = \frac{1}{2} - (-2\sin 2x) = \frac{1}{2} + 2\sin 2x$

Start by differentiating $\frac{1}{2}x - \cos 2x$.

Let $\frac{dy}{dx} = 0$ and solve for $x$:

Stationary points occur when $\frac{dy}{dx} = 0$.

← Year 1, Chapter 12

$\frac{1}{2} + 2\sin 2x = 0$

$2\sin 2x = -\frac{1}{2}$

$\sin 2x = -\frac{1}{4}$

$2x = 3.394..., 6.030...$

$x = 1.70, 3.02$ (3 s.f.)

$0 \leqslant x \leqslant \pi$ so the range for $2x$ is $0 \leqslant 2x \leqslant 2\pi$.

**Watch out** Whenever you are using calculus, you must work in **radians**.

When $x = 1.70$:

$y = \frac{1}{2}(1.70) - \cos(2 \times 1.70) = 1.82$ (3 s.f.)

When $x = 3.02$:

$y = \frac{1}{2}(3.02) - \cos(2 \times 3.02) = 0.539$ (3 s.f.)

Substitute $x$ values into $y = \frac{1}{2}x - \cos 2x$ to find the corresponding $y$ values.

The stationary points of $y = \frac{1}{2}x - \cos 2x$ in the interval $0 \leqslant x \leqslant \pi$ are (1.70, 1.82) and (3.02, 0.539).

## Exercise 9A

(P) **1 a** Given that $f(x) = \cos x$, show that

$$f'(x) = \lim_{h \to 0}\left(\left(\frac{\cos h - 1}{h}\right)\cos x - \frac{\sin h}{h}\sin x\right)$$

**b** Hence prove that $f'(x) = -\sin x$.

**Problem-solving**

Use the definition of the derivative and the addition formula for $\cos(A + B)$.

**2** Differentiate:

**a** $y = 2\cos x$ **b** $y = 2\sin\frac{1}{2}x$ **c** $y = \sin 8x$ **d** $y = 6\sin\frac{2}{3}x$

**3** Find $f'(x)$ given that:

**a** $f(x) = 2\cos x$ **b** $f(x) = 6\cos\frac{5}{6}x$ **c** $f(x) = 4\cos\frac{1}{2}x$ **d** $f(x) = 3\cos 2x$

**4** Find $\frac{dy}{dx}$ given that:

**a** $y = \sin 2x + \cos 3x$ **b** $y = 2\cos 4x - 4\cos x + 2\cos 7x$

**c** $y = x^2 + 4\cos 3x$ **d** $y = \frac{1 + 2x\sin 5x}{x}$

**5** A curve has equation $y = x - \sin 3x$. Find the stationary points of the curve in the interval $0 \leqslant x \leqslant \pi$.

**6** Find the gradient of the curve $y = 2\sin 4x - 4\cos 2x$ at the point where $x = \frac{\pi}{2}$

(P) **7** A curve has the equation $y = 2\sin 2x + \cos 2x$. Find the stationary points of the curve in the interval $0 \leqslant x \leqslant \pi$.

(E/P) **8** A curve has the equation $y = \sin 5x + \cos 3x$. Find the equation of the tangent to the curve at the point $(\pi, -1)$. **(4 marks)**

(E/P) **9** A curve has the equation $y = 2x^2 - \sin x$. Show that the equation of the normal to the curve at the point with $x$-coordinate $\pi$ is

$$x + (4\pi + 1)y - \pi(8\pi^2 + 2\pi + 1) = 0$$

**(7 marks)**

(E/P) **10** Prove, from first principles, that the derivative of $\sin x$ is $\cos x$.

You may assume the formula for $\sin(A + B)$ and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$. **(5 marks)**

**Challenge**

Prove, from first principles, that the derivative of $\sin(kx)$ is $k\cos(kx)$.

You may assume the formula for $\sin(A + B)$ and that as $h \to 0$, $\frac{\sin kh}{h} \to k$ and $\frac{\cos kh - 1}{h} \to 0$.

## 9.2 Differentiating exponentials and logarithms

You need to be able to differentiate expressions involving exponentials and logarithms.

- **If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$**
- **If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$**

You can use the derivative of $e^{kx}$ to find the derivative of $a^{kx}$ where $a$ is any positive real number.

**Watch out** For any real constant, $k$, $\ln kx = \ln k + \ln x$. Since $\ln k$ is also a constant, the derivative of $\ln kx$ is also $\frac{1}{x}$

### Example 4

Show that the derivative of $a^x$ is $a^x \ln a$.

Let $y = a^x$

$= e^{\ln(a^x)}$

$= e^{x \ln a}$

$\frac{dy}{dx} = \ln a \, e^{x \ln a}$

$= \ln a \, e^{\ln(a^x)}$

$= a^x \ln a$

You could also use the laws of logs like this:
$\ln y = \ln a^x = x \ln a \Rightarrow y = e^{x \ln a}$

← Year 1, Chapter 14

$\ln a$ is just a constant so use the standard result for the derivative of $e^{kx}$ with $k = \ln a$.

**Online** Explore the function $a^x$ and its derivative using technology.

- **If $y = a^{kx}$, where $k$ is a real constant and $a > 0$, then $\frac{dy}{dx} = a^{kx} k \ln a$**

### Example 5

Find $\frac{dy}{dx}$ given that:

**a** $y = e^{3x} + 2^{3x}$ **b** $y = \ln(x^3) + \ln 7x$ **c** $y = \frac{2 - 3e^{7x}}{4e^{3x}}$

**a** $y = e^{3x} + 2^{3x}$

$\frac{dy}{dx} = 3e^{3x} + 2^{3x}(3 \ln 2)$

Differentiate each term separately using the standard results for $e^{kx}$ with $k = 3$, and $a^{kx}$ with $a = 2$ and $k = 3$.

**b** $y = \ln(x^3) + \ln 7x$

$y = 3 \ln x + \ln 7 + \ln x = 4 \ln x + \ln 7$

Rewrite $y$ using the laws of logs.

$\frac{dy}{dx} = 4 \times \frac{1}{x} + 0 = \frac{4}{x}$

Use the standard result for $\ln x$. $\ln 7$ is a constant, so it disappears when you differentiate.

**c** $y = \frac{2 - 3e^{7x}}{4e^{3x}}$

$= \frac{1}{2}e^{-3x} - \frac{3}{4}e^{4x}$

Divide each term in the numerator by the denominator.

$\frac{dy}{dx} = \frac{1}{2} \times (-3e^{-3x}) - \frac{3}{4} \times 4e^{4x}$

$= -\frac{3}{2}e^{-3x} - 3e^{4x}$

Differentiate each term separately using the standard result for $e^{kx}$.

## Exercise 9B

1 a Find $\frac{dy}{dx}$ for each of the following:

a $y = 4e^{7x}$  b $y = 3^x$  c $y = \left(\frac{1}{2}\right)^x$  d $y = \ln 5x$

e $y = 4\left(\frac{1}{3}\right)^x$  f $y = \ln(2x^3)$  g $y = e^{3x} - e^{-3x}$  h $y = \frac{(1 + e^x)^2}{e^x}$

2 Find $f'(x)$ given that:

a $f(x) = 3^{4x}$  b $f(x) = \left(\frac{3}{2}\right)^{2x}$  c $f(x) = 2^{4x} + 4^{2x}$  d $f(x) = \frac{2^{7x} + 8^x}{4^{2x}}$

**Hint** In parts **c** and **d**, rewrite the terms so that they all have the same base and hence can be simplified.

3 Find the gradient of the curve $y = (e^{2x} - e^{-2x})^2$ at the point where $x = \ln 3$.

E 4 Find the equation of the tangent to the curve $y = 2^x + 2^{-x}$ at the point $\left(2, \frac{17}{4}\right)$. **(6 marks)**

E/P 5 A curve has the equation $y = e^{2x} - \ln x$. Show that the equation of the tangent at the point with $x$-coordinate 1 is

$$y = (2e^2 - 1)x - e^2 + 1$$

**(6 marks)**

6 A particular radioactive isotope has an activity, $R$ millicuries at time $t$ days, given by the equation $R = 200 \times 0.9^t$. Find the value of $\frac{dR}{dt}$, when $t = 8$.

P 7 The population of Cambridge was 37 000 in 1900, and was about 109 000 in 2000. Given that the population, $P$, at a time $t$ years after 1900 can be modelled using the equation $P = P_0k^t$,

a find the values of $P_0$ and $k$

b evaluate $\frac{dP}{dt}$ in the year 2000

c interpret your answer to part **b** in the context of the model.

P 8 A student is attempting to differentiate $\ln kx$. The student writes:

$y = \ln kx$, so $\frac{dy}{dx} = k \ln kx$

Explain the mistake made by the student and state the correct derivative.

E/P 9 Prove that the derivative of $a^{kx}$ is $a^{kx} k \ln a$. You may assume that the derivative of $e^{kx}$ is $ke^{kx}$. **(4 marks)**

P **10** $f(x) = e^{2x} - \ln(x^2) + 4, x > 0$

**a** Find $f'(x)$. **(3 marks)**

The curve with equation $y = f(x)$ has a gradient of 2 at point $P$. The $x$-coordinate of $P$ is $a$.

**b** Show that $a(e^{2a} - 1) = 1$. **(2 marks)**

P **11** A curve $C$ has equation

$$y = 5\sin 3x + 2\cos 3x, -\pi \leqslant x \leqslant \pi$$

**a** Show that the point $P$ (0, 2) lies on $C$. **(1 mark)**

**b** Find an equation of the normal to the curve $C$ at $P$. **(5 marks)**

P **12** The point $P$ lies on the curve with equation $y = 2(3^{4x})$. The $x$-coordinate of $P$ is 1.
Find an equation of the normal to the curve at the point $P$ in the form $y = ax + b$, where $a$ and $b$ are constants to be found in exact form. **(5 marks)**

**Challenge**

A curve $C$ has the equation $y = e^{4x} - 5x$. Find the equation of the tangent to $C$ that is parallel to the line $y = 3x + 4$.

## 9.3 The chain rule

You can use the chain rule to differentiate composite functions, or functions of another function.

- **The chain rule is:**

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

**where $y$ is a function of $u$ and $u$ is another function of $x$.**

**Example 6**

Given that $y = (3x^4 + x)^5$, find $\frac{dy}{dx}$ using the chain rule.

Let $u = 3x^4 + x$:

$\frac{du}{dx} = 12x^3 + 1$ — Differentiate $u$ with respect to $x$ to get $\frac{du}{dx}$

$y = u^5$

$\frac{dy}{du} = 5u^4$ — Substitute $u$ into the equation for $y$ and differentiate with respect to $u$ to get $\frac{dy}{du}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 5u^4(12x^3 + 1)$$

$\frac{dy}{dx} = 5(3x^4 + x)^4(12x^3 + 1)$ — Use $u = 3x^4 + x$ to write your final answer in terms of $x$ only.

## Example 7

Given that $y = \sin^4 x$, find $\frac{dy}{dx}$

$y = \sin^4 x = (\sin x)^4$

Let $u = \sin x$:

$\frac{du}{dx} = \cos x$

Differentiate $u$ with respect to $x$ to get $\frac{du}{dx}$

$y = u^4$

$\frac{dy}{du} = 4u^3$

Substitute $u$ into the equation for $y$ and differentiate with respect to $u$ to get $\frac{dy}{du}$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 4u^3(\cos x)$

$\frac{dy}{dx} = 4\sin^3 x \cos x$

Substitute $u = \sin x$ back into $\frac{dy}{dx}$ to get an answer in terms of $x$ only.

You can write the chain rule using function notation:

- **The chain rule enables you to differentiate a function of a function. In general,**
  - **if $y = (f(x))^n$ then $\frac{dy}{dx} = n(f(x))^{n-1} f'(x)$**
  - **if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x))g'(x)$**

## Example 8

Given that $y = \sqrt{5x^2 + 1}$, find $\frac{dy}{dx}$ at (4, 9).

$y = \sqrt{5x^2 + 1}$

This is $y = (f(x))^n$ with $f(x) = 5x^2 + 1$ and $n = \frac{1}{2}$

So $\frac{dy}{dx} = \frac{1}{2}(f(x))^{-\frac{1}{2}} f'(x)$.

Let $f(x) = 5x^2 + 1$

Then $f'(x) = 10x$

Using the chain rule:

$\frac{dy}{dx} = \frac{1}{2}(5x^2 + 1)^{-\frac{1}{2}} \times 10x$

$= 5x(5x^2 + 1)^{-\frac{1}{2}}$

At (4, 9), $\frac{dy}{dx} = 5(4)(5(4)^2 + 1)^{-\frac{1}{2}} = \frac{20}{9}$

Substitute $x = 4$ into $\frac{dy}{dx}$ to find the required value.

The following particular case of the chain rule is useful for differentiating functions that are not in the form $y = \mathrm{f}(x)$.

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$

**Hint** This is because: $\dfrac{\mathrm{d}y}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{\mathrm{d}y}{\mathrm{d}y} = 1$

## Example 9

Find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point (2, 1) on the curve with equation $y^3 + y = x$.

$\dfrac{dx}{dy} = 3y^2 + 1$ — Start with $x = y^3 + y$ and differentiate with respect to $y$.

$\therefore \dfrac{dy}{dx} = \dfrac{1}{3y^2 + 1}$ — Use $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$

$= \dfrac{1}{4}$ — Substitute y = 1.

## Exercise 9C

**1** Differentiate:

**a** $(1 + 2x)^4$ **b** $(3 - 2x^2)^{-5}$ **c** $(3 + 4x)^{\frac{1}{2}}$ **d** $(6x + x^2)^7$

**e** $\dfrac{1}{3 + 2x}$ **f** $\sqrt{7 - x}$ **g** $4(2 + 8x)^4$ **h** $3(8 - x)^{-6}$

**2** Differentiate:

**a** $\mathrm{e}^{\cos x}$ **b** $\cos(2x - 1)$ **c** $\sqrt{\ln x}$ **d** $(\sin x + \cos x)^5$

**e** $\sin(3x^2 - 2x + 1)$ **f** $\ln(\sin x)$ **g** $2\mathrm{e}^{\cos 4x}$ **h** $\cos(\mathrm{e}^{2x} + 3)$

**3** Given that $y = \dfrac{1}{(4x + 1)^2}$ find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$.

**4** A curve $C$ has equation $y = (5 - 2x)^3$. Find the tangent to the curve at the point $P$ with $x$-coordinate 1. **(7 marks)**

**5** Given that $y = (1 + \ln 4x)^{\frac{3}{2}}$, find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \frac{1}{4}\mathrm{e}^3$. **(5 marks)**

**6** Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following curves, giving your answers in terms of $y$.

**a** $x = y^2 + y$ **b** $x = \mathrm{e}^y + 4y$ **c** $x = \sin 2y$ **d** $4x = \ln y + y^3$

(P) **7** Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point (8, 2) on the curve with equation $3y^2 - 2y = x$.

**Problem-solving**

Your expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ will be in terms of $y$. Remember to substitute the $y$-coordinate into the expression to find the gradient.

(P) **8** Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $\left(\frac{5}{2}, 4\right)$ on the curve with equation $y^{\frac{1}{2}} + y^{-\frac{1}{2}} = x$.

**9** **a** Differentiate $\mathrm{e}^y = x$ with respect to $y$.

**b** Hence, prove that if $y = \ln x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$

(E/P) **10** The curve $C$ has equation $x = 4\cos 2y$.

**a** Show that the point $Q\left(2, \frac{\pi}{6}\right)$ lies on $C$. **(1 mark)**

**b** Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{4\sqrt{3}}$ at $Q$. **(4 marks)**

**c** Find an equation of the normal to $C$ at $Q$. Give your answer in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are exact constants. **(4 marks)**

**11** Differentiate:

**a** $\sin^2 3x$ **b** $\mathrm{e}^{(x+1)^2}$ **c** $\ln(\cos x)^2$

**d** $\frac{1}{3 + \cos 2x}$ **e** $\sin\left(\frac{1}{x}\right)$

(E/P) **12** The curve $C$ has equation $y = \frac{4}{(2-4x)^2}$, $x \neq \frac{1}{2}$

The point $A$ on $C$ has $x$-coordinate 3.
Find an equation of the normal to $C$ at $A$ in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are integers. **(7 marks)**

(E/P) **13** Find the exact value of the gradient of the curve with equation $y = 3^{x^3}$ at the point with coordinates (1, 3). **(4 marks)**

**Challenge**

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given that:

**a** $y = \sqrt{\sin\sqrt{x}}$ **b** $\ln y = \sin^3(3x + 4)$

## 9.4 The product rule

You need to be able to differentiate the product of two functions.

- **If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$,**

  **where $u$ and $v$ are functions of $x$.**

The product rule in function notation is:

- **If $f(x) = g(x)h(x)$ then $f'(x) = g(x)h'(x) + h(x)g'(x)$**

**Watch out** Make sure you can spot the difference between a product of two functions and a function of a function. A product is two separate functions multiplied together.

### Example 10

Given that $f(x) = x^2\sqrt{3x-1}$, find $f'(x)$.

Let $u = x^2$ and $v = \sqrt{3x-1} = (3x-1)^{\frac{1}{2}}$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3 \times \frac{1}{2}(3x-1)^{-\frac{1}{2}}$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$f'(x) = x^2 \times \frac{3}{2}(3x-1)^{-\frac{1}{2}} + \sqrt{3x-1} \times 2x$

$= \frac{3x^2 + 12x^2 - 4x}{2\sqrt{3x-1}}$

$= \frac{15x^2 - 4x}{2\sqrt{3x-1}}$

$= \frac{x(15x-4)}{2\sqrt{3x-1}}$

Write out your functions $u$, $v$, $\frac{du}{dx}$ and $\frac{dv}{dx}$ before substituting into the product rule. Use the chain rule to differentiate $(3x-1)^{\frac{1}{2}}$

Substitute $u$, $v$, $\frac{du}{dx}$ and $\frac{dv}{dx}$

### Example 11

Given that $y = e^{4x}\sin^2 3x$, show that $\frac{dy}{dx} = e^{4x}\sin 3x\,(A\cos 3x + B\sin 3x)$, where $A$ and $B$ are constants to be determined.

Let $u = e^{4x}$ and $v = \sin^2 3x = (\sin 3x)^2$

$\frac{du}{dx} = 4e^{4x}$ and $\frac{dv}{dx} = 2(\sin 3x) \times (3\cos 3x)$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = e^{4x} \times (6\sin 3x\cos 3x) + \sin^2 3x \times 4e^{4x}$

$= 6e^{4x}\sin 3x\cos 3x + 4e^{4x}\sin^2 3x$

$= e^{4x}\sin 3x\,(6\cos 3x + 4\sin 3x)$

This is in the required form with $A = 6$ and $B = 4$.

Write out $u$ and $v$ and find $\frac{du}{dx}$ and $\frac{dv}{dx}$

Use the chain rule to find $\frac{dv}{dx}$

Write out the product rule before substituting.

**Problem-solving**

Write out the value of any constants you have determined at the end of your working.
You can use this to check that your answer is in the required form.

## Exercise 9D

1 Differentiate:

a $x(1 + 3x)^5$ b $2x(1 + 3x^2)^3$ c $x^3(2x + 6)^4$ d $3x^2(5x - 1)^{-1}$

2 Differentiate:

a $e^{-2x}(2x - 1)^5$ b $\sin 2x \cos 3x$ c $e^x \sin x$ d $\sin(5x)\ln(\cos x)$

3 a Find the value of $\frac{dy}{dx}$ at the point (1, 8) on the curve with equation $y = x^2(3x - 1)^3$.

b Find the value of $\frac{dy}{dx}$ at the point (4, 36) on the curve with equation $y = 3x(2x + 1)^{\frac{1}{2}}$.

c Find the value of $\frac{dy}{dx}$ at the point $\left(2, \frac{1}{5}\right)$ on the curve with equation $y = (x - 1)(2x + 1)^{-1}$.

4 Find the stationary points of the curve $C$ with the equation $y = (x - 2)^2(2x + 3)$.

5 A curve $C$ has equation $y = \left(x - \frac{\pi}{2}\right)^5 \sin 2x,\ 0 < x < \pi$. Find the gradient of the curve at the point with $x$-coordinate $\frac{\pi}{4}$

E/P 6 A curve $C$ has equation $y = x^2\cos(x^2)$. Find the equation of the tangent to the curve $C$ at the point $P\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ in the form $ax + by + c = 0$ where $a$, $b$ and $c$ are exact constants. **(7 marks)**

E/P 7 Given that $y = 3x^2(5x - 3)^3$, show that

$$\frac{dy}{dx} = Ax(5x - 3)^n(Bx + C)$$

where $n$, $A$, $B$ and $C$ are constants to be determined. **(4 marks)**

E 8 A curve $C$ has equation $y = (x + 3)^2 e^{3x}$.

a Find $\frac{dy}{dx}$, using the product rule for differentiation. **(3 marks)**

b Find the gradient of $C$ at the point where $x = 2$. **(3 marks)**

E 9 Differentiate with respect to $x$:

a $(2\sin x - 3\cos x)\ln 3x$ **(3 marks)**

b $x^4 e^{7x-3}$ **(3 marks)**

E 10 Find the value of $\frac{dy}{dx}$ at the point where $x = 1$ on the curve with equation

$$y = x^5\sqrt{10x + 6}$$ **(6 marks)**

**Challenge**

Find $\frac{dy}{dx}$ for the following functions.

**a** $y = e^x \sin^2 x \cos x$ **b** $y = x(4x-3)^6(1-4x)^9$

## 9.5 The quotient rule

You need to be able to differentiate the quotient of two functions.

- **If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where $u$ and $v$ are functions of $x$.**

The quotient rule in function notation is:

- **If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$**

**Watch out** There is a minus sign in the numerator, so the order of the functions is important.

**Example 12**

Given that $y = \frac{x}{2x+5}$ find $\frac{dy}{dx}$

Let $u = x$ and $v = 2x + 5$:

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 2$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{(2x+5) \times 1 - x \times 2}{(2x+5)^2}$

$= \frac{5}{(2x+5)^2}$

Let $u$ be the numerator and let $v$ be the denominator.

Recognise that $y$ is a quotient and use the quotient rule.

Simplify the numerator of the fraction.

**Example 13**

A curve $C$ with equation $y = \frac{\sin x}{e^{2x}}$, $0 < x < \pi$, has a stationary point at $P$. Find the coordinates of $P$. Give your answer to 3 significant figures.

**Online** Explore the graph of this function using technology.

Let $u = \sin x$ and $v = e^{2x}$.

$\frac{du}{dx} = \cos x$ and $\frac{dv}{dx} = 2e^{2x}$

Write out $u$ and $v$ and find $\frac{du}{dx}$ and $\frac{dv}{dx}$ before using the quotient rule.

Using the quotient rule,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Write out the rule before substituting.

$$\frac{dy}{dx} = \frac{e^{2x}\cos x - \sin x(2e^{2x})}{(e^{2x})^2}$$

$$= \frac{e^{2x}\cos x - 2e^{2x}\sin x}{e^{4x}}$$

$$= \frac{e^{2x}(\cos x - 2\sin x)}{e^{4x}}$$

$$= e^{-2x}(\cos x - 2\sin x)$$

Simplify your expression for $\frac{dy}{dx}$ as much as possible.

When $\frac{dy}{dx} = 0$:

$P$ is a stationary point so $\frac{dy}{dx} = 0$.

$e^{-2x}(\cos x - 2\sin x) = 0$

$e^{-2x} = 0$ or $\cos x - 2\sin x = 0$

$e^{-2x} = 0$ has no solution.

$\cos x - 2\sin x = 0$

$\cos x = 2\sin x$

$\frac{1}{2} = \tan x$

$x = 0.464$ (3 s.f.)

**Problem-solving**

If the product of two factors is equal to 0 then one of the factors must be equal to 0.

This is the only solution in the range $0 < x < \pi$.

$y = \frac{\sin x}{e^{2x}}$

$y = \frac{\sin(0.464)}{e^{2\times 0.464}} = 0.177$ (3 s.f.)

Substitute $x$ into $y$ to find the $y$-coordinate of the stationary point.

So the coordinates of $P$ are (0.464, 0.177).

## Exercise 9E

**1** Differentiate:

**a** $\frac{5x}{x+1}$ **b** $\frac{2x}{3x-2}$ **c** $\frac{x+3}{2x+1}$ **d** $\frac{3x^2}{(2x-1)^2}$ **e** $\frac{6x}{(5x+3)^{\frac{1}{2}}}$

**2** Differentiate:

**a** $\frac{e^{4x}}{\cos x}$ **b** $\frac{\ln x}{x+1}$ **c** $\frac{e^{-2x}+e^{2x}}{\ln x}$ **d** $\frac{(e^x+3)^3}{\cos x}$ **e** $\frac{\sin^2 x}{\ln x}$

**3** Find the value of $\frac{dy}{dx}$ at the point $\left(1, \frac{1}{4}\right)$ on the curve with equation $y = \frac{x}{3x+1}$

**4** Find the value of $\frac{dy}{dx}$ at the point (12, 3) on the curve with equation $y = \frac{x+3}{(2x+1)^{\frac{1}{2}}}$

5 Find the stationary point(s) of the curve $C$ with equation $y = \frac{e^{2x+3}}{x}$, $x \neq 0$.

E 6 Find the equation of the tangent to the curve $y = \frac{e^{\frac{1}{3}x}}{x}$ at the point $\left(3, \frac{1}{3}e\right)$. **(7 marks)**

7 Find the exact value of $\frac{dy}{dx}$ at the point $x = \frac{\pi}{9}$ on the curve with equation $y = \frac{\ln x}{\sin 3x}$

P 8 The curve $C$ has equation $x = \frac{e^y}{3 + 2y}$

**a** Find the coordinates of the point $P$ where the curve cuts the $x$-axis. **(1 mark)**

**b** Find an equation of the normal to the curve at $P$, giving your answer in the form $y = mx + c$, where $m$ and $c$ are constants to be found. **(6 marks)**

E 9 Differentiate $\frac{x^4}{\cos 3x}$ with respect to $x$. **(4 marks)**

P 10 A curve $C$ has equation $y = \frac{e^{2x}}{(x-2)^2}$, $x \neq 2$.

**a** Show that

$$\frac{dy}{dx} = \frac{Ae^{2x}(Bx - C)}{(x-2)^3}$$

where $A$, $B$ and $C$ are integers to be found. **(4 marks)**

**b** Find the equation of the tangent of $C$ at the point $x = 1$. **(3 marks)**

11 Given that

$$f(x) = \frac{2x}{x+5} + \frac{6x}{x^2 + 7x + 10}, x > 0$$

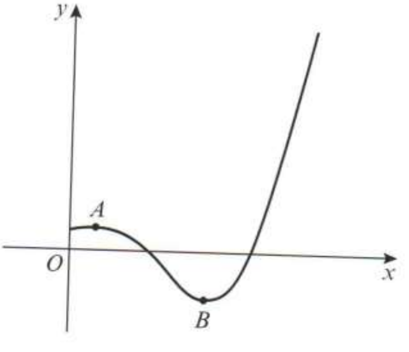
**a** show that $f(x) = \frac{2x}{x+2}$ **(4 marks)**

**b** Hence find $f'(3)$. **(3 marks)**

12 The diagram shows a sketch of the curve with equation $y = f(x)$, where

$$f(x) = \frac{2\cos 2x}{e^{2-x}}, 0 < x < \pi$$

The curve has a maximum turning point at $A$ and a minimum turning point and $B$ as shown in the diagram.




**a** Show that the $x$-coordinates of point $A$ and point $B$ are solutions to the equation $\tan 2x = \frac{1}{2}$ **(4 marks)**

**b** Find the range of $f(x)$. **(2 marks)**

## 9.6 Differentiating trigonometric functions

You can combine all the above rules and apply them to trigonometric functions to obtain standard results.

### Example 14

If $y = \tan x$, find $\frac{dy}{dx}$

$y = \tan x = \frac{\sin x}{\cos x}$

You can write $\tan x$ as $\frac{\sin x}{\cos x}$ and then use the quotient rule.

Let $u = \sin x$ and $v = \cos x$

$\frac{du}{dx} = \cos x$ and $\frac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\cos x \times \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$\frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

Use the identity $\cos^2 x + \sin^2 x \equiv 1$.

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x$$

You can generalise this method to differentiate $\tan kx$:

- **If $y = \tan kx$, then $\frac{dy}{dx} = k\sec^2 kx$**

### Example 15

Differentiate **a** $y = x\tan 2x$ **b** $y = \tan^4 x$

**a** $y = x\tan 2x$

This is a product.
Use $u = x$ and $v = \tan 2x$, together with the product rule.

$$\frac{dy}{dx} = x \times 2\sec^2 2x + \tan 2x$$
$$= 2x\sec^2 2x + \tan 2x$$

**b** $y = \tan^4 x = (\tan x)^4$

$$\frac{dy}{dx} = 4(\tan x)^3(\sec^2 x)$$

Use the chain rule with $u = \tan x$.

$$= 4\tan^3 x\sec^2 x$$

## Example 16

Show that if $y = \text{cosec}\, x$, then $\frac{dy}{dx} = -\text{cosec}x \cot x$.

$y = \text{cosec}\, x = \frac{1}{\sin x}$

Let $u = 1$ and $v = \sin x$

$\frac{du}{dx} = 0$ and $\frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x}$

$\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x}$

$\frac{dy}{dx} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\text{cosec}\, x \cot x$

Use the quotient rule with $u = 1$ and $v = \sin x$.
$u = 1$ is a constant so $\frac{du}{dx} = 0$.

Rearrange your answer into the desired form using the definitions of cosec and cot. ← Section 6.1

You can use similar techniques to differentiate sec $x$ and cot $x$ giving you the following general results:

- **If $y = \text{cosec}\, kx$, then $\frac{dy}{dx} = -k\, \text{cosec}\, kx \cot kx$**
- **If $y = \sec kx$, then $\frac{dy}{dx} = k \sec kx \tan kx$**
- **If $y = \cot kx$, then $\frac{dy}{dx} = -k\, \text{cosec}^2\, kx$**

**Watch out** While the standard results for tan, cosec, sec and cot are given in the formulae booklet, learning these results will enable you to differentiate a wide range of functions quickly and confidently.

## Example 17

Differentiate: **a** $y = \frac{\text{cosec}\, 2x}{x^2}$ **b** $y = \sec^3 x$

**a** $y = \frac{\text{cosec}\, 2x}{x^2}$

So $\frac{dy}{dx} = \frac{x^2(-2\text{cosec}\, 2x \cot 2x) - \text{cosec}\, 2x \times 2x}{x^4}$

$= \frac{-2\text{cosec}\, 2x(x \cot 2x + 1)}{x^3}$

Use the quotient rule with $u = \text{cosec}\, 2x$ and $v = x^2$.

**b** $y = \sec^3 x = (\sec x)^3$

$\frac{dy}{dx} = 3(\sec x)^2 (\sec x \tan x)$

$= 3\sec^3 x \tan x$

Use the chain rule with $u = \sec x$.

You can use the rule $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ to differentiate arcsin $x$, arccos $x$ and arctan $x$.

## Example 18

Show that the derivative of arcsin $x$ is $\frac{1}{\sqrt{1 - x^2}}$

Let $y = \arcsin x$

So $x = \sin y$ — arcsin is the inverse function of sin, so if $y = \arcsin x$ then $x = \sin y$. ← Section 6.5

$\frac{dx}{dy} = \cos y$ — Differentiate $x$ with respect to $y$.

$\frac{dy}{dx} = \frac{1}{\cos y}$ — Use $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$. This gives you an expression for $\frac{dy}{dx}$ in terms of $y$.

$\sin^2 y + \cos^2 y \equiv 1$

$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$ — Since $x = \sin y$, $x^2 = \sin^2 y$.

So $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

**Problem-solving**

Use the identity $\sin^2 \theta + \cos^2 \theta \equiv 1$ to write $\cos y$ in terms of $\sin y$. This will enable you to find an expression for $\frac{dy}{dx}$ in terms of $x$.

You can use similar techniques to differentiate arccos $x$ and arctan $x$ giving you the following results:

- If $y = \arccos x$, then $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$
- If $y = \arctan x$, then $\frac{dy}{dx} = \frac{1}{1 + x^2}$

## Exercise 9F

**1** Differentiate:

**a** $y = \tan 3x$ **b** $y = 4\tan^3 x$ **c** $y = \tan(x - 1)$ **d** $y = x^2 \tan \frac{1}{2}x + \tan\left(x - \frac{1}{2}\right)$

**2** Differentiate:

**a** $\cot 4x$ **b** $\sec 5x$ **c** $\operatorname{cosec} 4x$ **d** $\sec^2 3x$

**e** $x \cot 3x$ **f** $\frac{\sec^2 x}{x}$ **g** $\operatorname{cosec}^3 2x$ **h** $\cot^2(2x - 1)$

**3** Find the function $f'(x)$ where $f(x)$ is:

**a** $(\sec x)^{\frac{1}{2}}$ **b** $\sqrt{\cot x}$ **c** $\operatorname{cosec}^2 x$ **d** $\tan^2 x$

**e** $\sec^3 x$ **f** $\cot^3 x$

**4** Find $f'(x)$ where $f(x)$ is:

**a** $x^2 \sec 3x$ **b** $\frac{\tan 2x}{x}$ **c** $\frac{x^2}{\tan x}$ **d** $e^x \sec 3x$

**e** $\frac{\ln x}{\tan x}$ **f** $\frac{e^{\tan x}}{\cos x}$

P **5** The curve $C$ has equation

$$y = \frac{1}{\cos x \sin x}, \ 0 < x \leqslant \pi$$

**a** Find $\frac{dy}{dx}$ **(4 marks)**

**b** Determine the number of stationary points of the curve $C$. **(2 marks)**

**c** Find the equation of the tangent at the point where $x = \frac{\pi}{3}$, giving your answer in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are exact constants to be determined. **(3 marks)**

P **6** Show that if $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$. **(5 marks)**

P **7** Show that if $y = \cot x$ then $\frac{dy}{dx} = -\text{cosec}^2 x$. **(5 marks)**

P **8** Assuming standard results for $\sin x$ and $\cos x$, prove that:

**a** the derivative of $\arccos x$ is $-\frac{1}{\sqrt{1 - x^2}}$

**b** the derivative of $\arctan x$ is $\frac{1}{1 + x^2}$

P **9** Given that $x = \text{cosec } 5y$,

**a** find $\frac{dy}{dx}$ in terms of $y$. **(2 marks)**

**b** Hence find $\frac{dy}{dx}$ in terms of $x$. **(4 marks)**

## 9.7 Parametric differentiation

When functions are defined parametrically, you can find the gradient at a given point without converting into Cartesian form. You can use a variation of the chain rule:

- **If $x$ and $y$ are given as functions of a parameter, $t$:** $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

**Hint** You can obtain this from writing $\frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dt}$

### Example 19

Find the gradient at the point $P$ where $t = 2$, on the curve given parametrically by

$x = t^3 + t, \quad y = t^2 + 1, \quad t \in \mathbb{R}$

$\frac{dx}{dt} = 3t^2 + 1, \frac{dy}{dt} = 2t$

First differentiate $x$ and $y$ with respect to the parameter $t$.

$\frac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2 + 1}$

This rule will give the gradient function, $\frac{dy}{dx}$, in terms of the **parameter**, $t$.

When $t = 2$, $\frac{dy}{dx} = \frac{4}{13}$

Substitute $t = 2$ into $\frac{2t}{3t^2 + 1}$

So the gradient at $P$ is $\frac{4}{13}$

### Example 20

Find the equation of the normal at the point $P$ where $\theta = \frac{\pi}{6}$, to the curve with parametric equations $x = 3\sin\theta$, $y = 5\cos\theta$.

$\frac{dx}{d\theta} = 3\cos\theta, \frac{dy}{d\theta} = -5\sin\theta \quad 0 \leqslant \theta < 2\pi$

First differentiate $x$ and $y$ with respect to the parameter $\theta$.

$\therefore \quad \frac{dy}{dx} = \frac{-5\sin\theta}{3\cos\theta}$

At point $P$, where $\theta = \frac{\pi}{6}$,

$\frac{dy}{dx} = \dfrac{-5 \times \frac{1}{2}}{3 \times \frac{\sqrt{3}}{2}} = \frac{-5}{3\sqrt{3}}$

Use the chain rule, $\frac{dy}{d\theta} \div \frac{dx}{d\theta}$, and substitute $\theta = \frac{\pi}{6}$

**Online** Explore the graph of this curve and the normal at this point using technology.

The gradient of the normal at $P$ is $\frac{3\sqrt{3}}{5}$,

and at $P$, $x = \frac{3}{2}$, $y = \frac{5\sqrt{3}}{2}$

The equation of the normal is

$$y - \frac{5\sqrt{3}}{2} = \frac{3\sqrt{3}}{5}\left(x - \frac{3}{2}\right)$$

$\therefore\ 5y = 3\sqrt{3}x + 8\sqrt{3}$

The normal is perpendicular to the curve, so its gradient is $-\frac{1}{m}$ where $m$ is the gradient of the curve at that point.

You need to find the coordinates of $P$. Substitute $\theta = \frac{\pi}{6}$ into each of the parametric equations.
← Section 8.1

Use the equation for a line in the form $y - y_1 = m(x - x_1)$

## Exercise 9G

**1** Find $\frac{dy}{dx}$ for each of the following, leaving your answer in terms of the parameter $t$.

**a** $x = 2t, y = t^2 - 3t + 2$
**b** $x = 3t^2, y = 2t^3$
**c** $x = t + 3t^2, y = 4t$
**d** $x = t^2 - 2, y = 3t^5$
**e** $x = \frac{2}{t}, y = 3t^2 - 2$
**f** $x = \frac{1}{2t - 1}, y = \frac{t^2}{2t - 1}$
**g** $x = \frac{2t}{1 + t^2}, y = \frac{1 - t^2}{1 + t^2}$
**h** $x = t^2e^t, y = 2t$
**i** $x = 4\sin 3t, y = 3\cos 3t$
**j** $x = 2 + \sin t, y = 3 - 4\cos t$
**k** $x = \sec t, y = \tan t$
**l** $x = 2t - \sin 2t, y = 1 - \cos 2t$
**m** $x = e^t - 5, y = \ln t, t > 0$
**n** $x = \ln t, y = t^2 - 64, t > 0$
**o** $x = e^{2t} + 1, y = 2e^t - 1, -1 < t < 1$

(P) **2 a** Find the equation of the tangent to the curve with parametric equations $x = 3 - 2\sin t$, $y = t\cos t$, at the point $P$, where $t = \pi$.

**b** Find the equation of the tangent to the curve with parametric equations $x = 9 - t^2$, $y = t^2 + 6t$, at the point $P$, where $t = 2$.

(P) **3 a** Find the equation of the normal to the curve with parametric equations $x = e^t$, $y = e^t + e^{-t}$, at the point $P$, where $t = 0$.

**b** Find the equation of the normal to the curve with parametric equations $x = 1 - \cos 2t$, $y = \sin 2t$, at the point $P$, where $t = \frac{\pi}{6}$

(P) **4** Find the points of zero gradient on the curve with parametric equations

$$x = \frac{t}{1 - t}, \quad y = \frac{t^2}{1 - t}, \quad t \neq 1$$

You do not need to establish whether they are maximum or minimum points.

(P) **5** The curve $C$ has parametric equations $x = e^{2t}$, $y = e^t - 1$, $t \in \mathbb{R}$.

**a** Find the equation of the tangent to $C$ at the point $A$ where $t = \ln 2$.

**b** Show that the curve $C$ has no stationary points.

E/P **6** The curve $C$ has parametric equations

$$x = \frac{t^2 - 3t - 4}{t}, \quad y = 2t, \quad t > 0$$

The line $l_1$ is a tangent to $C$ and is parallel to the line with equation $y = x + 5$.
Find the equation of $l_1$. **(8 marks)**

E/P **7** A curve has parametric equations

$$x = 2\sin^2 t, \quad y = 2\cot t, \quad 0 < t < \frac{\pi}{2}$$

**a** Find an expression for $\frac{dy}{dx}$ in terms of the parameter $t$. **(4 marks)**

**b** Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{6}$ **(4 marks)**

E/P **8** The curve $C$ has parametric equations

$$x = 4\sin t, \qquad y = 2\operatorname{cosec} 2t, \qquad 0 \leqslant t \leqslant \pi$$

The point $A$ lies on $C$ and has coordinates $\left(2\sqrt{3}, \frac{4\sqrt{3}}{3}\right)$.

**a** Find the value of $t$ at the point $A$. **(2 marks)**

The line $l$ is a normal to $C$ at $A$.

**b** Show that an equation for $l$ is $9x + 12y - 34\sqrt{3} = 0$. **(6 marks)**

E/P **9** The curve $C$ has parametric equations

$$x = t^2 + t, \quad y = t^2 - 10t + 5, \quad t \in \mathbb{R}$$

where $t$ is a parameter. Given that at point $P$, the gradient of $C$ is 2,

**a** find the coordinates of $P$ **(4 marks)**

**b** find the equation of the tangent to $C$ at point $P$ **(3 marks)**

**c** show that the tangent to $C$ at point $P$ does not intersect the curve again. **(5 marks)**

**Problem-solving**

Substitute the equations for $x$ and $y$ into the equation of your tangent, and show that the resulting quadratic equation has one unique root.

E/P **10** The curve $C$ has parametric equations

$$x = 2\sin t, \quad y = \sqrt{2}\cos 2t, \quad 0 < t < \pi$$

**a** Find an expression for $\frac{dy}{dx}$ in terms of $t$. **(2 marks)**

The point $A$ lies on $C$ where $t = \frac{\pi}{3}$. The line $l$ is the normal to $C$ at $A$.

**b** Find an equation for $l$ in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are exact constants to be found. **(5 marks)**

**c** Prove that the line $l$ does not intersect the curve anywhere other than at point $A$. **(6 marks)**

E/P **11** A curve has parametric equations

$$x = \cos t, \quad y = \tfrac{1}{2}\sin 2t, \quad 0 \leqslant t < 2\pi$$

**a** Find an expression for $\frac{dy}{dx}$ in terms of $t$. **(2 marks)**

**b** Find an equation of the tangent to the curve at point $A$ where $t = \frac{\pi}{6}$ **(4 marks)**

The lines $l_1$ and $l_2$ are two further distinct tangents to the curve. Given that $l_1$ and $l_2$ are both parallel to the tangent to the curve at point $A$,

**c** find an equation of $l_1$ and an equation of $l_2$ **(6 marks)**

## 9.8 Implicit differentiation

Some equations are difficult to rearrange into the form $y = f(x)$ or $x = f(y)$. You can sometimes differentiate these equations **implicitly** without rearranging them.

**Notation** An equation in the form $y = f(x)$ is given **explicitly**.

Equations which involve functions of both $x$ and $y$ such as $x^2 + 2xy = 3$ or $\cos(x + y) = 2x$ are called **implicit** equations.

In general, from the chain rule:

- $$\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$$

The following two specific results are useful for implicit differentiation:

- $$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$
- $$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$$

When you differentiate implicit equations your expression for $\frac{dy}{dx}$ will usually be given in terms of **both $x$ and $y$**.

**Watch out** You need to pay careful attention to the variable you are differentiating with respect to.

## Example 21

Find $\frac{dy}{dx}$ in terms of $x$ and $y$ where $x^3 + x + y^3 + 3y = 6$.

$$3x^2 + 1 + 3y^2\frac{dy}{dx} + 3\frac{dy}{dx} = 0$$

Differentiate the expression term by term with respect to $x$.

Use $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$ with $n = 3$.

$$\frac{dy}{dx}(3y^2 + 3) = -3x^2 - 1$$

Divide both sides by $3y^2 + 3$ and factorise.

$$\frac{dy}{dx} = -\frac{3x^2 + 1}{3(1 + y^2)}$$

Then make $\frac{dy}{dx}$ the subject of the formula.

## Example 22

Given that $4xy^2 + \frac{6x^2}{y} = 10$, find the value of $\frac{dy}{dx}$ at the point (1, 1).

$$\left(4x \times 2y\frac{dy}{dx} + 4y^2\right) + \left(\frac{12x}{y} - \frac{6x^2}{y^2}\frac{dy}{dx}\right) = 0$$

Differentiate each term with respect to $x$.

Use the product rule on each term, expressing $\frac{6x^2}{y}$ as $6x^2y^{-1}$.

Substitute $x = 1$, $y = 1$ to give

Find the value of $\frac{dy}{dx}$ at (1, 1) by substituting $x = 1$, $y = 1$.

$$\left(8\frac{dy}{dx} + 4\right) + \left(12 - 6\frac{dy}{dx}\right) = 0$$

Substitute before rearranging, as this simplifies the working.

$$16 + 2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -8$$

Solve to find the value of $\frac{dy}{dx}$ at this point.

## Example 23

Find the value of $\frac{dy}{dx}$ at the point (1, 1) where $e^{2x} \ln y = x + y - 2$.

$e^{2x} \times \frac{1}{y}\frac{dy}{dx} + \ln y \times 2e^{2x} = 1 + \frac{dy}{dx}$ — Differentiate each term with respect to $x$.

Use the product rule applied to the term on the left hand side of the equation, noting that $\ln y$ differentiates to give $\frac{1}{y}\frac{dy}{dx}$

Substitute $x = 1$, $y = 1$ to give

$$e^2 \times \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\therefore \quad (e^2 - 1)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^2 - 1}$$

Rearrange to make $\frac{dy}{dx}$ the subject of the formula.

## Exercise 9H

(P) **1** By writing $u = y^n$, and using the chain rule, show that $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$

(P) **2** Use the product rule to show that $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$.

(P) **3** Find an expression in terms of $x$ and $y$ for $\frac{dy}{dx}$, given that:

**a** $x^2 + y^3 = 2$
**b** $x^2 + 5y^2 = 14$
**c** $x^2 + 6x - 8y + 5y^2 = 13$
**d** $y^3 + 3x^2y - 4x = 0$
**e** $3y^2 - 2y + 2xy = x^3$
**f** $x = \frac{2y}{x^2 - y}$
**g** $(x - y)^4 = x + y + 5$
**h** $e^x y = xe^y$
**i** $\sqrt{xy} + x + y^2 = 0$

(P) **4** Find the equation of the tangent to the curve with implicit equation $x^2 + 3xy^2 - y^3 = 9$ at the point (2, 1).

(P) **5** Find the equation of the normal to the curve with implicit equation $(x + y)^3 = x^2 + y$ at the point (1, 0).

(P) **6** Find the coordinates of the points of zero gradient on the curve with implicit equation $x^2 + 4y^2 - 6x - 16y + 21 = 0$.

**Problem-solving**

Find $\frac{dy}{dx}$ then set the numerator equal to 0 to find the $x$-coordinate at the points of 0 gradient. You need to find two corresponding $y$-coordinates.

(E/P) **7** A curve $C$ is described by the equation

$$2x^2 + 3y^2 - x + 6xy + 5 = 0$$

Find an equation of the tangent to $C$ at the point $(1, -2)$, giving your answer in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are integers. **(7 marks)**

(E/P) **8** A curve $C$ has equation

$$3^x = y - 2xy$$

Find the exact value of $\frac{dy}{dx}$ at the point on $C$ with coordinates $(2, -3)$. **(7 marks)**

(E/P) **9** Find the gradient of the curve with equation

$$\ln(y^2) = \tfrac{1}{2}x \ln(x - 1), \quad x > 1, \quad y > 0$$

at the point on the curve where $x = 4$. Give your answer as an exact value. **(7 marks)**

(E/P) **10** A curve $C$ satisfies $\sin x + \cos y = 0.5$, where $-\pi < x < \pi$ and $-\pi < y < \pi$.

**a** Find an expression for $\frac{dy}{dx}$ **(2 marks)**

**b** Find the coordinates of the stationary points on $C$. **(5 marks)**

(E/P) **11** The curve $C$ has the equation $ye^{-3x} - 3x = y^2$.

**a** Find $\frac{dy}{dx}$ in terms of $x$ and $y$. **(5 marks)**

**b** Show that the equation of the tangent to $C$ at the origin, $O$, is $y = 3x$. **(4 marks)**

**Challenge**

The curve $C$ has implicit equation $6x + y^2 + 2xy = x^2$.

**a** Show that there are no points on the curve such that $\frac{dy}{dx} = 0$.

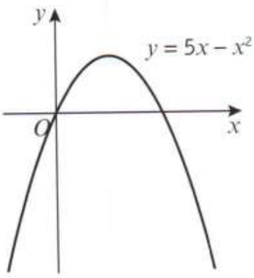
**b** Find the coordinates of the two points on $C$ such that $\frac{dx}{dy} = 0$.

## 9.9 Using second derivatives

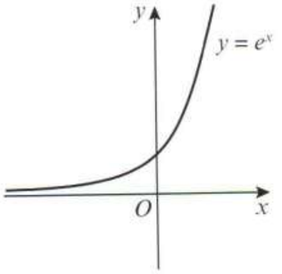
You can use the second derivative to determine whether a curve is **concave** or **convex** on a given domain.

- **The function $f(x)$ is concave on a given interval if and only if $f''(x) \leqslant 0$ for every value of $x$ in that interval.**
- **The function $f(x)$ is convex on a given interval if and only if $f''(x) \geqslant 0$ for every value of $x$ in that interval.**

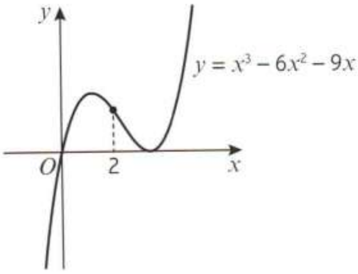
**Links** To find the second derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$, you differentiate **twice** with respect to $x$.
← Year 1, Chapter 12




$\frac{d^2y}{dx^2} = -2$ so the curve is concave for all $x \in \mathbb{R}$.




$\frac{d^2y}{dx^2} = e^x$ which is always positive, so the curve is convex for all $x \in \mathbb{R}$.




$\frac{d^2y}{dx^2} = 6x - 12$ so the curve is concave for all $x \leqslant 2$ and convex for $x \geqslant 2$.

### Example 24

Find the interval on which the function $f(x) = x^3 + 4x + 3$ is concave.

$f(x) = x^3 + 4x + 3$
$f'(x) = 3x^2 + 4$
$f''(x) = 6x$
For $f(x)$ to be concave, $f''(x) \leqslant 0$
$6x \leqslant 0$
$x \leqslant 0$
So $f(x)$ is concave for all $x \leqslant 0$.

Differentiate twice to get an expression for $f''(x)$.

Write down the condition for a concave function in your working.

**Notation** You can also write this interval as $(-\infty, 0]$.

## Example 25

Show that the function $f(x) = e^{2x} + x^2$ is convex for all real values of $x$.

$f(x) = e^{2x} + x^2$
$f'(x) = 2e^{2x} + 2x$
$f''(x) = 4e^{2x} + 2$

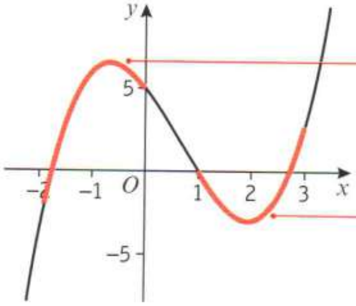
Differentiate twice to get an expression for $f''(x)$.

$e^{2x} > 0$ for all $x \in \mathbb{R}$, so $4e^{2x} + 2 > 2$ for all $x \in \mathbb{R}$

Hence $f''(x) \geqslant 0$, so f is convex for all $x \in \mathbb{R}$.

**Problem-solving** Write down the condition for a convex function and a conclusion.

The point at which a curve changes from being concave to convex (or vice versa) is called a **point of inflection**. The diagram shows the curve with equation $y = x^3 - 2x^2 - 4x + 5$.




In the interval [−2, 0] the curve is concave.

In the interval [1, 3], the curve is convex.

At some point between 0 and 1 the curve changes from being concave to being convex. This is the point of inflection.

- **A point of inflection is a point at which $f''(x)$ changes sign.**

To find a point of inflection you need to show that $f''(x) = 0$ at that point, and that it has different signs on either side of that point.

**Watch out** A point of inflection does not have to be a stationary point.

## Example 26

The curve $C$ has equation $y = x^3 - 2x^2 - 4x + 5$.

**a** Show that $C$ is concave on the interval [−2, 0] and convex on the interval [1, 3].

**b** Find the coordinates of the point of inflection.

**a** $\frac{dy}{dx} = 3x^2 - 4x - 4$

$\frac{d^2y}{dx^2} = 6x - 4$

Differentiate $y = x^3 - 2x^2 - 4x + 5$ with respect to $x$ twice.

$\frac{d^2y}{dx^2} = 6x - 4 \leqslant 0$ for all $-2 \leqslant x \leqslant 0$.

Consider the value of $6x - 4$ on the interval [−2, 0]. $6x - 4$ is a linear function. When $x = -2$, $\frac{d^2y}{dx^2} = -16$ and when $x = 0$, $\frac{d^2y}{dx^2} = -4$, so $\frac{d^2y}{dx^2} \leqslant 0$ on [0, 2].

Therefore, $y = x^3 - 2x^2 - 4x + 5$ is concave on the interval $[-2, 0]$.

$\frac{d^2y}{dx^2} = 6x - 4 \geqslant 0$ for all $1 \leqslant x \leqslant 3$.

Consider the value of $6x - 4$ on the interval $[1, 3]$. When $x = 1$, $\frac{d^2y}{dx^2} = 2$ and when $x = 3$, $\frac{d^2y}{dx^2} = 14$.

Therefore, $y = x^3 - 2x^2 - 4x + 5$ is convex on the interval $[1, 3]$.

b $\frac{d^2y}{dx^2} = 6x - 4 = 0$

Find the point where $f''(x) = 0$. You have already determined that $f''(x)$ changes sign on either side of this point.

$6x = 4$

$x = \frac{4}{6} = \frac{2}{3}$

Substitute $x$ into $y$ gives

$y = \left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 5 = \frac{47}{27}$

So the point of inflection of the curve $C$ is $\left(\frac{2}{3}, \frac{47}{27}\right)$.

**Online** Explore the solution to this example graphically using technology.

## Exercise 9I

(P) **1** For each of the following functions, find the interval on which the function is:

**i** convex **ii** concave

**a** $f(x) = x^3 - 3x^2 + x - 2$ **b** $f(x) = x^4 - 3x^3 + 2x - 1$ **c** $f(x) = \sin x, 0 < x < 2\pi$

**d** $f(x) = -x^2 + 3x - 7$ **e** $f(x) = e^x - x^2$ **f** $f(x) = \ln x, x > 0$

(P) **2** $f(x) = \arcsin x, -1 < x < 1$

**a** Show that $f'(x) = \frac{1}{\sqrt{1 - x^2}}$

**b** Hence show that $f(x)$ is concave on the interval $(-1, 0)$.

**c** Show that $f(x)$ is convex on the interval $(0, 1)$.

**d** Hence deduce the point of inflection of f.

(P) **3** Find any point(s) of inflection of the following functions.

**a** $f(x) = \cos^2 x - 2\sin x, 0 < x < 2\pi$ **b** $f(x) = -\frac{x^3 - 2x^2 + x - 1}{x - 2}, x \neq 2$

**c** $f(x) = \frac{x^3}{x^2 - 4}, x \neq \pm 2$

(P) **4** $f(x) = 2x^2 \ln x, x > 0$

Show that f has exactly one point of inflection and determine the value of $x$ at this point.

(P) **5** The curve $C$ has equation $y = e^x(x^2 - 2x + 2)$.

**a** Find the exact coordinates of the stationary point on $C$ and determine its nature.

**b** Find the coordinates of any non-stationary points of inflection on $C$.

(P) **6** The curve $C$ has equation $y = xe^x$.

**a** Find the exact coordinates of the stationary point on $C$ and determine its nature.

**b** Find the coordinates of any non-stationary points of inflection on $C$.

**c** Hence sketch the graph of $y = xe^x$.

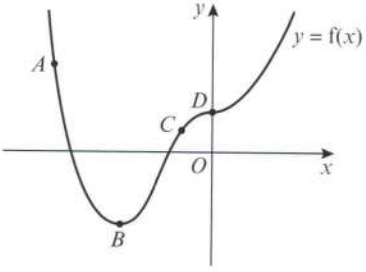
**Problem-solving**

Consider how $C$ behaves for very large positive and negative values of $x$.

(P) **7** For each point on the graph, state whether:

**i** $f'(x)$ is positive, negative or zero

**ii** $f''(x)$ is positive, negative or zero




(P) **8** $f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

Prove that $f(x)$ has exactly one point of inflection, at the origin.

(E) **9** Given that $y = x(3x - 1)^5$,

**a** find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ **(4 marks)**

**b** find the points of inflection of $y$. **(4 marks)**

(E/P) **10** A student is attempting to find the points of inflection on the curve $C$ with equation

$$y = (x - 5)^4$$

The attempt is shown below:

$$\frac{dy}{dx} = 4(x-5)^3$$

$$\frac{d^2y}{dx^2} = 12(x-5)^2$$

When $\frac{d^2y}{dx^2} = 0$,

$12(x-5)^2 = 0$

$(x-5)^2 = 0$

$x - 5 = 0$

$x = 5$

Therefore, the curve $C$ has a point of inflection at $x = 5$.

**a** Identify the mistake made by the student. **(2 marks)**

**b** Write down the coordinates of the stationary point on $C$ and determine its nature. **(2 marks)**

E/P **11** A curve $C$ has equation

$$y = \frac{1}{3}x^2 \ln x - 2x + 5, \ x > 0$$

Show that the curve $C$ is convex for all $x \geqslant e^{-\frac{3}{2}}$. **(5 marks)**

**Challenge**

**1** Prove that every cubic curve has exactly one point of inflection.

**2** The curve $C$ has equation $y = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$

**a** Show that $C$ has at most two points of inflection.

**b** Prove that if $3b^2 < 8ac$, then $C$ has no points of inflection.

## 9.10 Rates of change

- **You can use the chain rule to connect rates of change in situations involving more than two variables.**

### Example 27

Given that the area of a circle $A$ cm² is related to its radius $r$ cm by the formula $A = \pi r^2$, and that the rate of change of its radius in cm s$^{-1}$ is given by $\frac{dr}{dt} = 5$, find $\frac{dA}{dt}$ when $r = 3$.

$A = \pi r^2$

$\therefore \quad \frac{dA}{dr} = 2\pi r$

Using $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi r \times 5$

$= 30\pi$, when $r = 3$.

**Problem-solving**

In order to be able to apply the chain rule to find $\frac{dA}{dt}$ you need to know $\frac{dA}{dr}$. You can find it by differentiating $A = \pi r^2$ with respect to $r$.

You should use the chain rule, giving the derivative which you need to find in terms of known derivatives.

### Example 28

The volume of a hemisphere $V$ cm³ is related to its radius $r$ cm by the formula $V = \frac{2}{3}\pi r^3$ and the total surface area $S$ cm² is given by the formula $S = \pi r^2 + 2\pi r^2 = 3\pi r^2$. Given that the rate of increase of volume, in cm³ s$^{-1}$, $\frac{dV}{dt} = 6$, find the rate of increase of surface area $\frac{dS}{dt}$

$V = \frac{2}{3}\pi r^3$ and $S = 3\pi r^2$

This is area of circular base plus area of curved surface.

$\frac{dV}{dr} = 2\pi r^2$ and $\frac{dS}{dr} = 6\pi r$

As $V$ and $S$ are functions of $r$, find $\frac{dV}{dr}$ and $\frac{dS}{dr}$

$$\text{Now } \frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$$
$$= 6\pi r \times \frac{1}{2\pi r^2} \times 6$$
$$= \frac{18}{r}$$

Use the chain rule together with the property that $\frac{dr}{dV} = 1 \div \frac{dV}{dr}$

An equation which involves a rate of change is called a **differential equation**. You can formulate differential equations from information given in a question.

**Links** You can use integration to solve differential equations. → Section 11.10

## Example 29

In the decay of radioactive particles, the rate at which particles decay is proportional to the number of particles remaining. Write down a differential equation for the rate of change of the number of particles.

Let $N$ be the number of particles and let $t$ be time. The rate of change of the number of particles $\frac{dN}{dt}$ is proportional to $N$.

i.e. $\frac{dN}{dt} = -kN$, where $k$ is a positive constant.

The minus sign arises because the number of particles is decreasing.

$\frac{dN}{dt} \propto N$ so you can write $\frac{dN}{dt} = kN$ where $k$ is the constant of proportion.

## Example 30

Newton's law of cooling states that the rate of loss of temperature of a body is proportional to the excess temperature of the body over its surroundings. Write an equation that expresses this law.

Let the temperature of the body be $\theta$ degrees and the time be $t$ seconds.

The rate of change of the temperature $\frac{d\theta}{dt}$ is proportional to $\theta - \theta_0$, where $\theta_0$ is the temperature of the surroundings.

i.e. $\frac{d\theta}{dt} = -k(\theta - \theta_0)$, where $k$ is a positive constant.

$\theta - \theta_0$ is the difference between the temperature of the body and that of its surroundings.

The minus sign arises because the temperature is decreasing. The question mentions loss of temperature.

## Example 31

The head of a snowman of radius $R$ cm loses volume by evaporation at a rate proportional to its surface area. Assuming that the head is spherical, that the volume of a sphere is $\frac{4}{3}\pi R^3$ cm$^3$ and that the surface is $4\pi R^2$ cm$^2$, write down a differential equation for the rate of change of radius of the snowman's head.

The first sentence tells you that $\frac{dV}{dt} = -kA$, where $V$ cm$^3$ is the volume, $t$ seconds is time, $k$ is a positive constant and $A$ cm$^2$ is the surface area of the snowman's head.

Since $V = \frac{4}{3}\pi R^3$

The question asks for a differential equation in terms of $R$, so you need to use the expression for $V$ in terms of $R$.

$\frac{dV}{dR} = 4\pi R^2$

$\therefore \quad \frac{dV}{dt} = \frac{dV}{dR} \times \frac{dR}{dt} = 4\pi R^2 \times \frac{dR}{dt}$

The chain rule is used here because this is a related rate of change.

But as $\frac{dV}{dt} = -kA$

Use the expression for $A$ in terms of $R$.

$4\pi R^2 \times \frac{dR}{dt} = -k \times 4\pi R^2$

Divide both sides by the common factor $4\pi R^2$.

$\therefore \quad \frac{dR}{dt} = -k$

This gives the rate of change of radius as required.

## Exercise 9J

(P) **1** Given that $A = \frac{1}{4}\pi r^2$ and that $\frac{dr}{dt} = 6$, find $\frac{dA}{dt}$ when $r = 2$.

(P) **2** Given that $y = xe^x$ and that $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = 2$.

(P) **3** Given that $r = 1 + 3\cos\theta$ and that $\frac{d\theta}{dt} = 3$, find $\frac{dr}{dt}$ when $\theta = \frac{\pi}{6}$

(P) **4** Given that $V = \frac{1}{3}\pi r^3$ and that $\frac{dV}{dt} = 8$, find $\frac{dr}{dt}$ when $r = 3$.

(P) **5** A population is growing at a rate which is proportional to the size of the population. Write down a differential equation for the growth of the population.

(P) **6** A curve $C$ has equation $y = f(x)$, $y > 0$. At any point $P$ on the curve, the gradient of $C$ is proportional to the product of the $x$- and the $y$-coordinates of $P$. The point $A$ with coordinates $(4, 2)$ is on $C$ and the gradient of $C$ at $A$ is $\frac{1}{2}$

Show that $\frac{dy}{dx} = \frac{xy}{16}$

(P) **7** Liquid is pouring into a container at a constant rate of $30\,\text{cm}^3\,\text{s}^{-1}$. At time $t$ seconds liquid is leaking from the container at a rate of $\frac{2}{15}V\,\text{cm}^3\,\text{s}^{-1}$, where $V\,\text{cm}^3$ is the volume of the liquid in the container at that time.

Show that $-15\frac{dV}{dt} = 2V - 450$.

(P) **8** An electrically-charged body loses its charge, $Q$ coulombs, at a rate, measured in coulombs per second, proportional to the charge $Q$.
Write down a differential equation in terms of $Q$ and $t$ where $t$ is the time in seconds since the body started to lose its charge.

(P) **9** The ice on a pond has a thickness $x$ mm at a time $t$ hours after the start of freezing. The rate of increase of $x$ is inversely proportional to the square of $x$.
Write down a differential equation in terms of $x$ and $t$.

(P) **10** The radius of a circle is increasing at a constant rate of 0.4 cm per second.

**a** Find $\frac{dC}{dt}$, where $C$ is the circumference of the circle, and interpret this value in the context of the model.

**b** Find the rate at which the area of the circle is increasing when the radius is 10 cm.

**c** Find the radius of the circle when its area is increasing at the rate of $20\,\text{cm}^2$ per second.

(P) **11** The volume of a cube is decreasing at a constant rate of $4.5\,\text{cm}^3$ per second. Find:

**a** the rate at which the length of one side of the cube is decreasing when the volume is $100\,\text{cm}^3$

**b** the volume of the cube when the length of one side is decreasing at the rate of 2 mm per second.

(P) **12** Fluid flows out of a cylindrical tank with constant cross section. At time $t$ minutes, $t > 0$, the volume of fluid remaining in the tank is $V\,\text{m}^3$. The rate at which the fluid flows in $\text{m}^3\,\text{min}^{-1}$ is proportional to the square root of $V$.

Show that the depth, $h$ metres, of fluid in the tank satisfies the differential equation $\frac{dh}{dt} = -k\sqrt{h}$, where $k$ is a positive constant.

(P) **13** At time, $t$ seconds, the surface area of a cube is $A\,\text{cm}^2$ and the volume is $V\,\text{cm}^3$.
The surface area of the cube is expanding at a constant rate of $2\,\text{cm}^2\,\text{s}^{-1}$.

**a** Write an expression for $V$ in terms of $A$.

**b** Find an expression for $\frac{dV}{dA}$

**c** Show that $\frac{dV}{dt} = \frac{1}{2}V^{\frac{1}{3}}$

(P) **14** An inverted conical funnel is full of salt. The salt is allowed to leave by a small hole in the vertex. It leaves at a constant rate of $6\,\text{cm}^3\,\text{s}^{-1}$.
Given that the angle of the cone between the slanting edge and the vertical is 30°, show that the volume of the salt is $\frac{1}{9}\pi h^3$, where $h$ is the height of salt at time $t$ seconds. Show that the rate of change of the height of the salt in the funnel is inversely proportional to $h^2$. Write down a differential equation relating $h$ and $t$.

## Mixed Exercise 9

E 1 Differentiate with respect to $x$:

a $\ln x^2$ **(3 marks)**

b $x^2 \sin 3x$ **(4 marks)**

E/P 2 a Given that $2y = x - \sin x \cos x$, $0 < x < 2\pi$, show that $\frac{dy}{dx} = \sin^2 x$. **(4 marks)**

b Find the coordinates of the points of inflection of the curve. **(4 marks)**

E 3 Differentiate, with respect to $x$:

a $\frac{\sin x}{x}$, $x > 0$ **(4 marks)**

b $\ln \frac{1}{x^2 + 9}$ **(4 marks)**

E/P 4 $f(x) = \frac{x}{x^2 + 2}$, $x \in \mathbb{R}$

a Given that $f(x)$ is increasing on the interval $[-k, k]$, find the largest possible value of $k$. **(4 marks)**

b Find the exact coordinates of the points of inflection of $f(x)$. **(5 marks)**

E/P 5 The function f is defined for positive real values of $x$ by

$f(x) = 12 \ln x + x^{\frac{3}{2}}$

a Find the set of values of $x$ for which $f(x)$ is an increasing function of $x$. **(4 marks)**

b Find the coordinates of the point of inflection of the function f. **(4 marks)**

E/P 6 Given that a curve has equation $y = \cos^2 x + \sin x$, $0 < x < 2\pi$, find the coordinates of the stationary points of the curve. **(6 marks)**

E/P 7 The maximum point on the curve with equation $y = x\sqrt{\sin x}$, $0 < x < \pi$, is the point $A$. Show that the $x$-coordinate of point $A$ satisfies the equation $2 \tan x + x = 0$. **(5 marks)**

E 8 $f(x) = e^{0.5x} - x^2$, $x \in \mathbb{R}$

a Find $f'(x)$. **(3 marks)**

b By evaluating $f'(6)$ and $f'(7)$, show that the curve with equation $y = f(x)$ has a stationary point at $x = p$, where $6 < p < 7$. **(2 marks)**

E/P 9 $f(x) = e^{2x} \sin 2x$, $0 < x < \pi$

a Use calculus to find the coordinates of the turning points on the graph of $y = f(x)$. **(6 marks)**

b Show that $f''(x) = 8e^{2x} \cos 2x$. **(4 marks)**

c Hence, or otherwise, determine which turning point is a maximum and which is a minimum. **(3 marks)**

d Find the points of inflection of $f(x)$. **(2 marks)**

E/P **10** The curve $C$ has equation $y = 2e^x + 3x^2 + 2$. Find the equation of the normal to $C$ at the point where the curve intercepts the $y$-axis. Give your answer in the form $ax + by + c = 0$ where $a$, $b$ and $c$ are integers to be found. **(5 marks)**

E/P **11** The curve $C$ has equation $y = f(x)$, where

$$f(x) = 3\ln x + \frac{1}{x}, x > 0$$

The point $P$ is a stationary point on $C$.

**a** Calculate the $x$-coordinate of $P$. **(4 marks)**

The point $Q$ on $C$ has $x$-coordinate 1.

**b** Find an equation for the normal to $C$ at $Q$. **(4 marks)**

E/P **12** The curve $C$ has equation $y = e^{2x}\cos x$.

**a** Show that the turning points on $C$ occur when $\tan x = 2$. **(4 marks)**

**b** Find an equation of the tangent to $C$ at the point where $x = 0$. **(4 marks)**

E/P **13** Given that $x = y^2 \ln y$, $y > 0$,

**a** find $\frac{dx}{dy}$ **(4 marks)**

**b** Use your answer to part **a** to find in terms of e, the value of $\frac{dy}{dx}$ at $y = e$. **(2 marks)**

E/P **14** A curve has equation $f(x) = (x^3 - 2x)e^{-x}$.

**a** Find $f'(x)$. **(4 marks)**

The normal to $C$ at the origin $O$ intersects $C$ again at $P$.

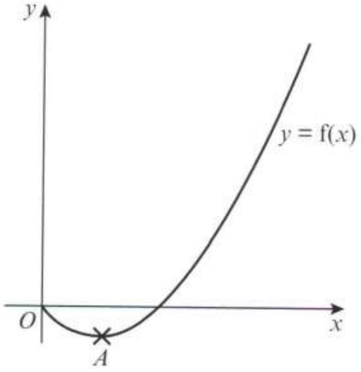
**b** Show that the $x$-coordinate of $P$ is the solution to the equation $2x^2 = e^x + 4$. **(6 marks)**

E/P **15** The diagram shows part of the curve with equation $y = f(x)$ where $f(x) = x(1 + x)\ln x$, $x > 0$

The point $A$ is the minimum point of the curve.

**a** Find $f'(x)$. **(4 marks)**

**b** Hence show that the $x$-coordinate of $A$ is the solution to the equation $x = e^{-\frac{1+x}{1+2x}}$ **(4 marks)**




E/P **16** The curve $C$ is given by the equations

$$x = 4t - 3, \quad y = \frac{8}{t^2}, \quad t > 0$$

where $t$ is a parameter.

At $A$, $t = 2$. The line $l$ is the normal to $C$ at $A$.

**a** Find $\frac{dy}{dx}$ in terms of $t$. **(4 marks)**

**b** Hence find an equation of $l$. **(3 marks)**

P **17** The curve $C$ is given by the equations $x = 2t$, $y = t^2$, where $t$ is a parameter.
Find an equation of the normal to $C$ at the point $P$ on $C$ where $t = 3$. **(7 marks)**

P **18** The curve $C$ has parametric equations

$$x = t^3, \quad y = t^2, \quad t > 0$$

Find an equation of the tangent to $C$ at $A$ (1, 1). **(7 marks)**

P **19** A curve $C$ is given by the equations

$$x = 2\cos t + \sin 2t, \quad y = \cos t - 2\sin 2t, \quad 0 < t < \pi$$

where $t$ is a parameter.

**a** Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of $t$. **(3 marks)**

**b** Find the value of $\frac{dy}{dx}$ at the point $P$ on $C$ where $t = \frac{\pi}{4}$ **(3 marks)**

**c** Find an equation of the normal to the curve at $P$. **(3 marks)**

P **20** A curve is given by $x = 2t + 3$, $y = t^3 - 4t$, where $t$ is a parameter. The point $A$ has parameter $t = -1$ and the line $l$ is the tangent to $C$ at $A$. The line $l$ also cuts the curve at $B$.

**a** Show that an equation for $l$ is $2y + x = 7$. **(6 marks)**

**b** Find the value of $t$ at $B$. **(5 marks)**

P **21** A car has value £$V$ at time $t$ years. A model for $V$ assumes that the rate of decrease of $V$ at time $t$ is proportional to $V$. Form an appropriate differential equation for $V$.

P **22** In a study of the water loss of picked leaves the mass, $M$ grams, of a single leaf was measured at times, $t$ days, after the leaf was picked. It was found that the rate of loss of mass was proportional to the mass $M$ of the leaf.
Write down a differential equation for the rate of change of mass of the leaf.

P **23** In a pond the amount of pondweed, $P$, grows at a rate proportional to the amount of pondweed already present in the pond. Pondweed is also removed by fish eating it at a constant rate of $Q$ per unit of time.
Write down a differential equation relating $P$ to $t$, where $t$ is the time which has elapsed since the start of the observation.

P **24** A circular patch of oil on the surface of some water has radius $r$ and the radius increases over time at a rate inversely proportional to the radius.
Write down a differential equation relating $r$ and $t$, where $t$ is the time which has elapsed since the start of the observation.

P **25** A metal bar is heated to a certain temperature, then allowed to cool down and it is noted that, at time $t$, the rate of loss of temperature is proportional to the difference between the temperature of the metal bar, $\theta$, and the temperature of its surroundings $\theta_0$.
Write down a differential equation relating $\theta$ and $t$.

E/P **26** The curve $C$ has parametric equations

$$x = 4\cos 2t, \quad y = 3\sin t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$A$ is the point $\left(2, \frac{3}{2}\right)$, and lies on $C$.

**a** Find the value of $t$ at the point $A$. **(2 marks)**

**b** Find $\frac{dy}{dx}$ in terms of $t$. **(3 marks)**

**c** Show that an equation of the normal to $C$ at $A$ is $6y - 16x + 23 = 0$. **(4 marks)**

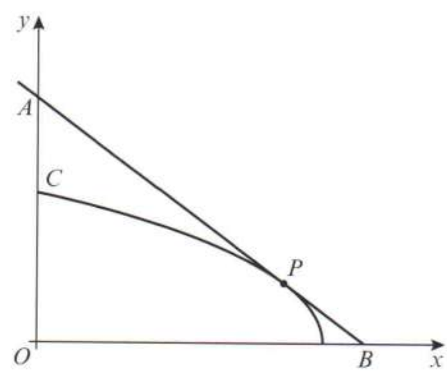
The normal at $A$ cuts $C$ again at the point $B$.

**d** Find the $y$-coordinate of the point $B$. **(6 marks)**

E/P **27** The diagram shows the curve $C$ with parametric equations

$$x = a\sin^2 t, \quad y = a\cos t, \quad 0 \leqslant t \leqslant \tfrac{1}{2}\pi$$

where $a$ is a positive constant. The point $P$ lies on $C$ and has coordinates $\left(\frac{3}{4}a, \frac{1}{2}a\right)$.




**a** Find $\frac{dy}{dx}$, giving your answer in terms of $t$. **(4 marks)**

**b** Find an equation of the tangent to $C$ at $P$. **(4 marks)**

The tangent to $C$ at $P$ cuts the coordinate axes at points $A$ and $B$.

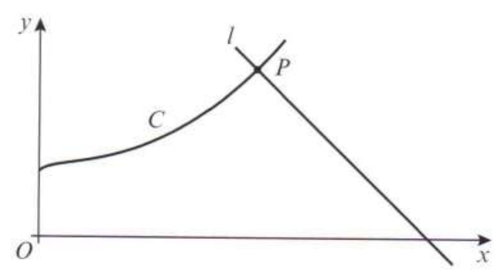
**c** Show that the triangle $AOB$ has area $ka^2$ where $k$ is a constant to be found. **(2 marks)**

E/P **28** This graph shows part of the curve $C$ with parametric equations

$$x = (t + 1)^2, \quad y = \tfrac{1}{2}t^3 + 3, \quad t > -1$$

$P$ is the point on the curve where $t = 2$.
The line $l$ is the normal to $C$ at $P$.

Find the equation of $l$. **(7 marks)**




E/P **29** Find the gradient of the curve with equation $5x^2 + 5y^2 - 6xy = 13$ at the point $(1, 2)$. **(7 marks)**

E/P **30** Given that $e^{2x} + e^{2y} = xy$, find $\frac{dy}{dx}$ in terms of $x$ and $y$. **(7 marks)**

E/P **31** Find the coordinates of the turning points on the curve $y^3 + 3xy^2 - x^3 = 3$. **(7 marks)**

E/P **32** **a** If $(1 + x)(2 + y) = x^2 + y^2$, find $\frac{dy}{dx}$ in terms of $x$ and $y$. **(4 marks)**

**b** Find the gradient of the curve $(1 + x)(2 + y) = x^2 + y^2$ at each of the two points where the curve meets the $y$-axis. **(3 marks)**

**c** Show also that there are two points at which the tangents to this curve are parallel to the $y$-axis. **(4 marks)**

E/P **33** A curve has equation $7x^2 + 48xy - 7y^2 + 75 = 0$. $A$ and $B$ are two distinct points on the curve and at each of these points the gradient of the curve is equal to $\frac{2}{11}$. Use implicit differentiation to show that the straight line passing through $A$ and $B$ has equation $x + 2y = 0$. **(6 marks)**

E/P **34** Given that $y = x^x$, $x > 0$, $y > 0$, by taking logarithms show that

$$\frac{dy}{dx} = x^x(1 + \ln x)$$ **(6 marks)**

E/P **35** **a** Given that $a^x \equiv e^{kx}$, where $a$ and $k$ are constants, $a > 0$ and $x \in \mathbb{R}$, prove that $k = \ln a$. **(2 marks)**

**b** Hence, using the derivative of $e^{kx}$, prove that when $y = 2^x$

$$\frac{dy}{dx} = 2^x \ln 2$$ **(4 marks)**

**c** Hence deduce that the gradient of the curve with equation $y = 2^x$ at the point $(2, 4)$ is $\ln 16$. **(3 marks)**

E/P **36** A population $P$ is growing at the rate of 9% each year and at time $t$ years may be approximated by the formula

$$P = P_0(1.09)^t, \; t \geqslant 0$$

where $P$ is regarded as a continuous function of $t$ and $P_0$ is the population at time $t = 0$.

**a** Find an expression for $t$ in terms of $P$ and $P_0$. **(2 marks)**

**b** Find the time $T$ years when the population has doubled from its value at $t = 0$, giving your answer to 3 significant figures. **(4 marks)**

**c** Find, as a multiple of $P_0$, the rate of change of population $\frac{dP}{dt}$ at time $t = T$. **(4 marks)**

E/P **37** A curve $C$ has equation

$$y = \ln(\sin x), \quad 0 < x < \pi$$

**a** Find the stationary point of the curve $C$. **(6 marks)**

**b** Show that the curve $C$ is concave at all values of $x$ in its given domain. **(3 marks)**

E/P **38** The mass of a radioactive substance $t$ years after first being observed is modelled by the equation

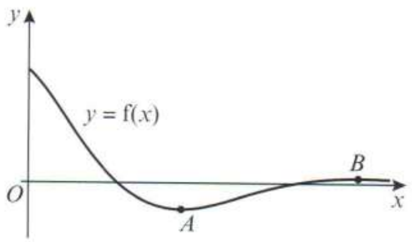
$$m = 40e^{-0.244t}$$

**a** Find the mass of the substance nine months after it was first observed. **(2 marks)**

**b** Find $\frac{dm}{dt}$ **(2 marks)**

**c** With reference to the model, interpret the significance of the sign of the value of $\frac{dm}{dt}$ found in part **b**. **(1 mark)**

E/P **39** The curve C with equation $y = f(x)$ is shown in the diagram, where $f(x) = \frac{\cos 2x}{e^x}$, $0 \leqslant x \leqslant \pi$




The curve has a local minimum at $A$ and a local maximum at $B$.

**a** Show that the $x$-coordinates of $A$ and $B$ satisfy the equation $\tan 2x = -0.5$ and hence find the coordinates of $A$ and $B$. **(6 marks)**

**b** Using your answer to part **a**, find the coordinates of the maximum and minimum turning points on the curve with equation $y = 2 + 4f(x - 4)$. **(3 marks)**

**c** Determine the values of $x$ for which f($x$) is concave. **(5 marks)**

## Challenge

The curve $C$ has parametric equations

$$y = 2\sin 2t, \quad x = 5\cos\left(t + \frac{\pi}{12}\right), \quad 0 \leqslant t \leqslant 2\pi$$

**a** Find $\frac{dy}{dx}$ in terms of $t$.

**b** Find the coordinates of the points on $C$ where $\frac{dy}{dx} = 0$.

**c** Find the coordinates of any points where the curve cuts or intersects the coordinate axes, and determine the gradient of the curve at these points.

**d** Find the coordinates of the points on $C$ where $\frac{dx}{dy} = 0$.

**e** Hence sketch $C$.

**Hint** The points on $C$ where $\frac{dx}{dy} = 0$ correspond to points where a tangent to the curve would be a vertical line.

## Summary of key points

**1** For small angles, measured in radians:

- $\sin x \approx x$
- $\cos x \approx 1 - \frac{1}{2}x^2$

**2**
- If $y = \sin kx$, then $\frac{dy}{dx} = k\cos kx$
- If $y = \cos kx$, then $\frac{dy}{dx} = -k\sin kx$

**3**
- If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$
- If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

**4** If $y = a^{kx}$, where $k$ is a real constant and $a > 0$, then $\frac{dy}{dx} = a^{kx}k\ln a$

**5** The **chain rule** is: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

where $y$ is a function of $u$ and $u$ is another function of $x$.

**6** The chain rule enables you to differentiate a function of a function. In general,

- if $y = (f(x))^n$ then $\frac{dy}{dx} = n(f(x))^{n-1}f'(x)$
- if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x))g'(x)$

**7** $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

**8** The **product rule**:

- If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, where $u$ and $v$ are functions of $x$.
- If $f(x) = g(x)h(x)$ then $f'(x) = g(x)h'(x) + h(x)g'(x)$

**9** The **quotient rule**:

- If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where $u$ and $v$ are functions of $x$.
- If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

**10** • If $y = \tan kx$, then $\frac{dy}{dx} = k\sec^2 kx$

• If $y = \operatorname{cosec} kx$, then $\frac{dy}{dx} = -k\operatorname{cosec} kx \cot kx$

• If $y = \sec kx$, then $\frac{dy}{dx} = k\sec kx \tan kx$

• If $y = \cot kx$, then $\frac{dy}{dx} = -k\operatorname{cosec}^2 kx$

**11** If $x$ and $y$ are given as functions of a parameter, $t$: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

**12** • $\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$

• $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$

• $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$

**13** • The function $f(x)$ is **concave** on a given interval if and only if $f''(x) \leqslant 0$ for every value of $x$ in that interval.

• The function $f(x)$ is **convex** on a given interval if and only if $f''(x) \geqslant 0$ for every value of $x$ in that interval.

**14** A **point of inflection** is a point at which $f''(x)$ changes sign.

**15** You can use the chain rule to connect rates of change in situations involving more than two variables.