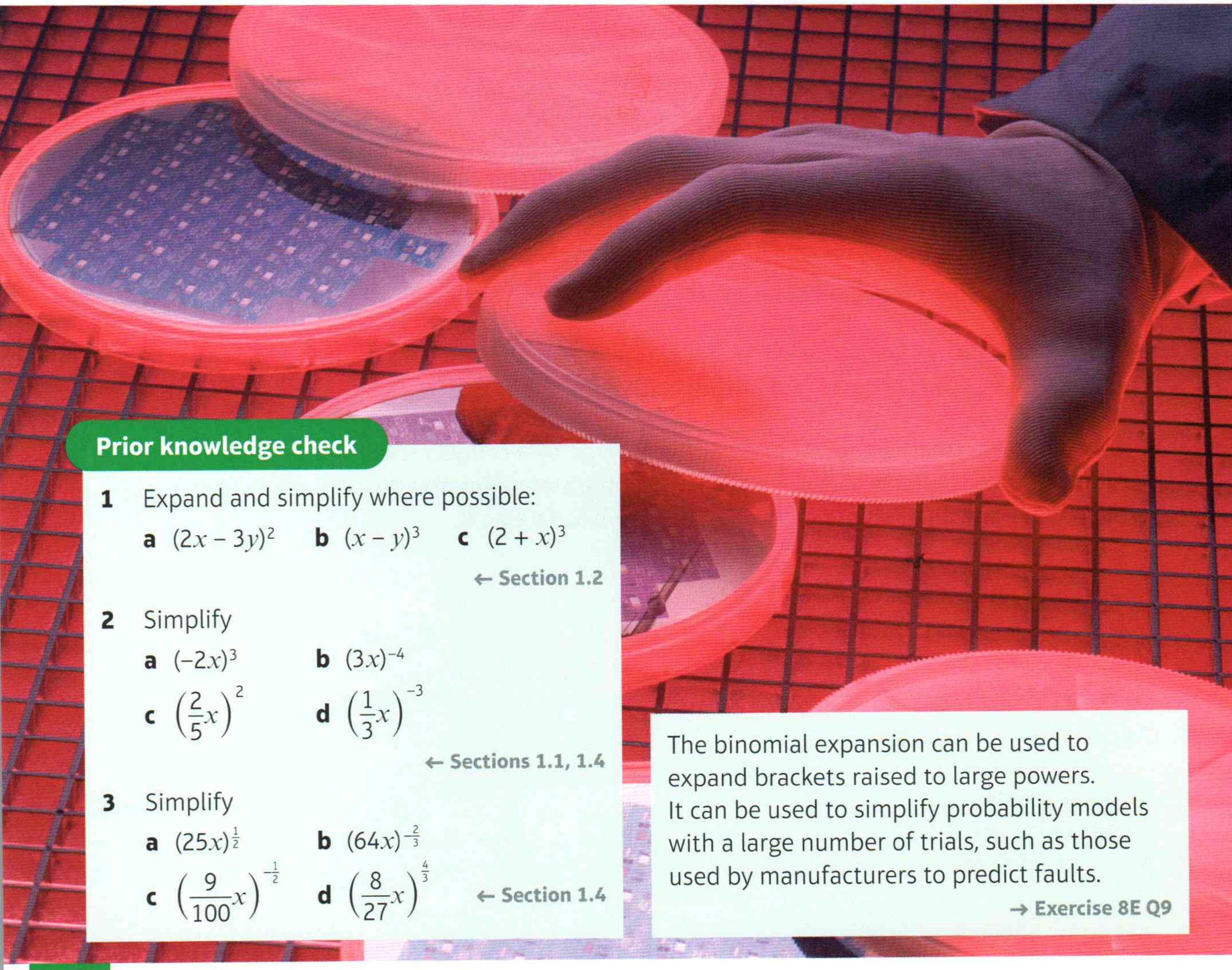
8 The binomial expansion

Objectives

After completing this chapter you should be able to:

- Use Pascal's triangle to identify binomial coefficients and use them to expand simple binomial expressions → pages 159–161
- Use combinations and factorial notation → pages 161–163
- Use the binomial expansion to expand brackets → pages 163–165
- Find individual coefficients in a binomial expansion → pages 165–167
- Make approximations using the binomial expansion → pages 167–169



Prior knowledge check

1 Expand and simplify where possible:

a $(2x - 3y)^2$ **b** $(x - y)^3$ **c** $(2 + x)^3$

← Section 1.2

2 Simplify

a $(-2x)^3$ **b** $(3x)^{-4}$

c $\left(\frac{2}{5}x\right)^2$ **d** $\left(\frac{1}{3}x\right)^{-3}$

← Sections 1.1, 1.4

3 Simplify

a $(25x)^{\frac{1}{2}}$ **b** $(64x)^{-\frac{2}{3}}$

c $\left(\frac{9}{100}x\right)^{-\frac{1}{2}}$ **d** $\left(\frac{8}{27}x\right)^{\frac{4}{3}}$

← Section 1.4

The binomial expansion can be used to expand brackets raised to large powers. It can be used to simplify probability models with a large number of trials, such as those used by manufacturers to predict faults.

→ Exercise 8E Q9

8.1 Pascal's triangle

You can use **Pascal's triangle** to quickly expand expressions such as $(x + 2y)^3$.

Consider the expansions of $(a + b)^n$ for $n = 0, 1, 2, 3$ and 4:

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a + 1b$$
$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

Each coefficient is the sum of the two coefficients immediately above it.

Every term in the expansion of $(a + b)^n$ has total index n:
In the $6a^2b^2$ term the total index is $2 + 2 = 4$.
In the $4ab^3$ term the total index is $1 + 3 = 4$.

The coefficients in the expansions form a pattern that is known as Pascal's triangle.

- **Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers on the next row.**

Here are the first 7 rows of Pascal's triangle:

1
1 1
1 + 2 1
1 3 + 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The third row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^2$.

- **The $(n + 1)$th row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^n$.**

Example 1

Use Pascal's triangle to find the expansions of:

a $(x + 2y)^3$ **b** $(2x - 5)^4$

a $(x + 2y)^3$

The coefficients are 1, 3, 3, 1 so:

$$(x + 2y)^3 = 1x^3 + 3x^2(2y) + 3x(2y)^2 + 1(2y)^3$$
$$= x^3 + 6x^2y + 12xy^2 + 8y^3$$

Index = 3 so look at the 4th row of Pascal's triangle to find the coefficients.

This is the expansion of $(a + b)^3$ with $a = x$ and $b = 2y$. Use brackets to make sure you don't make a mistake. $(2y)^2 = 4y^2$.

b $(2x - 5)^4$

The coefficients are 1, 4, 6, 4, 1 so:

$(2x - 5)^4 = 1(2x)^4 + 4(2x)^3(-5)^1$
$+ 6(2x)^2(-5)^2 + 4(2x)^1(-5)^3$
$+ 1(-5)^4$
$= 16x^4 - 160x^3 + 600x^2$
$- 1000x + 625$

Index = 4 so look at the 5th row of Pascal's triangle.

This is the expansion of $(a + b)^4$ with $a = 2x$ and $b = -5$.

Be careful with the negative numbers.

Example 2

The coefficient of x^2 in the expansion of $(2 - cx)^3$ is 294. Find the possible value(s) of the constant c.

Problem-solving

If there is an unknown in the original expression, you might be able to form an equation involving that unknown.

The coefficients are 1, 3, 3, 1:

The term in x^2 is $3 \times 2(-cx)^2 = 6c^2x^2$

So $6c^2 = 294$

$c^2 = 49$

$c = \pm 7$

Index = 3 so use the 4th row of Pascal's triangle.

From the expansion of $(a + b)^3$ the x^2 term is $3ab^2$ where $a = 2$ and $b = -cx$.

Form and solve an equation in c.

Exercise 8A

1 State the row of Pascal's triangle that would give the coefficients of each expansion:

a $(x + y)^3$ **b** $(3x - 7)^{15}$ **c** $\left(2x + \frac{1}{2}\right)^n$ **d** $(y - 2x)^{n+4}$

2 Write down the expansion of:

a $(x + y)^4$ **b** $(p + q)^5$ **c** $(a - b)^3$ **d** $(x + 4)^3$
e $(2x - 3)^4$ **f** $(a + 2)^5$ **g** $(3x - 4)^4$ **h** $(2x - 3y)^4$

3 Find the coefficient of x^3 in the expansion of:

a $(4 + x)^4$ **b** $(1 - x)^5$ **c** $(3 + 2x)^3$ **d** $(4 + 2x)^5$
e $(2 + x)^6$ **f** $\left(4 - \frac{1}{2}x\right)^4$ **g** $(x + 2)^5$ **h** $(3 - 2x)^4$

(P) 4 Fully expand the expression $(1 + 3x)(1 + 2x)^3$.

Problem-solving

Expand $(1 + 2x)^3$, then multiply each term by 1 and by $3x$.

(P) 5 Expand $(2 + y)^3$. Hence or otherwise, write down the expansion of $(2 + x - x^2)^3$ in ascending powers of x.

(P) 6 The coefficient of x^2 in the expansion of $(2 + ax)^3$ is 54. Find the possible values of the constant a.

P 7 The coefficient of x^2 in the expansion of $(2 - x)(3 + bx)^3$ is 45. Find possible values of the constant b.

P 8 Work out the coefficient of x^2 in the expansion of $(p - 2x)^3$. Give your answer in terms of p.

P 9 After 5 years, the value of an investment of £500 at an interest rate of X% per annum is given by:

$$500\left(1 + \frac{X}{100}\right)^5$$

Find an approximation for this expression in the form $A + BX + CX^2$, where A, B and C are constants to be found. You can ignore higher powers of X.

Challenge

Find the constant term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^3$.

8.2 Factorial notation

You can use combinations and factorial notation to help you expand binomial expressions. For larger indices, it is quicker than using Pascal's triangle.

Notation You say 'n factorial'. By definition, $0! = 1$.

Using **factorial notation** $3 \times 2 \times 1 = 3!$

- **You can use factorial notation and your calculator to find entries in Pascal's triangle quickly.**
  - **The number of ways of choosing r items from a group of n items is written as ^nC_r or $\binom{n}{r}$:**
    $$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$
  - **The rth entry in the nth row of Pascal's triangle is given by $^{n-1}C_{r-1} = \binom{n-1}{r-1}$**

Notation You can say 'n choose r' for ^nC_r. It is sometimes written without superscripts and subscripts as nCr.

Example 3

Calculate:

a $5!$ **b** $^5C_2$ **c** the 6th entry in the 10th row of Pascal's triangle

Online Use the ^nC_r and ! functions on your calculator to answer this question.

a $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

b $^5C_2 = \frac{5!}{2!3!} = \frac{120}{12} = 10$

You can calculate $^5C_2$ by using the ^nC_r function on your calculator.

$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{5!}{2!(5-2)!}$$

c $^9C_5 = 126$

The rth entry in the nth row is $^{n-1}C_{r-1}$.

In the expansion of $(a + b)^9$ this would give the term $126a^4b^5$.

Exercise 8B

1 Work out:

a $4!$ b $9!$ c $\frac{10!}{7!}$ d $\frac{15!}{13!}$

2 Without using a calculator, work out:

a $\binom{4}{2}$ b $\binom{6}{4}$ c ${}^{6}C_{3}$ d $\binom{5}{4}$ e ${}^{10}C_{8}$ f $\binom{9}{5}$

3 Use a calculator to work out:

a $\binom{15}{6}$ b ${}^{10}C_{7}$ c $\binom{20}{10}$ d $\binom{20}{17}$ e ${}^{14}C_{9}$ f ${}^{18}C_{5}$

4 Write each value a to d from Pascal's triangle using ${}^{n}C_{r}$ notation:

1

1 1

1 2 1

1 3 3 1

1 a 6 4 1

1 5 b 10 5 1

1 6 c d 15 6 1

5 Work out the 5th number on the 12th row from Pascal's triangle.

6 The 11th row of Pascal's triangle is shown below.

1 10 45 … …

a Find the next two values in the row.

b Hence find the coefficient of x^3 in the expansion of $(1 + 2x)^{10}$.

7 The 14th row of Pascal's triangle is shown below.

1 13 78 … …

a Find the next two values in the row.

b Hence find the coefficient of x^4 in the expansion of $(1 + 3x)^{13}$.

8 The probability of throwing exactly 10 heads when a fair coin is tossed 20 times is given by $\binom{20}{10}0.5^{20}$. Calculate the probability and describe the likelihood of this occurring.

(P) 9 Show that:

a ${}^{n}C_{1} = n$ b ${}^{n}C_{2} = \frac{n(n-1)}{2}$

(E) 10 Given that $\binom{50}{13} = \frac{50!}{13!a!}$, write down the value of a. **(1 mark)**

(E) 11 Given that $\binom{35}{p} = \frac{35!}{p!18!}$, write down the value of p. **(1 mark)**

Challenge

a Work out $^{10}C_3$ and $^{10}C_7$

b Work out $^{14}C_5$ and $^{14}C_9$

c What do you notice about your answers to parts **a** and **b**?

d Prove that $^nC_r = {}^nC_{n-r}$

8.3 The binomial expansion

A binomial expression has two terms. The binomial expansion allows you to expand powers of binomial expressions. For example, in the expansion of $(a + b)^5 = (a + b)(a + b)(a + b)(a + b)(a + b)$ the term a^2b^3 occurs $\binom{5}{3}$ times. This is because you need to choose b 3 times from the 5 brackets. You can do this in $\binom{5}{3}$ ways so when the expansion is simplified, the term in a^2b^3 is $\binom{5}{3}a^2b^3$.

- **The binomial expansion is:**

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Notation $n \in \mathbb{N}$ means that n must be a member of the **natural numbers**. This is all the **positive integers**.

Example 4

Use the binomial theorem to find the expansion of $(3 - 2x)^5$.

$$(3-2x)^5 = 3^5 + \binom{5}{1}3^4(-2x) + \binom{5}{2}3^3(-2x)^2 + \binom{5}{3}3^2(-2x)^3 + \binom{5}{4}3^1(-2x)^4 + (-2x)^5$$
$$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$$

There will be 6 terms.
Each term has a total index of 5.
Use $(a + b)^n$ where $a = 3$, $b = -2x$ and $n = 5$.

There are $\binom{5}{2}$ ways of choosing two '$-2x$' terms from five brackets.

Online Work out each coefficient quickly using the ^nC_r and power functions on your calculator.

Example 5

Find the first four terms in the binomial expansion of:

a $(1 + 2x)^{10}$ **b** $\left(10 - \frac{1}{2}x\right)^6$

a $(1 + 2x)^{10}$

$= 1^{10} + \binom{10}{1}1^9(2x) + \binom{10}{2}1^8(2x)^2 + \binom{10}{3}1^7(2x)^3 + \ldots$

$= 1 + 20x + 180x^2 + 960x^3 + \ldots$

b $\left(10 - \frac{1}{2}x\right)^6$

$= 10^6 + \binom{6}{1}10^5\left(-\frac{1}{2}x\right) + \binom{6}{2}10^4\left(-\frac{1}{2}x\right)^2 + \binom{6}{3}10^3\left(-\frac{1}{2}x\right)^3 + \ldots$

$= 1000000 - 300000x + 37500x^2 - 2500x^3 + \ldots$

Notation This is sometimes called the expansion in **ascending powers of** x.

Write each coefficient in its simplest form.

Exercise 8C

1 Write down the expansion of the following:

a $(1 + x)^4$ **b** $(3 + x)^4$ **c** $(4 - x)^4$ **d** $(x + 2)^6$ **e** $(1 + 2x)^4$ **f** $\left(1 - \frac{1}{2}x\right)^4$

2 Use the binomial theorem to find the first four terms in the expansion of:

a $(1 + x)^{10}$ **b** $(1 - 2x)^5$ **c** $(1 + 3x)^6$ **d** $(2 - x)^8$ **e** $\left(2 - \frac{1}{2}x\right)^{10}$ **f** $(3 - x)^7$

3 Use the binomial theorem to find the first four terms in the expansion of:

a $(2x + y)^6$ **b** $(2x + 3y)^5$ **c** $(p - q)^8$ **d** $(3x - y)^6$ **e** $(x + 2y)^8$ **f** $(2x - 3y)^9$

4 Use the binomial expansion to find the first four terms, in ascending powers of x, of:

a $(1 + x)^8$ **b** $(1 - 2x)^6$ **c** $\left(1 + \frac{x}{2}\right)^{10}$

d $(1 - 3x)^5$ **e** $(2 + x)^7$ **f** $(3 - 2x)^3$

g $(2 - 3x)^6$ **h** $(4 + x)^4$ **i** $(2 + 5x)^7$

Hint Your answers should be in the form $a + bx + cx^2 + dx^3$ where a, b, c and d are numbers.

E **5** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(2 - x)^6$ and simplify each term. **(4 marks)**

E **6** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(3 - 2x)^5$ giving each term in its simplest form. **(4 marks)**

E/P **7** Find the binomial expansion of $\left(x + \frac{1}{x}\right)^5$ giving each term in its simplest form. **(4 marks)**

Challenge

a Show that $(a + b)^4 - (a - b)^4 = 8ab(a^2 + b^2)$.

b Given that $82\,896 = 17^4 - 5^4$, write 82 896 as a product of its prime factors.

8.4 Solving binomial problems

You can use the general term of the binomial expansion to find individual coefficients in a binomial expansion.

■ **In the expansion of $(a + b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.**

Example 6

a Find the coefficient of x^4 in the binomial expansion of $(2 + 3x)^{10}$.

b Find the coefficient of x^3 in the binomial expansion of $(2 + x)(3 - 2x)^7$.

a x^4 term $= \binom{10}{4}2^6(3x)^4$

Use the general term. The power is 10, so $n = 10$, and you need to find the x^4 term so $r = 4$.

There are $\binom{10}{4}$ ways of choosing 4 '$3x$' terms from 10 brackets.

$= 210 \times 64 \times 81x^4$

$= 1088640x^4$

The coefficient of x^4 in the binomial expansion of $(2 + 3x)^{10}$ is 1088640.

b $(3 - 2x)^7$

First find the first four terms of the binomial expansion of $(3 - 2x)^7$.

$= 3^7 + \binom{7}{1}3^6(-2x) + \binom{7}{2}3^5(-2x)^2 + \binom{7}{3}3^4(-2x)^3 + \ldots$

$= 2187 - 10206x + 20412x^2 - 22680x^3 + \ldots$

$(2 + x)(2187 - 10206x + 20412x^2 - 22680x^3 + \ldots)$

Now expand the brackets $(2 + x)(3 - 2x)^7$.

x^3 term $= 2 \times (-22680x^3) + x \times 20412x^2$

There are two ways of making the x^3 term: (constant term × x^3 term) and (x term × x^2 term).

$= -24948x^3$

The coefficient of x^3 in the binomial expansion of $(2 + x)(3 - 2x)^7$ is −24948.

Example 7

$g(x) = (1 + kx)^{10}$, where k is a constant.

Given that the coefficient of x^3 in the binomial expansion of $g(x)$ is 15, find the value of k.

x^3 term $= \binom{10}{3}1^7(kx)^3 = 15x^3$

$a = 1$, $b = kx$, $n = 10$ and $r = 3$.

$120k^3x^3 = 15x^3$

$k^3x^3 = \frac{15}{120}x^3$

$k^3x^3 = \frac{1}{8}x^3$

$k^3 = \frac{1}{8}$, $k = \sqrt[3]{\frac{1}{8}}$

$k = \frac{1}{2}$

Example 8

a Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1 + qx)^8$, where q is a non-zero constant.

b Given that, in the expansion of $(1 + qx)^8$, the coefficient of x is $-r$ and the coefficient of x^2 is $7r$, find the value of q and the value of r.

a $(1 + qx)^8$

$= 1^8 + \binom{8}{1}1^7(qx)^1 + \binom{8}{2}1^6(qx)^2 + \dots$

$= 1 + 8qx + 28q^2x^2 + \dots$

b $8q = -r$ and $28q^2 = 7r$

$8q = -4q^2$ — Using $28q^2 = 7r$, $r = 4q^2$ and $-r = -4q^2$.

$4q^2 + 8q = 0$

$4q(q + 2) = 0$

$q = -2$, $r = 16$ — q is non-zero so $q = -2$.

Problem-solving

There are two unknowns in this expression. Your expansion will be in terms of q and x.

Exercise 8D

1 Find the coefficient of x^3 in the binomial expansion of:

a $(3 + x)^5$ **b** $(1 + 2x)^5$ **c** $(1 - x)^6$ **d** $(3x + 2)^5$

e $(1 + x)^{10}$ **f** $(3 - 2x)^6$ **g** $(1 + x)^{20}$ **h** $(4 - 3x)^7$

i $\left(1 - \frac{1}{2}x\right)^6$ **j** $\left(3 + \frac{1}{2}x\right)^7$ **k** $\left(2 - \frac{1}{2}x\right)^8$ **l** $\left(5 + \frac{1}{4}x\right)^5$

(P) **2** The coefficient of x^2 in the expansion of $(2 + ax)^6$ is 60. Find two possible values of the constant a.

Problem-solving

$a = 2$, $b = ax$, $n = 6$. Use brackets when you substitute ax.

(P) **3** The coefficient of x^3 in the expansion of $(3 + bx)^5$ is -720. Find the value of the constant b.

(P) **4** The coefficient of x^3 in the expansion of $(2 + x)(3 - ax)^4$ is 30. Find the three possible values of the constant a.

(E/P) **5** When $(1 - 2x)^p$ is expanded, the coefficient of x^2 is 40. Given that $p > 0$, use this information to find:

a the value of the constant p **(6 marks)**

b the coefficient of x **(1 mark)**

c the coefficient of x^3 **(2 marks)**

Problem-solving

You will need to use the definition of $\binom{n}{r}$ to find an expression for $\binom{p}{2}$.

(E/P) **6** **a** Find the first three terms, in ascending powers of x, of the binomial expansion of $(5 + px)^{30}$, where p is a non-zero constant. **(2 marks)**

b Given that in this expansion the coefficient of x^2 is 29 times the coefficient of x work out the value of p. **(4 marks)**

E/P **7** **a** Find the first four terms, in ascending powers of x, of the binomial expansion of $(1 + qx)^{10}$, where q is a non-zero constant. **(2 marks)**

b Given that in the expansion of $(1 + qx)^{10}$ the coefficient of x^3 is 108 times the coefficient of x, work out the value of q. **(4 marks)**

E/P **8** **a** Find the first three terms, in ascending powers of x of the binomial expansion of $(1 + px)^{11}$, where p is a constant. **(2 marks)**

b The first 3 terms in the same expansion are 1, $77x$ and qx^2, where q is a constant. Find the value of p and the value of q. **(4 marks)**

E/P **9** **a** Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1 + px)^{15}$, where p is a non-zero constant. **(2 marks)**

b Given that, in the expansion of $(1 + px)^{15}$, the coefficient of x is $(-q)$ and the coefficient of x^2 is $5q$, find the value of p and the value of q. **(4 marks)**

E/P **10** In the binomial expansion of $(1 + x)^{30}$, the coefficients of x^9 and x^{10} are p and q respectively. Find the value of $\frac{q}{p}$. **(4 marks)**

Challenge

Find the coefficient of x^4 in the binomial expansion of: **a** $(3 - 2x^2)^9$ **b** $\left(\frac{5}{x} + x^2\right)^8$

8.5 Binomial estimation

In engineering and science, it is often useful to find simple **approximations** for complicated functions. If the value of x is less than 1, then x^n gets smaller as n gets larger. If x is small you can sometimes **ignore large powers** of x to approximate a function or estimate a value.

Example 9

a Find the first four terms of the binomial expansion, in ascending powers of x, of $\left(1 - \frac{x}{4}\right)^{10}$.

b Use your expansion to estimate the value of 0.975^{10}, giving your answer to 4 decimal places.

a $\left(1 - \frac{x}{4}\right)^{10}$

$= 1^{10} + \binom{10}{1}1^9\left(-\frac{x}{4}\right) + \binom{10}{2}1^8\left(-\frac{x}{4}\right)^2 + \binom{10}{3}1^7\left(-\frac{x}{4}\right)^3 + \ldots$

$= 1 - 2.5x + 2.8125x^2 - 1.875x^3 + \ldots$

b We want $\left(1-\frac{x}{4}\right) = 0.975$

$\frac{x}{4} = 0.025$

$x = 0.1$

Substitute $x = 0.1$ into the expansion for $\left(1-\frac{x}{4}\right)^{10}$ from part **a**:

$0.975^{10} \approx 1 - 0.25 + 0.028125 - 0.001875$

$= 0.77625$

$0.975^{10} \approx 0.7763$ to 4 d.p.

Online Use technology to find the values of x for which the first four terms of this expansion give a good approximation to the value of the function.

Calculate the value of x.

Substitute $x = 0.1$ into your expansion.

Using a calculator, $0.975^{10} = 0.776\,329\,62$. So approximation is correct to 4 decimal places.

Exercise 8E

1 **a** Find the first four terms of the binomial expansion, in ascending powers of x, of $\left(1-\frac{x}{10}\right)^6$.

b By substituting an appropriate value for x, find an approximate value for 0.99^6.

2 **a** Write down the first four terms of the binomial expansion of $\left(2+\frac{x}{5}\right)^{10}$.

b By substituting an appropriate value for x, find an approximate value for 2.1^{10}.

(P) **3** If x is so small that terms of x^3 and higher can be ignored, show that:

$$(2 + x)(1 - 3x)^5 \approx 2 - 29x + 165x^2$$

Hint Start by using the binomial expansion to expand $(1 - 3x)^5$. You can ignore terms of x^3 and higher so you only need to expand up to and including the x^2 term.

(P) **4** If x is so small that terms of x^3 and higher can be ignored, and

$$(2 - x)(3 + x)^4 \approx a + bx + cx^2$$

find the values of the constants a, b and c.

Problem-solving Find the first 3 terms in the expansion of $(2 - x)(3 + x)^4$, compare with $a + bx + cx^2$ and write down the values of a, b and c.

5 **a** Write down the first four terms in the expansion of $(1 + 2x)^8$.

b By substituting an appropriate value of x (which should be stated), find an approximate value of 1.02^8.

6 $f(x) = (1 - 5x)^{30}$

a Find the first four terms, in ascending powers of x, in the binomial expansion of $f(x)$.

b Use your answer to part **a** to estimate the value of $(0.995)^{30}$, giving your answer to 6 decimal places.

c Use your calculator to evaluate 0.995^{30} and calculate the percentage error in your answer to part **b**.

(E/P) **7** **a** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $\left(3-\frac{x}{5}\right)^{10}$, giving each term in its simplest form. **(4 marks)**

b Explain how you would use your expansion to give an estimate for the value of 2.98^{10}. **(1 mark)**

E 8 a Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1 - 3x)^5$. Give each term in its simplest form. **(4 marks)**

b If x is small, so that x^2 and higher powers can be ignored, show that $(1 + x)(1 - 3x)^5 \approx 1 - 14x$. **(2 marks)**

P 9 A microchip company models the probability of having no faulty chips on a single production run as:

$$\text{P(no fault)} = (1 - p)^n, \; p < 0.001$$

where p is the probability of a single chip being faulty, and n being the total number of chips produced.

a State why the model is restricted to small values of p. **(1 mark)**

b Given that $n = 200$, find an approximate expression for P(no fault) in the form $a + bp + cp^2$. **(2 marks)**

c The company wants to achieve a 92% likelihood of having no faulty chips on a production run of 200 chips. Use your answer to part **b** to suggest a maximum value of p for this to be the case. **(4 marks)**

Mixed exercise 8

P 1 The 16th row of Pascal's triangle is shown below.

1 15 105 … …

a Find the next two values in the row.

b Hence find the coefficient of x^3 in the expansion of $(1 + 2x)^{15}$.

E 2 Given that $\binom{45}{17} = \frac{45!}{17!a!}$, write down the value of a. **(1 mark)**

3 20 people play a game at a school fete.
The probability that exactly n people win a prize is modelled as $\binom{20}{n}p^n(1 - p)^{20-n}$, where p is the probability of any one person winning.
Calculate the probability of:

a 5 people winning when $p = \frac{1}{2}$

b nobody winning when $p = 0.7$

c 13 people winning when $p = 0.6$

Give your answers to 3 significant figures.

E 4 When $\left(1 - \frac{3}{2}x\right)^p$ is expanded in ascending powers of x, the coefficient of x is -24.

a Find the value of p. **(2 marks)**

b Find the coefficient of x^2 in the expansion. **(3 marks)**

c Find the coefficient of x^3 in the expansion. **(1 mark)**

E 5 Given that:

$$(2 - x)^{13} \equiv A + Bx + Cx^2 + \ldots$$

find the values of the integers A, B and C. **(4 marks)**

E **6** **a** Expand $(1 - 2x)^{10}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient in the expansion. **(4 marks)**

b Use your expansion to find an approximation of 0.98^{10}, stating clearly the substitution which you have used for x. **(3 marks)**

E **7** **a** Use the binomial series to expand $(2 - 3x)^{10}$ in ascending powers of x up to and including the term in x^3, giving each coefficient as an integer. **(4 marks)**

b Use your series expansion, with a suitable value for x, to obtain an estimate for 1.97^{10}, giving your answer to 2 decimal places. **(3 marks)**

E/P **8** **a** Expand $(3 + 2x)^4$ in ascending powers of x, giving each coefficient as an integer. **(4 marks)**

b Hence, or otherwise, write down the expansion of $(3 - 2x)^4$ in ascending powers of x. **(2 marks)**

c Hence by choosing a suitable value for x show that $(3 + 2\sqrt{2})^4 + (3 - 2\sqrt{2})^4$ is an integer and state its value. **(2 marks)**

E/P **9** The coefficient of x^2 in the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, where n is a positive integer, is 7.

a Find the value of n. **(2 marks)**

b Using the value of n found in part **a**, find the coefficient of x^4. **(4 marks)**

E **10** **a** Use the binomial theorem to expand $(3 + 10x)^4$ giving each coefficient as an integer. **(4 marks)**

b Use your expansion, with an appropriate value for x, to find the exact value of 1003^4. State the value of x which you have used. **(3 marks)**

E **11** **a** Expand $(1 + 2x)^{12}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient. **(4 marks)**

b By substituting a suitable value for x, which must be stated, into your answer to part **a**, calculate an approximate value of 1.02^{12}. **(3 marks)**

c Use your calculator, writing down all the digits in your display, to find a more exact value of 1.02^{12}. **(1 mark)**

d Calculate, to 3 significant figures, the percentage error of the approximation found in part **b**. **(1 mark)**

E/P **12** Expand $\left(x - \frac{1}{x}\right)^5$, simplifying the coefficients. **(4 marks)**

E/P **13** In the binomial expansion of $(2k + x)^n$, where k is a constant and n is a positive integer, the coefficient of x^2 is equal to the coefficient of x^3.

a Prove that $n = 6k + 2$. **(3 marks)**

b Given also that $k = \frac{2}{3}$, expand $(2k + x)^n$ in ascending powers of x up to and including the term in x^3, giving each coefficient as an exact fraction in its simplest form. **(4 marks)**

(P) **14** **a** Expand $(2 + x)^6$ as a binomial series in ascending powers of x, giving each coefficient as an integer. **(4 marks)**

b By making suitable substitutions for x in your answer to part **a**, show that $(2 + \sqrt{3})^6 - (2 - \sqrt{3})^6$ can be simplified to the form $k\sqrt{3}$, stating the value of the integer k. **(3 marks)**

(P) **15** The coefficient of x^2 in the binomial expansion of $(2 + kx)^8$, where k is a positive constant, is 2800.

a Use algebra to calculate the value of k. **(2 marks)**

b Use your value of k to find the coefficient of x^3 in the expansion. **(4 marks)**

(P) **16** **a** Given that

$$(2 + x)^5 + (2 - x)^5 \equiv A + Bx^2 + Cx^4,$$

find the value of the constants A, B and C. **(4 marks)**

b Using the substitution $y = x^2$ and your answers to part **a**, solve

$$(2 + x)^5 + (2 - x)^5 = 349.$$

(3 marks)

(P) **17** In the binomial expansion of $(2 + px)^5$, where p is a constant, the coefficient of x^3 is 135. Calculate:

a the value of p, **(4 marks)**

b the value of the coefficient of x^4 in the expansion. **(2 marks)**

(P) **18** Find the constant term in the expansion of $\left(\frac{x^2}{2} - \frac{2}{x}\right)^9$.

(P) **19** **a** Find the first three terms, in ascending powers of x of the binomial expansion of $(2 + px)^7$, where p is a constant. **(2 marks)**

The first 3 terms are 128, $2240x$ and qx^2, where q is a constant.

b Find the value of p and the value of q. **(4 marks)**

(P) **20** **a** Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1 - px)^{12}$, where p is a non-zero constant. **(2 marks)**

b Given that, in the expansion of $(1 - px)^{12}$, the coefficient of x is q and the coefficient of x^2 is $6q$, find the value of p and the value of q. **(4 marks)**

(P) **21** **a** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $\left(2 + \frac{x}{2}\right)^7$, giving each term in its simplest form. **(4 marks)**

b Explain how you would use your expansion to give an estimate for the value of 2.05^7. **(1 mark)**

(P) **22** $g(x) = (4 + kx)^5$, where k is a constant.

Given that the coefficient of x^3 in the binomial expansion of $g(x)$ is 20, find the value of k. **(3 marks)**

Challenge

1 $f(x) = (2 - px)(3 + x)^5$ where p is a constant.
There is no x^2 term in the expansion of $f(x)$.
Show that $p = \frac{4}{3}$

2 Find the coefficient of x^2 in the expansion of $(1 + 2x)^8(2 - 5x)^7$.

Summary of key points

1 Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers on the next row.

2 The $(n + 1)$th row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^n$.

3 $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$.

4 You can use factorial notation and your calculator to find entries in Pascal's triangle quickly.

- The number of ways of choosing r items from a group of n items is written as ${}^nC_r$ or $\binom{n}{r}$: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
- The rth entry in the nth row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$.

5 The binomial expansion is:

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \ (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

6 In the expansion of $(a + b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.

7 If x is small, the first few terms in the binomial expansion can be used to find an approximate value for a complicated expression.