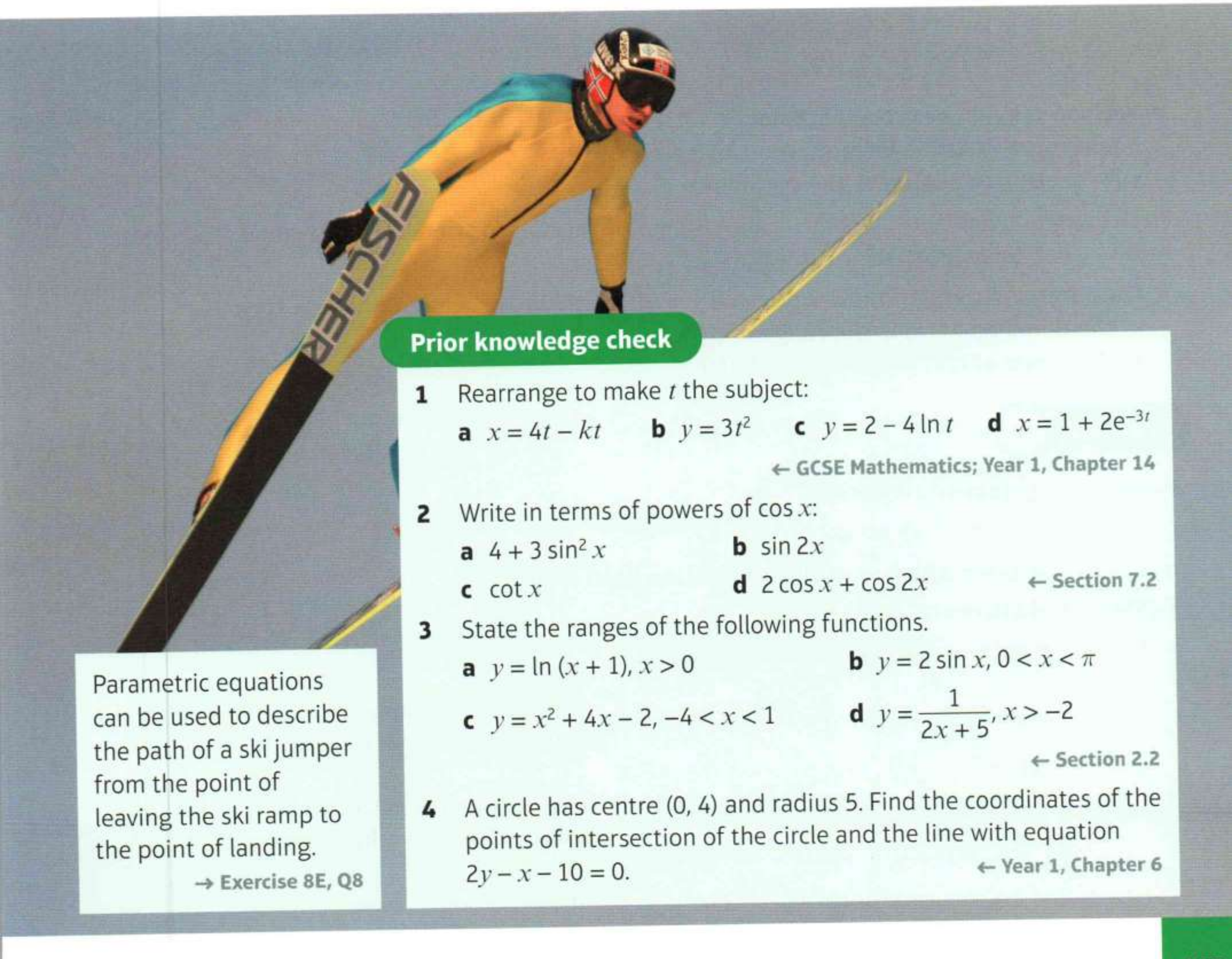
Parametric equations

8

Objectives

After completing this chapter you should be able to:

- Convert parametric equations into Cartesian form by substitution → pages 198–202
- Convert parametric equations into Cartesian form using trigonometric identities → pages 202–206
- Understand and use parametric equations of curves and sketch parametric curves → pages 206–208
- Solve coordinate geometry problems involving parametric equations → pages 209–213
- Use parametric equations in modelling in a variety of contexts → pages 213–220



Parametric equations can be used to describe the path of a ski jumper from the point of leaving the ski ramp to the point of landing.

→ Exercise 8E, Q8

Prior knowledge check

1 Rearrange to make t the subject:

a $x = 4t - kt$ **b** $y = 3t^2$ **c** $y = 2 - 4\ln t$ **d** $x = 1 + 2e^{-3t}$

← GCSE Mathematics; Year 1, Chapter 14

2 Write in terms of powers of $\cos x$:

a $4 + 3\sin^2 x$ **b** $\sin 2x$

c $\cot x$ **d** $2\cos x + \cos 2x$

← Section 7.2

3 State the ranges of the following functions.

a $y = \ln(x + 1), x > 0$ **b** $y = 2\sin x, 0 < x < \pi$

c $y = x^2 + 4x - 2, -4 < x < 1$ **d** $y = \frac{1}{2x + 5}, x > -2$

← Section 2.2

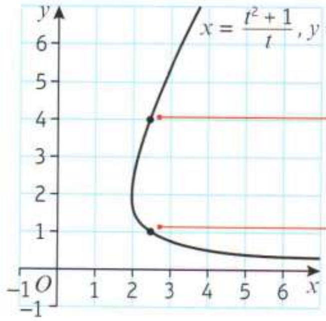
4 A circle has centre (0, 4) and radius 5. Find the coordinates of the points of intersection of the circle and the line with equation $2y - x - 10 = 0$.

← Year 1, Chapter 6

8.1 Parametric equations

You can write the x- and y-coordinates of each point on a curve as functions of a third variable. This variable is called a parameter and is often represented by the letter t.

- **A curve can be defined using parametric equations $x = p(t)$ and $y = q(t)$. Each value of the parameter, t, defines a point on the curve with coordinates $(p(t), q(t))$.**



These are the parametric equations of the curve. The domain of the parameter tells you the values of t you would need to substitute to find the coordinates of the points on the curve.

When $t = 2$, $x = \frac{2^2+1}{2} = 2.5$ and $y = 2 \times 2 = 4$.
This corresponds to the point (2.5, 4).

When $t = 0.5$, $x = \frac{0.5^2+1}{0.5} = 2.5$ and $y = 2 \times 0.5 = 1$.
This corresponds to the point (2.5, 1).

Watch out The value of the parameter t is generally not equal to either the x- or the y-coordinate, and more than one point on the curve can have the same x-coordinate.

- **You can convert between parametric equations and Cartesian equations by using substitution to eliminate the parameter.**

Notation A Cartesian equation in two dimensions involves the variables x and y only.

You can use the domain and range of the parametric functions to find the domain and range of the resulting Cartesian function.

- **For parametric equations $x = p(t)$ and $y = q(t)$ with Cartesian equation $y = f(x)$:**
  - **the domain of $f(x)$ is the range of $p(t)$**
  - **the range of $f(x)$ is the range of $q(t)$**

Example 1

A curve has parametric equations

$$x = 2t, \quad y = t^2, \quad -3 < t < 3$$

a Find a Cartesian equation of the curve in the form $y = f(x)$.

b State the domain and range of $f(x)$.

c Sketch the curve within the given domain for t.

a $x = 2t$ so $t = \frac{x}{2}$ (1)

$y = t^2$ (2)

A Cartesian equation only involves the variables x and y, so you need to eliminate t.

Substitute (1) into (2):

Rearrange one equation into the form $t = \ldots$ then substitute into the other equation.

$y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$

This is a quadratic curve.

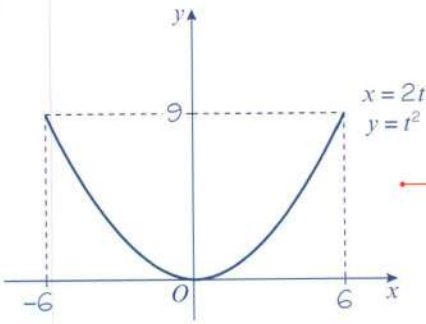
b $x = 2t, -3 < t < 3$

So the domain of $f(x)$ is $-6 < x < 6$.

$y = t^2, -3 < t < 3$

So the range of $f(x)$ is $0 \leqslant y < 9$.

c



The domain of f is the range of the parametric function for x. The range of $x = 2t$ over the domain $-3 < t < 3$ is $-6 < x < 6$. ← Section 2.1

The range of f is the range of the parametric function for y. Choose your inequalities carefully. $y = t^2$ can equal 0 in the interval $-3 < t < 3$, so use $\leqslant$, but it cannot equal 9, so use $<$.

The curve is a graph of $y = \frac{1}{4}x^2$. Use your answers to part **b** to help with your sketch.

Watch out Pay careful attention to the **domain** when sketching parametric curves. The curve is only defined for $-3 < t < 3$, or for $-6 < x < 6$. You should not draw any points on the curve outside that range.

Example 2

A curve has parametric equations

$$x = \ln(t + 3), \quad y = \frac{1}{t + 5}, \quad t > -2$$

a Find a Cartesian equation of the curve of the form $y = f(x)$, $x > k$ where k is a constant to be found.

b Write down the range of $f(x)$.

Online Sketch this parametric curve using technology.

a $x = \ln(t + 3)$

$e^x = t + 3$

So $e^x - 3 = t$

Substitute $t = e^x - 3$ into

$$y = \frac{1}{t + 5} = \frac{1}{e^x - 3 + 5} = \frac{1}{e^x + 2}$$

When $t = -2$: $x = \ln(t + 3) = \ln 1 = 0$

As t increases $\ln(t + 3)$ increases, so the range of the parametric function for x is $x > 0$.

The Cartesian equation is

$$y = \frac{1}{e^x + 2}, x > 0$$

b When $t = -2$: $y = \frac{1}{t + 5} = \frac{1}{3}$

As t increases y decreases, but is always positive, so the range of the parametric function for y is $0 < y < \frac{1}{3}$

The range of $f(x)$ is $0 < y < \frac{1}{3}$

e^x is the inverse function of $\ln x$.

Rearrange the equation for x into the form $t = \ldots$ then substitute into the equation for y.

To find the domain for $f(x)$, consider the range of values x can take for values of $t > -2$.

You need to consider what value x takes when $t = -2$ **and** what happens when t increases.

The range of f is the range of values y can take within the given range of the parameter.

You could also find the range of $f(x)$ by considering the domain of $f(x)$. $f(0) = \frac{1}{3}$ and $f(x)$ decreases as x increases, so $0 < y < \frac{1}{3}$. ← Section 2.1

Exercise 8A

1 Find a Cartesian equation for each of these parametric equations, giving your answer in the form $y = \text{f}(x)$. In each case find the domain and range of $\text{f}(x)$.

a $x = t - 2, \quad y = t^2 + 1, \quad -4 \leqslant t \leqslant 4$

b $x = 5 - t, \quad y = t^2 - 1, \quad t \in \mathbb{R}$

c $x = \dfrac{1}{t}, \quad y = 3 - t, \quad t \neq 0$

Notation If the domain of t is given as $t \neq 0$, this implies that t can take any value in $\mathbb{R}$ other than 0.

d $x = 2t + 1, \quad y = \dfrac{1}{t}, \quad t > 0$

e $x = \dfrac{1}{t - 2}, \quad y = t^2, \quad t > 2$

f $x = \dfrac{1}{t + 1}, \quad y = \dfrac{1}{t - 2}, \quad t > 2$

2 For each of these parametric curves:

i find a Cartesian equation for the curve in the form $y = \text{f}(x)$ giving the domain on which the curve is defined

ii find the range of $\text{f}(x)$.

a $x = 2\ln(5 - t), \quad y = t^2 - 5, \quad t < 4$

b $x = \ln(t + 3), \quad y = \dfrac{1}{t + 5}, \quad t > -2$

c $x = \text{e}^t, \quad y = \text{e}^{3t}, \quad t \in \mathbb{R}$

(P) 3 A curve C is defined by the parametric equations $x = \sqrt{t}, \quad y = t(9 - t), \quad 0 \leqslant t \leqslant 5$.

a Find a Cartesian equation of the curve in the form $y = \text{f}(x)$, and determine the domain and range of $\text{f}(x)$.

b Sketch C showing clearly the coordinates of any turning points, endpoints and intersections with the coordinate axes.

Problem-solving

$y = t(9 - t)$ is a quadratic with a negative t^2 term and roots at $t = 0$ and $t = 9$. It will take its **maximum** value when $t = 4.5$.

4 For each of the following parametric curves:

i find a Cartesian equation for the curve in the form $y = \text{f}(x)$

ii find the domain and range of $\text{f}(x)$

iii sketch the curve within the given domain of t.

a $x = 2t^2 - 3, \quad y = 9 - t^2, \quad t > 0$

b $x = 3t - 1, \quad y = (t - 1)(t + 2), \quad -4 < t < 4$

c $x = t + 1, \quad y = \dfrac{1}{t - 1}, \quad t \in \mathbb{R}, \quad t \neq 1$

d $x = \sqrt{t} - 1, \quad y = 3\sqrt{t}, \quad t > 0$

e $x = \ln(4 - t), \quad y = t - 2, \quad t < 3$

(P) **5** The curves C_1 and C_2 are defined by the following parametric equations.

$$C_1: \quad x = 1 + 2t, \quad y = 2 + 3t \quad 2 < t < 5 \qquad\qquad C_2: \quad x = \frac{1}{2t-3}, \quad y = \frac{t}{2t-3} \quad 2 < t < 3$$

a Show that both curves are segments of the same straight line.

b Find the length of each line segment.

Notation Straight lines and line segments can be referred to as 'curves' in coordinate geometry.

(E/P) **6** A curve C has parametric equations

$$x = \frac{3}{t} + 2, \quad y = 2t - 3 - t^2, \quad t \in \mathbb{R}, \quad t \neq 0$$

a Determine the ranges of x and y in the given domain of t. **(3 marks)**

b Show that the Cartesian equation of C can be written in the form

$$y = \frac{A(x^2 + bx + c)}{(x-2)^2}$$

where A, b and c are integers to be determined. **(3 marks)**

7 A curve has parametric equations

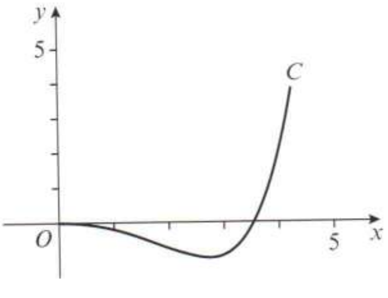
$$x = \ln(t+3), \quad y = \frac{1}{t+5}, \quad t > -2$$

a Show that a Cartesian equation of this curve is $y = \mathrm{f}(x)$, $x > k$ where k is a constant to be found.

b Write down the range of $\mathrm{f}(x)$.

(E/P) **8** A diagram shows a curve C with parametric equations

$$x = 3\sqrt{t}, \quad y = t^3 - 2t, \quad 0 \leqslant t \leqslant 2$$



a Find a Cartesian equation of the curve in the form $y = \mathrm{f}(x)$, and state the domain of $\mathrm{f}(x)$. **(3 marks)**

b Show that $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ when $t = \sqrt{\frac{2}{3}}$ **(3 marks)**

c Hence determine the range of $\mathrm{f}(x)$. **(2 marks)**

(E/P) **9** A curve C has parametric equations

$$x = t^3 - t, \quad y = 4 - t^2, \quad t \in \mathbb{R}$$

a Show that the Cartesian equation of C can be written in the form

$$x^2 = (a - y)(b - y)^2$$

where a and b are integers to be determined. **(3 marks)**

b Write down the maximum value of the y-coordinate for any point on this curve. **(2 marks)**

Challenge

A curve C has parametric equations

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}, \quad t \in \mathbb{R}$$

a Show that a Cartesian equation for this curve is $x^2 + y^2 = 1$.

b Hence describe C.

8.2 Using trigonometric identities

You can use trigonometric identities to convert trigonometric parametric equations into Cartesian form. In this chapter you will always consider angles measured in radians.

Example 3

A curve has parametric equations $x = \sin t + 2, \quad y = \cos t - 3, \quad t \in \mathbb{R}$

a Show that a Cartesian equation of the curve is $(x - 2)^2 + (y + 3)^2 = 1$.

b Hence sketch the curve.

a $x = \sin t + 2$

So $\sin t = x - 2$ (1)

$y = \cos t - 3$

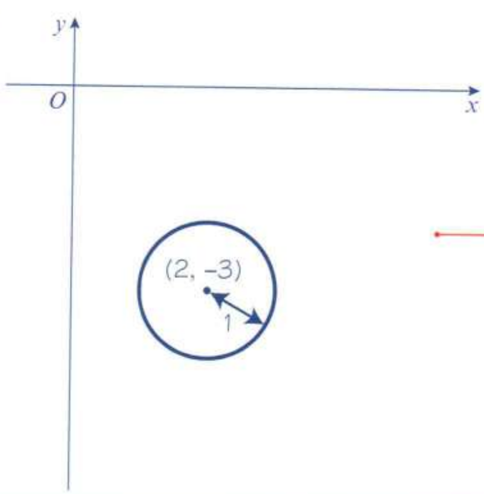
$\cos t = y + 3$ (2)

Substitute (1) and (2) into

$\sin^2 t + \cos^2 t \equiv 1$

$(x - 2)^2 + (y + 3)^2 = 1$

b



Problem-solving

If you can write expressions for $\sin t$ and $\cos t$ in terms of x and y then you can use the identity $\sin^2 t + \cos^2 t \equiv 1$ to eliminate the parameter, t.

← Year 1, Chapter 10

Your equations in (1) and (2) are in terms of $\sin t$ and $\cos t$ so you need to square them when you substitute. Make sure you square the whole expression.

$(x - a)^2 + (y - b)^2 = r^2$ is the equation of a circle with centre (a, b) and radius r.

So the curve is a circle with centre $(2, -3)$ and radius 1.

← Year 1, Chapter 6

Example 4

A curve is defined by the parametric equations

$$x = \sin t, \quad y = \sin 2t, \quad -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$$

a Find a Cartesian equation of the curve in the form

$y = f(x), \quad -k \leqslant x \leqslant k$

stating the value of the constant k.

b Write down the range of $f(x)$.

Online You can graph the parametric equations using technology.

a $y = \sin 2t$

$= 2\sin t \cos t$

$= 2x\cos t \qquad (1)$

Use the identity $\sin 2t \equiv 2\sin t\cos t$, then substitute $x = \sin t$. ← Section 7.2

$\sin^2 t + \cos^2 t \equiv 1$

Use the identity $\sin^2 t + \cos^2 t \equiv 1$ together with $x = \sin t$ to find an expression for $\cos t$ in terms of x.

$\cos^2 t \equiv 1 - \sin^2 t$

$= 1 - x^2$

$\cos t = \sqrt{1 - x^2} \qquad (2)$

Substitute (2) into (1): $y = 2x\sqrt{1 - x^2}$

Watch out Be careful when taking square roots. In this case you don't need to consider the negative square root because $\cos t$ is positive for all values in the domain of the parameter.

When $t = -\frac{\pi}{2}$, $x = \sin\left(-\frac{\pi}{2}\right) = -1$

When $t = \frac{\pi}{2}$, $x = \sin\left(\frac{\pi}{2}\right) = 1$

The Cartesian equation is $y = 2x\sqrt{1 - x^2}$, $-1 \leqslant x \leqslant 1$ so $k = 1$.

To find the domain of $f(x)$, consider the range of $x = \sin t$ for the values of the parameter given.

b $-1 \leqslant y \leqslant 1$

Within $-\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$, $y = \sin 2t$ takes a minimum value of -1 and a maximum value of 1.

Example 5

A curve C has parametric equations

$$x = \cot t + 2 \quad y = \operatorname{cosec}^2 t - 2, \quad 0 < t < \pi$$

a Find the equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined.

b Hence, sketch the curve.

a $x = \cot t + 2$

$\cot t = x - 2$ (1)

$y = \text{cosec}^2 t - 2$

$\text{cosec}^2 t = y + 2$ (2)

Substitute (1) and (2) into

$1 + \cot^2 t \equiv \text{cosec}^2 t$

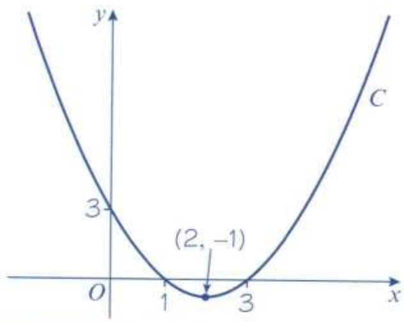
$1 + (x - 2)^2 = y + 2$

$1 + x^2 - 4x + 4 = y + 2$

$y = x^2 - 4x + 3$

The range of $x = \cot t + 2$ over the domain $0 < t < \pi$ is all of the real numbers, so the domain of f(x) is $x \in \mathbb{R}$.

b $y = x^2 - 4x + 3 = (x - 3)(x - 1)$ is a quadratic with roots at $x = 3$ and $x = 1$ and y-intercept 3. The minimum point is $(2, -1)$.



Problem-solving

The parametric equations involve $\cot t$ and $\text{cosec}^2 t$ so you can use the identity $1 + \cot^2 t \equiv \text{cosec}^2 t$. ← Section 6.4

Rearrange to find expressions for $\cot t$ and $\text{cosec}^2 t$ in terms of x and y.

Expand and rearrange to make y the subject. You could also write the equation as: $y = (x - 2)^2 - 1$

This is the completed square form which is useful when sketching the curve.

Consider the range of values taken by x over the domain of the parameter. The curve is defined on all of the real numbers, so it is the whole quadratic curve

Online Explore this curve graphically using technology.

Exercise 8B

1 Find the Cartesian equation of the curves given by the following parametric equations:

a $x = 2\sin t - 1,\quad y = 5\cos t + 4,\quad 0 < t < 2\pi$

b $x = \cos t,\quad y = \sin 2t,\quad 0 < t < 2\pi$

c $x = \cos t,\quad y = 2\cos 2t,\quad 0 < t < 2\pi$

d $x = \sin t,\quad y = \tan 2t,\quad 0 < t < \frac{\pi}{2}$

e $x = \cos t + 2,\quad y = 4\sec t,\quad 0 < t < \frac{\pi}{2}$

f $x = 3\cot t,\quad y = \text{cosec}\, t,\quad 0 < t < \pi$

2 A circle has parametric equations $x = \sin t - 5,\quad y = \cos t + 2$

a Find a Cartesian equation of the circle.

b Write down the radius and the coordinates of the centre of the circle.

c Write down a suitable domain of t which defines one full revolution around the circle.

Problem-solving

Think about how x and y change as t varies.

3 A circle has parametric equations $x = 4\sin t + 3,\quad y = 4\cos t - 1$. Find the radius and the coordinates of the centre of the circle.

4 A curve is given by the parametric equation $x = \cos t - 2$, $y = \sin t + 3$, $-\pi < t < \pi$.
Sketch the curve.

(P) 5 Find the Cartesian equation of the curves given by the following parametric equations.

a $x = \sin t$, $y = \sin\left(t + \frac{\pi}{4}\right)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

b $x = 3\cos t$, $y = 2\cos\left(t + \frac{\pi}{6}\right)$, $0 < t < \frac{\pi}{3}$

c $x = \sin t$, $y = 3\sin(t + \pi)$, $0 < t < 2\pi$

Hint Use the addition formulae and exact values.

(E) 6 The curve C has parametric equations

$$x = 8\cos t, \quad y = \frac{1}{4}\sec^2 t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

a Find a Cartesian equation of C. **(4 marks)**

b Sketch the curve C on the appropriate domain. **(3 marks)**

(E) 7 A curve has parametric equations

$$x = 3\cot^2 2t, \quad y = 3\sin^2 2t, \quad 0 < t \leqslant \frac{\pi}{4}$$

Find a Cartesian equation of the curve in the form $y = f(x)$. State the domain on which $f(x)$ is defined. **(6 marks)**

(E/P) 8 A curve C has parametric equations

$$x = \frac{1}{3}\sin t, \quad y = \sin 3t, \quad 0 < t < \frac{\pi}{2}$$

a Show that the Cartesian equation of the curve is given by

$$y = ax(1 - bx^2)$$

where a and b are integers to be found. **(5 marks)**

b State the domain and range of $y = f(x)$ in the given domain of t. **(2 marks)**

(E/P) 9 Show that the curve with parametric equations

$$x = 2\cos t, \quad y = \sin\left(t - \frac{\pi}{6}\right), \quad 0 < t < \pi$$

can be written in the form

$$y = \frac{1}{4}\left(\sqrt{12 - 3x^2} - x\right), -2 < x < 2$$

(6 marks)

(E/P) 10 A curve has parametric equations

$$x = \tan^2 t + 5, \quad y = 5\sin t, \quad 0 < t < \frac{\pi}{2}$$

a Find the Cartesian equation of the curve in the form $y^2 = f(x)$. **(4 marks)**

b Determine the possible values of x and y in the given domain of t. **(2 marks)**

E/P **11** A curve C has parametric equations

$$x = \tan t, \quad y = 3\sin(t - \pi), \quad 0 < t < \frac{\pi}{2}$$

Find a Cartesian equation of C. **(4 marks)**

Challenge

The curve C is given by the parametric equations:

$$x = \frac{1}{2}\cos 2t, \quad y = \sin\left(t + \frac{\pi}{6}\right), \quad 0 < t < 2\pi$$

Show that a Cartesian equation for C is $(4y^2 - 2 + 2x)^2 + 12x^2 - 3 = 0$.

8.3 Curve sketching

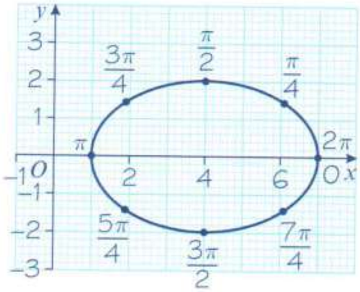
Most parametric curves do not result in curves you will recognise and can sketch easily. You can plot any parametric curve by substituting values of the parameter into each equation.

Example 6

Draw the curve given by parametric equations

$$x = 3\cos t + 4, \quad y = 2\sin t, \quad 0 \leqslant t \leqslant 2\pi$$

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$x = 3\cos t + 4$	7	6.12	4	1.88	1	1.88	4	6.12	7
$y = 2\sin t$	0	1.41	2	1.41	0	−1.41	−2	−1.41	0



This parametric curve has Cartesian equation $\left(\frac{x-4}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

This isn't a form of curve that you need to be able to recognise, but you can plot the curve using a table of values.

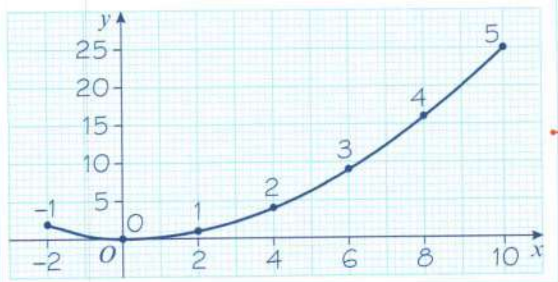
Choose values for t covering the domain of t. For each value of t, substitute to find corresponding values for x and y which will be the coordinates of points on the curve.

Plot the points and draw the curve through the points. The curve is an ellipse.

Example 7

Draw the curve given by the parametric equations $x = 2t$, $y = t^2$, for $-1 \leqslant t \leqslant 5$.

t	-1	0	1	2	3	4	5
$x = 2t$	-2	0	2	4	6	8	10
$y = t^2$	1	0	1	4	9	16	25



Online Use technology to graph the parametric equations.

Only calculate values of x and y for values of t in the given domain.

This is a 'partial' graph of the quadratic equation

$$y = \frac{x^2}{4}$$

You could also plot this curve by converting to Cartesian form and considering the domain of and range of the Cartesian function.

The domain is $-2 \leqslant x \leqslant 10$ and the range is $0 \leqslant y \leqslant 25$.

Exercise 8C

1 A curve is given by the parametric equations

$$x = 2t, \quad y = \frac{5}{t}, \quad t \neq 0$$

Copy and complete the table and draw a graph of the curve for $-5 \leqslant t \leqslant 5$.

t	-5	-4	-3	-2	-1	-0.5	0.5	1	2	3	4	5
$x = 2t$	-10	-8				-1						
$y = \frac{5}{t}$	-1	-1.25					10					

2 A curve is given by the parametric equations

$$x = t^2, \quad y = \frac{t^3}{5}$$

Copy and complete the table and draw a graph of the curve for $-4 \leqslant t \leqslant 4$.

t	-4	-3	-2	-1	0	1	2	3	4
$x = t^2$	16								
$y = \frac{t^3}{5}$	-12.8								

3 A curve is given by parametric equations

$$x = \tan t + 1, \quad y = \sin t, \quad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{3}$$

Copy and complete the table and draw a graph of the curve for the given domain of t.

t	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$x = \tan t + 1$	0			1				
$y = \sin t$				0				

4 Sketch the curves given by these parametric equations:

a $x = t - 2, \quad y = t^2 + 1, \quad -4 \leqslant t \leqslant 4$

b $x = 3\sqrt{t}, \quad y = t^3 - 2t, \quad 0 \leqslant t \leqslant 2$

c $x = t^2, \quad y = (2 - t)(t + 3), \quad -5 \leqslant t \leqslant 5$

d $x = 2\sin t - 1, \quad y = 5\cos t + 1, \quad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{4}$

e $x = \sec^2 t - 3, \quad y = 2\sin t + 1, \quad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{2}$

f $x = t - 3\cos t, \quad y = 1 + 2\sin t, \quad 0 \leqslant t \leqslant 2\pi$

(E) **5** The curve C has parametric equations

$$x = 3 - t, \quad y = t^2 - 2, \quad -2 \leqslant t \leqslant 3$$

a Find a Cartesian equation of C in the form $y = f(x)$. **(4 marks)**

b Sketch the curve C on the appropriate domain. **(3 marks)**

(E/P) **6** The curve C has parametric equations

$$x = 9\cos t - 2, \quad y = 9\sin t + 1, \quad -\frac{\pi}{6} \leqslant t \leqslant \frac{\pi}{2}$$

a Show that the Cartesian equation of C can be written as

$$(x + a)^2 + (y + b)^2 = c$$

where a, b and c are integers to be determined. **(4 marks)**

b Sketch the curve C on the given domain of t. **(3 marks)**

c Find the length of C. **(2 marks)**

Challenge

Sketch the curve given by the parametric equations on the given domain of t:

$$x = \frac{9t}{1 + t^3}, \quad y = \frac{9t^2}{1 + t^3}, \quad t \neq -1$$

Comment on the behaviour of the curve as t approaches -1 from the positive direction and from the negative direction.

8.4 Points of intersection

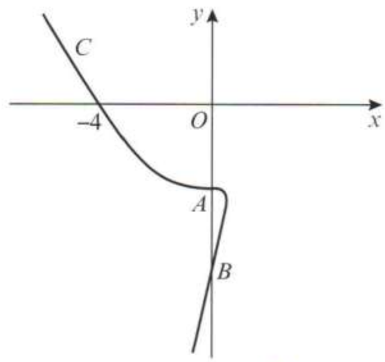
You need to be able to solve coordinate geometry problems involving parametric equations.

Example 8

The diagram shows a curve C with parametric equations $x = at^2 + t$, $y = a(t^3 + 8)$, $t \in \mathbb{R}$, where a is a non-zero constant. Given that C passes through the point $(-4, 0)$,

a find the value of a

b find the coordinates of the points A and B where the curve crosses the y-axis.



a At point $(-4, 0)$, $x = -4$ and $y = 0$

Hence

$-4 = at^2 + t$ (1)

$0 = a(t^3 + 8)$ (2)

Solving equation (2) for t:

$0 = a(t^3 + 8)$

$0 = t^3 + 8$

$-8 = t^3$

$-2 = t$

So, at the point $(-4, 0)$, $t = -2$.

Since $t = -2$ at $(-4, 0)$, then, from equation (1),

$-4 = a(-2)^2 + (-2)$

$-4 = 4a - 2$

$-2 = 4a$

$-\frac{1}{2} = a$

b At points A and B, the x-coordinate is 0:

$0 = -\frac{1}{2}t^2 + t$

$0 = t\left(-\frac{1}{2}t + 1\right)$

$t = 0$ or $-\frac{1}{2}t + 1 = 0$

$t = 2$

At $t = 0$,

$y = -\frac{1}{2}(0^3 + 8)$

$= -4$

At $t = 2$,

$y = -\frac{1}{2}(2^3 + 8)$

$= -8$

Therefore,

A is $(0, -4)$ and B is $(0, -8)$.

Online Explore curves with parametric equations of this form using technology.

The point $(-4, 0)$ lies on the curve. You can use this to write two equations for t.

Since a is non-zero, the factor $(t^3 + 8)$ must equal 0.

The value of t is the same in both equations at any given point on the curve.

Substitute $t = -2$ into equation (1).

Substitute $x = 0$ into the parametric equation for x. You now know that $a = -\frac{1}{2}$

Solve this quadratic equation to find the two values of t corresponding to points a and b.

Substitute each value for the parameter into the parametric equation for y to find the y-coordinates at these points.

You already know that these t-values will give you an x-coordinate of 0. Use the diagram to work out which point is A and which point is B.

Example 9

A curve is given parametrically by the equations $x = t^2$, $y = 4t$. The line $x + y + 4 = 0$ meets the curve at A. Find the coordinates of A.

$x + y + 4 = 0$

Substitute:

$t^2 + 4t + 4 = 0$

$(t + 2)^2 = 0$

$t + 2 = 0$

So $t = -2$

Substitute:

$x = t^2$

$= (-2)^2$

$= 4$

$y = 4t$

$= 4(-2)$

$= -8$

The coordinates of A are $(4, -8)$.

Find the value of t at A.
Solve the equations simultaneously.
Substitute $x = t^2$ and $y = 4t$ into $x + y + 4 = 0$.

Factorise.

Take the square root of each side.

Find the coordinates of A.
Substitute $t = -2$ into the parametric equations.

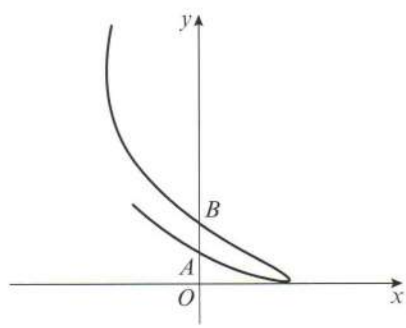
Example 10

The diagram shows a curve C with parametric equations

$$x = \cos t + \sin t, \quad y = \left(t - \frac{\pi}{6}\right)^2, \quad -\frac{\pi}{2} < t < \frac{4\pi}{3}$$

a Find the point where the curve intersects the line $y = \pi^2$.

b Find the coordinates of the points A and B where the curve cuts the y-axis.



a Curve crosses the line $y = \pi^2$ when

$\left(t - \frac{\pi}{6}\right)^2 = \pi^2$

$t - \frac{\pi}{6} = \pm\pi$

$t = \frac{7\pi}{6}$ or $-\frac{5\pi}{6}$

Reject $t = -\frac{5\pi}{6}$ since this is outside of the domain of t.

When $t = \frac{7\pi}{6}$,

For all points on the line $y = \pi^2$. Substitute this into the parametric equation for y and solve to find t.

You are taking square roots of both sides so consider the positive and negative values.

$-\frac{\pi}{2} < t < \frac{4\pi}{3}$ so only one of these solutions is a valid value for t.

$x = \cos\left(\frac{7\pi}{6}\right) + \sin\left(\frac{7\pi}{6}\right) = -\frac{1+\sqrt{3}}{2}$

Substitute into the parametric equation for x. Find an exact value for x.

The point of intersection is $\left(-\frac{1+\sqrt{3}}{2}, \pi^2\right)$

b Curve cuts the y-axis when $x = 0$. So,

$\cos t + \sin t = 0$

$\sin t = -\cos t$

$\tan t = -1$

Substitute $x = 0$ into the parametric equation for x and solve the resulting trigonometric equation.

Since, $-\frac{\pi}{2} < t < \frac{4\pi}{3}$

$t = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$

Consider the range of the parameter. There are two solutions to $\tan t = -1$ in this range. These correspond to the two points of intersection.

At $t = -\frac{\pi}{4}$, $y = \left(-\frac{\pi}{4} - \frac{\pi}{6}\right)^2 = \frac{25\pi^2}{144}$

At $t = \frac{3\pi}{4}$, $y = \left(\frac{3\pi}{4} - \frac{\pi}{6}\right)^2 = \frac{49\pi^2}{144}$

Substitute each value of t into the equation for y to find the y-coordinates.

A is $\left(0, \frac{25\pi^2}{144}\right)$ and B is $\left(0, \frac{49\pi^2}{144}\right)$.

Problem-solving

When you are given a sketch diagram in a question, you can't read off values, but you can check whether your answers have the correct sign. The y-coordinates at both points of intersection should be positive.

Exercise 8D

1 Find the coordinates of the point(s) where the following curves meet the x-axis.

a $x = 5 + t, \quad y = 6 - t$

b $x = 2t + 1, \quad y = 2t - 6$

c $x = t^2, \quad y = (1 - t)(t + 3)$

d $x = \frac{1}{t}, \quad y = (t - 1)(2t - 1), \quad t \neq 0$

e $x = \frac{2t}{1 + t}, \quad y = t - 9, \quad t \neq -1$

2 Find the coordinates of the point(s) where the following curves meet the y-axis.

a $x = 2t, \quad y = t^2 - 5$

b $x = 3t - 4, \quad y = \frac{1}{t^2}, \quad t \neq 0$

c $x = t^2 + 2t - 3, \quad y = t(t - 1)$

d $x = 27 - t^3, \quad y = \frac{1}{t - 1}, \quad t \neq 1$

e $x = \frac{t - 1}{t + 1}, \quad y = \frac{2t}{t^2 + 1}, \quad t \neq -1$

3 A curve has parametric equations $x = 4at^2, \quad y = a(2t - 1)$, where a is a constant. The curve passes through the point $(4, 0)$. Find the value of a.

4 A curve has parametric equations $x = b(2t - 3), \quad y = b(1 - t^2)$, where b is a constant. The curve passes through the point $(0, -5)$. Find the value of b.

5 Find the coordinates of the point of intersection of the line with parametric equations $x = 3t + 2, y = 1 - t$ and the line $y + x = 2$.

6 Find the values of t at the points of intersection of the line $4x - 2y - 15 = 0$ with the parabola $x = t^2, \quad y = 2t$ and give the coordinates of these points.

(P) 7 Find the points of intersection of the parabola $x = t^2$, $y = 2t$ with the circle $x^2 + y^2 - 9x + 4 = 0$.

8 Find the coordinates of the point(s) where the following curves meet the x-axis and the y-axis.

a $x = t^2 - 1, \quad y = \cos t, \quad 0 < t < \pi$

b $x = \sin 2t, \quad y = 2\cos t + 1, \quad \pi < t < 2\pi$

c $x = \tan t, \quad y = \sin t - \cos t, \quad 0 < t < \frac{\pi}{2}$

9 Find the coordinates of the point(s) where the following curves meet the x-axis and the y-axis.

a $x = e^t + 5, \quad y = \ln t, \quad t > 0$

b $x = \ln t, \quad y = t^2 - 64, \quad t > 0$

c $x = e^{2t} + 1, \quad y = 2e^t - 1, \quad -1 < t < 1$

10 Find the values of t at the points of intersection of the line $y = -3x + 2$ and the curve with parametric equations $x = t^2$, $y = t$, and give the coordinates of these points.

11 Find the value(s) of t at the point of intersection of the line $y = x - \ln 3$ and the curve with parametric equations $x = \ln(t - 1)$, $y = \ln(2t - 5)$, $t > \frac{5}{2}$, and give the exact coordinates of this point.

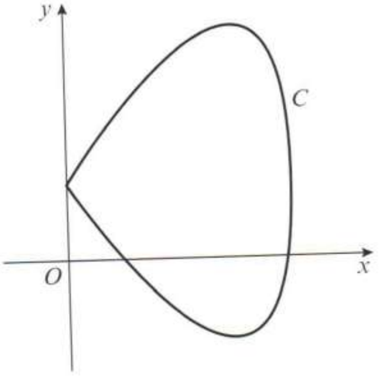
(E) 12 A curve C has parametric equations

$$x = 6\cos t, \quad y = 4\sin 2t + 2, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

a Find the coordinates of the points where the curve intersects the x-axis. **(4 marks)**

b Show that the curve crosses the line $y = 4$ where $t = \frac{\pi}{12}$ and $t = \frac{5\pi}{12}$ **(3 marks)**

c Hence determine the coordinates of points where $y = 4$ intersects the curve. **(2 marks)**



(E/P) 13 Show that the line with equation $y = 2x - 5$ does not intersect the curve with parametric equations $x = 2t, \quad y = 4t(t - 1)$. **(4 marks)**

Problem-solving

Consider the discriminant after substituting.

(E/P) 14 The curve C has parametric equations $x = \sin t, \quad y = \cos 2t + 1, \quad 0 \leqslant t \leqslant 2\pi$.
Given that the line $y = k$, where k is a constant, intersects the curve,

a show that $0 \leqslant k \leqslant 2$ **(3 marks)**

b show that if the line $y = k$ is a tangent to the curve, then $k = 2$. **(3 marks)**

E 15 The curve C has parametric equations $x = e^{2t}$, $y = e^t - 1$. The straight line l passes through the points A and B where $t = \ln 2$ and $t = \ln 3$ respectively.

a Find the points A and B. **(3 marks)**

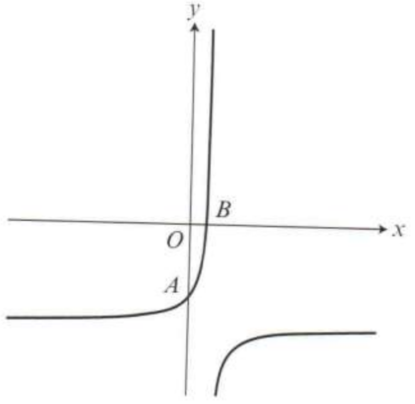
b Show that the gradient of the line l is $\frac{1}{5}$ **(2 marks)**

c Hence, find the equation for line l in the form $ax + by + c = 0$. **(2 marks)**

P 16 The curve C has parametric equations $x = \sin t$, $y = \cos t$. The straight line l passes through the points A and B where $t = \frac{\pi}{6}$ and $t = \frac{\pi}{2}$ respectively. Find an equation for the line l in the form $ax + by + c = 0$. **(7 marks)**

P 17 The diagram shows the curve C with parametric equations

$$x = \frac{t-1}{t}, \quad y = t - 4, \quad t \neq 0$$



The curve crosses the y-axis and the x-axis at points A and B respectively.

a Find the coordinates of A and B. **(4 marks)**

The line l_1 intersects the curve at points A and B. The lines l_2 and l_3 are parallel to l_1 and are distinct tangents to the curve.

b Show that the two possible equations for l_2 and l_3 are

$$y = 4x - 4 \text{ and } y = 4x - 12$$

(6 marks)

c Find the coordinates of the point where each tangent meets C. **(4 marks)**

Challenge

The curve C_1 has parametric equations

$$x = e^{2t}, \quad y = 2t + 1$$

The curve C_2 has parametric equations

$$x = e^t, \quad y = 1 + t^2$$

Find the coordinates of the points at which these two curves intersect.

8.5 Modelling with parametric equations

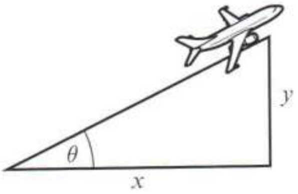
You can use parametric equations to model real-life situations. In mechanics you will use parametric equations with **time** as a parameter to model motion in two dimensions.

Example 11

A plane's position at time t seconds after take-off can be modelled with the following parametric equations:

$$x = (v\cos\theta)t\text{ m},\quad y = (v\sin\theta)t\text{ m},\quad t > 0$$

where v is the speed of the plane, θ is the angle of elevation of its path, x is the horizontal distance travelled and y is the vertical distance travelled, relative to a fixed origin.



When the plane has travelled 600 m horizontally, it has climbed 120 m

a Find the angle of elevation, θ.

Given that the plane's speed is 50 m s^{-1},

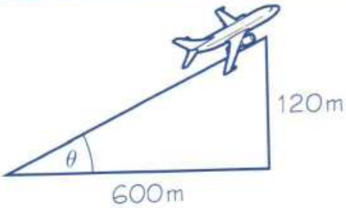
b find the parametric equations for the plane's motion

c find the vertical height of the plane after 10 seconds

d show that the plane's motion is a straight line

e explain why the domain of t, $t > 0$, is not realistic.

a



Angle of elevation

$\theta = \tan^{-1}\left(\frac{120}{600}\right) = 11.3°$ (1 d.p.)

The model assumes that the angle of elevation will stay constant so the ratio will always be the same regardless of how far along the journey the plane is.

b $x = (v\cos\theta)t$

$= (50 \times \cos 11.3\ldots)t = 49.0t$ m (3 s.f.)

$y = (v\sin\theta)t$

$= (50 \times \sin 11.3\ldots)t = 9.81t$ m (3 s.f.)

Substitute $v = 50$ and $\theta = 11.3$ into the equations for x and y. The units of length, metres, are given with the model.

c At $t = 10$,

$y = 9.81t = 9.81 \times 10 = 98.1$ m

So, the plane has climbed 98.1 m after 10 seconds.

Substitute $t = 10$ into y, as y represents the vertical height.

d $x = 49t$

So, $\frac{x}{49} = t$ (1)

$y = 9.81t$ (2)

Find a Cartesian equation for the plane's path. Rearrange the equation for x to make t the subject.

So,

$y = 9.81 \times \frac{x}{49} = 0.2x$

Substitute t from (1) into (2).

Since this is a linear equation, the motion of the plane is a straight line with gradient 0.2

The gradient in this context represents the height gained for every metre travelled horizontally.

e $t > 0$ is not realistic as this would mean the plane would continue climbing forever at the same speed and with the same angle of elevation.

Problem-solving

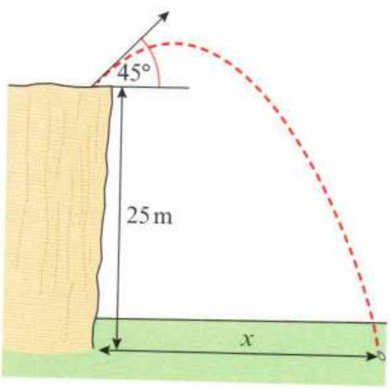
If you have to comment on a modelling assumption or range of validity, consider whether the assumption is realistic given the context of the question. Make sure you refer to the real-life situation being modelled in your answer.

Example 12

A stone is thrown from the top of a 25 m high cliff with an initial speed of 5 m s^{-1} at an angle of 45°. Its position after t seconds can be described using the following parametric equations

$$x = \frac{5\sqrt{2}}{2}t\text{ m}, \quad y = \left(-4.9t^2 + \frac{5\sqrt{2}}{2}t + 25\right)\text{ m}, \quad 0 \leqslant t \leqslant k$$

where x is the horizontal distance from the point of projection, y is the vertical distance from the ground and k is a constant.



Given that the model is valid from the time the stone is thrown to the time it hits the ground,

a find the value of k

b find the horizontal distance travelled by the stone once it hits the floor.

a The stone hits the ground when $y = 0$:

$$-4.9t^2 + \frac{5\sqrt{2}}{2}t + 25 = 0$$

$$t = \frac{-\frac{5\sqrt{2}}{2} \pm \sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 - 4(-4.9)(25)}}{2(-4.9)}$$

Use the quadratic formula on your calculator to find two solutions for t.

$t = -1.926\ldots$ or $t = 2.648\ldots$

$t \geqslant 0$, so the stone hits the ground at $t = 2.648\ldots$

So $k = 2.65$ (2 d.p.)

The model is only valid for $t \geqslant 0$ so disregard the negative solution.

b When $t = 2.648\ldots$

$$x = \frac{5\sqrt{2}}{2}t = \frac{5\sqrt{2}}{2} \times 2.648\ldots$$

Substitute this value of t into the parametric equation for x.

$= 9.362\ldots$ m

So the horizontal distance travelled by the stone is 9.36 m (2 d.p.).

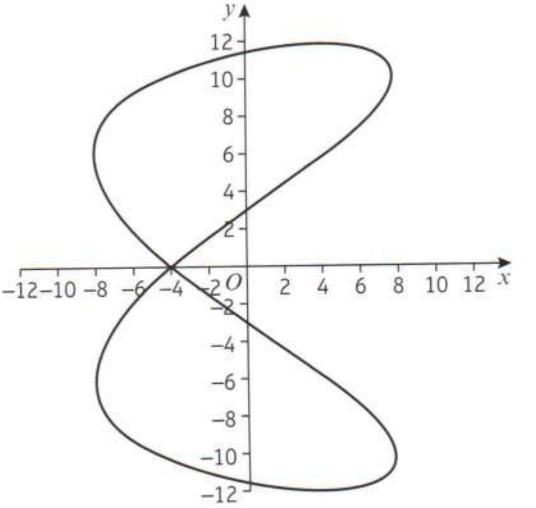
Online Use the polynomial function on your calculator to solve the quadratic equation.

Example 13

The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations

$$x = 8\cos 20t, \quad y = 12\sin\left(10t - \frac{\pi}{3}\right), \quad t \geqslant 0$$

where x and y are measured in metres.



a Find the coordinates of the figure skater at the beginning of his motion.

b Find the coordinates of the point where the figure skater intersects his own path.

c Find the coordinates of the points where the path of the figure skater crosses the y-axis.

d Determine how long it takes the figure skater to complete one complete figure-of-eight motion.

a At $t = 0$,

$x = 8\cos 0 = 8$

$y = 12\sin\left(10 \times 0 - \frac{\pi}{3}\right) = 12\sin\left(-\frac{\pi}{3}\right)$

$= -6\sqrt{3}$

The coordinates of the figure skater at the beginning of his motion are $(8, -6\sqrt{3})$.

> Substitute $t = 0$ into both equations to find the x and y coordinates.

b From the diagram, the figure skater intersects his own path on the x-axis, ie. when $y = 0$.

> Use the diagram to find information about the point of intersection.

$12\sin\left(10t - \frac{\pi}{3}\right) = 0$

$\sin\left(10t - \frac{\pi}{3}\right) = 0$

> Substitute $y = 0$ into the equation for y, and solve to find values of t in the domain $t \geqslant 0$.

$10t - \frac{\pi}{3} = 0, \pi, 2\pi, \ldots$

$10t = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \ldots$

$t = \frac{\pi}{30}, \frac{2\pi}{15}, \frac{7\pi}{30}, \ldots$

$x = 8\cos\left(20 \times \frac{\pi}{30}\right) = 8\cos\left(\frac{2\pi}{3}\right) = -4$

> There is only one point of intersection so choose any of these values of t. You can use one of the others to check: $8\cos\left(20 \times \frac{2\pi}{15}\right) = 8\cos\left(\frac{8\pi}{3}\right) = -4$

So, the figure skater intersects his own path at the point $(-4, 0)$.

c The figure skater crosses the y-axis when $x = 0$,

$0 = 8\cos 20t$

$0 = \cos 20t$

So, $20t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

Find solutions to $8\cos 20t = 0$ in the domain $t \geqslant 0$. There are 4 points of intersection so consider the first 4 solutions, and check that these give different values of y.

Substitute these values into y.

$20t = \frac{\pi}{2}$:

$y = 12\sin\left(\frac{1}{2} \times \frac{\pi}{2} - \frac{\pi}{3}\right) = 12\sin\left(-\frac{\pi}{12}\right)$

$= -3.11$ (2 d.p.)

Use your calculator to find the corresponding values of y. You can give your answers as decimals or as exact values: $12\sin\left(-\frac{\pi}{12}\right) = -3\sqrt{6} + 3\sqrt{2}$

$20t = \frac{3\pi}{2}$:

$y = 12\sin\left(\frac{1}{2} \times \frac{3\pi}{2} - \frac{\pi}{3}\right) = 12\sin\left(\frac{5\pi}{12}\right)$

$= 11.59$ (2 d.p.)

Online Find points of intersection of this curve with the coordinate axes using technology.

$20t = \frac{5\pi}{2}$:

$y = 12\sin\left(\frac{1}{2} \times \frac{5\pi}{2} - \frac{\pi}{3}\right) = 12\sin\left(\frac{11\pi}{12}\right)$

$= 3.11$ (2 d.p.)

$20t = \frac{7\pi}{2}$:

$y = 12\sin\left(\frac{1}{2} \times \frac{7\pi}{2} - \frac{\pi}{3}\right) = 12\sin\left(\frac{17\pi}{12}\right)$

$= -11.59$ (2 d.p.)

So the skater crosses the y-axis at (0, −3.11), (0, 11.59), (0, 3.11), (0, −11.59).

Check that these look sensible from the graph. The motion of the skater appears to be symmetrical about the x-axis so these look right.

d The period of $x = 8\cos 20t$ is $\frac{2\pi}{20}$,

so the skater returns to his x-position after $\frac{2\pi}{20}$ min, $\frac{4\pi}{20}$ min, …

The period of $a\cos(bx + c)$ is $\frac{2\pi}{b}$ and the period of $a\sin(bx + c)$ is $\frac{2\pi}{b}$

The period of $y = 12\sin\left(10t - \frac{\pi}{3}\right)$ is $\frac{2\pi}{10}$,

so the skater returns to his y-position after $\frac{2\pi}{10}$ min, $\frac{4\pi}{10}$ min, …

So the skater first completes a full figure-of-eight motion after $\frac{2\pi}{10}$ mins $= 0.628\ldots$ mins or 38 seconds (2 s.f.).

Problem-solving

In order for the figure skater to return to his starting position, **both** parametric equations must complete full periods. This occurs at the **least common multiple** of the two periods.

Exercise 8E

(P) **1** A river flows from north to south. The position at time t seconds of a rowing boat crossing the river from west to east is modelled by the parametric equations

$$x = 0.9t\text{ m}, \quad y = -3.2t\text{ m}$$

where x is the distance travelled east and y is the distance travelled north.

Given that the river is 75 m wide,

a find the time taken to get to the other side

b find the distance the boat has been moved off-course due to the current

c show that the motion of the boat is a straight line

d determine the speed of the boat.

(P) **2** The position of a small plane coming into land at time t seconds after it has started its descent is modelled by the parametric equations

$$x = 80t, \quad y = -9.1t + 3000, \quad 0 \leqslant t < 330$$

where x is the horizontal distance travelled (in metres) and y is the vertical distance (in metres) of the plane above ground level.

a Find the initial height of the plane.

b Justify the choice of domain, $0 \leqslant t < 330$, for this model.

c Find the horizontal distance the plane travels between beginning its descent and landing.

(P) **3** A ball is kicked from the ground with an initial speed of 20 m s^{-1} at an angle of 30°. Its position after t seconds can be described using the following parametric equations

$$x = 10\sqrt{3}\,t\text{ m}, \quad y = (-4.9t^2 + 10t)\text{ m}, \quad 0 \leqslant t \leqslant k$$

a Find the horizontal distance travelled by the ball when it hits the ground.

A player wants to head the ball when it is descending between 1.5 m and 2.5 m off the ground.

b Find the range of time after the ball has been kicked at which the player can head the ball.

c Find the closest horizontal distance from where the ball has been kicked at which the player can head the ball.

(P) **4** The path of a dolphin leaping out of the water can be modelled with the following parametric equations

$$x = 2t\text{ m}, \quad y = -4.9t^2 + 10t\text{ m}$$

where x is the horizontal distance from the point the dolphin jumps out of the water, y is the height above sea level of the dolphin and t is the time in seconds after the dolphin has started its jump.

a Find the time the dolphin takes to complete a single jump.

b Find the horizontal distance the dolphin travels during a single jump.

c Show that the dolphin's path is modelled by a quadratic curve.

d Find the maximum height of the dolphin.

(P) **5** The path of a car on a Ferris wheel at time t minutes is modelled using the parametric equations

$$x = 12\sin t, \quad y = 12 - 12\cos t$$

where x is the horizontal distance in metres of the car from the start of the ride and y is the height in metres above ground level of the car.

a Show that the motion of the car is a circle with radius 12 m.

b Hence, find the maximum height of the car during the journey.

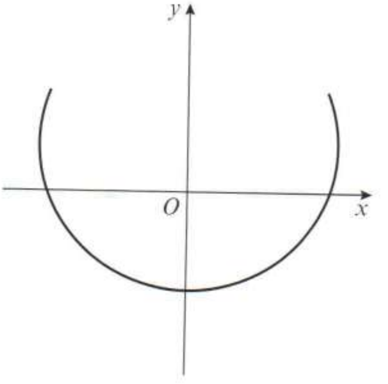
c Find the time taken to complete one revolution of the Ferris wheel and hence calculate the average speed of the car.

(P) **6** The cross-section of a bowl design is given by the following parametric equations

$$x = t - 4\sin t, \quad y = 1 - 2\cos t, \quad -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$$

a Find the length of the opening of the bowl. **(3 marks)**

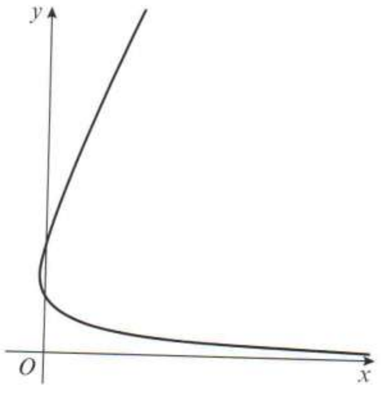
b Given that the cross-section of the bowl crosses the y-axis at its deepest point, find the depth of the bowl. **(4 marks)**



7 A particle is moving in the xy-plane such that its position after time t seconds relative to a fixed origin O is given by the parametric equations

$$x = \frac{t^2 - 3t + 2}{t}, \quad y = 2t, \quad t > 0$$

The diagram shows the path of the particle.



a Find the distance from the origin to the particle at time $t = 0.5$.

b Find the coordinates of the points where the particle crosses the y-axis.

Another particle travels in the same plane with its path given by the equation $y = 2x + 10$.

c Show that the paths of these two particles never intersect.

8 The path of a ski jumper from the point of leaving the ramp to the point of landing is modelled using the parametric equations

$$x = 18t, \quad y = -4.9t^2 + 4t + 10, \quad 0 \leqslant t \leqslant k$$

where x is the horizontal distance in metres from the point of leaving the ramp and y is the height in metres above ground level of the ski jumper, after t seconds.

a Find the initial height of the ski jumper. **(1 mark)**

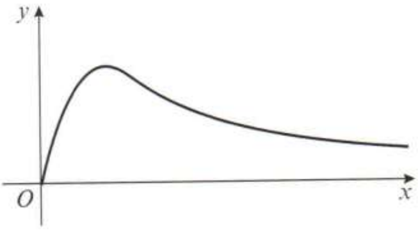
b Find the value of k and hence state the time taken for the ski jumper to complete her jump. **(3 marks)**

c Find the horizontal distance the ski jumper jumps. **(1 mark)**

d Show that the ski jumper's path is a parabola and find the maximum height above ground level of the ski jumper. **(5 marks)**

(P) **9** The profile of a hill climb in a bike race is modelled by the following parametric equations

$$x = 50\tan t\text{ m},\quad y = 20\sin 2t\text{ m},\quad 0 < t \leqslant \frac{\pi}{2}$$



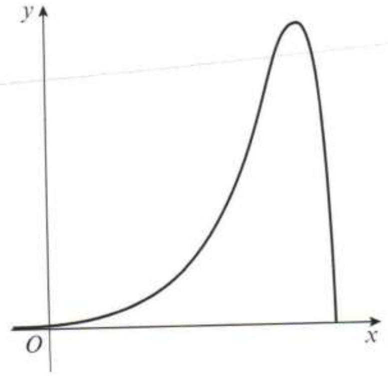
a Find the value of t at the highest point of the hill climb.

b Hence find the coordinates of the highest point.

c Find the coordinates when $t = 1$ and show that at this point, a cyclist will be descending.

(E/P) **10** A computer model for the shape of the path of a rollercoaster is given by the parametric equations

$$x = 5 + \ln t,\quad y = 5\sin 2t,\quad 0 < t \leqslant \frac{\pi}{2}$$



a Find the coordinates of the point where $t = \frac{\pi}{6}$ **(2 marks)**

Given that one unit on the model represents 5 m in real life,

b find the maximum height of the rollercoaster **(1 mark)**

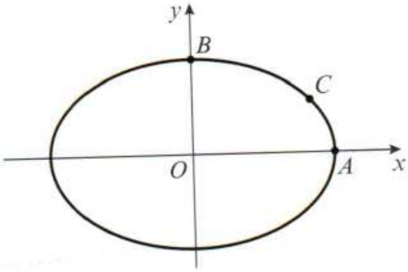
c find the horizontal distance covered during the descent of the rollercoaster. **(4 marks)**

d Hence, find the average gradient of the descent. **(1 mark)**

Mixed exercise 8

1 The diagram shows a sketch of the curve with parametric equations

$$x = 4\cos t,\quad y = 3\sin t,\quad 0 \leqslant t < 2\pi$$



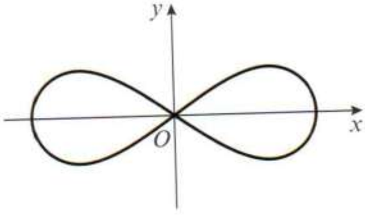
a Find the coordinates of the points A and B.

b The point C has parameter $t = \frac{\pi}{6}$. Find the exact coordinates of C.

c Find the Cartesian equation of the curve.

2 The diagram shows a sketch of the curve with parametric equations

$$x = \cos t,\quad y = \tfrac{1}{2}\sin 2t,\quad 0 \leqslant t < 2\pi$$



The curve is symmetrical about both axes.

Copy the diagram and label the points having parameters $t = 0$, $t = \frac{\pi}{2}$, $t = \pi$ and $t = \frac{3\pi}{2}$

P 3 A curve has parametric equations

$$x = e^{2t+1} + 1, \quad y = t + \ln 2, \quad t > 1$$

a Find a Cartesian equation of this curve in the form $y = f(x)$, $x > k$ where k is a constant to be found in exact form.

b Write down the range of $f(x)$, leaving your answer in exact form.

4 A curve has parametric equations

$$x = \frac{1}{2t+1}, \quad y = 2\ln\left(t + \frac{1}{2}\right), \quad t > \frac{1}{2}$$

Find a Cartesian equation of the curve in the form $y = f(x)$, and state the domain and range of $f(x)$.

5 A curve has parametric equations $x = \sin t$, $y = \cos 2t$, $0 \leqslant t < 2\pi$

a Find a Cartesian equation of the curve.

The curve cuts the x-axis at $(a, 0)$ and $(b, 0)$.

b Find the values of a and b.

6 A curve has parametric equations $x = \dfrac{1}{1+t}$, $y = \dfrac{1}{(1+t)(1-t)}$, $t > 1$

Express t in terms of x, and hence show that a Cartesian equation of the curve is $y = \dfrac{x^2}{2x-1}$

7 A circle has parametric equations $x = 4\sin t - 3$, $y = 4\cos t + 5$, $0 \leqslant t \leqslant 2\pi$

a Find a Cartesian equation of the circle.

b Draw a sketch of the circle.

c Find the exact coordinates of the points of intersection of the circle with the y-axis.

8 The curve C has parametric equations

$$x = \frac{2-3t}{1+t}, \quad y = \frac{3+2t}{1+t}, \quad 0 \leqslant t \leqslant 4$$

a Show that the curve C is part of a straight line. **(3 marks)**

b Find the length of this line segment. **(2 marks)**

9 A curve C has parametric equations

$$x = t^2 - 2, \quad y = 2t, \quad 0 \leqslant t \leqslant 2$$

a Find the Cartesian equation of C in the form $y = f(x)$. **(3 marks)**

b State the domain and range of $y = f(x)$ in the given domain of t. **(3 marks)**

c Sketch the curve in the given domain of t. **(2 marks)**

E/P **10** A curve C has parametric equations

$$x = 2\cos t, \quad y = 2\sin t - 5, \quad 0 \leqslant t \leqslant \pi$$

a Show that the curve C forms part of a circle. **(3 marks)**

b Sketch the curve in the given domain of t. **(3 marks)**

c Find the length of the curve in the given domain of t. **(3 marks)**

E/P **11** The curve C has parametric equations

$$x = t - 2, \quad y = t^3 - 2t^2, \quad t \in \mathbb{R}$$

a Find a Cartesian equation of C in the form $y = \mathrm{f}(x)$. **(3 marks)**

b Sketch the curve C. **(3 marks)**

E/P **12** Show that the line with equation $y = 4x + 20$ is a tangent to the curve with parametric equations $x = t - 3$, $y = 4 - t^2$. **(4 marks)**

E/P **13** The curve C has parametric equations $x = 2\ln t, \quad y = t^2 - 1, \quad t > 0$

a Find the coordinates of the point where the line $x = 5$ intersects the curve. Give your answer as exact values. **(4 marks)**

b Given that the line $y = k$ intersects the curve, find the range of values for k. **(3 marks)**

E/P **14** The diagram shows the curve C with parametric equations

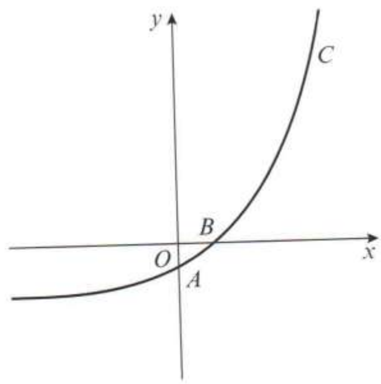
$$x = 1 + 2t, \quad y = 4^t - 1$$

The curve crosses the y-axis and the x-axis at points A and B respectively.

a Find the coordinates of A and B. **(4 marks)**

The line l intersects the curve at points A and B.

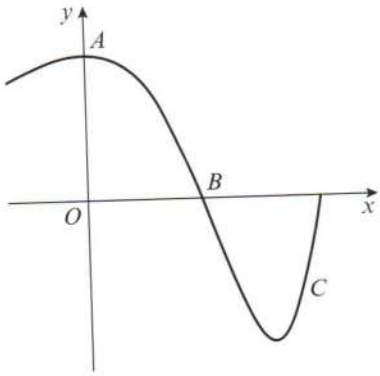
b Find the equation of l in the form $ax + by + c = 0$. **(3 marks)**



E/P **15** The diagram shows the curve C with parametric equations

$$x = \ln t - \ln\left(\frac{\pi}{2}\right), \quad y = \sin t, \quad 0 < t < 2\pi$$

The curve crosses the y-axis and the x-axis at points A and B respectively. The line l intersects the curve at points A and B. Find the equation of l in the form $ax + by + c = 0$. **(7 marks)**



(P) **16** A plane's position at time t seconds during its descent can be modelled with the following parametric equations

$$x = 80t, \quad y = 3000 - 30t, \quad 0 < t < k$$

where x is the horizontal distance travelled in metres and y is the vertical height of the plane in metres.

a Show that the plane's descent is a straight line. **(3 marks)**

Given that the model is valid until the plane is 30 m off the ground,

b find the value of k **(2 marks)**

c determine the distance travelled by the plane in this portion of its descent. **(3 marks)**

(P) **17** The path of an arrow path at time t seconds from being fired can be described using the following parametric equations

$$x = 50\sqrt{2}\,t, \quad y = 1.5 - 4.9t^2 + 50\sqrt{2}\,t, \quad 0 \leqslant t \leqslant k$$

where x is the horizontal distance from the archer in metres and y is the vertical height of the arrow above level ground.

a Find the furthest horizontal distance the arrow can travel.

A castle is located a horizontal distance of 1000 m from the archer's position. The height of the castle is 10 m.

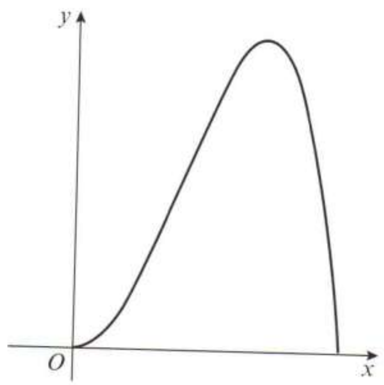
b Show that the arrow misses the castle.

c Find the distance the archer should step back so that he can hit the top of the castle.

(P) **18** A mountaineer's climb at time t hours can be modelled with the following parametric equations

$$x = 300\sqrt{t}, \quad y = 244t(4 - t), \quad 0 < t < k$$

where x represents the distance travelled horizontally in metres and y represents the height above sea level in metres.



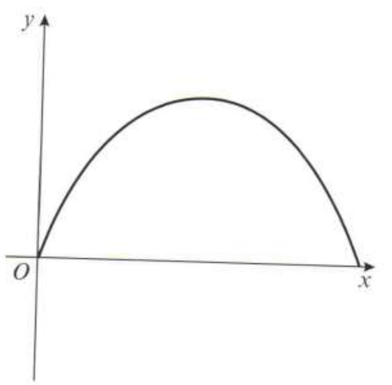
a Find the height of the peak and the time at which the mountaineer reaches it. **(3 marks)**

Given that the mountaineer completes her climb when she gets back to sea level,

b find the horizontal distance from the beginning of her climb to the end. **(2 marks)**

19 A bridge is designed using the following parametric equations:

$$x = \frac{4t}{\pi} - 2\sin t, \quad y = -\cos t, \quad \frac{\pi}{2} < t < \frac{3\pi}{2}$$



Given that 1 unit in the design is 10 m in real life,

a find the highest point of the bridge

b find the width of the widest river this bridge can cross.

20 A BMX cyclist's position on a ramp at time t seconds can be modelled with the parametric equations

$$x = 3(e^t - 1), \quad y = 10(t - 1)^2, \quad 0 \leqslant t \leqslant 1.3$$

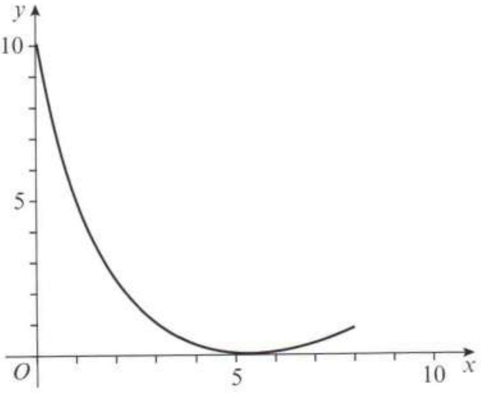
where x is the horizontal distance travelled in metres and y is the height above ground level in metres.

a Find the initial height of the cyclist.

b Find the time the cyclist is at her lowest height.

Given that after 1.3 seconds, the cyclist is at the end of the ramp,

c find the height at which the cyclist leaves the ramp.



Challenge

Two particles A and B move in the x-y plane, such that their positions relative to a fixed origin at a time t seconds are given, respectively, by the parametric equations:

$$A: x = \frac{2}{t}, \quad y = 3t + 1, \quad t > 0$$

$$B: x = 5 - 2t, \quad y = 2t^2 + 2k - 1, \quad t > 0$$

where k is a non-zero constant.

Given that the particles collide,

a find the value of k

b find the coordinates of the point of collision.

Summary of key points

1 A curve can be defined using parametric equations $x = \mathrm{p}(t)$ and $y = \mathrm{q}(t)$. Each value of the parameter, t, defines a point on the curve with coordinates $(\mathrm{p}(t), \mathrm{q}(t))$.

2 You can convert between parametric equations and Cartesian equations by using substitution to eliminate the parameter.

3 For parametric equations $x = \mathrm{p}(t)$ and $y = \mathrm{q}(t)$ with Cartesian equation $y = \mathrm{f}(x)$:
- the domain of $\mathrm{f}(x)$ is the range of $\mathrm{p}(t)$
- the range of $\mathrm{f}(x)$ is the range of $\mathrm{q}(t)$

4 You can use parametric equations to model real-life situations. In mechanics you will use parametric equations with time as a parameter to model motion in two dimensions.