8 Modelling in mechanics

Objectives

After completing this chapter you should be able to:

- Understand how the concept of a mathematical model applies to mechanics → pages 119–120
- Understand and be able to apply some of the common assumptions used in mechanical models → pages 120–122
- Know SI units for quantities and derived quantities used in mechanics → pages 122–124
- Know the difference between scalar and vector quantities → pages 125–127

Prior knowledge check

Give your answers correct to 3 s.f. where appropriate.

1 Solve these equations:

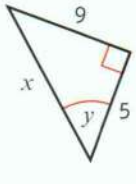
a $5x^2 - 21x + 4 = 0$ **b** $6x^2 + 5x = 21$

c $3x^2 - 5x - 4 = 0$ **d** $8x^2 - 18 = 0$

← GCSE Mathematics

2 Find the value of x and y in these right-angled triangles.

a



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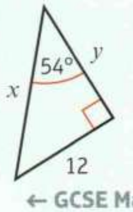


← GCSE Mathematics

3 Convert:

a $30\,\text{km}\,\text{h}^{-1}$ to $\text{cm}\,\text{s}^{-1}$ **b** $5\,\text{g}\,\text{cm}^{-3}$ to $\text{kg}\,\text{m}^{-3}$

← GCSE Mathematics

4 Write in standard form:

a 7 650 000 **b** 0.003 806

← GCSE Mathematics

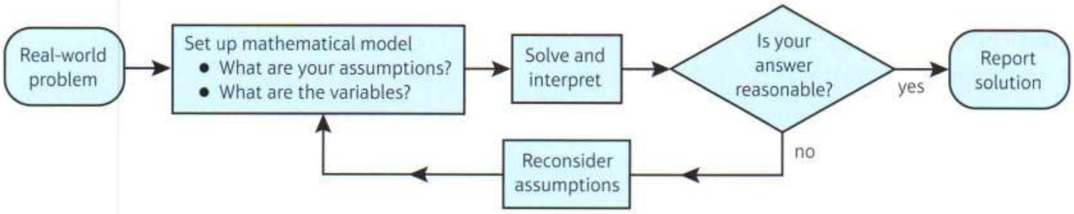
Mathematical models can be used to find solutions to **real-world problems** in many everyday situations. If you model a firework as a particle you can ignore the effects of wind and air resistance.

8.1 Constructing a model

Mechanics deals with motion and the action of forces on objects. Mathematical models can be constructed to simulate real-life situations, but in many cases it is necessary to simplify the problem by making assumptions so that it can be described using equations or graphs in order to solve it.

The solution to a mathematical model needs to be interpreted in the context of the original problem. It is possible your model may need to be refined and your assumptions reconsidered.

This flow chart summarises the mathematical modelling process.



Example 1

The motion of a basketball as it leaves a player's hand and passes through the net can be modelled using the equation $h = 2 + 1.1x - 0.1x^2$, where h m is the height of the basketball above the ground and x m is the horizontal distance travelled.

a Find the height of the basketball: **i** when it is released **ii** at a horizontal distance of 0.5 m.

b Use the model to predict the height of the basketball when it is at a horizontal distance of 15 m from the player.

c Comment on the validity of this prediction.

a **i** $x = 0$: $h = 2 + 0 + 0$
Height = 2 m

When the basketball is released at the start of the motion $x = 0$. Substitute $x = 0$ into the equation for h.

ii $x = 0.5$: $h = 2 + 1.1 \times 0.5 - 0.1 \times (0.5)^2$
Height = 2.525 m

Substitute $x = 0.5$ into the equation for h.

b $x = 15$: $h = 2 + 1.1 \times 15 - 0.1 \times (15)^2$
Height = −4 m

Substitute $x = 15$ into the equation for h.

c Height cannot be negative so the model is not valid when $x = 15$ m.

h represents the height of the basketball above the ground, so it is only valid if $h \geqslant 0$.

Exercise 8A

1 The motion of a golf ball after it is struck by a golfer can be modelled using the equation $h = 0.36x - 0.003x^2$, where h m is the height of the golf ball above the ground and x m is the horizontal distance travelled.

a Find the height of the golf ball when it is:
i struck **ii** at a horizontal distance of 100 m.

b Use the model to predict the height of the golf ball when it is 200 m from the golfer.

c Comment on the validity of this prediction.

2 A stone is thrown into the sea from the top of a cliff. The height of the stone above sea level, h m at time t s after it is thrown can be modelled by the equation $h = -5t^2 + 15t + 90$.

a Write down the height of the cliff above sea level.

b Find the height of the stone:

i when $t = 3$ **ii** when $t = 5$.

c Use the model to predict the height of the stone after 20 seconds.

d Comment on the validity of this prediction.

(P) **3** The motion of a basketball as it leaves a player's hand and passes through the net is modelled using the equation $h = 2 + 1.1x - 0.1x^2$, where h m is the height of the basketball above the ground and x m is the horizontal distance travelled.

a Find the two values of x for which the basketball is exactly 4 m above the ground.

This model is valid for $0 \leqslant x \leqslant k$, where k m is the horizontal distance of the net from the player. Given that the height of the net is 3 m:

b Find the value of k.

c Explain why the model is not valid for $x > k$.

(P) **4** A car accelerates from rest to 60 mph in 10 seconds. A quadratic equation of the form $d = kt^2$ can be used to model the distance travelled, d metres in time t seconds.

Problem-solving

Use the information given to work out the value of k.

a Given that when $t = 1$ second the distance travelled by the car is 13.2 metres, use the model to find the distance travelled when the car reaches 60 mph.

b Write down the range of values of t for which the model is valid.

(P) **5** The model for the motion of a golf ball given in question 1 is only valid when $h \geqslant 0$. Find the range of values of x for which the model is valid.

(P) **6** The model for the height of the stone above sea level given in question 2 is only valid from the time the stone is thrown until the time it enters the sea. Find the range of values of t for which the model is valid.

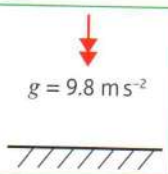
8.2 Modelling assumptions

Modelling assumptions can simplify a problem and allow you to analyse a real-life situation using known mathematical techniques.
You need to understand the significance of different modelling assumptions and how they affect the calculations in a particular problem.

Watch out

Modelling assumptions can affect the validity of a model. For example, when modelling the landing of a commercial flight, it would not be appropriate to ignore the effects of wind and air resistance.

These are some common models and modelling assumptions that you need to know.

Model	Modelling assumptions
Particle – Dimensions of the object are negligible.	• mass of the object is concentrated at a single point • rotational forces and air resistance can be ignored
Rod – All dimensions but one are negligible, like a pole or a beam.	• mass is concentrated along a line • no thickness • rigid (does not bend or buckle)
Lamina – Object with area but negligible thickness, like a sheet of paper.	• mass is distributed across a flat surface
Uniform body – Mass is distributed evenly.	• mass of the object is concentrated at a single point at the geometrical centre of the body – the **centre of mass**
Light object – Mass of the object is small compared to other masses, like a string or a pulley.	• treat object as having zero mass • tension the same at both ends of a light string
Inextensible string – A string that does not stretch under load.	• acceleration is the same in objects connected by a taut inextensible string
Smooth surface	• assume that there is no friction between the surface and any object on it
Rough surface – If a surface is not smooth, it is rough.	• objects in contact with the surface experience a frictional force if they are moving or are acted on by a force
Wire – Rigid thin length of metal.	• treated as one-dimensional
Smooth and light pulley – all pulleys you consider will be smooth and light.	• pulley has no mass • tension is the same on either side of the pulley
Bead – Particle with a hole in it for threading on a wire or string.	• moves freely along a wire or string • tension is the same on either side of the bead
Peg – A support from which a body can be suspended or rested.	• dimensionless and fixed • can be rough or smooth as specified in question
Air resistance – Resistance experienced as an object moves through the air.	• usually modelled as being negligible
Gravity – Force of attraction between all objects. Acceleration due to gravity is denoted by g. 	• assume that all objects with mass are attracted towards the Earth • Earth's gravity is uniform and acts vertically downwards • g is constant and is taken as 9.8 m s^{-2}, unless otherwise stated in the question

Example 2

A mass is attached to a length of string which is fixed to the ceiling.
The mass is drawn to one side with the string taut and allowed to swing.
State the effect of the following assumptions on any calculations made using this model:

a The string is light and inextensible. **b** The mass is modelled as a particle.

a Ignore the mass of the string and any stretching effect caused by the mass.

b Ignore the rotational effect of any external forces that are acting on it, and the effects of air resistance.

Exercise 8B

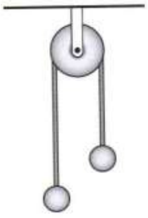
1 A football is kicked by the goalkeeper from one end of the football pitch.
State the effect of the following assumptions on any calculations made using this model:
  a The football is modelled as a particle.
  b Air resistance is negligible.

2 An ice puck is hit and slides across the ice.
State the effect of the following assumptions on any calculations made using this model:
  a The ice puck is modelled as a particle.
  b The ice is smooth.

3 A parachute jumper wants to model her descent from an aeroplane to the ground. She models herself and her parachute as particles connected by a light inextensible string. Explain why this may not be a suitable modelling assumption for this situation.

4 A fishing rod manufacturer constructs a mathematical model to predict the behaviour of a particular fishing rod. The fishing rod is modelled as a light rod.
  a Describe the effects of this modelling assumption.
  b Comment on its validity in this situation.

5 Make a list of the assumptions you might make to create simple models of the following:
  a The motion of a golf ball after it is hit
  b The motion of a child on a sledge going down a snow-covered hill
  c The motion of two objects of different masses connected by a string that passes over a pulley
  d The motion of a suitcase on wheels being pulled along a path by its handle.



8.3 Quantities and units

The International System of Units, (abbreviated **SI** from the French, Système international d'unités) is the modern form of the metric system. These **base** SI units are most commonly used in mechanics.

Quantity	Unit	Symbol
Mass	kilogram	kg
Length/displacement	metre	m
Time	seconds	s

Watch out

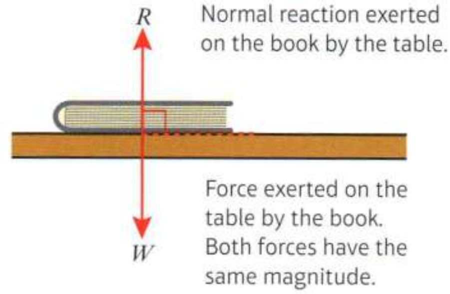
A common misconception is that kilograms measure weight not mass. However, **weight** is a **force** which is measured in **newtons (N)**.

These **derived** units are compound units built from the base units.

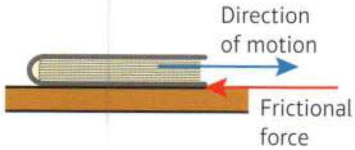
Quantity	Unit	Symbol
Speed/velocity	metres per second	$m\,s^{-1}$
Acceleration	metres per second per second	$m\,s^{-2}$
Weight/force	newton	N (= $kg\,m\,s^{-2}$)

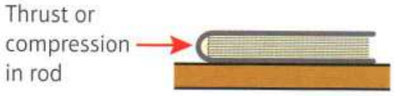
You will encounter a variety of forces in mechanics. These **force diagrams** show some of the most common forces.

- The **weight** (or gravitational force) of an object acts vertically downwards.
- The **normal reaction** is the force which acts perpendicular to a surface when an object is in contact with the surface. In this example the normal reaction is due to the weight of the book resting on the surface of the table.

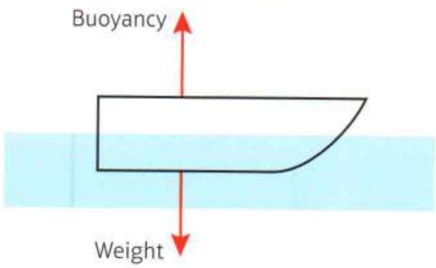


- The **friction** is a force which opposes the motion between two rough surfaces.
- If an object is being pulled along by a string, the force acting on the object is called the **tension** in the string.
- If an object is being pushed along using a light rod, the force acting on the object is called the **thrust** or **compression** in the rod.





- **Buoyancy** is the upward force on a body that allows it to float or rise when submerged in a liquid. In this example buoyancy acts to keep the boat afloat in the water.



- **Air resistance** opposes motion. In this example the weight of the parachutist acts vertically downwards and the air resistance acts vertically upwards.

Example 3

Write the following quantities in SI units.

a 4 km **b** 0.32 grams **c** 5.1×10^6 km h^{-1}

a $4\text{ km} = 4 \times 1000 = 4000\text{ m}$

The SI unit of length is the metre and 1 km = 1000 m.

b $0.32\text{ g} = 0.32 \div 1000 = 3.2 \times 10^{-4}\text{ kg}$

The SI unit of mass is the kg and 1 kg = 1000 g. The answer is given in standard form.

c $5.1 \times 10^6\text{ km h}^{-1} = 5.1 \times 10^6 \times 1000$
$= 5.1 \times 10^9\text{ m h}^{-1}$

The SI unit of speed is m s^{-1}. Convert from km h^{-1} to m h^{-1} by multiplying by 1000.

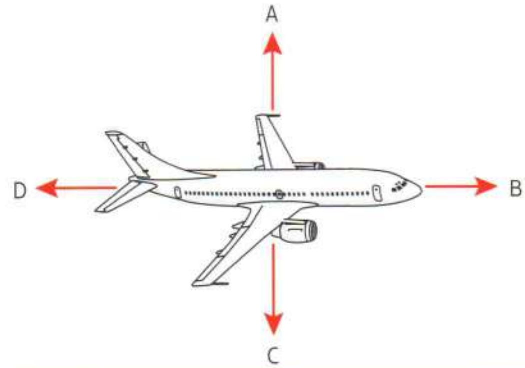
$5.1 \times 10^9 \div (60 \times 60) = 1.42 \times 10^6\text{ m s}^{-1}$

Convert from m h^{-1} to m s^{-1} by dividing by 60 × 60. The answer is given in standard form to 3 s.f.

Example 4

The force diagram shows an aircraft in flight.

Write down the names of the four forces shown on the diagram.



A upward thrust

B forward thrust

C weight

D air resistance

Also known as 'lift', this is the upward force that keeps the aircraft up in the air.

Also known as 'thrust', this is the force that propels the aircraft forward.

This is the gravitational force acting downwards on the aircraft.

Also known as 'drag', this is the force that acts in the opposite direction to the forward thrust.

Exercise 8C

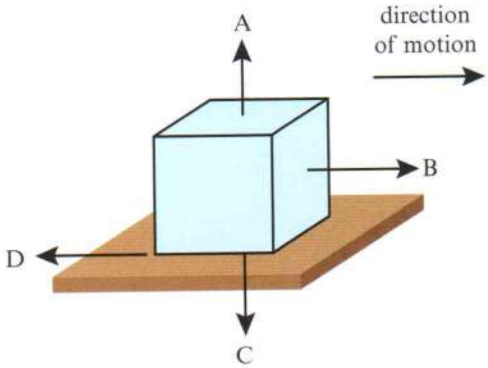
1 Convert to SI units:

a $65\,\text{km}\,\text{h}^{-1}$ **b** $15\,\text{g}\,\text{cm}^{-2}$ **c** 30 cm per minute

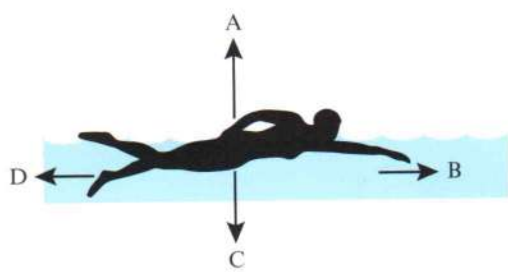
d $24\,\text{g}\,\text{m}^{-3}$ **e** $4.5 \times 10^{-2}\,\text{g}\,\text{cm}^{-3}$ **f** $6.3 \times 10^{-3}\,\text{kg}\,\text{cm}^{-2}$

2 Write down the names of the forces shown in each of these diagrams.

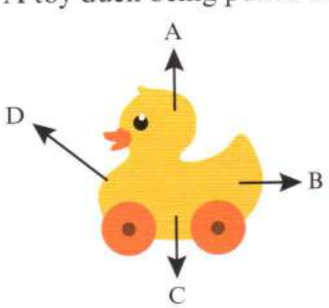
a A box being pushed along rough ground



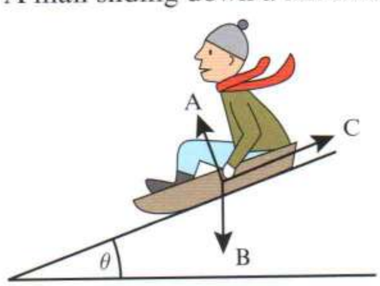
b A man swimming through the water



c A toy duck being pulled along by a string



d A man sliding down a hill on a sledge



8.4 Working with vectors

- **A vector is a quantity which has both magnitude and direction.**

These are examples of **vector** quantities.

Quantity	Description	Unit
Displacement	distance in a particular direction	metre (m)
Velocity	rate of change of displacement	metres per second ($m\,s^{-1}$)
Acceleration	rate of change of velocity	metres per second per second ($m\,s^{-2}$)
Force/weight	described by magnitude, direction and point of application	newton (N)

- **A scalar quantity has magnitude only.**

These are examples of **scalar** quantities.

Quantity	Description	Unit
Distance	measure of length	metre (m)
Speed	measure of how quickly a body moves	metres per second ($m\,s^{-1}$)
Time	measure of ongoing events taking place	second (s)
Mass	measure of the quantity of matter contained in an object	kilogram (kg)

Scalar quantities are always **positive**. When considering motion in a straight line (1-dimensional motion), **vector** quantities can be **positive** or **negative**.

Example 5

Fully describe the motion of the following particles:

	a	b	c	d
Velocity	+ve	+ve	−ve	−ve
Acceleration	+ve	−ve	−ve	+ve

Positive direction →

a The particle is moving to the right and its speed is increasing.

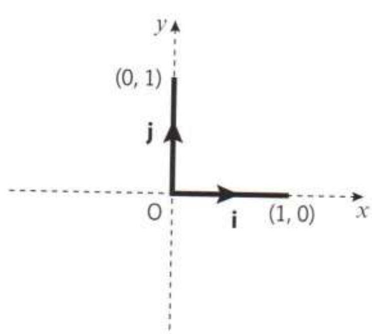
b The particle is moving to the right and its speed is decreasing.

c The particle is moving to the left and its speed is increasing.

d The particle is moving to the left and its speed is decreasing.

When the direction of the acceleration opposes the direction of motion, the particle is **slowing down**. This is also called **deceleration** or **retardation**.

You can describe vectors using **i**, **j** notation, where **i** and **j** are the unit vectors in the positive x and y directions.



Links When a vector is given in **i–j** notation you can:
- use Pythagoras' theorem to find its magnitude
- use trigonometry to work out its direction

← Pure Year 1, Chapter 11

- **Distance is the magnitude of the displacement vector.**
- **Speed is the magnitude of the velocity vector.**

Example 6

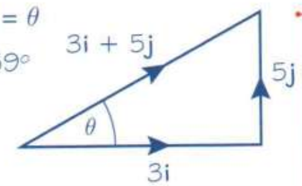
The velocity of a particle is given by $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$ m s^{-1}. Find:

a the speed of the particle

b the angle the direction of motion of the particle makes with the unit vector **i**.

a Speed $= |\mathbf{v}| = \sqrt{3^2 + 5^2} = \sqrt{34}$
$= 5.83$ m s^{-1}

b Angle made with $\mathbf{i} = \theta$
$\tan\theta = \frac{5}{3}$ so $\theta = 59°$



The speed of the particle is the magnitude of the vector **v**. This is written as $|\mathbf{v}|$.
In general, if $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ then $|\mathbf{v}| = \sqrt{a^2 + b^2}$

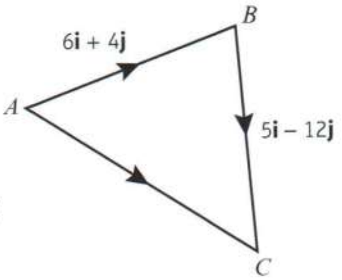
Draw a diagram. The direction of **v** can be found using trigonometry.

Example 7

A man walks from A to B and then from B to C.
His displacement from A to B is $6\mathbf{i} + 4\mathbf{j}$ km.
His displacement from B to C is $5\mathbf{i} - 12\mathbf{j}$ km.

a What is the magnitude of the displacement from A to C?

b What is the total distance the man has walked in getting from A to C?



a $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{AC} = \begin{pmatrix}6\\4\end{pmatrix} + \begin{pmatrix}5\\-12\end{pmatrix} = \begin{pmatrix}11\\-8\end{pmatrix}$

$|\overrightarrow{AC}| = \sqrt{11^2 + (-8)^2}$
$= 13.6$ km

b Total distance $= |\overrightarrow{AB}| + |\overrightarrow{BC}|$

$|\overrightarrow{AB}| = \sqrt{6^2 + 4^2} = 7.21$ km

$|\overrightarrow{BC}| = \sqrt{5^2 + (-12)^2} = 13$ km

total distance $= 7.21 + 13 = 20.21$ km

This is the triangle law for vector addition. Write $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in column vector form and add the **i** components and the **j** components to find $|\overrightarrow{AC}|$. Use Pythagoras to work out the magnitude of $\overrightarrow{AC}$.

Note that the distance from A to C is not the same as the total distance travelled which is $|\overrightarrow{AB}| + |\overrightarrow{BC}|$.

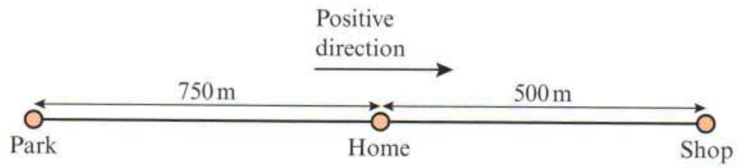
Online Check your answers by entering the vectors directly into your calculator.

Exercise 8D

1 A man walks from his home along a straight road to a shop with a speed of 2.1 m s^{-1} and walks home again at a speed of 1.8 m s^{-1}.

He then jogs along a straight road from his home to the park with a speed of 2.7 m s^{-1} and returns home at a speed of 2.5 m s^{-1}.

The park, the man's home and the shop all lie on a straight line, as shown in the diagram.



Taking the positive direction as shown in the diagram, state the man's:

a velocity on the journey from his home to the shop

b displacement from his home when he reaches the shop

c velocity on the journey from the shop to his home

d velocity on the journey from his home to the park

e displacement from his home when he reaches the park

f velocity on the journey from the park to his home.

2 The velocity of a car is given by $\mathbf{v} = 12\mathbf{i} - 10\mathbf{j}\text{ m s}^{-1}$. Find:

a the speed of the car

b the angle the direction of motion of the car makes with the unit vector $\mathbf{i}$.

3 The acceleration of a motorbike is given by $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}\text{ m s}^{-2}$. Find:

a the magnitude of the acceleration

b the angle the direction of the acceleration vector makes with the unit vector $\mathbf{j}$.

Problem-solving

Draw a sketch to help you find the direction. $\mathbf{j}$ acts in the positive y-direction, so the angle between $\mathbf{j}$ and the vector $3\mathbf{i} - 4\mathbf{j}$ will be obtuse.

4 A girl cycles from A to B and then from B to C.

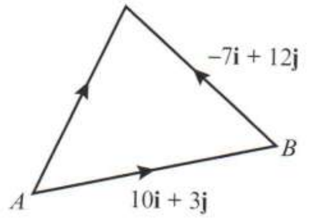
The displacement from A to B is $10\mathbf{i} + 3\mathbf{j}$ km.

The displacement from B to C is $-7\mathbf{i} + 12\mathbf{j}$ km.

a Find the magnitude of the displacement from A to C.

b Find the total distance the girl has cycled in getting from A to C.

c Work out the angle $\overrightarrow{AC}$ makes with the unit vector $\mathbf{i}$.



Mixed exercise 8

(P) **1** The motion of a cricket ball after it is hit until it lands on the cricket pitch can be modelled using the equation $h = \frac{1}{10}(24x - 3x^2)$, where h m is the vertical height of the ball above the cricket pitch and x m is the horizontal distance from where it was hit. Find:

Hint The path of the cricket ball is modelled as a quadratic curve. Draw a sketch for the model and use the symmetry of the curve.

a the vertical height of the ball when it is at a horizontal distance of 2 m from where it was hit

b the two horizontal distances for which the height of the ball was 2.1 m.

Given that the model is valid from when the ball is hit to when it lands on the cricket pitch:

c find the values of x for which the model is valid

d work out the maximum height of the cricket ball.

(P) **2** A diver dives from a diving board into a swimming pool with a depth of 4.5 m. The height of the diver above the water, h m, can be modelled using $h = 10 - 0.58x^2$ for $0 \leqslant x \leqslant 5$, where x m is the horizontal distance from the end of the diving board.

a Find the height of the diver when $x = 2$ m.

b Find the horizontal distance from the end of the diving board to the point where the diver enters the water.

In this model the diver is modelled as particle.

c Describe the effects of this modelling assumption.

d Comment on the validity of this modelling assumption for the motion of the diver after she enters the water.

3 Make a list of the assumptions you might make to create simple models of the following:

a The motion of a man skiing down a snow-covered slope.

b The motion of a yo-yo on a string.

In each case, describe the effects of the modelling assumptions.

4 Convert to SI units:

a 2.5 km per minute **b** $0.6\,\text{kg}\,\text{cm}^{-2}$ **c** $1.2 \times 10^3\,\text{g}\,\text{cm}^{-3}$

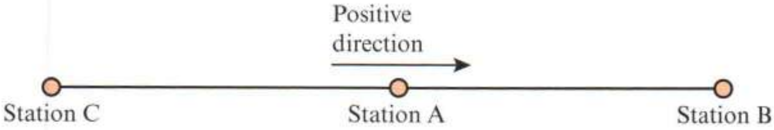
5 A man throws a bowling ball in a bowling alley.

a Make a list of the assumptions you might make to create a simple model of the motion of the bowling ball.

b Taking the direction that the ball travels in as the positive direction, state with a reason whether each of the following are likely to be positive or negative:

i the velocity **ii** the acceleration.

6 A train engine pulling a truck starts at station A then travels in a straight line to station B. It then moves back from station B to station A and on to station C as shown in the diagram.



Hint The **sign** of something means whether it is positive or negative.

What is the sign of the velocity and displacement on the journey from:

a station A to station B **b** station B to station A **c** station A to station C?

7 The acceleration of a boat is given by $\mathbf{a} = -0.05\mathbf{i} + 0.15\mathbf{j}\ \text{m s}^{-2}$. Find:

a the magnitude of the acceleration

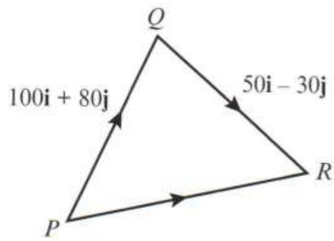
b the angle the direction of the acceleration vector makes with the unit vector $\mathbf{i}$.

8 The velocity of a toy car is given by $\mathbf{v} = 3.5\mathbf{i} - 2.5\mathbf{j}\ \text{m s}^{-1}$. Find:

a the speed of the toy car

b the angle the direction of motion of the toy car makes with the unit vector $\mathbf{j}$.

9 A plane flies from P to Q and then from Q to R.
The displacement from P to Q is $100\mathbf{i} + 80\mathbf{j}$ m.
The displacement from Q to R is $50\mathbf{i} - 30\mathbf{j}$ m.



a Find the magnitude of the displacement from P to R.

b Find the total distance the plane has travelled in getting from P to R.

c Find the angle the vector $\overrightarrow{PQ}$ makes with the unit vector $\mathbf{j}$.

Summary of key points

1. Mathematical models can be constructed to simulate real-life situations.
2. Modelling assumptions can be used to simplify your calculations.
3. The base SI units most commonly used in mechanics are:

Quantity	Unit	Symbol
Mass	kilogram	kg
Length/displacement	metre	m
Time	second	s

4. A vector is a quantity which has both magnitude and direction.
5. A scalar quantity has magnitude only.
6. Distance is the magnitude of the displacement vector.
7. Speed is the magnitude of the velocity vector.