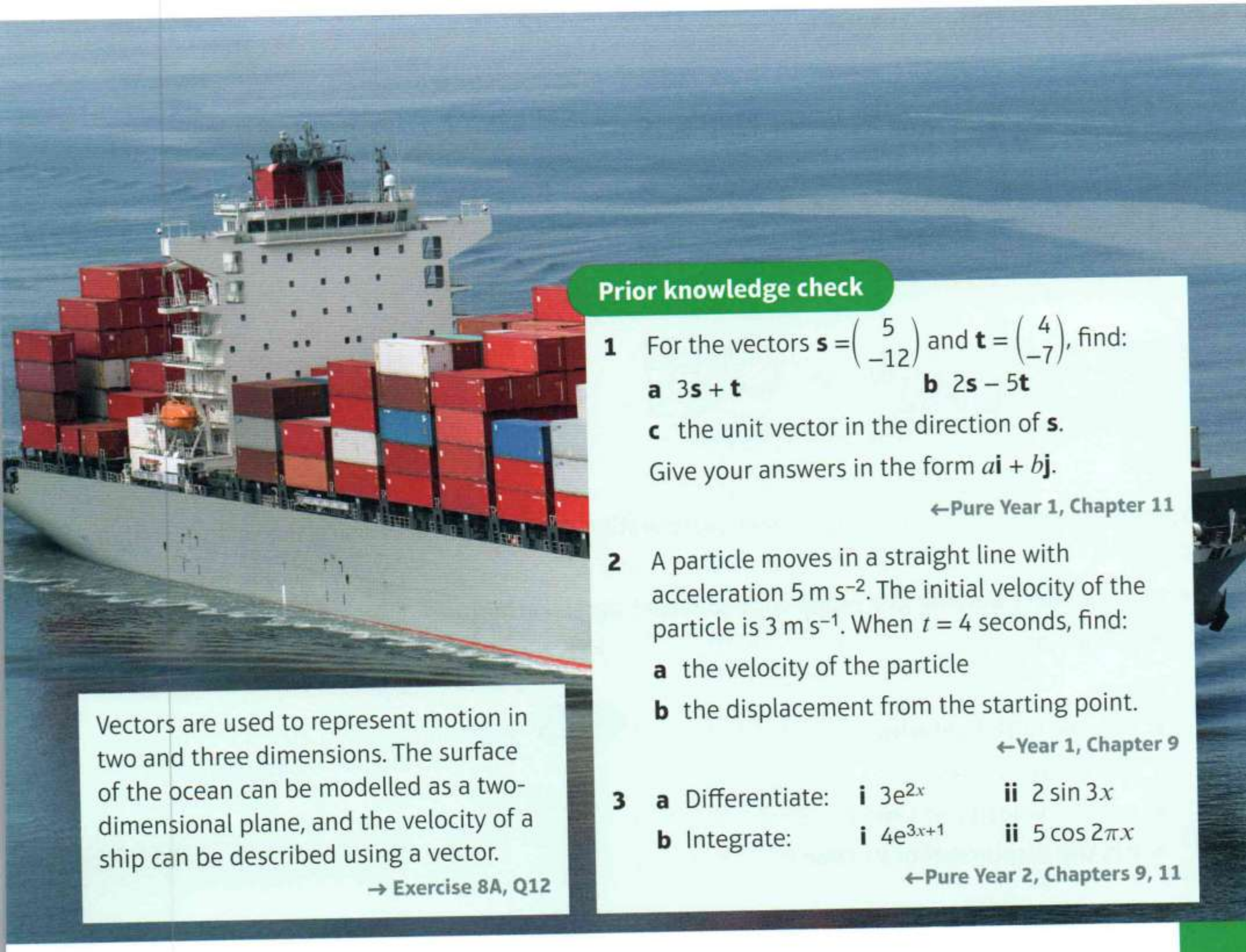
Further kinematics

8

Objectives

After completing this chapter you should be able to:

- Work with vectors for displacement, velocity and acceleration when using the vector equations of motion → pages 160–167
- Use calculus with harder functions of time involving variable acceleration → pages 167–170
- Differentiate and integrate vectors with respect to time → pages 171–177



Vectors are used to represent motion in two and three dimensions. The surface of the ocean can be modelled as a two-dimensional plane, and the velocity of a ship can be described using a vector.
→ Exercise 8A, Q12

Prior knowledge check

1 For the vectors $\mathbf{s} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$, find:

a $3\mathbf{s} + \mathbf{t}$ **b** $2\mathbf{s} - 5\mathbf{t}$

c the unit vector in the direction of **s**.

Give your answers in the form $a\mathbf{i} + b\mathbf{j}$.

←Pure Year 1, Chapter 11

2 A particle moves in a straight line with acceleration 5 m s^{-2}. The initial velocity of the particle is 3 m s^{-1}. When $t = 4$ seconds, find:

a the velocity of the particle

b the displacement from the starting point.

←Year 1, Chapter 9

3 **a** Differentiate: **i** $3e^{2x}$ **ii** $2\sin 3x$

b Integrate: **i** $4e^{3x+1}$ **ii** $5\cos 2\pi x$

←Pure Year 2, Chapters 9, 11

8.1 Vectors in kinematics

You can use two-dimensional vectors to describe motion in a plane.

- **If a particle starts from the point with position vector $\mathbf{r}_0$ and moves with constant velocity $\mathbf{v}$, then its displacement from its initial position at time t is $\mathbf{v}t$ and its position vector $\mathbf{r}$ is given by $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$.**

Notation In this equation $\mathbf{r}$, $\mathbf{r}_0$ and $\mathbf{v}$ are vectors and t is a scalar. Displacement, velocity and acceleration can be given using **i**-**j** notation, or as column vectors.

Unless otherwise informed, you should assume that **i** and **j** are unit vectors due east and north respectively.

Example 1

A particle starts from the point with position vector $(3\mathbf{i} + 7\mathbf{j})$ m and moves with constant velocity $(2\mathbf{i} - \mathbf{j})$ m s^{-1}.

a Find the position vector of the particle 4 seconds later.

b Find the time at which the particle is due east of the origin.

a $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$ — Write down the formula.

$= (3\mathbf{i} + 7\mathbf{j}) + 4(2\mathbf{i} - \mathbf{j})$

$= (3\mathbf{i} + 7\mathbf{j}) + (8\mathbf{i} - 4\mathbf{j})$

$= (11\mathbf{i} + 3\mathbf{j})$ m — With a vector quantity you still need to give units.

b $\mathbf{r} = (3\mathbf{i} + 7\mathbf{j}) + (2\mathbf{i} - \mathbf{j})t$ — Use $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$

$= (3 + 2t)\mathbf{i} + (7 - t)\mathbf{j}$

$7 - t = 0$ so $t = 7$ seconds.

Problem-solving When the particle is due east of the origin, the displacement will only have an **i**-component. Set the coefficient of **j** equal to 0 and solve the equation to find t.

Online Explore the solution to this example using technology.

You can solve questions involving constant acceleration in two dimensions using the vector equations of motion.

- **For an object moving in a plane with constant acceleration:**
  - $\mathbf{v} = \mathbf{u} + \mathbf{a}t$
  - $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

  where
  - **$\mathbf{u}$ is the initial velocity**
  - **$\mathbf{a}$ is the acceleration**
  - **$\mathbf{v}$ is the velocity at time t**
  - **$\mathbf{r}$ is the displacement at time t**

Links These are the vector equivalents of the *suvat* formulae for motion in **one** dimension:

$v = u + at$

$s = ut + \frac{1}{2}at^2$ ← Year 1, Sections 9.3, 9.4

Example 2

A particle P has velocity $(-3\mathbf{i} + \mathbf{j})\ \text{m s}^{-1}$ at time $t = 0$. The particle moves with constant accleration $\mathbf{a} = (2\mathbf{i} + 3\mathbf{j})\ \text{m s}^{-2}$. Find the speed of the particle and the bearing on which it is travelling at time $t = 3$ seconds.

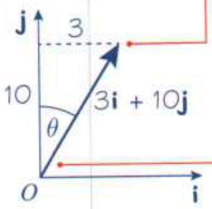
$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$= (-3\mathbf{i} + \mathbf{j}) + 3 \times (2\mathbf{i} + 3\mathbf{j})$
$= (-3\mathbf{i} + \mathbf{j}) + (6\mathbf{i} + 9\mathbf{j})$
$= (3\mathbf{i} + 10\mathbf{j})\ \text{m s}^{-1}$

Use $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ with $\mathbf{u} = (-3\mathbf{i} + \mathbf{j})\ \text{m s}^{-1}$ and $\mathbf{a} = (2\mathbf{i} + 3\mathbf{j})\ \text{m s}^{-2}$.

So the speed of $P = \sqrt{3^2 + 10^2} = \sqrt{109}$
$= 10.4\ \text{m s}^{-1}$ to 3 s.f.

The speed of P is the magnitude of its velocity.

Components of velocity at time $t = 3$ seconds:



$\tan\theta = \frac{3}{10} \Rightarrow \theta = 16.7°$

The bearing at time $t = 3$ seconds is 017°.

The direction of the velocity vector tells you the direction in which the object is travelling at that time.

The bearing is measured clockwise from the north vector **j**.

Example 3

An ice skater is skating on a large flat ice rink. At time $t = 0$ the skater is at a fixed point O and is travelling with velocity $(2.4\mathbf{i} - 0.6\mathbf{j})\ \text{m s}^{-1}$.

At time $t = 20$ s the skater is travelling with velocity $(-5.6\mathbf{i} + 3.4\mathbf{j})\ \text{m s}^{-1}$.

Relative to O, the skater has position vector $\mathbf{s}$ at time t seconds.

Modelling the ice skater as a particle with constant acceleration, find:

a the acceleration of the ice skater

b an expression for $\mathbf{s}$ in terms of t

c the time at which the skater is directly north-east of O.

A second skater travels so that she has position vector $\mathbf{r} = (1.1t - 6)\mathbf{j}$ m relative to O at time t.

d Show that the two skaters will meet.

a Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$,

$$\begin{pmatrix}-5.6\\3.4\end{pmatrix} = \begin{pmatrix}2.4\\-0.6\end{pmatrix} + 20\mathbf{a}$$

It is often easier to work using column vectors rather than **i**–**j** notation.

$$20\mathbf{a} = \begin{pmatrix}-5.6 - 2.4\\3.4 + 0.6\end{pmatrix} = \begin{pmatrix}-8\\4\end{pmatrix}$$

$$\mathbf{a} = \frac{1}{20}\begin{pmatrix}-8\\4\end{pmatrix} = \begin{pmatrix}-0.4\\0.2\end{pmatrix}\text{m s}^{-2}$$

or $\mathbf{a} = (-0.4\mathbf{i} + 0.2\mathbf{j})\ \text{m s}^{-2}$

Unless otherwise instructed you can give your answer either in column vector or **i**–**j** form. Remember to include units with your answer.

b Using $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

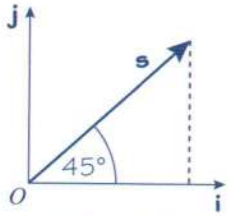
$$\mathbf{s} = \begin{pmatrix} 2.4 \\ -0.6 \end{pmatrix}t + \frac{1}{2}\begin{pmatrix} -0.4 \\ 0.2 \end{pmatrix}t^2$$

$$= \begin{pmatrix} 2.4t - 0.2t^2 \\ -0.6t + 0.1t^2 \end{pmatrix}\text{m}$$

or $\mathbf{s} = ((2.4t - 0.2t^2)\mathbf{i} + (0.1t^2 - 0.6t)\mathbf{j})$ m

Online Explore the solution to this example using technology.

c



Problem-solving

When the skater is directly north-east of O, the **i** and **j** components of the displacement must be equal. Set these components equal and solve the corresponding equation to find the value of t.

$\mathbf{s} = (2.4t - 0.2t^2)\mathbf{i} + (0.1t^2 - 0.6t)\mathbf{j}$

$2.4t - 0.2t^2 = 0.1t^2 - 0.6t$

$3t - 0.3t^2 = 0$

$3t(1 - 0.1t) = 0$

either $3t = 0$ so ~~$t = 0$~~

or $1 - 0.1t = 0$ so $t = 10$ seconds

At $t = 0$, the skater is at O and both components are equal to 0, so reject this solution.

d For the second skater: $\mathbf{r} = (1.1t - 6)\mathbf{j}$

$(2.4t - 0.2t^2)\mathbf{i} + (0.1t^2 - 0.6t)\mathbf{j} = (1.1t - 6)\mathbf{j}$

Equate the displacement vectors for each skater. If the skaters meet, there will be a value of t for which $\mathbf{s} = \mathbf{r}$.

i: $2.4t - 0.2t^2 = 0$

$24t - 2t^2 = 0$

$2t(12 - t) = 0$ either $t = 0$ or $t = 12$

Equate coefficients of **i**.

j: When $t = 0$, $0.1t^2 - 0.6t = 0$

and $1.1t - 6 = -6$

so skaters do not meet at $t = 0$.

When $t = 12$ seconds, $0.1t^2 - 0.6t = 7.2$

and $1.1t - 6 = 7.2$

Check whether the **j** components are equal at either $t = 0$ or $t = 12$.

So the two skaters will meet when $t = 12$ seconds.

You could also equate the **j** coefficients and solve:

$0.1t^2 - 0.6t = 1.1t - 6$

$t^2 - 17t + 60 = 0$

$(t - 12)(t - 5) = 0$

So $t = 12$ or $t = 5$.

$t = 12$ satisfies both equations so the skaters meet at this time.

Exercise 8A

For all questions in this exercise, take **i** and **j** to be the unit vectors due east and north respectively.

1 A particle P starts at the point with position vector $\mathbf{r}_0$. P moves with constant velocity $\mathbf{v}$ m s^{-1}. After t seconds, P is at the point with position vector $\mathbf{r}$.

a Find $\mathbf{r}$ if $\mathbf{r}_0 = 2\mathbf{i}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$, and $t = 4$.

b Find $\mathbf{r}$ if $\mathbf{r}_0 = 3\mathbf{i} - \mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$, and $t = 5$.

c Find $\mathbf{r}_0$ if $\mathbf{r} = 4\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$, and $t = 3$.

d Find $\mathbf{r}_0$ if $\mathbf{r} = -2\mathbf{i} + 5\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$, and $t = 6$.

e Find $\mathbf{v}$ if $\mathbf{r}_0 = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{r} = 8\mathbf{i} - 7\mathbf{j}$, and $t = 3$.

f Find t if $\mathbf{r}_0 = 4\mathbf{i} + \mathbf{j}$, $\mathbf{r} = 12\mathbf{i} - 11\mathbf{j}$, and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$.

2 A radio-controlled boat starts from position vector $(10\mathbf{i} - 5\mathbf{j})$ m relative to a fixed origin and travels with constant velocity, passing a point with position vector $(-2\mathbf{i} + 9\mathbf{j})$ m after 4 seconds. Find the speed and bearing of the boat.

3 A clockwork mouse starts from a point with position vector $(-2\mathbf{i} + 3\mathbf{j})$ m relative to a fixed origin and moves in a straight line with a constant speed of $4\,\text{m s}^{-1}$. Find the time taken for the mouse to travel to the point with position vector $(6\mathbf{i} - 3\mathbf{j})$ m.

4 A helicopter starts from the point with position vector $\begin{pmatrix}120\\-10\end{pmatrix}$ m relative to a fixed origin, and moves with constant velocity $\begin{pmatrix}-30\\40\end{pmatrix}\text{m s}^{-1}$. Find:

a the position vector of the helicopter t seconds later

b the time at which the helicopter is due north of the origin.

Hint When the helicopter is due north of the origin, the **i**-component of its position vector will be 0.

(P) **5** At time $t = 0$, the particle P is at the point with position vector $4\mathbf{i}$, and moving with constant velocity $\mathbf{i} + \mathbf{j}\,\text{m s}^{-1}$. A second particle Q is at the point with position vector $-3\mathbf{j}$ and moving with velocity $\mathbf{v}\,\text{m s}^{-1}$. After 8 seconds, the paths of P and Q meet. Find the speed of Q.

(P) **6** At noon, a ferry F is 400 m due north of an observation point O and is moving with a constant velocity of $(7\mathbf{i} + 7\mathbf{j})\,\text{m s}^{-1}$, and a speedboat S is 500 m due east of O, moving with a constant velocity of $(-3\mathbf{i} + 15\mathbf{j})\,\text{m s}^{-1}$.

a Write down the position vectors of F and S at time t seconds after noon.

b Show that F and S will collide, and find the position vector of the point of collision.

7 A particle starts at rest and moves with constant acceleration. After 5 seconds its velocity is $\begin{pmatrix}3\\4\end{pmatrix}\text{m s}^{-1}$.

a Find the acceleration of the particle.

b The displacement vector of the particle from its starting position after 5 seconds.

8 An object moves with constant acceleration so that its velocity changes from $(15\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$ to $(5\mathbf{i} - 3\mathbf{j})\,\text{m s}^{-1}$ in 4 seconds. Find:

a the acceleration of the particle

Given that the initial position vector of the particle relative to a fixed origin O is $10\mathbf{i} - 8\mathbf{j}$ m,

b find the position vector of the particle after t seconds.

9 A plane moves with constant acceleration $\begin{pmatrix}-1\\1.5\end{pmatrix}\text{m s}^{-2}$.

When $t = 0$, the velocity of the plane is $\begin{pmatrix}70\\-30\end{pmatrix}\text{m s}^{-1}$. Find:

a the velocity of the plane after 10 seconds

b the distance of the plane from its starting point after 10 seconds.

(P) **10** A model boat moves with constant acceleration $(0.2\mathbf{i} + 0.6\mathbf{j})\text{ m s}^{-2}$. After 20 seconds its velocity is $(4\mathbf{i} + 3\mathbf{j})\text{ m s}^{-1}$. Find the displacement vector of the boat from its starting position after 20 seconds.

(P) **11** A particle A starts at the point with position vector $12\mathbf{i} + 12\mathbf{j}$. The initial velocity of A is $(-\mathbf{i} + \mathbf{j})\text{ m s}^{-1}$, and it has constant acceleration $(2\mathbf{i} - 4\mathbf{j})\text{ m s}^{-2}$. Another particle, B, has initial velocity $\mathbf{i}\text{ m s}^{-1}$ and constant acceleration $2\mathbf{j}\text{ m s}^{-2}$. After 3 seconds the two particles collide. Find:

a the speeds of the two particles when they collide

b the position vector of the point where the two particles collide

c the position vector of B's starting point.

(E/P) **12** A ship is moving such that at time 12:00 its position is O and its velocity is $(-4\mathbf{i} + 8\mathbf{j})\text{ km h}^{-1}$. At 14:00, the ship is travelling with velocity $(-2\mathbf{i} - 6\mathbf{j})\text{ km h}^{-1}$.

Relative to O, the ship has displacement $\mathbf{s}$ at time t hours after 12:00 where $t \geqslant 0$.

Modelling the ship as a particle with constant acceleration, find:

a the acceleration of the ship **(2 marks)**

b an expression for $\mathbf{s}$ in terms of t **(2 marks)**

c the time at which the ship is directly south-west of O. **(3 marks)**

At time t hours after 12:00, another ship has displacement $\mathbf{r} = (40 - 25t)\mathbf{j}$ relative to O.

d Find the position vector of the point where the two ships meet. **(4 marks)**

(E/P) **13** A particle moves so that its position vector, in metres, relative to a fixed origin O at time t seconds is $\mathbf{r} = (2t^2 - 3)\mathbf{i} + (7 - 4t)\mathbf{j}$, where $t \geqslant 0$.

a Show that the particle is north-east of O when $t^2 + 2t - 5 = 0$. **(2 marks)**

b Hence determine the distance of the particle from O when it is north-east of O, giving your answer correct to 3 significant figures. **(3 marks)**

A second particle moves with constant acceleration $(3a\mathbf{i} - 2a\mathbf{j})\text{ m s}^{-2}$. When $t = 0$ the velocity of the particle is $(5\mathbf{i} + 6\mathbf{j})\text{ m s}^{-1}$ and its position vector relative to O is $5\mathbf{j}$ m. When $t = 2$ seconds the particle is travelling with velocity $(b\mathbf{i} + 2b\mathbf{j})\text{ m s}^{-1}$.

c Find the speed and direction of the particle when $t = 2$. **(6 marks)**

d Find the distance between the two particles at this time. **(4 marks)**

Challenge

During an air show, a stunt aeroplane passes over a control tower with velocity $(20\mathbf{i} - 100\mathbf{j})\text{ m s}^{-1}$, and flies in a horizontal plane with constant acceleration $6\mathbf{j}\text{ m s}^{-2}$. A second aeroplane passes over the same control tower at time t seconds later, where $t > 0$, travelling with velocity $(70\mathbf{i} + 40\mathbf{j})\text{ m s}^{-1}$. The second aeroplane is flying in a higher horizontal plane with constant acceleration $-8\mathbf{j}\text{ m s}^{-2}$.

Given that the two aeroplanes pass directly over one another in their subsequent motion, find the value of t.

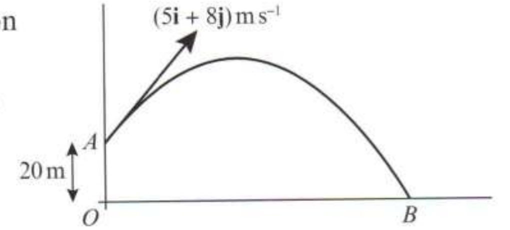
8.2 Vector methods with projectiles

Projectile motion is motion in a vertical plane with constant acceleration. Hence you can analyse it using the vector equations of motion. When using vectors with projectile questions you should consider **i** and **j** to be the unit vectors horizontally and vertically, unless you are told otherwise.

Links You can also analyse projectile motion by considering the horizontal and vertical components of velocity separately. ← Chapter 6

Example 4

A ball is struck by a racket from a point A which has position vector $20\mathbf{j}$ m relative to a fixed origin O. Immediately after being struck, the ball has velocity $(5\mathbf{i} + 8\mathbf{j})\text{ m s}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors horizontally and vertically respectively. After being struck, the ball travels freely under gravity until it strikes the ground at point B.



a Find the speed of the ball 1.5 seconds after being struck.

b Find an expression for the position vector, $\mathbf{r}$, of the ball relative to O at time t seconds.

c Hence determine the distance OB.

a $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$= (5\mathbf{i} + 8\mathbf{j}) + (-9.8\mathbf{j})t$

$= 5\mathbf{i} + (8 - 9.8t)\mathbf{j}\text{ m s}^{-1}$

Acceleration due to gravity acts vertically downwards, and has vector $-9.8\mathbf{j}\text{ m s}^{-2}$.

When $t = 1.5$: $\mathbf{v} = 5\mathbf{i} - 6.7\mathbf{j}\text{ m s}^{-1}$

Speed $= |\mathbf{v}| = \sqrt{5^2 + 6.7^2}$

$= 8.4\text{ m s}^{-1}$ (2 s.f.)

Speed is the magnitude of the velocity vector.

b Displacement relative to A:

$\mathbf{r}_A = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$= (5\mathbf{i} + 8\mathbf{j})t + \frac{1}{2}(-9.8\mathbf{j})t^2$ m

Position vector relative to O:

$\mathbf{r} = \mathbf{r}_A + 20\mathbf{j}$

$= (5\mathbf{i} + 8\mathbf{j})t + \frac{1}{2}(-9.8\mathbf{j})t^2 + 20\mathbf{j}$

$= (5t)\mathbf{i} + (8t - 4.9t^2 + 20)\mathbf{j}$ m

Watch out The equation $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ gives you the displacement relative to the starting position of the ball, A. To find the position vector of the ball relative to O you need to add on the position vector of A.

Rearrange so that you have separate expressions for the **i**-component and the **j**-component.

c When **j**-component is 0:

$8t - 4.9t^2 + 20 = 0$

For all the points on the horizontal axes, the vertical component of the position vector is 0.

$t = -1.362...$ or $t = 2.995...$

Use your calculator to solve the quadratic equation, then take the positive answer.

$OB = 5t = 5 \times 2.995... = 15$ m (2 s.f.)

The horizontal component of the position vector tells you the horizontal distance from O.

Exercise 8B

For all questions in this exercise **i** and **j** are unit vectors horizontally and vertically respectively. Unless stated otherwise, take $g = 9.8\,\text{m s}^{-2}$.

1 A particle P is projected from the origin with velocity $(12\mathbf{i} + 24\mathbf{j})\,\text{m s}^{-1}$. The particle moves freely under gravity. Find:

a the position vector of P after 3 s

b the speed of P after 3 s.

2 In this question use $g = 10\,\text{m s}^{-2}$

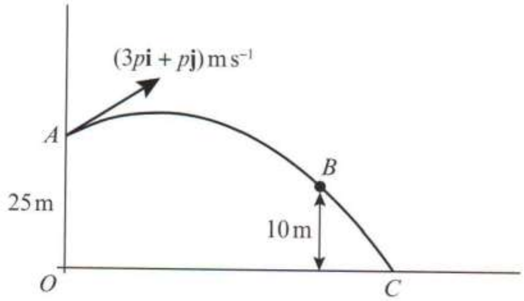
A particle P is projected from the origin with velocity $(4\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$. The particle moves freely under gravity. Find:

a the position vector of P after t s

b the greatest height of the particle.

Hint When the particle is at its greatest height, the **j**-component of the velocity will be 0.

E/P **3** A ball is projected from a point A at the top of a cliff, with position vector $25\mathbf{j}$ m relative to the base of the cliff O. The base of the cliff is at sea level. The velocity of projection is $(3p\mathbf{i} + p\mathbf{j})\,\text{m s}^{-1}$, where p is a constant. After 2 seconds, the ball passes a point B with position vector $(q\mathbf{i} + 10\mathbf{j})$ m, where q is a constant, before hitting the sea at point C. The ball is modelled as a particle moving freely under gravity and the sea is modelled as a horizontal plane.



a Suggest, with reasons, which of these two modelling assumptions is most realistic. **(2 marks)**

b Find the velocity vector of the ball at point B. **(6 marks)**

A remote-control boat leaves O at the same time the ball is projected, and travels in a straight line towards C with constant acceleration. Given that the ball lands on the boat,

c find the acceleration of the boat. **(6 marks)**

E **4** A particle P is projected with velocity $(3u\mathbf{i} + 4u\mathbf{j})\,\text{m s}^{-1}$ from a fixed point O on horizontal ground. Given that P strikes the ground at a point 750 m from O,

a show that $u = 17.5$ **(6 marks)**

b calculate the greatest height above the ground reached by P **(3 marks)**

c find the angle the direction of motion of P makes with **i** when $t = 5$. **(4 marks)**

E **5** A particle is projected with velocity $(8\mathbf{i} + 10\mathbf{j})\,\text{m s}^{-1}$ from a point O at the top of a cliff and moves freely under gravity. Six seconds after projection, the particle strikes the sea at the point S. Calculate:

a the horizontal distance between O and S **(2 marks)**

b the vertical distance between O and S. **(3 marks)**

At time T seconds after projection, the particle is moving with velocity $(8\mathbf{i} - 14.5\mathbf{j})\,\text{m s}^{-1}$.

c Find the value of T and the position vector, relative to O, of the particle at this instant. **(6 marks)**

E/P **6** In this question use $g = 10\,\text{m}\,\text{s}^{-2}$

A body B is projected from a fixed point O on horizontal ground with velocity $a\mathbf{i} + b\mathbf{j}\,\text{m}\,\text{s}^{-1}$, where a and b are positive constants. The body moves freely under gravity until it hits the ground at the point P, where it immediately comes to rest.

The position vector of a point on the path of B relative to O is $(x\mathbf{i} + y\mathbf{j})$ m.

a Show that $y = \frac{bx}{a} - \frac{5x^2}{a^2}$ **(5 marks)**

Given that $a = 8$, $OP = X$ m and the maximum vertical height of B above the ground is Y m,

b find, in terms of b,

i X **ii** Y **(6 marks)**

8.3 Variable acceleration in one dimension

The equations of motion for constant acceleration allow you to write velocity and displacement as functions of time.

When a body experiences **variable acceleration** you can model the acceleration as a function of time. You can use calculus to describe the relationship between displacement, velocity and acceleration.

Links Velocity, v, is the rate of change of displacement, s

$v = \frac{ds}{dt}$ $\quad s = \int v\,dt$

Acceleration, a, is the rate of change of velocity, v

$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ $\quad v = \int a\,dt$ ← Year 1, Section 11.4

You need to be able to use any of the functions and techniques from your A level course to analyse motion in a straight line.

Example 5

A particle is moving in a straight line with acceleration at time t seconds given by

$a = \cos 2\pi t\,\text{m}\,\text{s}^{-2}$, where $t \geqslant 0$

The velocity of the particle at time $t = 0$ is $\frac{1}{2\pi}\,\text{m}\,\text{s}^{-1}$. Find:

a an expression for the velocity at time t seconds

b the maximum speed

c the distance travelled in the first 3 seconds.

a $v = \int \cos 2\pi t\,dt$

$= \frac{1}{2\pi}\sin 2\pi t + c$

When $t = 0$, $v = \frac{1}{2\pi}$ so $c = \frac{1}{2\pi}$

$v = \frac{1}{2\pi}\sin 2\pi t + \frac{1}{2\pi}\,\text{m}\,\text{s}^{-1}$

b Maximum speed $= \frac{1}{2\pi} \times 1 + \frac{1}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}\,\text{m}\,\text{s}^{-1}$

$\int \cos at\,dt = \frac{1}{a}\sin at + c$ ← Pure Year 2, Section 11.1

Substitute $t = 0$, $v = \frac{1}{2\pi}$ into the equation to find c.

The maximum value of $\sin x$ is 1.

c $s = \frac{1}{2\pi}\int_0^3 (\sin 2\pi t + 1)\,dt$

$= \frac{1}{2\pi}\left[-\frac{1}{2\pi}\cos 2\pi t + t\right]_0^3$

$= \frac{1}{2\pi}\left[\left(-\frac{1}{2\pi} + 3\right) - \left(-\frac{1}{2\pi}\right)\right]$

$= \frac{3}{2\pi}$ m or 0.477 m (3 s.f.)

To find the distance travelled in the first 3 seconds integrate v between $t = 0$ and $t = 3$:
$\int_0^3 \left(\frac{1}{2\pi}\sin 2\pi t + \frac{1}{2\pi}\right)dt = \frac{1}{2\pi}\int_0^3 (\sin 2\pi t + 1)\,dt$

Example 6

A particle of mass 6 kg is moving on the positive x-axis. At time t seconds the displacement, s, of the particle from the origin is given by

$$s = 2t^{\frac{3}{2}} + \frac{e^{-2t}}{3}\text{ m, where } t \geqslant 0$$

a Find the velocity of the particle when $t = 1.5$.

Given that the particle is acted on by a single force of variable magnitude F N which acts in the direction of the positive x-axis,

b find the value of F when $t = 2$.

a $v = \frac{ds}{dt} = 3t^{\frac{1}{2}} - \frac{2e^{-2t}}{3}$ m s^{-1}

When $t = 1.5$ seconds:

$v = 3 \times 1.5^{0.5} - \frac{2e^{-3}}{3}$

$= 3.64$ m s^{-1} (3 s.f.)

b $a = \frac{dv}{dt} = 1.5t^{-0.5} + \frac{4e^{-2t}}{3}$ m s^{-2}

When $t = 2$ seconds:

$a = 1.5 \times 2^{-0.5} + \frac{4e^{-4}}{3}$

$= 1.0850\ldots$ m s^{-2}

$F = ma = 6 \times 1.0850\ldots = 6.51$ N (3 s.f.)

If $y = ae^{kt}$ then $\frac{dy}{dt} = kae^{kt}$

← Pure Year 2, Section 9.2

Differentiate to find an expression for v and substitute $t = 1.5$.

Problem-solving

You know the mass of the particle, so if you find the acceleration you can use $F = ma$ to find the magnitude of the force acting on it. Differentiate the velocity, then substitute $t = 2$ to find the acceleration when $t = 2$ seconds.

Exercise 8C

1 A particle P moves in a straight line. The acceleration, a, of P at time t seconds is given by $a = 1 - \sin \pi t$ m s^{-2}, where $t \geqslant 0$.

When $t = 0$, the velocity of P is 0 m s^{-1} and its displacement is 0 m. Find expressions for:

a the velocity at time t seconds

b the displacement at time t seconds.

2 A particle moving in a straight line has acceleration a, given by

$$a = \sin 3\pi t \text{ m s}^{-2},\ t \geqslant 0$$

At time t seconds the particle has velocity v m s^{-1} and displacement s m. Given that when $t = 0$, $v = \frac{1}{3\pi}$ and $s = 1$, find:

a an expression for v in terms of t

b the maximum speed of the particle

c an expression for s in terms of t.

(P) **3** An object moves in a straight line from a point O. At time t seconds the object has acceleration, a, where

$$a = -\cos 4\pi t \text{ m s}^{-2},\ 0 \leqslant t \leqslant 4$$

When $t = 0$, the velocity of the object is 0 m s^{-1} and its displacement is 0 m. Find:

a an expression for the velocity at time t seconds

b the maximum speed of the object

c an expression for the displacement of the object at time t seconds

d the maximum distance of the object from O

e the number of times the object changes direction during its motion.

Problem-solving

In part **e**, consider the number of times the velocity changes sign.

4 A body, M, of mass 5 kg moves along the positive x-axis. The displacement, s, of the body at time t seconds is given by $s = 3t^{\frac{2}{3}} + 2e^{-3t}$ m, where $t \geqslant 0$.

Find:

a the velocity of M when $t = 0.5$

b the acceleration of M when $t = 3$.

Given that M is acted on by a single force of variable magnitude F N which acts in the direction of the x-axis,

c find the value of F when $t = 3$ seconds.

(P) **5** A particle P moves in a straight line so that, at time t seconds, its displacement, s m, from a fixed point O on the line is given by

$$s = \begin{cases} \frac{1}{2}t, & 0 \leqslant t \leqslant 6 \\ \sqrt{t+3}, & t > 6 \end{cases}$$

Find:

a the velocity of P when $t = 4$

b the velocity of P when $t = 22$.

(P) **6** A particle P moves in a straight line so that, at time t seconds, its displacement from a fixed point O on the line is given by

$$s = \begin{cases} 3^t + 3t, & 0 \leqslant t \leqslant 3 \\ 24t - 36, & 3 < t \leqslant 6 \\ -252 + 96t - 6t^2, & t > 6 \end{cases}$$

Find:

a the velocity of P when $t = 2$

b the velocity of P when $t = 10$

c the greatest positive displacement of P from O

d the values of s when the speed of P is $18\,\text{m}\,\text{s}^{-1}$.

(P) **7** A particle moves in a straight line. At time t seconds after it begins its motion, the acceleration of the particle is $3\sqrt{t}\,\text{m}\,\text{s}^{-2}$ where $t > 0$.

Given that after 1 second the particle is moving with velocity $2\,\text{m}\,\text{s}^{-1}$, find the time taken for the particle to travel 16 m.

(E/P) **8** A runner takes part in a race in which competitors have to sprint 200 m in a straight line. At time t seconds after starting, her displacement, s, from the starting position is modelled as:

$$s = k\sqrt{t},\ 0 \leqslant t \leqslant T$$

Given that the runner completes the race in 25 seconds,

a find the value of k and the value of T **(2 marks)**

b find the speed of the runner when she crosses the finish line **(3 marks)**

c criticise this model for small values of t. **(2 marks)**

(E/P) **9** A particle P is moving in a straight line. At time t seconds, where $t \geqslant 0$, the acceleration of P is $a\,\text{m}\,\text{s}^{-2}$ and the velocity $v\,\text{m}\,\text{s}^{-1}$ of P is given by

$$v = 2 + 8\sin kt$$

where k is a constant.

The initial acceleration of P is $4\,\text{m}\,\text{s}^{-2}$.

a Find the value of k. **(3 marks)**

Using the value of k found in part **a**,

b find, in terms of π, the values of t in the interval $0 \leqslant t \leqslant 4\pi$ for which $a = 0$ **(2 marks)**

c show that $4a^2 = 64 - (v - 2)^2$ **(5 marks)**

d find the maximum velocity and the maximum acceleration. **(2 marks)**

(E/P) **10** A particle P moves on the x-axis. At time t seconds the velocity of P is $v\,\text{m}\,\text{s}^{-1}$ in the direction of x increasing, where v is given by

$$v = \begin{cases} 10t - 2t^{\frac{3}{2}}, & 0 \leqslant t \leqslant 4 \\ 24 - \left(\dfrac{t-4}{2}\right)^4, & t > 4 \end{cases}$$

When $t = 0$, P is at the origin O.

Find:

a the greatest speed of P in the interval $0 \leqslant t \leqslant 4$ **(4 marks)**

b the distance of P from O when $t = 4$ **(3 marks)**

c the time at which P is instantaneously at rest for $t > 4$ **(1 mark)**

d the total distance travelled by P in the first 10 seconds of its motion. **(7 marks)**

8.4 Differentiating vectors

You can use calculus with vectors to solve problems involving motion in two dimensions with variable acceleration.

To differentiate a vector quantity in the form $f(t)\mathbf{i} + g(t)\mathbf{j}$ you differentiate each function of time separately.

- **If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, then** $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$

  and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$

Notation Dot notation is a short-hand for differentiation with respect to time:

$\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$

$\ddot{x} = \frac{d^2x}{dt^2}$ and $\ddot{y} = \frac{d^2y}{dt^2}$

Example 7

A particle P of mass 0.8 kg is acted on by a single force **F** N. Relative to a fixed origin O, the position vector of P at time t seconds is **r** metres, where

$\mathbf{r} = 2t^3\mathbf{i} + 50t^{-\frac{1}{2}}\mathbf{j},\ t \geqslant 0$

Find:

a the speed of P when $t = 4$

b the acceleration of P as a vector when $t = 2$

c **F** when $t = 2$.

a $\mathbf{v} = \dot{\mathbf{r}} = 6t^2\mathbf{i} - 25t^{-\frac{3}{2}}\mathbf{j}\ \text{m s}^{-1}$

When $t = 4$: $\mathbf{v} = \left(96\mathbf{i} - \frac{25}{8}\mathbf{j}\right)\text{m s}^{-1}$

Speed $= \sqrt{96^2 + \left(\frac{25}{8}\right)^2} = 96.1\ \text{m s}^{-1}$ (3 s.f.)

b $\mathbf{a} = \ddot{\mathbf{r}} = \left(12t\mathbf{i} + \frac{75}{2}t^{-\frac{5}{2}}\mathbf{j}\right)\text{m s}^{-2}$

When $t = 2$: $\mathbf{a} = 24\mathbf{i} + 6.6291\ldots\mathbf{j}\ \text{m s}^{-2}$

c $\mathbf{F} = m\mathbf{a} = 0.8(24\mathbf{i} + 6.6291\ldots\mathbf{j})$

$= (19.2\mathbf{i} + 5.30\mathbf{j})\ \text{N}$ (3 s.f.)

Differentiate $2t^3$ and $5t^{-\frac{1}{2}}$ separately to find the **i**- and **j**-components of the velocity.

The speed is the magnitude of **v**.

$\mathbf{a} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2}$ or alternatively, $\mathbf{a} = \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt}$

Use $\mathbf{F} = m\mathbf{a}$ and round each coefficient to 3 significant figures.

Exercise 8D

1 At time t seconds, a particle P has position vector **r** m with respect to a fixed origin O, where

$\mathbf{r} = (3t - 4)\mathbf{i} + (t^3 - 4t)\mathbf{j},\ t \geqslant 0$

Find:

a the velocity of P when $t = 3$

b the acceleration of P when $t = 3$.

2 A particle P of mass 3 grams moving in a plane is acted on by a force **F** N. Its velocity at time t seconds is given by $\mathbf{v} = (t^2\mathbf{i} + (2t - 3)\mathbf{j})\ \text{m s}^{-1},\ t \geqslant 0$.

Find **F** when $t = 4$.

(P) **3** In this question **i** and **j** are the unit vectors east and north respectively.

A particle P is moving in a plane. At time t seconds, the position vector of P, **r** m, relative to a fixed origin O is given by $\mathbf{r} = 5e^{-3t}\mathbf{i} + 2\mathbf{j}$, $t \geqslant 0$.

a Find the time at which the particle is directly north-east of O.

b Find the speed of the particle at this time.

c Explain why the particle is always moving directly west.

(E) **4** At time t seconds, a particle P has position vector **r** m with respect to a fixed origin O, where

$$\mathbf{r} = 4t^2\mathbf{i} + (24t - 3t^2)\mathbf{j},\ t \geqslant 0$$

a Find the speed of P when $t = 2$. **(3 marks)**

b Show that the acceleration of P is a constant and find the magnitude of this acceleration. **(3 marks)**

(E) **5** A particle P of mass 0.5 kg is initially at a fixed origin O. At time $t = 0$, P is projected from O and moves so that, at time t seconds after projection, its position vector **r** m relative to O is given by

$$\mathbf{r} = (t^3 - 12t)\mathbf{i} + (4t^2 - 6t)\mathbf{j},\ t \geqslant 0$$

Find:

a the speed of projection of P **(5 marks)**

b the value of t at the instant when P is moving parallel to **j** **(3 marks)**

c the position vector of P at the instant when P is moving parallel to **j**. **(3 marks)**

The motion of the particle is due to it being acted on by a single variable force, **F** N.

d Given that the mass of the particle is 0.5 kg, find the magnitude of **F** when $t = 5$ s. **(4 marks)**

(E/P) **6** A particle P is moving in a plane. At time t seconds, the position vector of P, **r** m, is given by $\mathbf{r} = (3t^2 - 6t + 4)\mathbf{i} + (t^3 + kt^2)\mathbf{j}$, where k is a constant.

When $t = 3$, the speed of P is $12\sqrt{5}\,\mathrm{m\,s^{-1}}$.

a Find the two possible values of k. **(6 marks)**

b For each of these values of k, find the magnitude of the acceleration of P when $t = 1.5$. **(4 marks)**

(E) **7** Relative to a fixed origin O, the position vector of a particle P at time t seconds is **r** metres, where

$$\mathbf{r} = 6t^2\mathbf{i} + t^{\frac{5}{2}}\mathbf{j},\ t \geqslant 0$$

At the instant when $t = 4$, find:

a the speed of P **(5 marks)**

b the acceleration of P, giving your answer as a vector. **(2 marks)**

(E/P) **8** A particle P moves in a horizontal plane. At time t seconds, the position vector of P is **r** metres relative to a fixed origin O where **r** is given by

$$\mathbf{r} = (18t - 4t^3)\mathbf{i} + ct^2\mathbf{j},\ t \geqslant 0,$$

where c is a positive constant. When $t = 1.5$, the speed of P is $15\,\mathrm{m\,s^{-1}}$. Find:

a the value of c **(6 marks)**

b the acceleration of P when $t = 1.5$. **(3 marks)**

E **9** At time t seconds, a particle P has position vector $\mathbf{r}$ metres relative to a fixed origin O, where

$\mathbf{r} = (2t^2 - 3t)\mathbf{i} + (5t + t^2)\mathbf{j},\ t \geqslant 0$

Show that the acceleration of P is constant and find its magnitude. **(5 marks)**

E/P **10** A particle P moves in a horizontal plane. At time t seconds, the position vector of P is $\mathbf{r}$ metres relative to a fixed origin O, and $\mathbf{r}$ is given by $\mathbf{r} = (20t - 2t^3)\mathbf{i} + kt^2\mathbf{j},\ t \geqslant 0$, where k is a positive constant. When $t = 2$, the speed of P is $16\,\mathrm{m\,s^{-1}}$. Find:

a the value of k **(6 marks)**

b the acceleration of P at the instant when it is moving parallel to $\mathbf{j}$. **(4 marks)**

8.5 Integrating vectors

You can integrate vectors in the form $f(t)\mathbf{i} + g(t)\mathbf{j}$ by integrating each function of time separately.

- $\mathbf{v} = \int \mathbf{a}\,dt$ **and** $\mathbf{r} = \int \mathbf{v}\,dt$

Watch out When you integrate a vector, the constant of integration will also be a vector. Write it in the form $\mathbf{c} = p\mathbf{i} + q\mathbf{j}$.

Example 8

A particle P is moving in a plane. At time t seconds, its velocity $\mathbf{v}\,\mathrm{m\,s^{-1}}$ is given by

$\mathbf{v} = 3t\,\mathbf{i} + \frac{1}{2}t^2\mathbf{j},\ t \geqslant 0$

When $t = 0$, the position vector of P with respect to a fixed origin O is $(2\mathbf{i} - 3\mathbf{j})$ m. Find the position vector of P at time t seconds.

$$\mathbf{r} = \int \mathbf{v}\,dt = \int\left(3t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}\right)dt$$
$$= \frac{3t^2}{2}\mathbf{i} + \frac{t^3}{6}\mathbf{j} + \mathbf{c}$$

You integrate $3t$ and $\frac{1}{2}t^2$ in the usual way, using $\int t^n\,dt = \frac{t^{n+1}}{n+1}$. You must include the constant of integration, which is a vector, $\mathbf{c}$.

When $t = 0$, $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j}$:

$$2\mathbf{i} - 3\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{c}$$
$$\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$$

You are given an **initial condition** (or **boundary condition**) which allows you to find $\mathbf{c}$. Substitute $t = 0$ and $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j}$ into the integrated expression and solve to find $\mathbf{c}$.

← Pure Year 1, Section 13.3

Hence

$$\mathbf{r} = \frac{3t^2}{2}\mathbf{i} + \frac{t^3}{6}\mathbf{j} + 2\mathbf{i} - 3\mathbf{j} = \left(\frac{3t^2}{2} + 2\right)\mathbf{i} + \left(\frac{t^3}{6} - 3\right)\mathbf{j}$$

The position vector of P at time t seconds is $\left(\left(\frac{3t^2}{2} + 2\right)\mathbf{i} + \left(\frac{t^3}{6} - 3\right)\mathbf{j}\right)$ m.

Collect together the terms in $\mathbf{i}$ and $\mathbf{j}$ to complete your answer.

Example 9

A particle P is moving in a plane so that, at time t seconds, its acceleration is $(4\mathbf{i} - 2t\mathbf{j})\,\mathrm{m\,s^{-2}}$. When $t = 3$, the velocity of P is $6\mathbf{i}\,\mathrm{m\,s^{-1}}$ and the position vector of P is $(20\mathbf{i} + 3\mathbf{j})$ m with respect to a fixed origin O. Find:

a the angle between the direction of motion of P and $\mathbf{i}$ when $t = 2$

b the distance of P from O when $t = 0$.

a $\mathbf{v} = \int \mathbf{a}\,dt = \int (4\mathbf{i} - 2t\mathbf{j})\,dt$

$= 4t\mathbf{i} - t^2\mathbf{j} + \mathbf{c}$

When $t = 3$, $\mathbf{v} = 6\mathbf{i}$:

$6\mathbf{i} = 12\mathbf{i} - 9\mathbf{j} + \mathbf{c}$

$\mathbf{c} = -6\mathbf{i} + 9\mathbf{j}$

Hence

$\mathbf{v} = 4t\mathbf{i} - t^2\mathbf{j} - 6\mathbf{i} + 9\mathbf{j}$

$= ((4t - 6)\mathbf{i} + (9 - t^2)\mathbf{j})\,\text{m s}^{-1}$

When $t = 2$:

$\mathbf{v} = (8 - 6)\mathbf{i} + (9 - 4)\mathbf{j} = 2\mathbf{i} + 5\mathbf{j}\,\text{m s}^{-1}$

The angle $\mathbf{v}$ makes with $\mathbf{i}$ is given by

$\tan\theta = \frac{5}{2} \Rightarrow \theta \approx 68.2°$.

When $t = 2$, the angle between the direction of motion of P and $\mathbf{i}$ is 68.2° (1 d.p.)

b $\mathbf{r} = \int \mathbf{v}\,dt = \int ((4t - 6)\mathbf{i} + (9 - t^2)\mathbf{j})\,dt$

$= (2t^2 - 6t)\mathbf{i} + \left(9t - \frac{t^3}{3}\right)\mathbf{j} + \mathbf{d}$

When $t = 3$, $\mathbf{r} = 20\mathbf{i} + 3\mathbf{j}$:

$20\mathbf{i} + 3\mathbf{j} = (18 - 18)\mathbf{i} + (27 - 9)\mathbf{j} + \mathbf{d}$

$= 18\mathbf{j} + \mathbf{d}$

$\mathbf{d} = 20\mathbf{i} - 15\mathbf{j}$

Hence

$\mathbf{r} = ((2t^2 - 6t)\mathbf{i} + \left(9t - \frac{t^3}{3}\right)\mathbf{j} + 20\mathbf{i} - 15\mathbf{j})\,\text{m}$

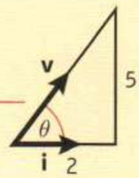
When $t = 0$, $\mathbf{r} = (20\mathbf{i} - 15\mathbf{j})\,\text{m}$:

$OP = |20\mathbf{i} - 15\mathbf{j}| = \sqrt{20^2 + 15^2} = 25\,\text{m}$

When $t = 0$, the distance of P from O is 25 m.

The direction of motion of P is the direction of the velocity vector of P. Your first step is to find the velocity by integrating the acceleration.

You then use the fact that the velocity is $6\mathbf{i}\,\text{m s}^{-1}$ when $t = 3$ to find the constant of integration.



You find the angle the velocity vector makes with **i** using trigonometry.

You find the position vector by integrating the velocity vector. Remember to include the constant of integration.

The constant of integration is a vector. This constant is different from the constant in part **a** so you should give it a different letter.

Watch out Read the question carefully to work out whether you need to find a vector or a scalar quantity. The **distance** from O is the magnitude of the displacement vector, so use Pythagoras' Theorem.

Example 10

The velocity of a particle P at time t seconds is $((3t^2 - 8)\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$. When $t = 0$, the position vector of P with respect to a fixed origin O is $(2\mathbf{i} - 4\mathbf{j})\,\text{m}$.

a Find the position vector of P after t seconds.

A second particle Q moves with constant velocity $(8\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$. When $t = 0$, the position vector of Q with respect to the fixed origin O is $2\mathbf{i}\,\text{m}$.

b Prove that P and Q collide.

a Let the position vector of P after t seconds be $\mathbf{p}$ metres.

$\mathbf{p} = \int \mathbf{v}\,dt = \int((3t^2 - 8)\mathbf{i} + 5\mathbf{j})\,dt$

$= (t^3 - 8t)\mathbf{i} + 5t\mathbf{j} + \mathbf{c}$

When $t = 0$, $\mathbf{p} = 2\mathbf{i} - 4\mathbf{j}$:

$2\mathbf{i} - 4\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{c} \Rightarrow \mathbf{c} = 2\mathbf{i} - 4\mathbf{j}$

Hence

$\mathbf{p} = (t^3 - 8t)\mathbf{i} + 5t\mathbf{j} + 2\mathbf{i} - 4\mathbf{j}$

$= (t^3 - 8t + 2)\mathbf{i} + (5t - 4)\mathbf{j}$

The position vector of P after t seconds is $((t^3 - 8t + 2)\mathbf{i} + (5t - 4)\mathbf{j})$ m.

> There are two position vectors in this question and to write them both as **r** m would be confusing. It is sensible to write the position vector of P as **p** m and the position vector of Q as **q** m.

b Let the position vector of Q after t seconds be $\mathbf{q}$ m.

$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$

$\mathbf{q} = 2\mathbf{i} + (8\mathbf{i} + 4\mathbf{j})t = (8t + 2)\mathbf{i} + 4t\mathbf{j}$

> Use the equation for the position vector of a particle moving with constant velocity. You could also integrate $8\mathbf{i} + 4\mathbf{j}$ with the boundary condition $\mathbf{q} = 2\mathbf{i}$ when $t = 0$.

Equating the position vectors of P and Q:

$(t^3 - 8t + 2)\mathbf{i} + (5t - 4)\mathbf{j} = (8t + 2)\mathbf{i} + 4t\mathbf{j}$

Equate coefficients of $\mathbf{j}$: $5t - 4 = 4t$

$\Rightarrow t = 4$

Check with coefficients of $\mathbf{i}$:

When $t = 4$, $t^3 - 8t + 2 = 4^3 - 8(4) + 2$

$= 34$

and $8t + 2 = 8(4) + 2 = 34$

So the particles will collide when $t = 4$ seconds.

> **Problem-solving**
> Equate the position vectors for each particle. If they collide there will be a single value of t for which $\mathbf{p} = \mathbf{q}$. This means that the coefficients of **i** will be equal **and** the coefficients of **j** will be equal.

> The coefficient of **i** involves a t^3 term so it is easier to start by equating the **j** components.

> Now check **i** as well, as the particles only collide if **both** coefficients match.

Exercise 8E

E **1** A particle P starts from rest at a fixed origin O. The acceleration of P at time t seconds (where $t \geqslant 0$) is $(6t^2\mathbf{i} + (8 - 4t^3)\mathbf{j})\,\text{m s}^{-2}$. Find:

a the velocity of P when $t = 2$ **(3 marks)**

b the position vector of P when $t = 4$. **(3 marks)**

E **2** A particle P is moving in a plane with velocity $\mathbf{v}\,\text{m s}^{-1}$ at time t seconds where

$$\mathbf{v} = (3t^2 + 2)\mathbf{i} + (6t - 4)\mathbf{j},\ t \geqslant 0$$

When $t = 2$, P has position vector $9\mathbf{j}$ m with respect to a fixed origin O. Find:

a the distance of P from O when $t = 0$ **(4 marks)**

b the acceleration of P at the instant when it is moving parallel to the vector **i**. **(4 marks)**

E **3** At time t seconds, where $t \geqslant 0$, the particle P is moving in a plane with velocity $\mathbf{v}\,\text{m s}^{-1}$ and acceleration $\mathbf{a}\,\text{m s}^{-2}$, where $\mathbf{a} = (2t - 4)\mathbf{i} + 6\sin t\,\mathbf{j}$.

Given that P is instantaneously at rest when $t = \frac{\pi}{2}$ seconds, find:

a $\mathbf{v}$ in terms of π and t **(5 marks)**

b the exact speed of P when $t = \frac{3\pi}{2}$ **(3 marks)**

E/P **4** At time t seconds (where $t \geqslant 0$), the particle P is moving in a plane with acceleration $\mathbf{a}\,\text{m s}^{-2}$, where

$$\mathbf{a} = (5t - 3)\mathbf{i} + (8 - t)\mathbf{j}$$

When $t = 0$, the velocity of P is $(2\mathbf{i} - 5\mathbf{j})\,\text{m s}^{-1}$. Find:

a the velocity of P after t seconds **(3 marks)**

b the value of t for which P is moving parallel to $\mathbf{i} - \mathbf{j}$ **(4 marks)**

c the speed of P when it is moving parallel to $\mathbf{i} - \mathbf{j}$. **(3 marks)**

E/P **5** At time t seconds (where $t \geqslant 0$), a particle P is moving in a plane with acceleration $(2\mathbf{i} - 2t\mathbf{j})\,\text{m s}^{-2}$. When $t = 0$, the velocity of P is $2\mathbf{j}\,\text{m s}^{-1}$ and the position vector of P is $6\mathbf{i}\,\text{m}$ with respect to a fixed origin P.

a Find the position vector of P at time t seconds. **(5 marks)**

At time t seconds (where $t \geqslant 0$), a second particle Q is moving in the plane with velocity $((3t^2 - 4)\mathbf{i} - 2t\mathbf{j})\,\text{m s}^{-1}$. The particles collide when $t = 3$.

b Find the position vector of Q at time $t = 0$. **(4 marks)**

E **6** At time $t = 0$ a particle P is at rest at a point with position vector $(4\mathbf{i} - 6\mathbf{j})\,\text{m}$ with respect to a fixed origin O. The acceleration of P at time t seconds (where $t \geqslant 0$) is $((4t - 3)\mathbf{i} - 6t^2\mathbf{j})\,\text{m s}^{-2}$. Find:

a the velocity of P when $t = \frac{1}{2}$ **(5 marks)**

b the position vector of P when $t = 6$. **(5 marks)**

E **7** At time t seconds (where $t \geqslant 0$) the particle P is moving in a plane with acceleration $\mathbf{a}\,\text{m s}^{-2}$, where $\mathbf{a} = (8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j}$.

When $t = 2$, the velocity of P is $(16\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$. Find:

a the velocity of P after t seconds **(4 marks)**

b the value of t when P is moving parallel to $\mathbf{i}$. **(3 marks)**

E/P **8** At time t seconds the velocity of a particle P is $((4t - 3)\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$. When $t = 0$, the position vector of P is $(\mathbf{i} + 2\mathbf{j})$ m, relative to a fixed origin O.

a Find an expression for the position vector of P at time t seconds. **(4 marks)**

A second particle Q moves with constant velocity $(5\mathbf{i} + k\mathbf{j})$ m s^{-1}.
When $t = 0$, the position vector of Q is $(11\mathbf{i} + 5\mathbf{j})\,\text{m}$.

b Given that the particles P and Q collide, find:

i the value of k

ii the position vector of the point of collision. **(6 marks)**

Challenge

A particle P is moving in a plane. At time t seconds, P is moving with velocity $\mathbf{v}\,\mathrm{m\,s^{-1}}$, where $\mathbf{v} = 3t\cos t\mathbf{i} + 5t\mathbf{j}$. Given that P is initially at the point with position vector $4\mathbf{i} + \mathbf{j}$ m relative to a fixed origin O, find the position vector of P when $t = \frac{\pi}{2}$.

Mixed exercise 8

1 A constant force $\mathbf{F}$ N acts on a particle of mass 4 kg for 5 seconds. The particle was initially at rest, and after 5 seconds it has velocity $6\mathbf{i} - 8\mathbf{j}\,\mathrm{m\,s^{-1}}$. Find $\mathbf{F}$.

(P) 2 A force $2\mathbf{i} - \mathbf{j}$ N acts on a particle of mass 2 kg. If the initial velocity of the particle is $\mathbf{i} + 3\mathbf{j}\,\mathrm{m\,s^{-1}}$, find the distance of the particle from its initial position after 3 seconds.

3 In this question $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors due east and north respectively. At 2 pm the coastguard spots a rowing dinghy 500 m due south of a fixed observation point. The dinghy has constant velocity $(2\mathbf{i} + 3\mathbf{j})\,\mathrm{m\,s^{-1}}$.

a Find, in terms of t, the displacement vector of the dinghy relative to the observation point t seconds after 2 pm.

b Find the distance of the dinghy from the observation point at 2.05 pm.

(E/P) 4 In this question $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors due east and north respectively. At 8 am two ships A and B have position vectors $\mathbf{r}_A = (\mathbf{i} + 3\mathbf{j})$ km and $\mathbf{r}_B = (5\mathbf{i} - 2\mathbf{j})$ km relative to a fixed origin, O. Their velocities are $\mathbf{v}_A = (2\mathbf{i} - \mathbf{j})\,\mathrm{km\,h^{-1}}$ and $\mathbf{v}_B = (-\mathbf{i} + 4\mathbf{j})\,\mathrm{km\,h^{-1}}$ respectively.

a Write down the position vectors of A and B t hours later. **(3 marks)**

b Show that t hours after 8 am the displacement vector of B relative to A is given by

$$((4 - 3t)\mathbf{i} + (-5 + 5t)\mathbf{j})\text{ km}$$ **(2 marks)**

c Show that the two ships do not collide. **(3 marks)**

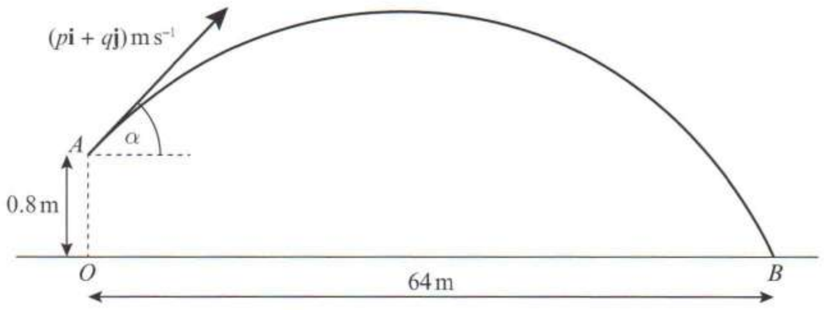
d Find the distance between A and B at 10 am. **(3 marks)**

5 A particle is projected with velocity $(8\mathbf{i} + 10\mathbf{j})\,\mathrm{m\,s^{-1}}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors horizontally and vertically respectively, from a point O at the top of a cliff and moves freely under gravity. Six seconds after projection, the particle strikes the sea at the point S. Calculate:

a the horizontal distance between O and S

b the vertical distance between O and S.

E/P **6**



A cricket ball is hit from a point A with velocity of $(p\mathbf{i} + q\mathbf{j})\,\text{m s}^{-1}$, at an angle α above the horizontal. $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors horizontally and vertically upwards respectively. The point A is 0.8 m vertically above the point O, which is on horizontal ground.

The ball takes 4 seconds to travel from A to B, where B is on the ground and $OB = 64\,\text{m}$, as shown in the diagram. By modelling the motion of the ball as that of a particle moving freely under gravity,

a find the value of p and the value of q **(5 marks)**

b find the initial speed of the ball **(2 marks)**

c find the exact value of $\tan\alpha$ **(1 mark)**

d find the length of time for which the cricket ball is at least 5 m above the ground. **(6 marks)**

e State an additional physical factor which may be taken into account in a refinement of the above model to make it more realistic. **(1 mark)**

E **7** A particle P moves in a straight line in such a way that, at time t seconds, its velocity, $\mathbf{v}\,\text{m s}^{-1}$, is given by

$$\mathbf{v} = \begin{cases} t\sqrt{14 + 2t^2}, & 0 \leqslant t \leqslant 5 \\ \dfrac{1000}{t^2}, & t > 5 \end{cases}$$

When $t = 0$, P is at the point O. Calculate the displacement of P from O:

a when $t = 5$ **(3 marks)**

b when $t = 6$. **(3 marks)**

E/P **8** A particle P of mass 0.4 kg is moving in a straight line under the action of a single variable force of magnitude F N. At time t seconds (where $t \geqslant 0$) the displacement x m of P from a fixed point O is given by $x = 2t + \frac{k}{t+1}$, where k is a constant. Given that when $t = 0$, the velocity of P is $6\,\text{m s}^{-1}$, find:

a the value of k **(5 marks)**

b the distance of P from O when $t = 0$ **(1 mark)**

c the value of F when $t = 3$. **(4 marks)**

E **9** A ball, attached to the end of an elastic string, is moving in a vertical line. The motion of the ball is modelled as a particle B moving along a vertical line so that its displacement, x m, from a fixed point O on the line at time t seconds is given by $x = 0.6\cos\left(\frac{\pi t}{3}\right)$. Find:

a the distance of B from O when $t = \frac{1}{2}$ **(2 marks)**

b the smallest positive value of t for which B is instantaneously at rest **(4 marks)**

c the magnitude of the acceleration of B when $t = 1$. Give your answer to 3 significant figures. **(3 marks)**

E **10** A light spot S moves along a straight line on a screen. At time $t = 0$, S is at a point O. At time t seconds (where $t \geqslant 0$) the distance, x cm, of S from O is given by $x = 4t\mathrm{e}^{-0.5t}$. Find:

a the acceleration of S when $t = \ln 4$ **(5 marks)**

b the greatest distance of S from O. **(2 marks)**

/P **11** Two particles P and Q move in a plane so that at time t seconds, where $t \geqslant 0$, P and Q have position vectors $\mathbf{r}_P$ metres and $\mathbf{r}_Q$ metres respectively, relative to a fixed origin O, where

$$\mathbf{r}_P = (3t^2 + 4)\mathbf{i} + \left(2t - \tfrac{1}{2}\right)\mathbf{j}$$

$$\mathbf{r}_Q = (t + 6)\mathbf{i} + \frac{3t^2}{2}\mathbf{j}$$

Find:

a the velocity vectors of P and Q at time t seconds **(5 marks)**

b the speed of P when $t = 2$ **(2 marks)**

c the value of t at the instant when the particles are moving parallel to one another. **(4 marks)**

d Show that the particles collide and find the position vector of their point of collision. **(6 marks)**

12 At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = (3t^2 - 4)\mathbf{i} + (8 - 4t^2)\mathbf{j}$$

a Show that the acceleration of P is a constant.

b Find the magnitude of the acceleration of P and the size of the angle which the acceleration makes with $\mathbf{j}$.

E **13** At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = 2\cos 3t\mathbf{i} - 2\sin 3t\mathbf{j}$$

a Find the velocity of P when $t = \frac{\pi}{6}$ **(5 marks)**

b Show that the magnitude of the acceleration of P is constant. **(4 marks)**

/P **14** A particle of mass 0.5 kg is acted upon by a variable force $\mathbf{F}$. At time t seconds, the velocity $\mathbf{v}$ m s^{-1} is given by $\mathbf{v} = (4ct - 6)\mathbf{i} + (7 - c)t^2\mathbf{j}$, where c is a constant.

a Show that $\mathbf{F} = (2c\mathbf{i} + (7 - c)t\mathbf{j})$ N. **(4 marks)**

b Given that when $t = 5$ the magnitude of $\mathbf{F}$ is 17 N, find the possible values of c. **(5 marks)**

E **15** At time t seconds (where $t \geqslant 0$) the particle P is moving in a plane with acceleration $\mathbf{a}$ m s^{-2}, where $\mathbf{a} = (8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j}$.

When $t = 2$, the velocity of P is $(16\mathbf{i} + 3\mathbf{j})$ m s^{-1}. Find:

a the velocity of P after t seconds **(3 marks)**

b the value of t when P is moving parallel to $\mathbf{i}$. **(4 marks)**

(E) **16** A particle P moves so that its acceleration $\mathbf{a}$ m s^{-2} at time t seconds, where $t \geqslant 0$, is given by

$$\mathbf{a} = 4t\mathbf{i} + 5t^{-\frac{1}{2}}\mathbf{j}$$

When $t = 0$, the velocity of P is $10\mathbf{i}$ m s^{-1}.

Find the speed of P when $t = 5$. **(6 marks)**

(E/P) **17** In this question $\mathbf{i}$ and $\mathbf{j}$ are horizontal unit vectors due east and due north respectively.

A clockwork train is moving on a flat, horizontal floor. At time $t = 0$, the train is at a fixed point O and is moving with velocity $3\mathbf{i} + 13\mathbf{j}$ m s^{-1}. The velocity of the train at time t seconds is $\mathbf{v}$ m s^{-1}, and its acceleration, $\mathbf{a}$ m s^{-2}, is given by $\mathbf{a} = 2t\mathbf{i} + 3\mathbf{j}$.

a Find $\mathbf{v}$ in terms of t. **(3 marks)**

b Find the value of t when the train is moving in a north-east direction. **(3 marks)**

Challenge

1 A particle moves on the positive x-axis such that its displacement, s m, from O at time t seconds is given by

$$s = (20 - t^2)\sqrt{t+1},\ t \geqslant 0$$

a State the initial displacement of the particle.

b Show that the particle changes direction exactly once and determine the time at which this occurs.

c Find the exact speed of the particle when it crosses O.

2 Relative to a fixed origin O, the particle R has position vector $\mathbf{r}$ metres at time t seconds, where

$$\mathbf{r} = (6\sin\omega t)\mathbf{i} + (4\cos\omega t)\mathbf{j}$$

and ω is a positive constant.

a Find $\dot{\mathbf{r}}$ and hence show that $v^2 = 2\omega^2(13 + 5\cos 2\omega t)$, where v m s^{-1} is the speed of R at time t seconds.

b Deduce that $4\omega \leqslant v \leqslant 6\omega$.

c At the instant when $t = \frac{\pi}{3\omega}$, find the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$, giving your answer in degrees to one decimal place.

Summary of key points

1 If a particle starts from the point with position vector $\mathbf{r}_0$ and moves with constant velocity $\mathbf{v}$, then its displacement from its initial position at time t is $\mathbf{v}t$ and its position vector $\mathbf{r}$ is given by $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$.

2 For an object moving in a plane with constant acceleration:

- $\mathbf{v} = \mathbf{u} + \mathbf{a}t$
- $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

where

- $\mathbf{u}$ is the initial velocity
- $\mathbf{a}$ is the acceleration
- $\mathbf{v}$ is the velocity at time t
- $\mathbf{r}$ is the displacement at time t.

3 If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, then $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$

and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$

4 $\mathbf{v} = \int \mathbf{a}\,dt$ and $\mathbf{r} = \int \mathbf{v}\,dt$