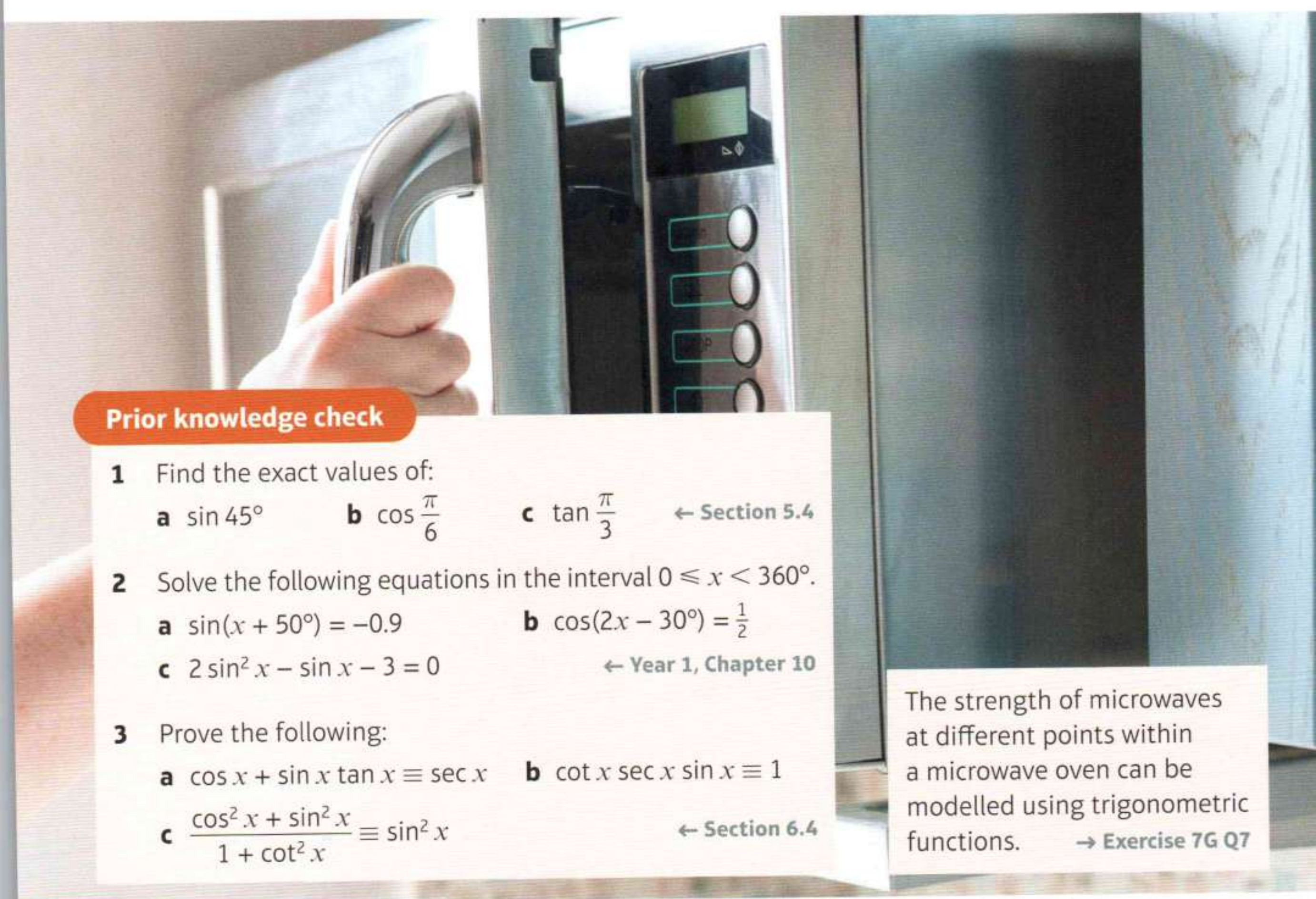
7 Trigonometry and modelling

Objectives

After completing this unit you should be able to:

- Prove and use the addition formulae → pages 167–173
- Understand and use the double-angle formulae → pages 174–177
- Solve trigonometric equations using the double-angle and addition formulae → pages 177–181
- Write expressions of the form $a\cos\theta \pm b\sin\theta$ in the forms $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$ → pages 181–186
- Prove trigonometric identities using a variety of identities → pages 186–189
- Use trigonometric functions to model real-life situations → pages 189–191



Prior knowledge check

1 Find the exact values of:

a $\sin 45°$ **b** $\cos\frac{\pi}{6}$ **c** $\tan\frac{\pi}{3}$ ← Section 5.4

2 Solve the following equations in the interval $0 \leqslant x < 360°$.

a $\sin(x + 50°) = -0.9$ **b** $\cos(2x - 30°) = \frac{1}{2}$

c $2\sin^2 x - \sin x - 3 = 0$ ← Year 1, Chapter 10

3 Prove the following:

a $\cos x + \sin x \tan x \equiv \sec x$ **b** $\cot x \sec x \sin x \equiv 1$

c $\dfrac{\cos^2 x + \sin^2 x}{1 + \cot^2 x} \equiv \sin^2 x$ ← Section 6.4

The strength of microwaves at different points within a microwave oven can be modelled using trigonometric functions. → Exercise 7G Q7

7.1 Addition formulae

The addition formulae for sine, cosine and tangent are defined as follows:

> **Notation** The addition formulae are sometimes called the **compound-angle formulae**.

- $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$
- $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$
- $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

$\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

You can prove these identities using geometric constructions.

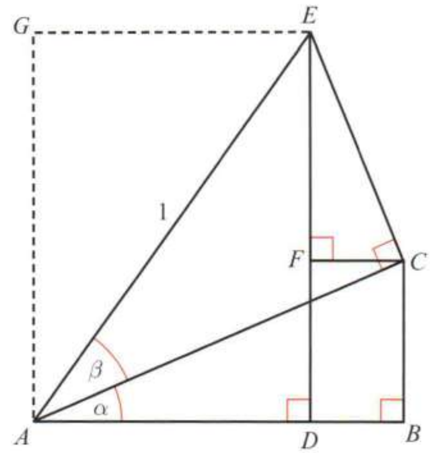
Example 1

In the diagram $\angle BAC = \alpha$, $\angle CAE = \beta$ and $AE = 1$. Additionally, lines AB and BC are perpendicular, lines AB and DE are perpendicular, lines AC and EC are perpendicular and lines EF and FC are perpendicular.

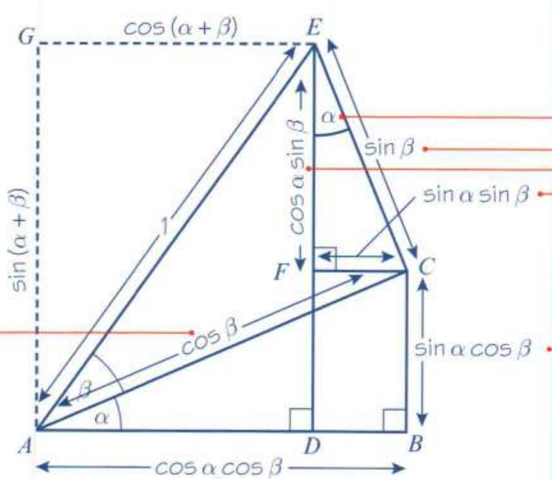
Use the diagram, together with known properties of sine and cosine, to prove the following identities:

a $\sin(\alpha + \beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$

b $\cos(\alpha + \beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$



The diagram can be labelled with the following lengths using the properties of sine and cosine.



$\angle ACF = \alpha \Rightarrow \angle FCE = 90° - \alpha$. So $\angle FEC = \alpha$.

In triangle ACE, $\sin\beta = \dfrac{EC}{AE} \Rightarrow \sin\beta = \dfrac{EC}{1}$
So $EC = \sin\beta$.

In triangle FEC, $\cos\alpha = \dfrac{FE}{EC} \Rightarrow \cos\alpha = \dfrac{FE}{\sin\beta}$
So $FE = \cos\alpha\sin\beta$.

In triangle FEC, $\sin\alpha = \dfrac{FC}{EC} \Rightarrow \sin\alpha = \dfrac{FC}{\sin\beta}$
So $FC = \sin\alpha\sin\beta$.

In triangle ABC, $\sin\alpha = \dfrac{BC}{AC} \Rightarrow \sin\alpha = \dfrac{BC}{\cos\beta}$
So $BC = \sin\alpha\cos\beta$.

In triangle ABC, $\cos\alpha = \dfrac{AB}{AC} \Rightarrow \cos\alpha = \dfrac{AB}{\cos\beta}$
So $AB = \cos\alpha\cos\beta$.

In triangle ACE, $\cos\beta = \dfrac{AC}{AE} \Rightarrow \cos\beta = \dfrac{AC}{1}$
So $AC = \cos\beta$.

a Using triangle ADE

$DE = \sin(\alpha + \beta)$

$AD = \cos(\alpha + \beta)$

$DE = DF + FE$

$\Rightarrow \sin(\alpha + \beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$ as required

b $AD = AB - DB$

$\Rightarrow \cos(\alpha + \beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$ as required

Substitute the lengths from the diagram.

Problem-solving

You are looking for a relationship involving $\sin(\alpha + \beta)$, so consider the right-angled triangle ADE with angle $(\alpha + \beta)$. You can see these relationships more easily on the diagram by looking at $AG = DE$ and $GE = AD$.

Online Explore the proof step-by-step using GeoGebra.

Example 2

Use the results from Example 1 to show that

a $\cos(A - B) \equiv \cos A\cos B + \sin A\sin B$

b $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

a Replace B by $-B$ in

$\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$

$\cos(A + (-B)) \equiv \cos A\cos(-B) - \sin A\sin(-B)$

$\cos(A - B) \equiv \cos A\cos B + \sin A\sin B$

$\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$

← Year 1, Chapter 9

b $\tan(A + B) \equiv \dfrac{\sin(A + B)}{\cos(A + B)}$

$\equiv \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$

Divide the numerator and denominator by $\cos A\cos B$.

$$\equiv \frac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}} + \dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A}\cancel{\cos B}}{\cancel{\cos A}\cancel{\cos B}} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

Cancel where possible.

$\equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ as required

Example 3

Prove that

$$\frac{\cos A}{\sin B} - \frac{\sin A}{\cos B} \equiv \frac{\cos(A + B)}{\sin B\cos B}$$

$$\text{LHS} \equiv \frac{\cos A}{\sin B} - \frac{\sin A}{\cos B}$$

$$\equiv \frac{\cos A \cos B}{\sin B \cos B} - \frac{\sin A \sin B}{\sin B \cos B}$$

$$\equiv \frac{\cos A \cos B - \sin A \sin B}{\sin B \cos B}$$

$$\equiv \frac{\cos(A + B)}{\sin B \cos B} \equiv \text{RHS}$$

Write both fractions with a common denominator.

Problem-solving

When proving an identity, always keep an eye on the final answer. This can act as a guide as to what to do next.

Use the addition formula in reverse:
$\cos A \cos B - \sin A \sin B \equiv \cos(A + B)$

Example 4

Given that $2\sin(x + y) = 3\cos(x - y)$, express $\tan x$ in terms of $\tan y$.

Expanding $\sin(x + y)$ and $\cos(x - y)$ gives

$$2\sin x \cos y + 2\cos x \sin y = 3\cos x \cos y + 3\sin x \sin y$$

so $\dfrac{2\sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} + \dfrac{2\cancel{\cos x} \sin y}{\cancel{\cos x} \cos y} = \dfrac{3\cancel{\cos x}\cancel{\cos y}}{\cancel{\cos x}\cancel{\cos y}} + \dfrac{3\sin x \sin y}{\cos x \cos y}$

$$2\tan x + 2\tan y = 3 + 3\tan x \tan y$$

$$2\tan x - 3\tan x \tan y = 3 - 2\tan y$$

$$\tan x(2 - 3\tan y) = 3 - 2\tan y$$

So $$\tan x = \frac{3 - 2\tan y}{2 - 3\tan y}$$

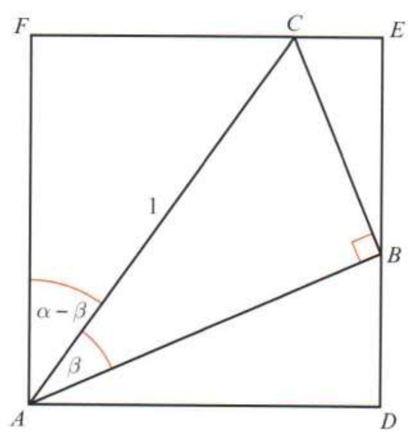
Remember $\tan x = \dfrac{\sin x}{\cos x}$
Dividing each term by $\cos x \cos y$ will produce $\tan x$ and $\tan y$ terms.

Collect all $\tan x$ terms on one side of the equation.

Factorise.

Exercise 7A

1 In the diagram $\angle BAC = \beta$, $\angle CAF = \alpha - \beta$ and $AC = 1$. Additionally lines AB and BC are perpendicular.



a Show each of the following:
- **i** $\angle FAB = \alpha$ **ii** $\angle ABD = \alpha$ and $\angle ECB = \alpha$
- **iii** $AB = \cos\beta$ **iv** $BC = \sin\beta$

b Use $\triangle ABD$ to write an expression for the lengths
- **i** AD **ii** BD

c Use $\triangle BEC$ to write an expression for the lengths
- **i** CE **ii** BE

d Use $\triangle FAC$ to write an expression for the lengths
- **i** FC **ii** FA

e Use your completed diagram to show that:
- **i** $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- **ii** $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(P) **2** Use the formulae for $\sin(A - B)$ and $\cos(A - B)$ to show that

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(P) **3** By substituting $A = P$ and $B = -Q$ into the addition formula for $\sin(A + B)$, show that $\sin(P - Q) \equiv \sin P \cos Q - \cos P \sin Q$.

(P) **4** A student makes the mistake of thinking that $\sin(A + B) \equiv \sin A + \sin B$.
Choose non-zero values of A and B to show that this identity is not true.

Watch out This is a common mistake. One counter-example is sufficient to disprove the statement.

(P) **5** Using the expansion of $\cos(A - B)$ with $A = B = \theta$, show that $\sin^2\theta + \cos^2\theta \equiv 1$.

(P) **6** **a** Use the expansion of $\sin(A - B)$ to show that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

b Use the expansion of $\cos(A - B)$ to show that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

(P) **7** Write $\sin\left(x + \frac{\pi}{6}\right)$ in the form $p\sin x + q\cos x$ where p and q are constants to be found.

(P) **8** Write $\cos\left(x + \frac{\pi}{3}\right)$ in the form $a\cos x + b\sin x$ where a and b are constants to be found.

(P) **9** Express the following as a single sine, cosine or tangent:

a $\sin 15° \cos 20° + \cos 15° \sin 20°$

b $\sin 58° \cos 23° - \cos 58° \sin 23°$

c $\cos 130° \cos 80° - \sin 130° \sin 80°$

d $\dfrac{\tan 76° - \tan 45°}{1 + \tan 76° \tan 45°}$

e $\cos 2\theta \cos\theta + \sin 2\theta \sin\theta$

f $\cos 4\theta \cos 3\theta - \sin 4\theta \sin 3\theta$

g $\sin\frac{1}{2}\theta \cos 2\frac{1}{2}\theta + \cos\frac{1}{2}\theta \sin 2\frac{1}{2}\theta$

h $\dfrac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta \tan 3\theta}$

i $\sin(A + B)\cos B - \cos(A + B)\sin B$

j $\cos\left(\frac{3x + 2y}{2}\right)\cos\left(\frac{3x - 2y}{2}\right) - \sin\left(\frac{3x + 2y}{2}\right)\sin\left(\frac{3x - 2y}{2}\right)$

(P) **10** Use the addition formulae for sine or cosine to write each of the following as a single trigonometric function in the form $\sin(x \pm \theta)$ or $\cos(x \pm \theta)$, where $0 < \theta < \frac{\pi}{2}$

a $\frac{1}{\sqrt{2}}(\sin x + \cos x)$ **b** $\frac{1}{\sqrt{2}}(\cos x - \sin x)$ **c** $\frac{1}{2}(\sin x + \sqrt{3}\cos x)$ **d** $\frac{1}{\sqrt{2}}(\sin x - \cos x)$

(P) **11** Given that $\cos y = \sin(x + y)$, show that $\tan y = \sec x - \tan x$.

(P) **12** Given that $\tan(x - y) = 3$, express $\tan y$ in terms of $\tan x$.

(P) **13** Given that $\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y)$, find the value of $\tan(x + y)$.

Hint First multiply out the brackets.

(P) **14** In each of the following, calculate the exact value of $\tan x$.

a $\tan(x - 45°) = \frac{1}{4}$ **b** $\sin(x - 60°) = 3\cos(x + 30°)$ **c** $\tan(x - 60°) = 2$

(E/P) **15** Given that $\tan\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$, show that $\tan x = 8 - 5\sqrt{3}$. **(3 marks)**

(E/P) **16** Prove that

$$\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right) = 0$$

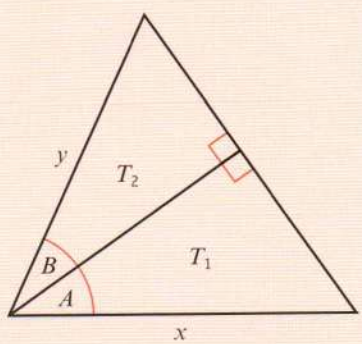
You must show each stage of your working. **(4 marks)**

Challenge

This triangle is constructed from two right-angled triangles T_1 and T_2.

a Find expressions involving x, y, A and B for:
- **i** the area of T_1
- **ii** the area of T_2
- **iii** the area of the large triangle.

b Hence prove that $\sin(A + B) = \sin A\cos B + \cos A\sin B$



Hint For part **a** your expressions should all involve **all four** variables. You will need to use the formula Area $= \frac{1}{2}ab\sin\theta$ in each case.

7.2 Using the angle addition formulae

The addition formulae can be used to find exact values of trigonometric functions of different angles.

Example 5

Show, using the formula for $\sin(A - B)$, that $\sin 15° = \frac{\sqrt{6} - \sqrt{2}}{4}$

$$\sin 15° = \sin(45° - 30°)$$
$$= \sin 45°\cos 30° - \cos 45°\sin 30°$$
$$= \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) - \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right)$$
$$= \tfrac{1}{4}(\sqrt{3}\sqrt{2} - \sqrt{2})$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

You know the exact values of sin and cos for many angles, e.g. 30°, 45°, 60°, 90°, 180° …, so write 15° using two of these angles. You could also use $\sin(60° - 45°)$.

Example 6

Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$ and B is obtuse, find the value of:

a $\cos(A - B)$ **b** $\tan(A + B)$ **c** $\operatorname{cosec}(A - B)$

a $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

You know $\sin A$ and $\cos B$, but need to find $\sin B$ and $\cos A$.

$\cos^2 A \equiv 1 - \sin^2 A = 1 - \left(-\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$

$\sin^2 B \equiv 1 - \cos^2 = 1 - \left(-\frac{12}{13}\right)^2 = 1 - \frac{144}{169} = \frac{25}{169}$

Use $\sin^2 x + \cos^2 x \equiv 1$ to determine $\cos A$ and $\sin B$.

$\cos A = \pm\frac{4}{5}$ $\sin B = \pm\frac{5}{13}$

$180° < A < 270°$ so $\cos A = -\frac{4}{5}$ B is obtuse so $\sin B = \frac{5}{13}$

Problem-solving

Remember there are two possible solutions to $\cos^2 A = \frac{16}{25}$. Use a CAST diagram to determine which one to use.

$\cos x$ is negative in the third quadrant, so choose the negative square root $-\frac{4}{5}$. $\sin x$ is positive in the second quadrant (obtuse angle) so choose the positive square root.

$\cos(A - B) = \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(+\frac{5}{13}\right)$

$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$

Substitute the values for $\sin A$, $\sin B$, $\cos A$ and $\cos B$ into the formula and then simplify.

b $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

So $\tan(A + B) = \dfrac{\frac{3}{4} + \left(-\frac{5}{12}\right)}{1 - \left(\frac{3}{4}\right)\left(-\frac{5}{12}\right)}$

$= \dfrac{\frac{1}{3}}{\frac{21}{16}} = \frac{1}{3} \times \frac{16}{21} = \frac{16}{63}$

$\tan A = \dfrac{\sin A}{\cos A} = \dfrac{-\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4}$

$\tan B = \dfrac{\sin B}{\cos B} = \dfrac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}$

c $\operatorname{cosec}(A - B) \equiv \dfrac{1}{\sin(A - B)}$

Remember $\operatorname{cosec} x = \dfrac{1}{\sin x}$

$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

$\sin(A - B) = \left(\frac{-3}{5}\right)\left(\frac{-12}{13}\right) - \left(\frac{-4}{5}\right)\left(\frac{5}{13}\right) = \frac{56}{65}$

$\operatorname{cosec}(A - B) = \dfrac{1}{\left(\frac{56}{65}\right)} = \frac{65}{56}$

Exercise 7B

1 Without using your calculator, find the exact value of:

a $\cos 15°$ **b** $\sin 75°$ **c** $\sin(120° + 45°)$ **d** $\tan 165°$

2 Without using your calculator, find the exact value of:

a $\sin 30° \cos 60° + \cos 30° \sin 60°$

b $\cos 110° \cos 20° + \sin 110° \sin 20°$

c $\sin 33° \cos 27° + \cos 33° \sin 27°$

d $\cos\frac{\pi}{8}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\sin\frac{\pi}{8}$

e $\sin 60° \cos 15° - \cos 60° \sin 15°$

f $\cos 70° (\cos 50° - \tan 70° \sin 50°)$

g $\dfrac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°}$

h $\dfrac{1 - \tan 15°}{1 + \tan 15°}$

i $\dfrac{\tan\frac{7\pi}{12} - \tan\frac{\pi}{3}}{1 + \tan\frac{7\pi}{12}\tan\frac{\pi}{3}}$

j $\sqrt{3}\cos 15° - \sin 15°$

(E) 3 a Express $\tan(45° + 30°)$ in terms of $\tan 45°$ and $\tan 30°$. **(2 marks)**

b Hence show that $\tan 75° = 2 + \sqrt{3}$. **(2 marks)**

(P) 4 Given that $\cot A = \frac{1}{4}$ and $\cot(A + B) = 2$, find the value of $\cot B$.

(E/P) 5 a Using $\cos(\theta + \alpha) \equiv \cos\theta\cos\alpha - \sin\theta\sin\alpha$, or otherwise, show that $\cos 105° = \dfrac{\sqrt{2} - \sqrt{6}}{4}$ **(4 marks)**

b Hence, or otherwise, show that $\sec 105° = -\sqrt{a}(1 + \sqrt{b})$, where a and b are constants to be found. **(3 marks)**

(P) 6 Given that $\sin A = \frac{4}{5}$ and $\sin B = \frac{1}{2}$, where A and B are both acute angles, calculate the exact value of:

a $\sin(A + B)$ b $\cos(A - B)$ c $\sec(A - B)$

(P) 7 Given that $\cos A = -\frac{4}{5}$, and A is an obtuse angle measured in radians, find the exact value of:

a $\sin A$ b $\cos(\pi + A)$ c $\sin\left(\frac{\pi}{3} + A\right)$ d $\tan\left(\frac{\pi}{4} + A\right)$

(P) 8 Given that $\sin A = \frac{8}{17}$, where A is acute, and $\cos B = -\frac{4}{5}$, where B is obtuse, calculate the exact value of:

a $\sin(A - B)$ b $\cos(A - B)$ c $\cot(A - B)$

(P) 9 Given that $\tan A = \frac{7}{24}$, where A is reflex, and $\sin B = \frac{5}{13}$, where B is obtuse, calculate the exact value of:

a $\sin(A + B)$ b $\tan(A - B)$ c $\operatorname{cosec}(A + B)$

(P) 10 Given that $\tan A = \frac{1}{5}$ and $\tan B = \frac{2}{3}$, calculate, without using your calculator, the value of $A + B$ in degrees, where:

a A and B are both acute,

b A is reflex and B is acute.

7.3 Double-angle formulae

You can use the addition formulae to derive the following double-angle formulae.

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$

Example 7

Use the double-angle formulae to write each of the following as a single trigonometric ratio.

a $\cos^2 50° - \sin^2 50°$ **b** $\dfrac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}}$ **c** $\dfrac{4\sin 70°}{\sec 70°}$

a $\cos^2 50° - \sin^2 50° = \cos(2 \times 50°)$
$= \cos 100°$

Use $\cos 2A \equiv \cos^2 A - \sin^2 A$ in reverse, with $A = 50°$.

b $\dfrac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} = \tan\left(2 \times \frac{\pi}{6}\right)$

Use $\tan 2A \equiv \dfrac{2\tan A}{1-\tan^2 A}$ in reverse, with $A = \frac{\pi}{6}$

$= \tan\frac{\pi}{3}$

c $\dfrac{4\sin 70°}{\sec 70°} = 4\sin 70° \cos 70°$

$\sec x = \dfrac{1}{\cos x}$ so $\cos x = \dfrac{1}{\sec x}$

$= 2(2\sin 70° \cos 70°)$

Recognise this is a multiple of $2\sin A \cos A$.

$= 2\sin(2 \times 70°) = 2\sin 140°$

Use $\sin 2A \equiv 2\sin A \cos A$ in reverse with $A = 70°$.

Example 8

Given that $x = 3\sin\theta$ and $y = 3 - 4\cos 2\theta$, eliminate θ and express y in terms of x.

The equations can be written as

$$\sin\theta = \frac{x}{3} \quad \cos 2\theta = \frac{3-y}{4}$$

As $\cos 2\theta \equiv 1 - 2\sin^2\theta$ for all values of θ,

$$\frac{3-y}{4} = 1 - 2\left(\frac{x}{3}\right)^2$$

θ has been eliminated from this equation. We still need to solve for y.

So $\dfrac{y}{4} = 2\left(\dfrac{x}{3}\right)^2 - \dfrac{1}{4}$

or $y = 8\left(\dfrac{x}{3}\right)^2 - 1$

The final answer should be in the form $y = \ldots$

Watch out Be careful with this manipulation. Many errors can occur in the early part of a solution.

Example 9

Given that $\cos x = \frac{3}{4}$, and that $180° < x < 360°$, find the exact value of:

a $\sin 2x$ **b** $\tan 2x$

a $\sin^2 A = 1 - \cos^2 A$

$= 1 - \left(\frac{3}{4}\right)^2$

$= \frac{7}{16}$

Use $\sin^2 A + \cos^2 A = 1$ to determine $\sin A$.

$180° < A < 360°$, so $\sin A = -\frac{\sqrt{7}}{4}$

$\sin A$ is negative in the third and fourth quadrants, so choose the negative square root.

$\sin 2x = 2 \sin x \cos x$

$= 2\left(-\frac{\sqrt{7}}{4}\right)\left(\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$

b $\tan x = \frac{\sin x}{\cos x} = \frac{\frac{-\sqrt{7}}{4}}{\frac{3}{4}}$

$= -\frac{\sqrt{7}}{3}$

Find $\tan x$ in simplified surd form, then substitute this value into the double-angle formula for $\tan 2x$.

$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} = \frac{-\frac{2\sqrt{7}}{3}}{1 - \frac{7}{9}}$

$= -\frac{2\sqrt{7}}{3} \times \frac{9}{2}$

$= -3\sqrt{7}$

Make sure you square all of $\tan x$ when working out $\tan^2 x$:

$\left(-\frac{\sqrt{7}}{3}\right)^2 = \frac{7}{9}$

Exercise 7C

(P) **1** Use the expansion of $\sin(A + B)$ to show that $\sin 2A \equiv 2 \sin A \cos A$.

Hint Set $B = A$.

(P) **2** **a** Using the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, show that $\cos 2A \equiv \cos^2 A - \sin^2 A$.

b Hence show that:

i $\cos 2A \equiv 2\cos^2 A - 1$

ii $\cos 2A \equiv 1 - 2\sin^2 A$

Problem-solving

Use $\sin^2 A + \cos^2 A \equiv 1$

(P) **3** Use the expansion of $\tan(A + B)$ to express $\tan 2A$ in terms of $\tan A$.

(P) **4** Write each of the following expressions as a single trigonometric ratio.

a $2\sin 10° \cos 10°$ **b** $1 - 2\sin^2 25°$ **c** $\cos^2 40° - \sin^2 40°$

d $\dfrac{2\tan 5°}{1 - \tan^2 5°}$ **e** $\dfrac{1}{2\sin(24.5)° \cos(24.5)°}$ **f** $6\cos^2 30° - 3$

g $\dfrac{\sin 8°}{\sec 8°}$ **h** $\cos^2\dfrac{\pi}{16} - \sin^2\dfrac{\pi}{16}$

(P) **5** Without using your calculator find the exact values of:

a $2\sin 22.5° \cos 22.5°$ **b** $2\cos^2 15° - 1$

c $(\sin 75° - \cos 75°)^2$ **d** $\dfrac{2\tan\frac{\pi}{8}}{1 - \tan^2\frac{\pi}{8}}$

(E/P) **6** **a** Show that $(\sin A + \cos A)^2 \equiv 1 + \sin 2A$. **(3 marks)**

b Hence find the exact value of $\left(\sin\dfrac{\pi}{8} + \cos\dfrac{\pi}{8}\right)^2$. **(2 marks)**

7 Write the following in their simplest form, involving only one trigonometric function:

a $\cos^2 3\theta - \sin^2 3\theta$ **b** $6\sin 2\theta \cos 2\theta$ **c** $\dfrac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$

d $2 - 4\sin^2\dfrac{\theta}{2}$ **e** $\sqrt{1 + \cos 2\theta}$ **f** $\sin^2\theta \cos^2\theta$

g $4\sin\theta \cos\theta \cos 2\theta$ **h** $\dfrac{\tan\theta}{\sec^2\theta - 2}$ **i** $\sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta$

(P) **8** Given that $p = 2\cos\theta$ and $q = \cos 2\theta$, express q in terms of p.

(P) **9** Eliminate θ from the following pairs of equations:

a $x = \cos^2\theta$, $y = 1 - \cos 2\theta$ **b** $x = \tan\theta$, $y = \cot 2\theta$

c $x = \sin\theta$, $y = \sin 2\theta$ **d** $x = 3\cos 2\theta + 1$, $y = 2\sin\theta$

(P) **10** Given that $\cos x = \frac{1}{4}$, find the exact value of $\cos 2x$.

(P) **11** Find the possible values of $\sin\theta$ when $\cos 2\theta = \frac{23}{25}$

(P) **12** Given that $\tan\theta = \frac{3}{4}$, and that θ is acute,

a find the exact value of: **i** $\tan 2\theta$ **ii** $\sin 2\theta$ **iii** $\cos 2\theta$

b deduce the value of $\sin 4\theta$.

(P) **13** Given that $\cos A = -\frac{1}{3}$, and that A is obtuse,

a find the exact value of: **i** $\cos 2A$ **ii** $\sin A$ **iii** $\operatorname{cosec} 2A$

b show that $\tan 2A = \dfrac{4\sqrt{2}}{7}$

(E/P) **14** Given that $\pi < \theta < \dfrac{3\pi}{2}$, find the value of $\tan\dfrac{\theta}{2}$ when $\tan\theta = \frac{3}{4}$ **(4 marks)**

(E/P) **15** Given that $\cos x + \sin x = m$ and $\cos x - \sin x = n$, where m and n are constants, write down, in terms of m and n, the value of $\cos 2x$. **(4 marks)**

(E/P) **16** In $\triangle PQR$, $PQ = 3\,\text{cm}$, $PR = 6\,\text{cm}$, $QR = 5\,\text{cm}$ and $\angle QPR = 2\theta$.

a Use the cosine rule to show that $\cos 2\theta = \frac{5}{9}$ **(3 marks)**

b Hence find the exact value of $\sin\theta$. **(2 marks)**

(E/P) **17** The line l, with equation $y = \frac{3}{4}x$, bisects the angle between the x-axis and the line $y = mx$, $m > 0$. Given that the scales on each axis are the same, and that l makes an angle θ with the x-axis,

a write down the value of $\tan\theta$ **(1 mark)**

b show that $m = \frac{24}{7}$ **(3 marks)**

(E/P) **18** **a** Use the identity $\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$, to show that $\cos 2A \equiv 2\cos^2 A - 1$. **(2 marks)**

The curves C_1 and C_2 have equations

C_1: $y = 4\cos 2x$
C_2: $y = 6\cos^2 x - 3\sin 2x$

b Show that the x-coordinates of the points where C_1 and C_2 intersect satisfy the equation

$\cos 2x + 3\sin 2x - 3 = 0$ **(3 marks)**

(P) **19** Use the fact that $\tan 2A \equiv \dfrac{\sin 2A}{\cos 2A}$ to derive the formula for $\tan 2A$ in terms of $\tan A$.

Hint Use the identities for $\sin 2A$ and $\cos 2A$ and then divide both the numerator and denominator by $\cos^2 A$.

7.4 Solving trigonometric equations

You can use the addition formulae and the double-angle formulae to help you solve trigonometric equations.

Example 10

Solve $4\cos(\theta - 30°) = 8\sqrt{2}\sin\theta$ in the range $0 \leqslant \theta \leqslant 360°$. Round your answer to 1 decimal place.

$4\cos(\theta - 30°) = 8\sqrt{2}\sin\theta$

$4\cos\theta\cos 30° + 4\sin\theta\sin 30° = 8\sqrt{2}\sin\theta$ — Use the formula for $\cos(A - B)$.

$4\cos\theta\left(\frac{\sqrt{3}}{2}\right) + 4\sin\theta\left(\frac{1}{2}\right) = 8\sqrt{2}\sin\theta$ — Substitute $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{1}{2}$

$2\sqrt{3}\cos\theta + 2\sin\theta = 8\sqrt{2}\sin\theta$

$2\sqrt{3}\cos\theta = (8\sqrt{2} - 2)\sin\theta$ — Gather cosine terms on the LHS and sine terms on the RHS of the equation.

$\frac{2\sqrt{3}}{8\sqrt{2} - 2} = \tan\theta$ — Divide both sides by $\cos\theta$ and by $(8\sqrt{2} - 2)$.

$\tan\theta = 0.3719\ldots$

$\theta = 20.4°, 200.4°$ — Use a CAST diagram or a sketch graph to find all the solutions in the given range.

Example 11

Solve $3\cos 2x - \cos x + 2 = 0$ for $0 \leqslant x \leqslant 360°$.

Using a double angle formula for $\cos 2x$

$$3\cos 2x - \cos x + 2 = 0$$

becomes

$$3(2\cos^2 x - 1) - \cos x + 2 = 0$$

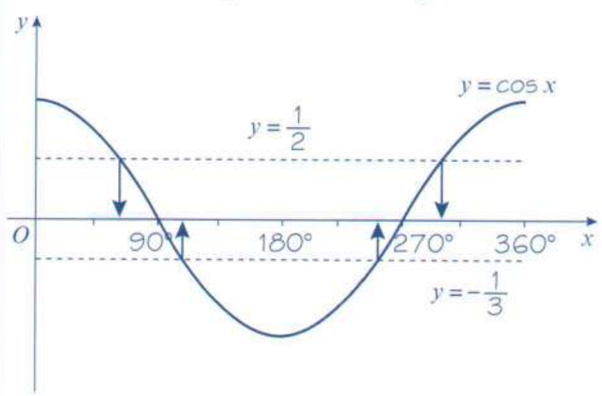
$$6\cos^2 x - 3 - \cos x + 2 = 0$$

$$6\cos^2 x - \cos x - 1 = 0$$

— This quadratic equation factorises: $6X^2 - X - 1 = (3X + 1)(2X - 1)$

So $(3\cos x + 1)(2\cos x - 1) = 0$

Solving: $\cos x = -\frac{1}{3}$ or $\cos x = \frac{1}{2}$



$\cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$ $\cos^{-1}\left(\frac{1}{2}\right) = 60°$

So $x = 60°, 109.5°, 250.5°, 300°$

Problem-solving

Choose the double angle formula for $\cos 2x$ which only involves $\cos x$:
$\cos 2x \equiv 2\cos^2 x - 1$
This will give you a quadratic equation in $\cos x$.

Example 12

Solve $2\tan 2y \tan y = 3$ for $0 \leqslant y \leqslant 2\pi$. Give your answers to 2 decimal places.

$2\tan 2y \tan y = 3$

$2\left(\dfrac{2\tan y}{1 - \tan^2 y}\right)\tan y = 3$

Use the double-angle identity for tan.

$\dfrac{4\tan^2 y}{1 - \tan^2 y} = 3$

$4\tan^2 y = 3 - 3\tan^2 y$

This is a quadratic equation in $\tan y$. Because there is a $\tan^2 y$ term but no $\tan y$ term you can solve it directly.

$7\tan^2 y = 3$

$\tan^2 y = \frac{3}{7}$

$\tan y = \pm\sqrt{\frac{3}{7}}$

$y = 0.58, 2.56, 3.72, 5.70$

Watch out Remember to include the positive and negative square roots.

Example 13

a By expanding $\sin(2A + A)$ show that $\sin 3A \equiv 3\sin A - 4\sin^3 A$.

b Hence, or otherwise, for $0 < \theta < 2\pi$, solve $16\sin^3\theta - 12\sin\theta - 2\sqrt{3} = 0$ giving your answers in terms of π.

a $\text{LHS} \equiv \sin 3A \equiv \sin(2A + A)$

$\equiv \sin 2A \cos A + \cos 2A \sin A$

Use the addition formula for $\sin(A + B)$.

$\equiv (2\sin A\cos A)\cos A + (1 - 2\sin^2 A)\sin A$

Subsitute for $\sin 2A$ and $\cos 2A$. As the answer is in terms of $\sin A$, $\cos 2A \equiv 1 - 2\sin^2 A$ is the best identity to use.

$\equiv 2\sin A\cos^2 A + \sin A - 2\sin^3 A$

$\equiv 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$

Use $\sin^2 A + \cos^2 A \equiv 1$ to substitute for $\cos^2 A$.

$\equiv 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$

$\equiv 3\sin A - 4\sin^3 A \equiv \text{RHS}$

b $16\sin^3\theta - 12\sin\theta - 2\sqrt{3} = 0$

$16\sin^3\theta - 12\sin\theta = 2\sqrt{3}$

$-4\sin 3\theta = 2\sqrt{3}$

$\sin 3\theta = -\dfrac{\sqrt{3}}{2}$

Problem-solving The question says 'hence' so look for an opportunity to use the identity you proved in part **a**. You need to multiply both sides of the identity by −4.

$3\theta = \dfrac{4\pi}{3}, \dfrac{5\pi}{3}, \dfrac{10\pi}{3}, \dfrac{11\pi}{3}, \dfrac{16\pi}{3}, \dfrac{17\pi}{3}$

Use a CAST diagram or a sketch graph to find all answers for 3θ. $0 < \theta < 2\pi$ so $0 < 3\theta < 6\pi$.

$\theta = \dfrac{4\pi}{9}, \dfrac{5\pi}{9}, \dfrac{10\pi}{9}, \dfrac{11\pi}{9}, \dfrac{16\pi}{9}, \dfrac{17\pi}{9}$

Exercise 7D

(P) **1** Solve, in the interval $0 \leqslant \theta < 360°$, the following equations. Give your answers to 1 d.p.

a $3\cos\theta = 2\sin(\theta + 60°)$ **b** $\sin(\theta + 30°) + 2\sin\theta = 0$

c $\cos(\theta + 25°) + \sin(\theta + 65°) = 1$ **d** $\cos\theta = \cos(\theta + 60°)$

(E/P) **2** **a** Show that $\sin\left(\theta + \frac{\pi}{4}\right) \equiv \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$ **(2 marks)**

b Hence, or otherwise, solve the equation $\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta) = \frac{1}{\sqrt{2}}$, $0 \leqslant \theta \leqslant 2\pi$. **(4 marks)**

c Use your answer to part **b** to write down the solutions to $\sin\theta + \cos\theta = 1$ over the same interval. **(2 marks)**

(P) **3** **a** Solve the equation $\cos\theta\cos 30° - \sin\theta\sin 30° = 0.5$, for $0 \leqslant \theta \leqslant 360°$.

b Hence write down, in the same interval, the solutions of $\sqrt{3}\cos\theta - \sin\theta = 1$.

(P) **4** **a** Given that $3\sin(x - y) - \sin(x + y) = 0$, show that $\tan x = 2\tan y$.

b Solve $3\sin(x - 45°) - \sin(x + 45°) = 0$, for $0 \leqslant x \leqslant 360°$.

(P) **5** Solve the following equations, in the intervals given.

a $\sin 2\theta = \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$ **b** $\cos 2\theta = 1 - \cos\theta$, $-180° < \theta \leqslant 180°$

c $3\cos 2\theta = 2\cos^2\theta$, $0 \leqslant \theta < 360°$ **d** $\sin 4\theta = \cos 2\theta$, $0 \leqslant \theta \leqslant \pi$

e $3\cos\theta - \sin\frac{\theta}{2} - 1 = 0$, $0 \leqslant \theta < 720°$ **f** $\cos^2\theta - \sin 2\theta = \sin^2\theta$, $0 \leqslant \theta \leqslant \pi$

g $2\sin\theta = \sec\theta$, $0 \leqslant \theta \leqslant 2\pi$ **h** $2\sin 2\theta = 3\tan\theta$, $0 \leqslant \theta < 360°$

i $2\tan\theta = \sqrt{3}(1 - \tan\theta)(1 + \tan\theta)$, $0 \leqslant \theta \leqslant 2\pi$ **j** $\sin^2\theta = 2\sin 2\theta$, $-180° < \theta < 180°$

k $4\tan\theta = \tan 2\theta$, $0 \leqslant \theta \leqslant 360°$

(E/P) **6** In $\triangle ABC$, $AB = 4$ cm, $AC = 5$ cm, $\angle ABC = 2\theta$ and $\angle ACB = \theta$. Find the value of θ, giving your answer, in degrees, to 1 decimal place. **(4 marks)**

(E/P) **7** **a** Show that $5\sin 2\theta + 4\sin\theta = 0$ can be written in the form $a\sin\theta(b\cos\theta + c) = 0$, stating the values of a, b and c. **(2 marks)**

b Hence solve, for $0 \leqslant \theta < 360°$, the equation $5\sin 2\theta + 4\sin\theta = 0$. **(4 marks)**

(E/P) **8** **a** Given that $\sin 2\theta + \cos 2\theta = 1$, show that $2\sin\theta(\cos\theta - \sin\theta) = 0$. **(2 marks)**

b Hence, or otherwise, solve the equation $\sin 2\theta + \cos 2\theta = 1$ for $0 \leqslant \theta < 360°$. **(4 marks)**

(E/P) **9** **a** Prove that $(\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$. **(4 marks)**

b Use the result to solve, for $0 \leqslant \theta < \pi$, the equation $\cos 2\theta - \sin 2\theta = \frac{1}{\sqrt{2}}$
Give your answers in terms of π. **(3 marks)**

(P) **10 a** Show that:

i $\sin\theta \equiv \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ **ii** $\cos\theta \equiv \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

b By writing the following equations as quadratics in $\tan\frac{\theta}{2}$, solve, in the interval $0 \leqslant \theta \leqslant 360°$:

i $\sin\theta + 2\cos\theta = 1$ **ii** $3\cos\theta - 4\sin\theta = 2$

(E/P) **11 a** Show that $3\cos^2 x - \sin^2 x \equiv 1 + 2\cos 2x$. **(3 marks)**

b Hence sketch, for $-\pi \leqslant x \leqslant \pi$, the graph of $y = 3\cos^2 x - \sin^2 x$, showing the coordinates of points where the curve meets the axes. **(3 marks)**

(E/P) **12 a** Express $2\cos^2\frac{\theta}{2} - 4\sin^2\frac{\theta}{2}$ in the form $a\cos\theta + b$, where a and b are constants. **(4 marks)**

b Hence solve $2\cos^2\frac{\theta}{2} - 4\sin^2\frac{\theta}{2} = -3$, in the interval $0 \leqslant \theta < 360°$. **(3 marks)**

(E/P) **13 a** Use the identity $\sin^2 A + \cos^2 A \equiv 1$ to show that $\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$. **(5 marks)**

b Deduce that $\sin^4 A + \cos^4 A \equiv \frac{1}{4}(3 + \cos 4A)$. **(3 marks)**

c Hence solve $8\sin^4\theta + 8\cos^4\theta = 7$, for $0 < \theta < \pi$. **(3 marks)**

Hint Start by squaring $(\sin^2 A + \cos^2 A)$.

(E/P) **14 a** By writing 3θ as $2\theta + \theta$, show that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. **(4 marks)**

b Hence, or otherwise, for $0 < \theta < \pi$, solve $6\cos\theta - 8\cos^3\theta + 1 = 0$ giving your answer in terms of π. **(5 marks)**

7.5 Simplifying $a\cos x \pm b\sin x$

You can use the addition formulae to simplify some trigonometric expressions:

- **For positive values of a and b,**
  - **$a\sin x \pm b\cos x$ can be expressed in the form $R\sin(x \pm \alpha)$**
  - **$a\cos x \pm b\sin x$ can be expressed in the form $R\cos(x \mp \alpha)$**

  with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

  where $R\cos\alpha = a$ and $R\sin\alpha = b$ and $R = \sqrt{a^2 + b^2}$.

Notation The symbol $\mp$ means that $a\cos x + b\sin x$ will be written in the form $R\cos(x - \alpha)$, and $a\cos x - b\sin x$ will be written in the form $R\cos(x + \alpha)$.

Use the addition formulae to expand $\sin(x \pm \alpha)$ or $\cos(x \mp \alpha)$, then equate coefficients.

Example 14

Show that you can express $3\sin x + 4\cos x$ in the form:

a $R\sin(x + \alpha)$

b $R\cos(x - \beta)$

where $R > 0$, $0 < \alpha < 90°$, $0 < \beta < 90°$ giving your values of R, α and β to 1 decimal place when appropriate.

a $R\sin(x + \alpha) \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

> Use $\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$ and multiply through by R.

Let $3\sin x + 4\cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

So $R\cos\alpha = 3$ and $R\sin\alpha = 4$

> Equate the coefficients of the $\sin x$ and $\cos x$ terms.

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{4}{3}$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$

> Divide the equations to eliminate R and use $\tan^{-1}$ to find α.

So $\alpha = 53.1°$ (1 d.p.)

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2$

> Square and add the equations to eliminate α and find R^2.

$R^2(\cos^2\alpha + \sin^2\alpha) = 25$

$R^2 = 25$, so $R = 5$

> Use $\sin^2\alpha + \cos^2\alpha \equiv 1$.

$3\sin x + 4\cos x \equiv 5\sin(x + 53.1°)$

b $R\cos(x - \beta) \equiv R\cos x\cos\beta + R\sin x\sin\beta$

> Use $\cos(A - B) \equiv \cos A\cos B + \sin A\sin B$ and multiply through by R.

Let $3\sin x + 4\cos x \equiv R\cos x\cos\beta + R\sin x\sin\beta$

So $R\cos\beta = 4$ and $R\sin\beta = 3$

> Equate the coefficients of the $\cos x$ and $\sin x$ terms.

$\frac{R\sin\beta}{R\cos\beta} = \tan\beta = \frac{3}{4}$

> Divide the equations to eliminate R.

So $\beta = 36.9°$ (1 d.p.)

$R^2\cos^2\beta + R^2\sin^2\beta = 3^2 + 4^2$

> Square and add the equations to eliminate α and find R^2.

$R^2(\cos^2\beta + \sin^2\beta) = 25$

$R^2 = 25$, so $R = 5$

> Remember $\sin^2\alpha + \cos^2\alpha \equiv 1$.

$3\sin x + 4\cos x \equiv 5\cos(x - 36.9°)$

Online Explore how you can transform the graphs of $y = \sin x$ and $y = \cos x$ to obtain the graph of $y = 3\sin x + 4\cos x$ using technology.

Example 15

a Show that you can express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$, $0 < \alpha < \frac{\pi}{2}$

b Hence sketch the graph of $y = \sin x - \sqrt{3}\cos x$.

a Set $\sin x - \sqrt{3}\cos x \equiv R\sin(x - \alpha)$

$\sin x - \sqrt{3}\cos x \equiv R\sin x\cos\alpha - R\cos x\sin\alpha$

Expand $\sin(x - \alpha)$ and multiply by R.

So $R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$

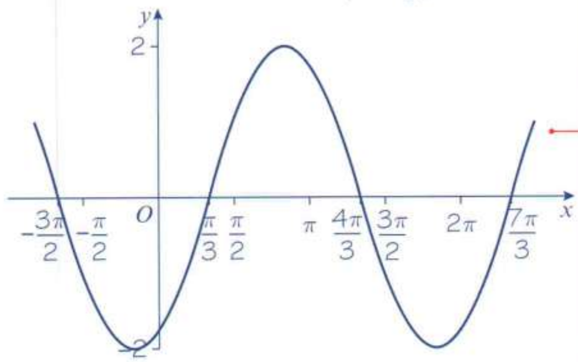
Equate the coefficients of $\sin x$ and $\cos x$ on both sides of the identity.

Dividing, $\tan\alpha = \sqrt{3}$, so $\alpha = \frac{\pi}{3}$

Squaring and adding: $R = 2$

So $\sin x - \sqrt{3}\cos x \equiv 2\sin\left(x - \frac{\pi}{3}\right)$

b $y = \sin x - \sqrt{3}\cos x \equiv 2\sin\left(x - \frac{\pi}{3}\right)$



You can sketch $y = 2\sin\left(x - \frac{\pi}{3}\right)$ by translating $y = \sin x$ by $\frac{\pi}{3}$ to the right and then stretching by a scale factor of 2 in the y-direction.

Example 16

a Express $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$, $0 < \alpha < 90°$.

b Hence solve, for $0 < \theta < 360°$, the equation $2\cos\theta + 5\sin\theta = 3$.

a Set $2\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 2$ and $R\sin\alpha = 5$

Equate the coefficients of $\sin x$ and $\cos x$ on both sides of the identity.

Dividing, $\tan\alpha = \frac{5}{2}$, so $\alpha = 68.2°$

Squaring and adding: $R = \sqrt{29}$

So $2\cos\theta + 5\sin\theta \equiv \sqrt{29}\cos(\theta - 68.2°)$

b $\sqrt{29}\cos(\theta - 68.2°) = 3$

Use the result from part **a**: $2\cos\theta + 5\sin\theta \equiv \sqrt{29}\cos(\theta - 68.2°)$.

So $\cos(\theta - 68.2°) = \frac{3}{\sqrt{29}}$

Divide both sides by $\sqrt{29}$.

$\cos^{-1}\left(\frac{3}{\sqrt{29}}\right) = 56.1...°$

As $0 < \theta < 360°$, the interval for $(\theta - 68.2°)$ is $-68.2° < \theta - 68.2° < 291.8°$.

$\frac{3}{\sqrt{29}}$ is positive, so solutions for $\theta - 68.2°$ are in the 1st and 4th quadrants.

So $\theta - 68.2° = -56.1...°, 56.1...°$

$\theta = 12.1°, 124.3°$ (to the nearest 0.1°)

Example 17

$f(\theta) = 12\cos\theta + 5\sin\theta$

a Write $f(\theta)$ in the form $R\cos(\theta - \alpha)$.

b Find the maximum value of $f(\theta)$ and the smallest positive value of θ at which it occurs.

Online Use technology to explore maximums and minimums of curves in the form $R\cos(\theta - \alpha)$.

a Set $12\cos\theta + 5\sin\theta \equiv R\cos(\theta - \alpha)$

So $12\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 12$ and $R\sin\alpha = 5$

$R = 13$ and $\tan\alpha = \frac{5}{12} \Rightarrow \alpha = 22.6°$

So $12\cos\theta + 5\sin\theta \equiv 13\cos(\theta - 22.6°)$

Equate $\sin x$ and $\cos x$ terms and then solve for R and α.

b The maximum value of $13\cos(\theta - 22.6°)$ is 13.

This occurs when $\cos(\theta - 22.6°) = 1$

$\theta - 22.6° = \ldots, -360°, 0°, 360°, \ldots$

The smallest positive value of θ is 22.6°.

The maximum value of $\cos x$ is 1 so the maximum value of $\cos(\theta - 22.6°)$ is also 1.

Solve the equation to find the smallest positive value of θ.

Exercise 7E

Unless otherwise stated, give all angles to 1 decimal place and write non-integer values of R in surd form.

1 Given that $5\sin\theta + 12\cos\theta \equiv R\sin(\theta + \alpha)$, find the value of R, $R > 0$, and the value of $\tan\alpha$.

2 Given that $\sqrt{3}\sin\theta + \sqrt{6}\cos\theta \equiv 3\cos(\theta - \alpha)$, where $0 < \alpha < 90°$, find the value of α.

3 Given that $2\sin\theta - \sqrt{5}\cos\theta \equiv -3\cos(\theta + \alpha)$, where $0 < \alpha < 90°$, find the value of α.

4 **a** Show that $\cos\theta - \sqrt{3}\sin\theta$ can be written in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

b Hence sketch the graph of $y = \cos\theta - \sqrt{3}\sin\theta$, $0 < \theta < \frac{\pi}{2}$, giving the coordinates of points of intersection with the axes.

(P) **5** **a** Express $7\cos\theta - 24\sin\theta$ in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

b The graph of $y = 7\cos\theta - 24\sin\theta$ meets the y-axis at P. State the coordinates of P.

c Write down the maximum and minimum values of $7\cos\theta - 24\sin\theta$.

d Deduce the number of solutions, in the interval $0 < \theta < 360°$, of the following equations:

i $7\cos\theta - 24\sin\theta = 15$ **ii** $7\cos\theta - 24\sin\theta = 26$ **iii** $7\cos\theta - 24\sin\theta = -25$

(E) **6** $f(\theta) = \sin\theta + 3\cos\theta$

Given $f(\theta) = R\sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$.

a Find the value of R and the value of α. **(4 marks)**

b Hence, or otherwise, solve $f(\theta) = 2$ for $0 \leqslant \theta < 360°$. **(3 marks)**

(E) 7 a Express $\cos 2\theta - 2\sin 2\theta$ in the form $R\cos(2\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the value of α to 3 decimal places. **(4 marks)**

b Hence, or otherwise, solve for $0 \leqslant \theta < \pi$, $\cos 2\theta - 2\sin 2\theta = -1.5$, rounding your answers to 2 decimal places. **(4 marks)**

(P) 8 Solve the following equations, in the intervals given in brackets.

a $6\sin x + 8\cos x = 5\sqrt{3}$, [0, 360°]

b $2\cos 3\theta - 3\sin 3\theta = -1$, [0, 90°]

c $8\cos\theta + 15\sin\theta = 10$, [0, 360°]

d $5\sin\frac{x}{2} - 12\cos\frac{x}{2} = -6.5$, [−360°, 360°]

(E/P) 9 a Express $3\sin 3\theta - 4\cos 3\theta$ in the form $R\sin(3\theta - \alpha)$, with $R > 0$ and $0 < \alpha < 90°$. **(3 marks)**

b Hence write down the minimum value of $3\sin 3\theta - 4\cos 3\theta$ and the value of θ at which it occurs. **(3 marks)**

c Solve, for $0 \leqslant \theta < 180°$, the equation $3\sin 3\theta - 4\cos 3\theta = 1$. **(3 marks)**

(E/P) 10 a Express $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$ in the form $a\sin 2\theta + b\cos 2\theta + c$, where a, b and c are constants to be found. **(3 marks)**

b Hence find the maximum and minimum values of $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$. **(4 marks)**

c Solve $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta = -1$ for $0 \leqslant \theta < 180°$, rounding your answers to 1 decimal place. **(4 marks)**

(P) 11 A class were asked to solve $3\cos\theta = 2 - \sin\theta$ for $0 \leqslant \theta < 360°$. One student expressed the equation in the form $R\cos(\theta - \alpha) = 2$, with $R > 0$ and $0 < \alpha < 90°$, and correctly solved the equation.

a Find the values of R and α and hence find her solutions.

Another student decided to square both sides of the equation and then form a quadratic equation in $\sin\theta$.

b Show that the correct quadratic equation is $10\sin^2\theta - 4\sin\theta - 5 = 0$.

c Solve this equation, for $0 \leqslant \theta < 360°$.

d Explain why not all of the answers satisfy $3\cos\theta = 2 - \sin\theta$.

(E/P) 12 a Given $\cot\theta + 2 = \operatorname{cosec}\theta$, show that $2\sin\theta + \cos\theta = 1$. **(4 marks)**

b Solve $\cot\theta + 2 = \operatorname{cosec}\theta$ for $0 \leqslant \theta < 360°$. **(3 marks)**

(E/P) 13 a Given $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1)\sin\theta = 2$, show that $\cos\theta + \sqrt{3}\sin\theta = 2$. **(4 marks)**

b Solve $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1)\sin\theta = 2$ for $0 \leqslant \theta \leqslant 2\pi$. **(2 marks)**

E/P **14** **a** Express $9\cos\theta + 40\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give the value of α to 3 decimal places. **(4 marks)**

b $g(\theta) = \dfrac{18}{50 + 9\cos\theta + 40\sin\theta}$, $0 \leqslant \theta \leqslant 360°$

Calculate:

i the minimum value of $g(\theta)$ **(2 marks)**

ii the smallest positive value of θ at which the minimum occurs. **(2 marks)**

E/P **15** $p(\theta) = 12\cos 2\theta - 5\sin 2\theta$

Given that $p(\theta) = R\cos(2\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$,

a find the value of R and the value of α. **(3 marks)**

b Hence solve the equation $12\cos 2\theta - 5\sin 2\theta = -6.5$ for $0 \leqslant \theta < 180°$. **(5 marks)**

c Express $24\cos^2\theta - 10\sin\theta\cos\theta$ in the form $a\cos 2\theta + b\sin 2\theta + c$, where a, b and c are constants to be found. **(3 marks)**

d Hence, or otherwise, find the minimum value of $24\cos^2\theta - 10\sin\theta\cos\theta$. **(2 marks)**

7.6 Proving trigonometric identities

You can use known trigonometric identities to prove other identities.

Example 18

a Show that $2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta \equiv \frac{1}{2}\sin 2\theta$.

b Show that $1 + \cos 4\theta \equiv 2\cos^2 2\theta$.

a $\sin 2A \equiv 2\sin A\cos A$

$\sin\theta \equiv 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ — Substitute $A = \frac{\theta}{2}$ into the formula for $\sin 2A$.

$\text{LHS} \equiv 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta$

$\equiv \sin\theta\cos\theta$ — Use the above result for $2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\equiv \frac{1}{2}\sin 2\theta$ — Remember $\sin 2\theta \equiv 2\sin\theta\cos\theta$.

$\equiv \text{RHS}$

b $\text{LHS} \equiv 1 + \cos 4\theta$

$\equiv 1 + 2\cos^2 2\theta - 1$ — Use $\cos 2A \equiv 2\cos^2 A - 1$ with $A = 2\theta$.

$\equiv 2\cos^2 2\theta$

$\equiv \text{RHS}$

Problem-solving

Always be aware that the addition formulae can be altered by making a substitution.

Example 19

Prove the identity $\tan 2\theta \equiv \dfrac{2}{\cot\theta - \tan\theta}$

$\text{LHS} \equiv \tan 2\theta \equiv \dfrac{2\tan\theta}{1-\tan^2\theta}$

Divide the numerator and denominator by $\tan\theta$.

So $\tan 2\theta \equiv \dfrac{2}{\frac{1}{\tan\theta} - \tan\theta}$

$\equiv \dfrac{2}{\cot\theta - \tan\theta}$

Problem-solving

Dividing the numerator and denominator by a common term can be helpful when trying to rearrange an expression into a required form.

Example 20

Prove that $\sqrt{3}\cos 4\theta + \sin 4\theta \equiv 2\cos\left(4\theta - \frac{\pi}{6}\right)$.

$\text{RHS} \equiv 2\cos\left(4\theta - \frac{\pi}{6}\right)$

$\equiv 2\cos 4\theta\cos\frac{\pi}{6} + 2\sin 4\theta\sin\frac{\pi}{6}$ — Use the addition formulae.

$\equiv 2\cos 4\theta\left(\frac{\sqrt{3}}{2}\right) + 2\sin 4\theta\left(\frac{1}{2}\right)$ — Write the exact values of $\cos\frac{\pi}{6}$ and $\sin\frac{\pi}{6}$

$\equiv \sqrt{3}\cos 4\theta + \sin 4\theta \equiv \text{LHS}$

Problem-solving

Sometimes it is easier to begin with the RHS of the identity.

Exercise 7F

(P) 1 Prove the following identities.

a $\dfrac{\cos 2A}{\cos A + \sin A} \equiv \cos A - \sin A$

b $\dfrac{\sin B}{\sin A} - \dfrac{\cos B}{\cos A} \equiv 2\operatorname{cosec} 2A \sin(B - A)$

c $\dfrac{1-\cos 2\theta}{\sin 2\theta} \equiv \tan\theta$

d $\dfrac{\sec^2\theta}{1-\tan^2\theta} \equiv \sec 2\theta$

e $2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta) \equiv \sin 2\theta$

f $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} \equiv 2$

g $\operatorname{cosec}\theta - 2\cot 2\theta\cos\theta \equiv 2\sin\theta$

h $\dfrac{\sec\theta - 1}{\sec\theta + 1} \equiv \tan^2\dfrac{\theta}{2}$

i $\tan\left(\dfrac{\pi}{4} - x\right) \equiv \dfrac{1-\sin 2x}{\cos 2x}$

(P) **2** Prove the identities:

a $\sin(A + 60°) + \sin(A - 60°) \equiv \sin A$

b $\dfrac{\cos A}{\sin B} - \dfrac{\sin A}{\cos B} \equiv \dfrac{\cos(A + B)}{\sin B \cos B}$

c $\dfrac{\sin(x + y)}{\cos x \cos y} \equiv \tan x + \tan y$

d $\dfrac{\cos(x + y)}{\sin x \sin y} + 1 \equiv \cot x \cot y$

e $\cos\left(\theta + \dfrac{\pi}{3}\right) + \sqrt{3}\sin\theta \equiv \sin\left(\theta + \dfrac{\pi}{6}\right)$

f $\cot(A + B) \equiv \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

g $\sin^2(45° + \theta) + \sin^2(45° - \theta) \equiv 1$

h $\cos(A + B)\cos(A - B) \equiv \cos^2 A - \sin^2 B$

(E/P) **3** **a** Show that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$. **(3 marks)**

b Hence find the value of $\tan 75° + \cot 75°$. **(2 marks)**

(E/P) **4** **a** Show that $\sin 3\theta \equiv 3\sin\theta\cos^2\theta - \sin^3\theta$. **(3 marks)**

b Show that $\cos 3\theta \equiv \cos^3\theta - 3\sin^2\theta\cos\theta$. **(3 marks)**

c Hence, or otherwise, show that $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ **(4 marks)**

d Given that θ is acute and that $\cos\theta = \frac{1}{3}$, show that $\tan 3\theta = \dfrac{10\sqrt{2}}{23}$ **(3 marks)**

5 **a** Using $\cos 2A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$, show that:

i $\cos^2\dfrac{x}{2} \equiv \dfrac{1 + \cos x}{2}$ **ii** $\sin^2\dfrac{x}{2} \equiv \dfrac{1 - \cos x}{2}$

b Given that $\cos\theta = 0.6$, and that θ is acute, write down the values of:

i $\cos\dfrac{\theta}{2}$ **ii** $\sin\dfrac{\theta}{2}$ **iii** $\tan\dfrac{\theta}{2}$

c Show that $\cos^4\dfrac{A}{2} \equiv \dfrac{1}{8}(3 + 4\cos A + \cos 2A)$.

(E/P) **6** Show that $\cos^4\theta \equiv \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$. You must show each stage of your working. **(6 marks)**

(E/P) **7** Prove that $\sin^2(x + y) - \sin^2(x - y) \equiv \sin 2x \sin 2y$. **(5 marks)**

(E/P) **8** Prove that $\cos 2\theta - \sqrt{3}\sin 2\theta \equiv 2\cos\left(2\theta + \dfrac{\pi}{3}\right)$. **(4 marks)**

(E/P) **9** Prove that $4\cos\left(2\theta - \dfrac{\pi}{6}\right) \equiv 2\sqrt{3} - 4\sqrt{3}\sin^2\theta + 4\sin\theta\cos\theta$. **(4 marks)**

(P) **10** Show that:

a $\cos\theta + \sin\theta \equiv \sqrt{2}\sin\left(\theta + \dfrac{\pi}{4}\right)$

b $\sqrt{3}\sin 2\theta - \cos 2\theta \equiv 2\sin\left(2\theta - \dfrac{\pi}{6}\right)$

Challenge

1 a Show that $\cos(A + B) - \cos(A - B) \equiv -2\sin A \sin B$.

b Hence show that $\cos P - \cos Q \equiv -2\sin\left(\frac{P + Q}{2}\right)\sin\left(\frac{P - Q}{2}\right)$.

c Express $3\sin x \sin 7x$ as the difference of cosines.

2 a Prove that $\sin P + \sin Q \equiv 2\sin\left(\frac{P + Q}{2}\right)\cos\left(\frac{P - Q}{2}\right)$.

b Hence, or otherwise, show that $2\sin\frac{11\pi}{24}\cos\frac{5\pi}{24} = \frac{\sqrt{3} + \sqrt{2}}{2}$

7.7 Modelling with trigonometric functions

You can use trigonometric functions to model real-life situations. In trigonometrical modelling questions you will often have to write the model using $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$ to find maximum or minimum values.

Example 21

The cabin pressure, P, in pounds per square inch (psi) on an aeroplane at cruising altitude can be modelled by the equation $P = 11.5 - 0.5\sin(t - 2)$, where t is the time in hours since the cruising altitude was first reached, and angles are measured in radians.

a State the maximum and the minimum cabin pressure.

b Find the time after reaching cruising altitude that the cabin first reaches a maximum pressure.

c Calculate the cabin pressure after 5 hours at a cruising altitude.

d Find all the times during the first 8 hours of cruising that the cabin pressure would be exactly 11.3 psi.

a Maximum pressure $= 11.5 - 0.5 \times (-1) = 12$ psi

Minimum pressure $= 11.5 - 0.5 \times 1 = 11$ psi

$-1 \leqslant \sin(t - 2) \leqslant 1$. Use the maximum and minimum values of the sine function to find the maximum and minimum pressure.

b $11.5 - 0.5\sin(t - 2) = 12$

Set the model equal to 12, the maximum pressure.

$-0.5\sin(t - 2) = 0.5$

$\sin(t - 2) = -1$

$t - 2 = \ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

Remember the model uses radians.

$t = 0.43$ hours $= 26$ min

Multiply 0.43 by 60 to get the time in minutes.

c $P = 11.5 - 0.5\sin(5 - 2)$

Substitute $t = 5$.

$= 11.5 - 0.070\ldots$

$= 11.43$ psi

Online Explore the solution to this modelling problem graphically using technology.

d $11.5 - 0.5\sin(t - 2) = 11.3$
$-0.5\sin(t - 2) = -0.2$
$\sin(t - 2) = 0.4$
$t - 2 = -3.553..., 0.4115..., 2.73..., 6.6947...$
$t = 2.41$ hours, 4.73 hours.
$t = 2$ h 25 min, 4 h 44 min

Set the model equal to 11.3.

Use $\sin^{-1}(0.4)$ to find the principal solution, then use the properties of the sine function to find other possible solutions in the range $0 \leqslant t \leqslant 8$.

$0 \leqslant t \leqslant 8$ so $-2 \leqslant t - 2 \leqslant 6$. There are two solutions in the required range.

Exercise 7G

(P) **1** The height, h, of a buoy on a boating lake can be modelled by $h = 0.25\sin(1800t)°$, where h is the height in metres above the buoy's resting position and t is the time in minutes.

a State the maximum height the buoy reaches above its resting position according to this model.

b Calculate the time, to the nearest tenth of a second, at which the buoy is first at a height of 0.1 metres.

c Calculate the time interval between successive minimum heights of the buoy.

(P) **2** The angle of displacement of a pendulum, θ, at time t seconds after it is released is modelled as $\theta = 0.03\cos(25t)$, where all angles are measured in radians.

a State the maximum displacement of the pendulum according to this model.

b Calculate the angle of displacement of the pendulum after 0.2 seconds.

c Find the time taken for the pendulum to return to its starting position.

d Find all the times in the first half second of motion that the pendulum has a displacement of 0.01 radians.

(P) **3** The price, P, of stock in pounds during a 9-hour trading window can be modelled by $P = 17.4 + 2\sin(0.7t - 3)$, where t is the time in hours after the stock market opens, and angles are measured in radians.

a State the beginning and end price of the stock.

b Calculate the maximum price of the stock and the time when it occurs.

c A day trader wants to sell the stock when it firsts shows a profit of £0.40 above the day's starting price. At what time should the trader sell the stock?

(P) **4** The temperature of an oven can be modelled by the equation $T = 225 - 0.3\sin(2x - 3)$, where T is the temperate in Celsius and x is the time in minutes after the oven first reaches the desired temperature, and angles are measured in radians.

a State the minimum temperature of the oven.

b Find the times during the first 10 minutes when the oven is at a minimum temperature.

c Calculate the time when the oven first reaches a temperature of 225.2 °C.

(E/P) **5** **a** Express $0.3\sin\theta - 0.4\cos\theta$ in the form $R\sin(\theta - \alpha)°$, where $R > 0$ and $0 < \alpha < 90°$. Give the value of α to 2 decimal places. **(4 marks)**

b **i** Find the maximum value of $0.3 \sin\theta - 0.4\cos\theta$. **(2 marks)**

ii Find the value of θ, for $0 < \theta < 180$ at which the maximum occurs. **(1 mark)**

Jack models the temperature in his house, T°C, on a particular day by the equation

$$T = 23 + 0.3\sin(18x)° - 0.4\cos(18x)°, \ x \geqslant 0$$

where x is the number of minutes since the thermostat was adjusted.

c Calculate the minimum value of T predicted by this model, and the value of x, to 2 decimal places, when this minimum occurs. **(3 marks)**

d Calculate, to the nearest minute, the times in the first hour when the temperature is predicted, by this model, to be exactly 23 °C. **(4 marks)**

E/P **6** **a** Express $65\cos\theta - 20\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the value of α correct to 4 decimal places. **(4 marks)**

A city wants to build a large circular wheel as a tourist attraction. The height of a tourist on the circular wheel is modelled by the equation

$$H = 70 - 65\cos 0.2t + 20\sin 0.2t$$

where H is the height of the tourist above the ground in metres, t is the number of minutes after boarding and the angles are given in radians. Find:

b the maximum height of the wheel **(2 marks)**

c the time for one complete revolution **(2 marks)**

d the number of minutes the tourist will be over 100 m above the ground in each revolution. **(4 marks)**

E/P **7** **a** Express $200\sin\theta - 150\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the value of α to 4 decimal places. **(4 marks)**

The electric field strength, E V/m, in a microwave of width 25 cm can be modelled using the equation

$$E = 1700 + 200\sin\left(\frac{4\pi x}{25}\right) - 150\cos\left(\frac{4\pi x}{25}\right)$$

where x is the distance in cm from the left hand edge of the microwave oven.

b **i** Calculate the maximum value of E predicted by this model.

ii Find the values of x, for $0 \leqslant x < 25$, where this maximum occurs. **(3 marks)**

c Food in the microwave will heat best when the electric field strength at the centre of the food is above 1800 V/m. Find the range of possible locations for the centre of the food. **(5 marks)**

Challenge

Look at the model for the electric field strength in a microwave oven given in question **7** above. For food of the same type and mass, the energy transferred by the oven is proportional to the square of the electric field strength. Given that a square of chocolate placed at a point of maximum field strength takes 20 seconds to melt,

a estimate the range of locations within the oven that an identical square of chocolate will take longer than 30 seconds to melt.

b State two limitations of the model.

Mixed exercise 7

P 1 Without using a calculator, find the value of:

a $\sin 40° \cos 10° - \cos 40° \sin 10°$ b $\frac{1}{\sqrt{2}}\cos 15° - \frac{1}{\sqrt{2}}\sin 15°$ c $\frac{1 - \tan 15°}{1 + \tan 15°}$

P 2 Given that $\sin x = \frac{1}{\sqrt{5}}$ where x is acute and that $\cos(x - y) = \sin y$, show that $\tan y = \frac{\sqrt{5} + 1}{2}$

P 3 The lines l_1 and l_2, with equations $y = 2x$ and $3y = x - 1$ respectively, are drawn on the same set of axes. Given that the scales are the same on both axes and that the angles l_1 and l_2 make with the positive x-axis are A and B respectively,

a write down the value of $\tan A$ and the value of $\tan B$;

b without using your calculator, work out the acute angle between l_1 and l_2.

P 4 In $\triangle ABC$, $AB = 5$ cm and $AC = 4$ cm, $\angle ABC = (\theta - 30°)$ and $\angle ACB = (\theta + 30°)$. Using the sine rule, show that $\tan\theta = 3\sqrt{3}$.

P 5 The first three terms of an arithmetic series are $\sqrt{3}\cos\theta$, $\sin(\theta - 30°)$ and $\sin\theta$, where θ is acute. Find the value of θ.

P 6 Two of the angles, A and B, in $\triangle ABC$ are such that $\tan A = \frac{3}{4}$, $\tan B = \frac{5}{12}$

a Find the exact value of: i $\sin(A + B)$ ii $\tan 2B$.

b By writing C as $180° - (A + B)$, show that $\cos C = -\frac{33}{65}$

P 7 The angles x and y are acute angles such that $\sin x = \frac{2}{\sqrt{5}}$ and $\cos y = \frac{3}{\sqrt{10}}$

a Show that $\cos 2x = -\frac{3}{5}$

b Find the value of $\cos 2y$.

c Show without using your calculator, that:

i $\tan(x + y) = 7$ ii $x - y = \frac{\pi}{4}$

P 8 Given that $\sin x \cos y = \frac{1}{2}$ and $\cos x \sin y = \frac{1}{3}$,

a show that $\sin(x + y) = 5\sin(x - y)$.

Given also that $\tan y = k$, express in terms of k:

b $\tan x$

c $\tan 2x$

E/P 9 a Given that $\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$, show that $\tan 2\theta = \frac{1}{\sqrt{3}}$ **(2 marks)**

b Hence solve, for $0 \leqslant \theta \leqslant \pi$, the equation $\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$. **(4 marks)**

E/P **10** **a** Show that $\cos 2\theta = 5\sin\theta$ may be written in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are constants to be found. **(3 marks)**

b Hence solve, for $-\pi \leqslant \theta \leqslant \pi$, the equation $\cos 2\theta = 5\sin\theta$. **(4 marks)**

E/P **11** **a** Given that $2\sin x = \cos(x - 60)°$, show that $\tan x = \dfrac{1}{4 - \sqrt{3}}$ **(4 marks)**

b Hence solve, for $0 \leqslant x \leqslant 360°$, $2\sin x = \cos(x - 60°)$, giving your answers to 1 decimal place. **(2 marks)**

E/P **12** **a** Given that $4\sin(x + 70°) = \cos(x + 20°)$, show that $\tan x = -\frac{3}{5}\tan 70°$. **(4 marks)**

b Hence solve, for $0 \leqslant x \leqslant 180°$, $4\sin(x + 70°) = \cos(x + 20°)$, giving your answers to 1 decimal place. **(3 marks)**

P **13** **a** Given that α is acute and $\tan\alpha = \frac{3}{4}$, prove that

$$3\sin(\theta + \alpha) + 4\cos(\theta + \alpha) \equiv 5\cos\theta$$

b Given that $\sin x = 0.6$ and $\cos x = -0.8$, evaluate $\cos(x + 270°)$ and $\cos(x + 540°)$.

E/P **14** **a** Prove, by counter-example, that the statement

$$\sec(A + B) \equiv \sec A + \sec B, \text{ for all } A \text{ and } B$$

is false. **(2 marks)**

b Prove that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$, $\theta \neq \dfrac{n\pi}{2}$, $n \in \mathbb{Z}$. **(4 marks)**

P **15** Using $\tan 2\theta \equiv \dfrac{2\tan\theta}{1 - \tan^2\theta}$ with an appropriate value of θ,

a show that $\tan\dfrac{\pi}{8} = \sqrt{2} - 1$.

b Use the result in **a** to find the exact value of $\tan\dfrac{3\pi}{8}$

E/P **16** **a** Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x - \alpha)$, with $R > 0$ and $0 < \alpha < 90°$. **(4 marks)**

b Hence sketch the graph of $y = \sin x - \sqrt{3}\cos x$, for $-360° \leqslant x \leqslant 360°$, giving the coordinates of all points of intersection with the axes. **(4 marks)**

E/P **17** Given that $7\cos 2\theta + 24\sin 2\theta \equiv R\cos(2\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$, find:

a the value of R and the value of α, to 2 decimal places **(4 marks)**

b the maximum value of $14\cos^2\theta + 48\sin\theta\cos\theta$. **(1 mark)**

c Solve the equation $7\cos 2\theta + 24\sin 2\theta = 12.5$, for $0 \leqslant \theta \leqslant \pi$, giving your answers to 2 decimal places. **(5 marks)**

E/P **18** **a** Express $1.5 \sin 2x + 2 \cos 2x$ in the form $R \sin(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving your values of R and α to 3 decimal places where appropriate. **(4 marks)**

b Express $3 \sin x \cos x + 4 \cos^2 x$ in the form $a \sin 2x + b \cos 2x + c$, where a, b and c are constants to be found. **(3 marks)**

c Hence, using your answer to part **a**, deduce the maximum value of $3 \sin x \cos x + 4 \cos^2 x$. **(1 mark)**

E/P **19** **a** Given that $\sin^2 \frac{\theta}{2} = 2 \sin \theta$, show that $\sqrt{17} \sin(\theta + \alpha) = 1$ and state the value of α, where $0 \leqslant \alpha \leqslant 90°$. **(3 marks)**

b Hence, or otherwise, solve $\sin^2 \frac{\theta}{2} = 2 \sin \theta$ for $0 \leqslant \theta \leqslant 360°$. **(4 marks)**

E/P **20** **a** Given that $2 \cos \theta = 1 + 3 \sin \theta$, show that $R \cos(\theta + \alpha) = 1$, where R and α are constants to be found, and $0 \leqslant \alpha \leqslant 90°$. **(2 marks)**

b Hence, or otherwise, solve $2 \cos \theta = 1 + 3 \sin \theta$ for $0 \leqslant \theta \leqslant 360°$. **(4 marks)**

P **21** Using known trigonometric identities, prove the following:

a $\sec \theta \operatorname{cosec} \theta \equiv 2 \operatorname{cosec} 2\theta$

b $\tan\left(\frac{\pi}{4} + x\right) - \tan\left(\frac{\pi}{4} - x\right) \equiv 2 \tan 2x$

c $\sin(x + y) \sin(x - y) \equiv \cos^2 y - \cos^2 x$

d $1 + 2 \cos 2\theta + \cos 4\theta \equiv 4 \cos^2 \theta \cos 2\theta$

E/P **22** **a** Use the double-angle formulae to prove that $\frac{1 - \cos 2x}{1 + \cos 2x} \equiv \tan^2 x$. **(4 marks)**

b Hence find, for $-\pi \leqslant x \leqslant \pi$, all the solutions of $\frac{1 - \cos 2x}{1 + \cos 2x} = 3$, leaving your answers in terms of π. **(2 marks)**

E/P **23** **a** Prove that $\cos^4 2\theta - \sin^4 2\theta \equiv \cos 4\theta$. **(4 marks)**

b Hence find, for $0 \leqslant \theta \leqslant 180°$, all the solutions of $\cos^4 2\theta - \sin^4 2\theta = \frac{1}{2}$ **(2 marks)**

E/P **24** **a** Prove that $\frac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan \theta$. **(4 marks)**

b Verify that $\theta = 180°$ is a solution of the equation $\sin 2\theta = 2 - 2 \cos 2\theta$. **(1 mark)**

c Using the result in part **a**, or otherwise, find the two other solutions, $0 < \theta < 360°$, of the equation $\sin 2\theta = 2 - 2 \cos 2\theta$. **(3 marks)**

E/P **25** The curve on an oscilloscope screen satisfies the equation $y = 2 \cos x - \sqrt{5} \sin x$.

a Express the equation of the curve in the form $y = R \cos(x + \alpha)$, where R and α are constants and $R > 0$ and $0 \leqslant \alpha < \frac{\pi}{2}$ **(4 marks)**

b Find the values of x, $0 \leqslant x < 2\pi$, for which $y = -1$. **(3 marks)**

E/P **26** **a** Express $1.4\sin\theta - 5.6\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90°$. Round R and α to 3 decimal places. **(4 marks)**

b Hence find the maximum value of $1.4\sin\theta - 5.6\cos\theta$ and the smallest positive value of θ for which this maximum occurs. **(3 marks)**

The length of daylight, $d(t)$ at a location in northern Scotland can be modelled using the equation

$$d(t) = 12 - 5.6\cos\left(\frac{360t}{365}\right)^\circ + 1.4\sin\left(\frac{360t}{365}\right)^\circ$$

where t is the numbers of days into the year.

c Calculate the minimum number of daylight hours in northern Scotland as given by this model. **(2 marks)**

d Find the value of t when this minimum number of daylight hours occurs. **(1 mark)**

E/P **27** **a** Express $12\sin x + 5\cos x$ in the form $R\sin(x + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90°$. Round α to 1 decimal place. **(4 marks)**

A runner's speed, v in m/s, in an endurance race can be modelled by the equation

$$v(x) = \frac{50}{12\sin\left(\frac{2x}{5}\right)^\circ + 5\cos\left(\frac{2x}{5}\right)^\circ}, \quad 0 \leqslant x \leqslant 300$$

where x is the time in minutes since the beginning of the race.

b Find the minimum value of v. **(2 marks)**

c Find the time into the race when this speed occurs. **(1 mark)**

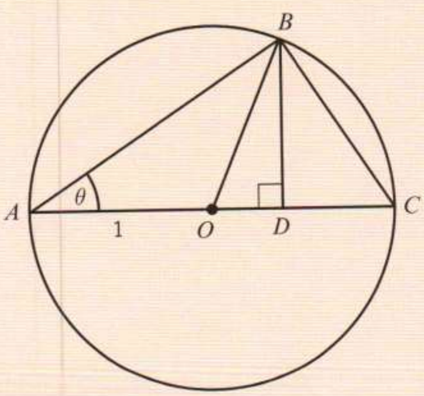
Challenge

1 Prove the identities:

a $\dfrac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} \equiv -\cot\theta$

b $\cos x + 2\cos 3x + \cos 5x \equiv 4\cos^2 x \cos 3x$

2 The points A, B and C lie on a circle with centre O and radius 1. AC is a diameter of the circle and point D lies on OC such that $\angle ODB = 90°$.



Use this construction to prove:

a $\sin 2\theta \equiv 2\sin\theta\cos\theta$ **b** $\cos 2\theta \equiv 2\cos^2\theta - 1$

Hint Find expressions for $\angle BOD$ and AB, then consider the lengths OD and DB.

Summary of key points

1 The **addition** (or compound-angle) formulae are:

- $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$ $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

2 The **double-angle** formulae are:

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$

3 For positive values of a and b,

- $a \sin x \pm b \cos x$ can be expressed in the form $R \sin(x \pm \alpha)$
- $a \cos x \pm b \sin x$ can be expressed in the form $R \cos(x \mp \alpha)$

with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

where $R \cos \alpha = a$ and $R \sin \alpha = b$ and $R = \sqrt{a^2 + b^2}$.