7 Applications of forces

Objectives

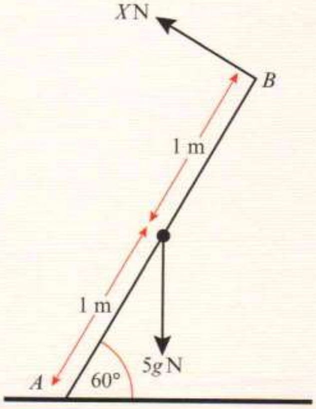
After completing this chapter you should be able to:

- Find an unknown force when a system is in equilibrium → pages 129–132
- Solve statics problems involving weight, tension and pulleys → pages 132–137
- Understand and solve problems involving limiting equilibrium → pages 137–146
- Solve problems involving motion on rough or smooth inclined planes → pages 147–150
- Solve problems involving connected particles that require the resolution of forces → pages 150–154

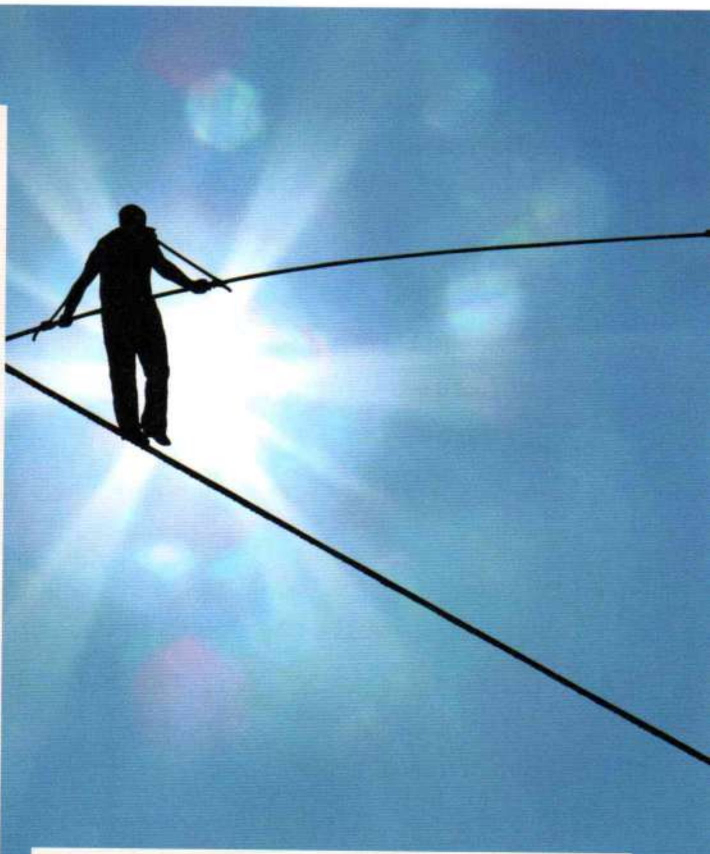
Prior knowledge check

1 A particle of mass 2 kg sits on a rough plane that is inclined at 45° to the horizontal. A force of 10 N acts parallel to and up the plane. Given that the particle is on the point of moving, work out the coefficient of friction μ. ← Section 5.3

2 A uniform rod AB of length 2 m and mass 5 kg rests in equilibrium at an angle of 60° to a horizontal surface. The rod is pivoted at A and a force of magnitude X N acts perpendicular to the rod at B. Find the value of X.



← Section 4.3



A tightrope walker uses a mathematical model to calculate the tension in his wire. This allows him to make sure that the wire is strong enough to hold his weight safely. → Mixed exercise, Q3

7.1 Static particles

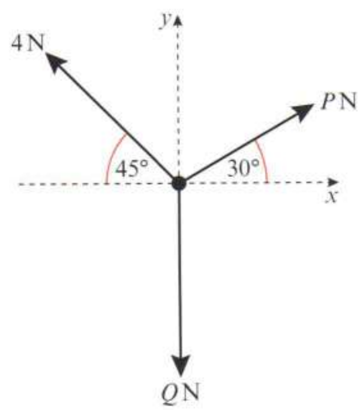
- **A particle or rigid body is in static equilibrium if it is at rest and the resultant force acting on the particle is zero.**

To solve problems in statics you should:
- Draw a diagram showing clearly the forces acting on the particle.
- Resolve the forces into horizontal and vertical components or, if the particle is on an inclined plane, into components parallel and perpendicular to the plane.
- Set the sum of the components in each direction equal to zero.
- Solve the resulting equations to find the unknown force(s).

Hint The particle is not accelerating, so $a = 0$. $F = ma = 0$

Example 1

The diagram shows a particle in equilibrium under the forces shown. By resolving horizontally and vertically find the magnitudes of the forces P and Q.



Method 1:

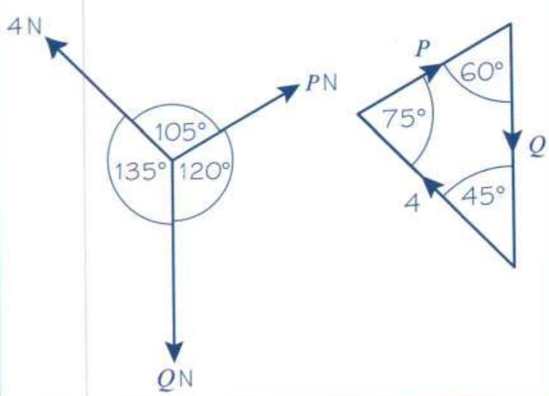
R(→), $P\cos 30° - 4\cos 45° = 0$

R(↑), $P\sin 30° + 4\sin 45° - Q = 0$

$$P = \frac{4\cos 45°}{\cos 30°} = 3.27 \text{ (3 s.f.)}$$

$$Q = P\sin 30° + 4\sin 45° = 4.46 \text{ (3 s.f.)}$$

Method 2:



Resolve horizontally and vertically. Equate the sum of the forces to zero as there is no acceleration (the particle is in equilibrium).

Solve the first equation to find P (as there is only one unknown quantity), and then use your value for P in the second equation to find Q.

If exact answers are required these would be $P = \frac{4\sqrt{6}}{3}$ and $Q = \frac{2(\sqrt{6} + 3\sqrt{2})}{3}$

Problem-solving

You can use a vector diagram to solve equilibrium problems involving three forces. Because the particle is in equilibrium, the three forces will form a **closed triangle**.

If the angle between forces on the force diagram is θ, the angle between those forces on the **triangle of forces** is $180° - \theta$.

The length of each side of the triangle is the magnitude of the force.

So $\frac{4}{\sin 60°} = \frac{P}{\sin 45°} = \frac{Q}{\sin 75°}$

$P = \frac{4 \sin 45°}{\sin 60°} = \frac{4\sqrt{6}}{3}$ N

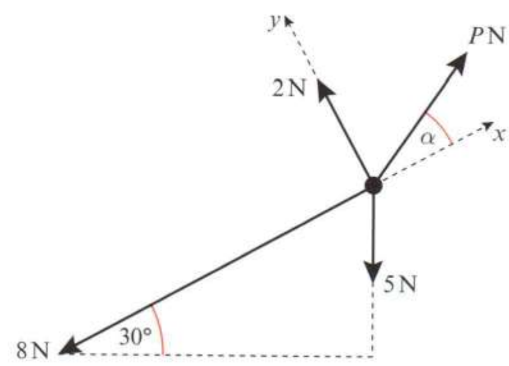
$Q = \frac{4 \sin 75°}{\sin 60°} = \frac{2(\sqrt{6} + 3\sqrt{2})}{3}$ N

Use the sine rule.

Watch out This method only works for a particle in equilibrium. If the resultant force is not zero, the vector diagram will not be a closed triangle.

Example 2

The diagram shows a particle in equilibrium on an inclined plane under the forces shown.
Find the magnitude of the force P and the size of the angle α.



R(↗), $P\cos\alpha - 8 - 5\sin 30° = 0$

∴ $P\cos\alpha = 8 + 5\sin 30°$ (1)

R(↖), $P\sin\alpha + 2 - 5\cos 30° = 0$

∴ $P\sin\alpha = 5\cos 30° - 2$ (2)

Divide equation (2) by equation (1) to give:

$\tan\alpha = \frac{5\cos 30° - 2}{8 + 5\sin 30°} = \frac{2.330}{10.5} = 0.222$

∴ $\alpha = 12.5°$ (3 s.f.)

Substitute into equation (2):

$P\sin 12.5°... = 5\cos 30° - 2$

∴ $P = 10.8$ (3 s.f.)

Resolve parallel to the plane. Take the direction up the plane as positive.

Rearrange the equation to make $P\cos\alpha$ the subject.

Resolve perpendicular to the plane. Rearrange the second equation to make $P\sin\alpha$ the subject.

After division use $\frac{\sin\alpha}{\cos\alpha} \equiv \tan\alpha$.

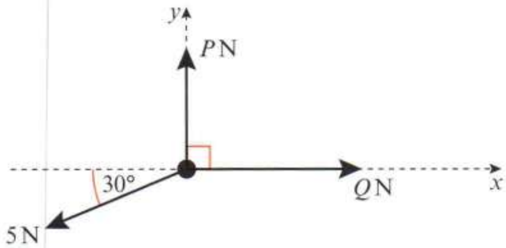
Use $\tan^{-1}$ and give your answer to three significant figures.

You could check your answers by substituting into equation (1).

Exercise 7A

1 Each of the following diagrams shows a particle in static equilibrium. For each particle:
- i resolve the components in the x-direction
- ii resolve the components in the y-direction
- iii find the magnitude of any unknown forces (marked P and Q) and the size of any unknown angles (marked θ).

a



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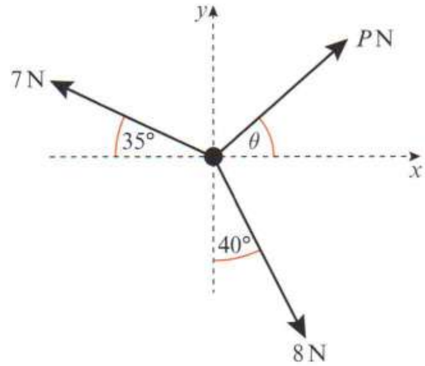

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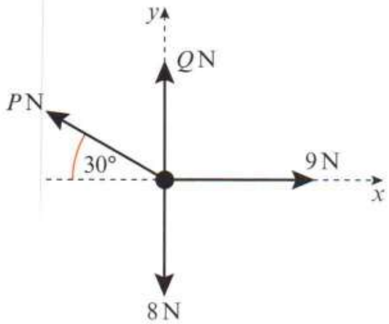

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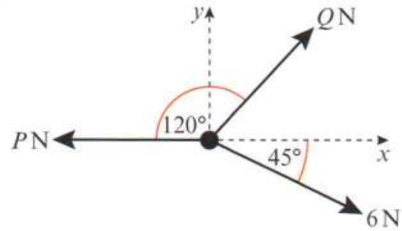
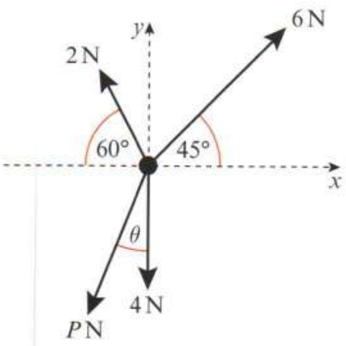
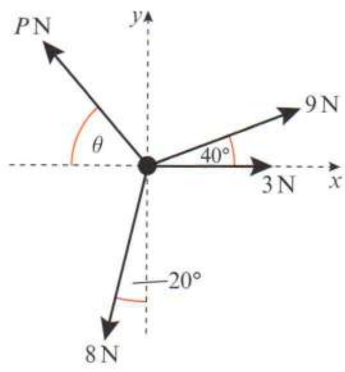


e



f

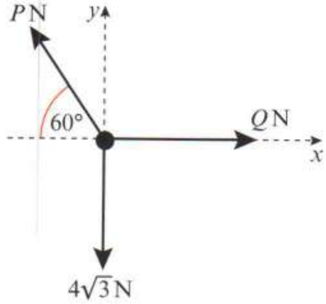


2 For each of the following particles in static equilibrium:

i draw a triangle of forces diagram.

ii Use trigonometry to find the magnitude of any unknown forces (marked P and Q) and the size of any unknown angles (marked θ).

a



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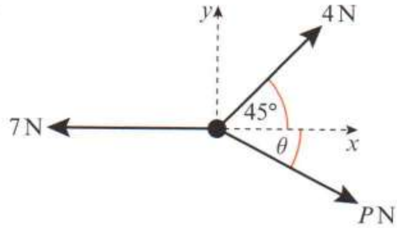
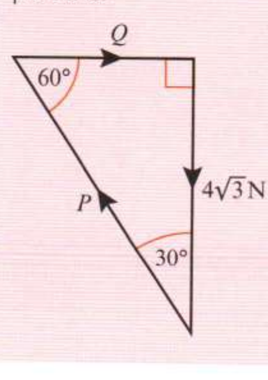
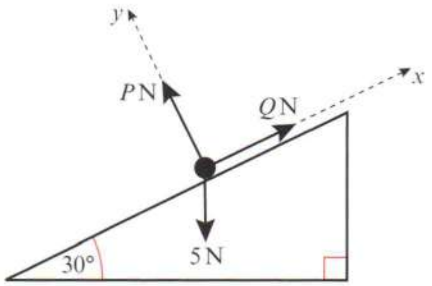


Hint The triangle of forces diagram for part **a** is:



3 Each of these particles rests in equilibrium on a sloping plane under the forces shown. In each case, find the magnitude of forces P and Q.

a



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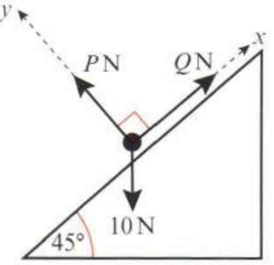

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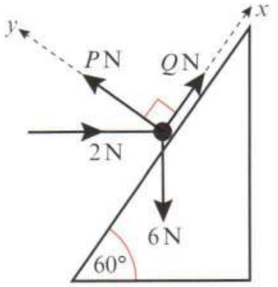

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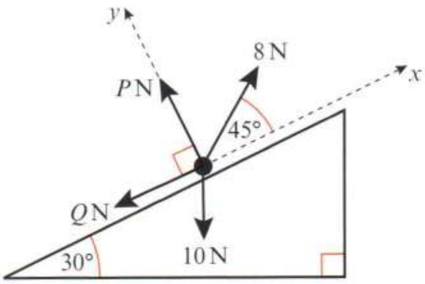

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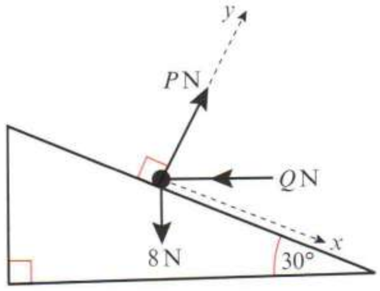
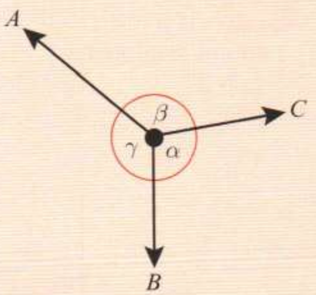


Challenge

The diagram shows three coplanar forces of A, B and C acting on a particle in equilibrium.



Show that $\frac{A}{\sin \alpha} = \frac{B}{\sin \beta} = \frac{C}{\sin \gamma}$

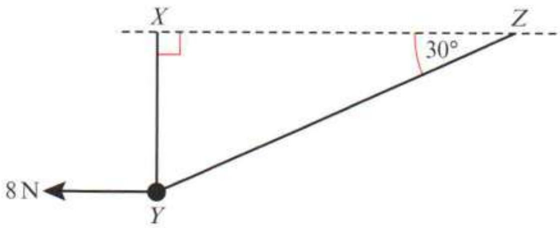
Notation This result is known as **Lami's Theorem**.

7.2 Modelling with statics

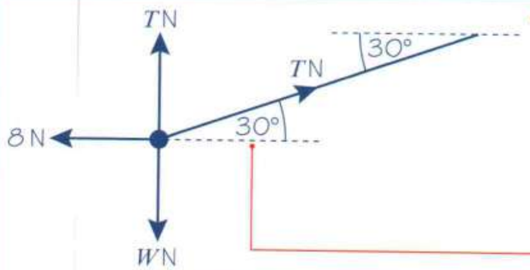
You can use force diagrams to model objects in static equilibrium, and to solve problems involving weight, tension and pulleys.

Example 3

A smooth bead Y is threaded on a light inextensible string. The ends of the string are attached to two fixed points, X and Z, on the same horizontal level. The bead is held in equilibrium by a horizontal force of magnitude 8 N acting parallel to ZX. The bead Y is vertically below X and $\angle XZY = 30°$ as shown in the diagram.



Find the tension in the string and the weight of the bead.



Draw a forces diagram.

Notation The bead is **smooth** so the tension in the string will be the same on either side of the bead.

This angle is 30° (alternate angles).

Let the weight be W N.

R(→), $T\cos 30° - 8 = 0$

$\therefore \quad T = \dfrac{8}{\cos 30°}$

Resolve horizontally and make T the subject of the formula.

$= \dfrac{16}{3}\sqrt{3} = 9.24$ (3 s.f.)

Give your answer to three significant figures as an approximation for g has not been used.

R(↑), $T + T\sin 30° - W = 0$

$\therefore \quad W = T(1 + \sin 30°)$

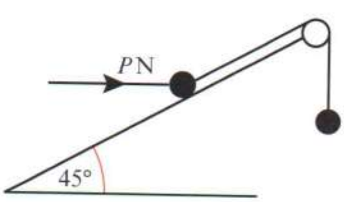
$= \dfrac{16}{3}\sqrt{3}\left(1 + \dfrac{1}{2}\right)$

$= 8\sqrt{3} = 13.9$ (3 s.f.)

Resolve vertically. Make W the subject of the formula and substitute for T.

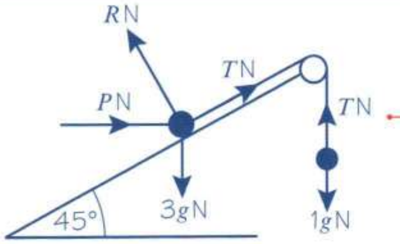
Example 4

A mass of 3 kg rests on the surface of a smooth plane which is inclined at an angle of 45° to the horizontal. The mass is attached to a cable which passes up the plane along the line of greatest slope and then passes over a smooth pulley at the top of the plane. The cable carries a mass of 1 kg freely suspended at the other end. The masses are modelled as particles, and the cable as a light inextensible string. There is a force of P N acting horizontally on the 3 kg mass and the system is in equilibrium.



Calculate: **a** the magnitude of P **b** the normal reaction between the mass and the plane.

c State how you have used the assumption that the pulley is smooth in your calculations.



Draw a diagram showing the forces acting on each particle. The tension, T N, will be the same throughout the string. The normal reaction, R N acts perpendicular to the plane. Show the weights $3g$ N and $1g$ N.

a Consider the 1 kg mass:

R(↑), $T - 1g = 0$ — Resolve vertically to obtain T.

$\therefore \quad T = g = 9.8$

Consider the 3 kg mass: — Resolve up the plane.

R(↗), $T + P\cos 45° - 3g\sin 45° = 0$ — R has no component in this direction as R is perpendicular to the plane.

$\therefore \quad P\cos 45° = 3g\sin 45° - T$

But $T = g$ — Substitute the value for T you found earlier.

$\therefore \quad P\cos 45° = 3g\sin 45° - g$ — Divide this equation by cos 45° and use the fact that $\frac{\sin 45°}{\cos 45°} = \tan 45° = 1$.

$$P = 3g - \frac{g}{\cos 45°}$$

$$= 3g - g\sqrt{2} = 16 \text{ (2 s.f.)}$$

Use the result that $\cos 45° = \sin 45° = \frac{1}{\sqrt{2}}$

b R(↖), $R - P\sin 45° - 3g\cos 45° = 0$ — Resolve perpendicular to the plane.

$\therefore \quad R = P\sin 45° + 3g\cos 45°$

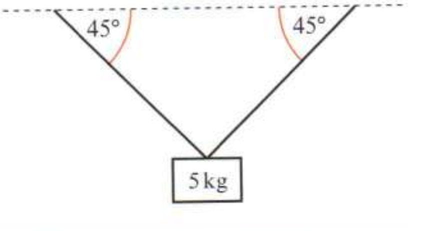
$$= 6g\frac{\sqrt{2}}{2} - g = 32 \text{ (2 s.f.)}$$

Substitute the value of P which you have found to evaluate R.

c The pulley is smooth so the tension in the string will be the same on both sides of the pulley.

Exercise 7B

1 A picture of mass 5 kg is suspended by two light inextensible strings, each inclined at 45° to the horizontal as shown. By modelling the picture as a particle find the tension in the strings when the system is in equilibrium.

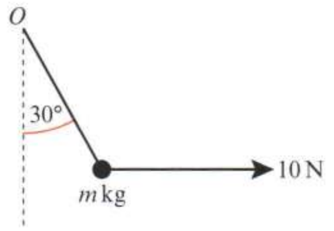


Problem-solving

This is a three-force problem involving an object in static equilibrium, so you could use a triangle of forces.

2 A particle of mass m kg is suspended by a single light inextensible string. The string is inclined at an angle of 30° to the vertical and the other end of the string is attached to a fixed point O. Equilibrium is maintained by a horizontal force of magnitude 10 N which acts on the particle, as shown in the diagram. Find:

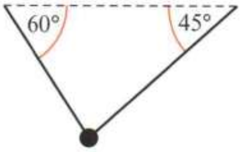
a the tension in the string **b** the value of m.



3 A particle of weight 12 N is suspended by a light inextensible string from a fixed point O. A horizontal force of 8 N is applied to the particle and the particle remains in equilibrium with the string at an angle θ to the vertical. Find:

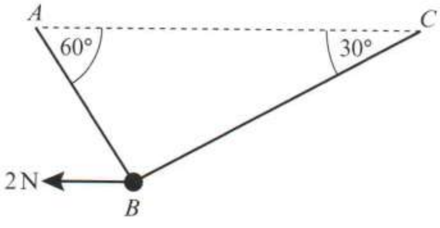
a the angle θ **b** the tension in the string.

4 A particle of mass 6 kg hangs in equilibrium, suspended by two light inextensible strings, inclined at 60° and 45° to the horizontal, as shown. Find the tension in each of the strings.



Hint The particle is attached **separately** to each string, so the tension in the two strings can be different.

(E) **5** A smooth bead B is threaded on a light inextensible string. The ends of the string are attached to two fixed points, A and C, on the same horizontal level. The bead is held in equilibrium by a horizontal force of magnitude 2 N acting parallel to CA. The sections of string make angles of 60° and 30° with the horizontal. Find:



a the tension in the string **(3 marks)**

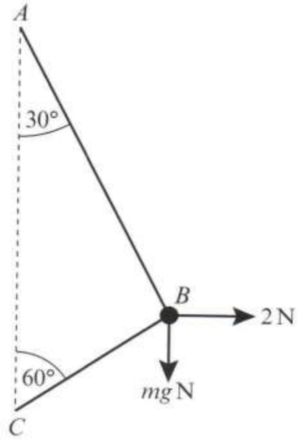
b the mass of the bead. **(4 marks)**

c State how you have used the modelling assumption that the bead is smooth in your calculations. **(1 mark)**

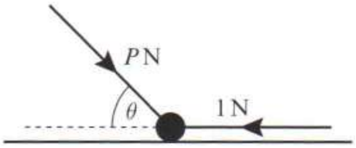
(E) **6** A smooth bead B is threaded on a light inextensible string. The ends of the string are attached to two fixed points A and C where A is vertically above C. The bead is held in equilibrium by a horizontal force of magnitude 2 N. The sections AB and BC of the string make angles of 30° and 60° with the vertical respectively. Find:

a the tension in the string **(3 marks)**

b the mass of the bead, giving your answer to the nearest gram. **(4 marks)**



(E) **7** A particle of weight 2 N rests on a smooth horizontal surface and remains in equilibrium under the action of the two external forces shown in the diagram. One is a horizontal force of magnitude 1 N and the other is a force of magnitude P N which acts at an angle θ to the horizontal, where $\tan\theta = \frac{12}{5}$. Find:



a the value of P **(3 marks)**

b the normal reaction between the particle and the surface. **(2 marks)**

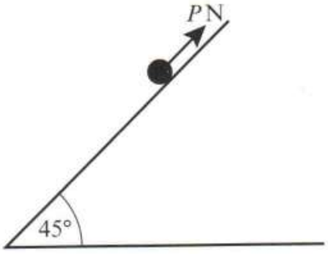
(P) **8** A particle A of mass m kg rests on a smooth horizontal table. The particle is attached by a light inextensible string to another particle B of mass $2m$ kg, which hangs over the edge of the table. The string passes over a smooth pulley, which is fixed at the edge of the table so that the string is horizontal between A and the pulley and then is vertical between the pulley and B. A horizontal force F N applied to A maintains equilibrium. The normal reaction between A and the table is R N.

a Find the values of F and R in terms of m.

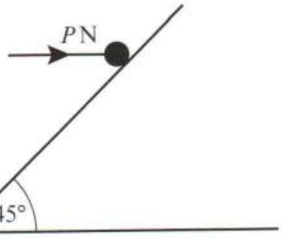
The pulley is now raised to a position above the edge of the table so that the string is inclined at 30° to the horizontal between A and the pulley. The string still hangs vertically between the pulley and B. A horizontal force F' N applied to A maintains equilibrium in this new situation. The normal reaction between A and the table is now R' N.

b Find, in terms of m, the values of F' and R'.

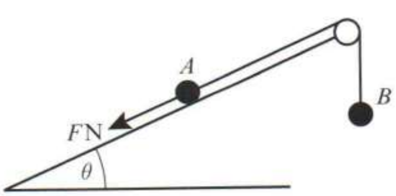
9 A particle of mass 2 kg rests on a smooth inclined plane, which makes an angle of 45° with the horizontal. The particle is maintained in equilibrium by a force P N acting up the line of greatest slope of the inclined plane, as shown in the diagram. Find the value of P.



10 A particle of mass 4 kg is held in equilibrium on a smooth plane which is inclined at 45° to the horizontal by a horizontal force of magnitude P N, as shown in the diagram. Find the value of P.



(E/P) **11** A particle A of mass 2 kg rests in equilibrium on a smooth inclined plane. The plane makes an angle θ with the horizontal, where $\tan\theta = \frac{3}{4}$.



The particle is attached to one end of a light inextensible string which passes over a smooth pulley, as shown in the diagram. The other end of the string is attached to a particle B of mass 5 kg. Particle A is also acted upon by a force of magnitude F N down the plane, along a line of greatest slope.

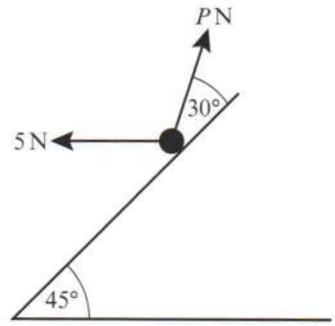
Find:

a the magnitude of the normal reaction between A and the plane **(5 marks)**

b the value of F. **(3 marks)**

c State how you have used the fact that the pulley is smooth in your calculations. **(1 mark)**

12 A particle of weight 20 N rests in equilibrium on a smooth inclined plane. It is maintained in equilibrium by the application of two external forces as shown in the diagram. One of the forces is a horizontal force of 5 N, the other is a force P N acting at an angle of 30° to the plane, as shown in the diagram. Find the magnitude of the normal reaction between the particle and the plane. **(8 marks)**



7.3 Friction and static particles

When a body is in static equilibrium under the action of a number of forces, including friction, you need to consider whether the body is on the point of moving or not.

In many cases the force of friction will be less than μR, as a smaller force is sufficient to prevent motion and to maintain static equilibrium. In these situations the equilibrium is not limiting.

- **The maximum value of the frictional force $F_{MAX} = \mu R$ is reached when the body you are considering is on the point of moving. The body is then said to be in limiting equilibrium.**
- **In general, the force of friction F is such that $F \leqslant \mu R$, and the direction of the frictional force is opposite to the direction in which the body would move if the frictional force were absent.**

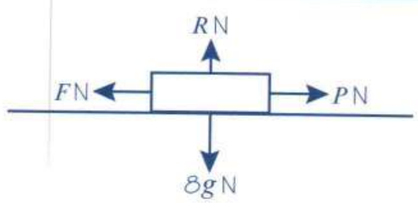
Example 5

A mass of 8 kg rests on a rough horizontal plane. The mass may be modelled as a particle, and the coefficient of friction between the mass and the plane is 0.5. Find the magnitude of the maximum force P N which acts on this mass without causing it to move if:

a the force P is horizontal

b the force P acts at an angle 60° above the horizontal.

a



This is an example of limiting equilibrium.

Draw a diagram showing the weight $8g$ N, the normal reaction R N, the force P N and the friction F N. The friction is in the opposite direction to force P N.

The question asks you for the maximum force before movement takes place.

R(↑), $R - 8g = 0$

$\therefore\ R = 8g$

As friction is limiting, $F = \mu R$

$\therefore\ F = 0.5 \times 8g$

$= 39.2$

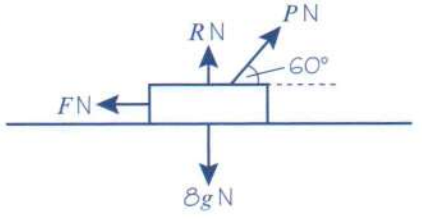
R(→), $P - F = 0$

$\therefore\ P = F = 39$ (2 s.f.)

For an object in limiting equilibrium, $F = F_{MAX}$

Give your answer to two significant figures.

b



Draw another diagram showing P at 60° above the horizontal.

Again this is limiting equilibrium.

R(↑), $R + P\sin 60° - 8g = 0$

$\therefore\ R = 8g - P\sin 60°$

As friction is limiting, $F = \mu R$

$\therefore\ F = 0.5\,(8g - P\sin 60°)$

R(→), $P\cos 60° - F = 0$

$\therefore\ P\cos 60° = 0.5\,(8g - P\sin 60°)$

$\therefore\ P\cos 60° + 0.5\,P\sin 60° = 0.5 \times 8g$

$\therefore\ P(\cos 60° + 0.5\sin 60°) = 4g$

$$\therefore\ P = \frac{4g}{\cos 60° + 0.5\sin 60°}$$

$P = 42$ (2 s.f.)

Express R in terms of P.

Use $F = \mu R$ with $\mu = 0.5$

As $F = P\cos 60°$ eliminate F from the previous equation.

Collect the terms in P and factorise to make P the subject.

Example 6

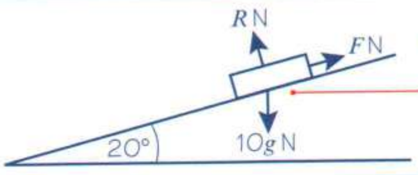
A box of mass 10 kg rests in limiting equilibrium on a rough plane inclined at 20° above the horizontal.

a Find the coefficient of friction between the box and the plane.

A horizontal force of magnitude P N is applied to the box. Given that the box remains in equilibrium,

b find the maximum possible value of P.

a



Model the box as a particle and draw a diagram showing the weight, the normal reaction and the force of friction.

The friction acts up the plane, as it acts in an opposite direction to the motion that would take place if there was no friction.

R(↖), $R - 10g\cos 20° = 0$
$\therefore\ R = 92.089...$
R(↗), $F - 10g\sin 20° = 0$
$\therefore\ F = 33.517...$

Resolve perpendicular and parallel to the plane.

Online Use the storage function to store exact values on your calculator.

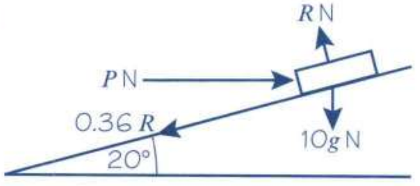
As the friction is limiting
$F = \mu R$
$\therefore\ 33.517... = \mu \times 92.089...$

Find R and F, then use $F = \mu R$ to find μ.

$\therefore\ \mu = \frac{33.517...}{92.089...} = 0.36$ (to 2 s.f.)

Give your answer to two significant figures and note that $\mu = \tan 20°$.

b



Watch out For the maximum possible value of P, the box will be on the point of moving up the slope, so the friction will act down the slope.

When P is at its maximum value,
$F = \mu R = 0.36R$
R(↗), $P\cos 20° - 0.36R - 10g\sin 20° = 0$ (1)
R(↖), $R - 10g\cos 20° - P\sin 20° = 0$ (2)

From (1): $R = \frac{P\cos 20° - 10g\sin 20°}{0.36}$

From (2) $R = P\sin 20° + 10g\cos 20°$

So $\frac{P\cos 20° - 10g\sin 20°}{0.36}$

$= P\sin 20° + 10g\cos 20°$

Eliminate R to find P.

$P = \frac{3.6g\cos 20° + 10g\sin 20°}{\cos 20° - 0.36\sin 20°}$

$= 82$ N (2 s.f.)

You have used $g = 9.8\text{ m s}^{-2}$ in your calculations, so round your final answer to 2 significant figures.

Exercise 7C

1 A book of mass 2 kg rests on a rough horizontal table. When a force of magnitude 8 N acts on the book, at an angle of 20° to the horizontal in an upward direction, the book is on the point of slipping.

Calculate, to three significant figures, the value of the coefficient of friction between the book and the table.

Hint 'On the point of slipping' means that the book is in limiting equilibrium.

2 A block of mass 4 kg rests on a rough horizontal table. When a force of 6 N acts on the block, at an angle of 30° to the horizontal in a downward direction, the block is on the point of slipping. Find the value of the coefficient of friction between the block and the table.

3 A block of weight 10 N is at rest on a rough horizontal surface. A force of magnitude 3 N is applied to the block at an angle of 60° above the horizontal in an upward direction. The coefficient of friction between the block and the surface is 0.3.

a Calculate the force of friction. **b** Determine whether the friction is limiting.

(P) 4 A packing crate of mass 10 kg rests on rough horizontal ground. It is filled with books which are evenly distributed through the crate. The coefficient of friction between the crate and the ground is 0.3.

a Find the mass of the books if the crate is in limiting equilibrium under the effect of a horizontal force of magnitude 147 N.

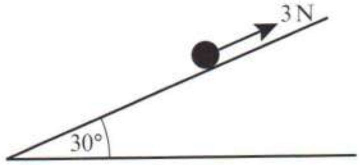
b State what modelling assumptions you have made.

5 A block of mass 2 kg rests on a rough horizontal plane. A force P acts on the block at an angle of 45° to the horizontal. The equilibrium is limiting, with $\mu = 0.3$.

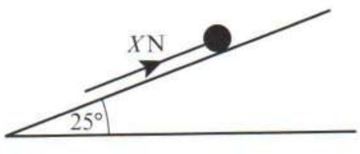
Find the magnitude of P if:

a P acts in a downward direction **b** P acts in an upward direction.

6 A particle of mass 0.3 kg is on a rough plane which is inclined at an angle 30° to the horizontal. The particle is held at rest on the plane by a force of magnitude 3 N acting up the plane, in a direction parallel to a line of greatest slope of the plane. The particle is on the point of slipping up the plane. Find the coefficient of friction between the particle and the plane.

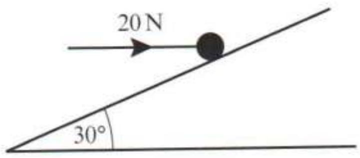


7 A particle of mass 1.5 kg rests in equilibrium on a rough plane under the action of a force of magnitude X N acting up a line of greatest slope of the plane. The plane is inclined at 25° to the horizontal. The particle is in limiting equilibrium and on the point of moving up the plane. The coefficient of friction between the particle and the plane is 0.25. Calculate:



a the normal reaction of the plane on the particle **b** the value of X.

(E/P) 8 A horizontal force of magnitude 20 N acts on a block of mass 1.5 kg, which is in equilibrium resting on a rough plane inclined at 30° to the horizontal. The line of action of the force is in the same vertical plane as the line of greatest slope of the inclined plane.

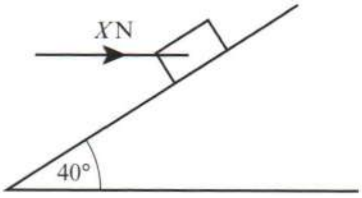


a Find the normal reaction between the block and the plane. **(4 marks)**

b Find the magnitude and direction of the frictional force acting on the block. **(3 marks)**

c Hence find the minimum value of the coefficient of friction between the block and the plane. **(2 marks)**

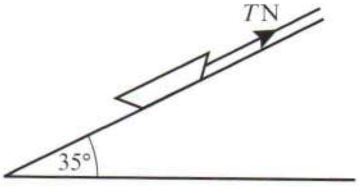
9 A box of mass 3 kg lies on a rough plane inclined at 40° to the horizontal. The box is held in equilibrium by means of a horizontal force of magnitude X N. The line of action of the force is in the same vertical plane as the line of greatest slope of the inclined plane. The coefficient of friction between the box and the plane is 0.3 and the box is in limiting equilibrium and is about to move up the plane.



a Find X. **(6 marks)**

b Find the normal reaction between the box and the plane. **(2 marks)**

10 A small child, sitting on a sledge, rests in equilibrium on an inclined slope. The sledge is held by a rope which lies along the slope and is under tension. The sledge is on the point of slipping down the plane. Modelling the child and sledge as a particle and the rope as a light inextensible string, calculate the tension in the rope, given that the mass of the child and sledge is 22 kg, the coefficient of friction is 0.125 and that the slope is a plane inclined at 35° to the horizontal.

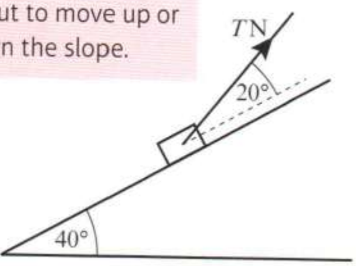


11 A box of mass 0.5 kg is placed on a plane which is inclined at an angle of 40° to the horizontal. The coefficient of friction between the box and the plane is $\frac{1}{5}$. The box is kept in equilibrium by a light string which lies in a vertical plane containing a line of greatest slope of the plane. The string makes an angle of 20° with the plane, as shown in the diagram. The box is in limiting equilibrium and may be modelled as a particle. The tension in the string is T N.

Find the range of possible values of T.

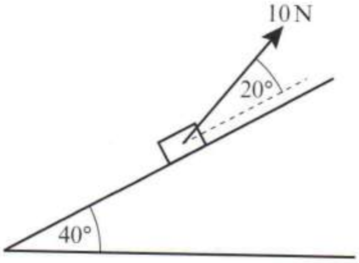
Problem-solving

The box might be about to move up or down the slope.



(8 marks)

12 A box of mass 1 kg is placed on a plane, which is inclined at an angle of 40° to the horizontal. The box is kept in equilibrium on the point of moving up the plane by a light string, which lies in a vertical plane containing a line of greatest slope of the plane. The string makes an angle of 20° with the plane, as shown in the diagram. The box is in limiting equilibrium and may be modelled as a particle. The tension in the string is 10 N and the coefficient of friction between the box and the plane is μ. Find μ.



(7 marks)

13 A box of mass 2 kg rests in limiting equilibrium on a rough plane angled at θ above the horizontal where $\tan\theta = \frac{3}{4}$. A horizontal force of magnitude P N acting into the plane is applied to the box. Given that the box remains in equilibrium, find the maximum possible value of P. **(8 marks)**

Problem-solving

First find the coefficient of friction between the box and the plane.

7.4 Static rigid bodies

If you need to consider the rotational forces acting on an object you can model it as a **rigid body**.

- **For a rigid body in static equilibrium:**
  - **the body is stationary**
  - **the resultant force in any direction is zero**
  - **the resultant moment is zero**

Links The moment of a force of magnitude F N about a point P is Fd, where d is the **perpendicular distance** from the line of action of the force to P. ← Section 4.1

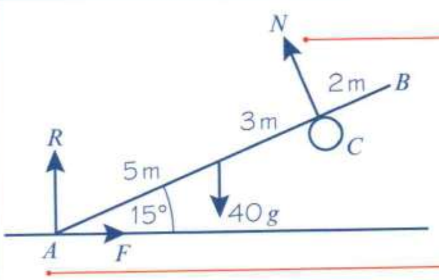
You will sometimes need to consider the moments acting on the body and the resultant force acting on the body separately.

Example 7

A uniform rod AB of mass 40 kg and length 10 m rests with the end A on rough horizontal ground. The rod rests against a smooth peg C where $AC = 8$ m. The rod is in limiting equilibrium at an angle of 15° to the horizontal. Find:

a the magnitude of the reaction at C

b the coefficient of friction between the rod and the ground.



Start with a diagram showing all the forces.

N, the reaction at C, is perpendicular to the rod. The peg is smooth, so there is no friction here.

At A there is a normal reaction and a frictional force. The peg is smooth so the only force stopping the rod from sliding is the frictional force at A.

a Taking moments about A:

$40g \times 5\cos 15° = N \times 8$

$N = \dfrac{200g\cos 15°}{8}$

$= 25g\cos 15°$

$= 236.65\ldots$ N

The reaction at C has magnitude 240 N (2 s.f.).

Problem-solving

Solve part **a** by taking moments. If you take moments about A you can ignore the frictional force. For part **b** you can resolve forces horizontally and vertically for the whole body.

b R(→), $F = N\cos 75° = 61.25$ N

R(↑), $R + N\cos 15° = 40g$

$R = 40g - N\cos 15° = 163.41\ldots$ N

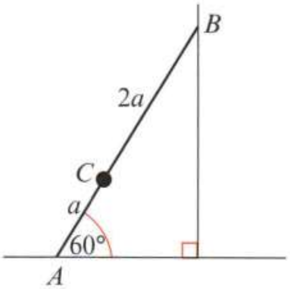
The rod is in limiting equilibrium, so

$F = \mu R,\ \mu = \dfrac{F}{R} = 0.37$ (2 s.f.)

Resolve horizontally and vertically.

Example 8

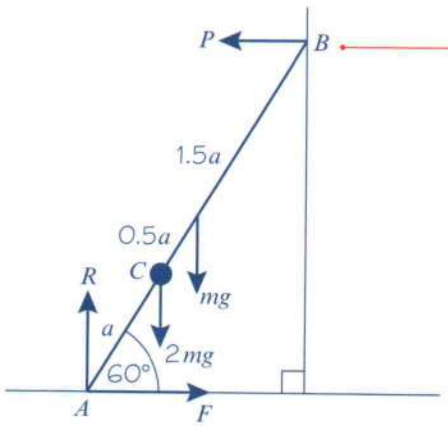
A ladder AB, of mass m and length $3a$, has one end A resting on rough horizontal ground. The other end B rests against a smooth vertical wall. A load of mass $2m$ is fixed on the ladder at the point C, where $AC = a$. The ladder is modelled as a uniform rod in a vertical plane perpendicular to the wall and the load is modelled as a particle. The ladder rests in limiting equilibrium at an angle of 60° with the ground.



a Find the coefficient of friction between the ladder and the ground.

b State how you have used the assumption that the ladder is uniform in your calculations.

a



The reaction at B is perpendicular to the wall. The wall is smooth, so there is no friction at B.

Watch out The reactions at the wall and the floor are different and so must be labelled with different letters.

Online Explore the forces in this question in a more detailed diagram using GeoGebra.

R(→), $F = P$

R(↑), $R = 2mg + mg = 3mg$

Resolve vertically and horizontally for the whole system.

Taking moments about B:

$2mg \times 2a\cos 60° + mg \times 1.5a\cos 60°$
$+ F \times 3a\sin 60° = R \times 3a\cos 60°$

You want to find $\mu = \dfrac{F}{R}$ so take moments at B. Remember to use the **perpendicular distance** from each force to B.

$5.5mg\cos 60° + 3F\sin 60°$
$= 3R\cos 60°$

You can divide both sides by a.

$$2.75mg + \frac{3\sqrt{3}}{2}F = 1.5R$$

Since $R = 3mg$, and $F = \mu R$ (as the ladder is in limiting equilibrium)

$$2.75mg + \frac{3\sqrt{3}}{2}\mu \times 3mg = 1.5 \times 3mg$$

$$\mu = \frac{4.5 - 2.75}{\left(\frac{9\sqrt{3}}{2}\right)} = \frac{7}{18\sqrt{3}} = 0.225 \text{ (3 s.f.)}$$

Divide through by mg and solve to find μ. The answer is independent of g so round to 3 s.f.

b The assumption that the ladder is uniform allows you to assume that its weight acts at its midpoint.

Problem-solving

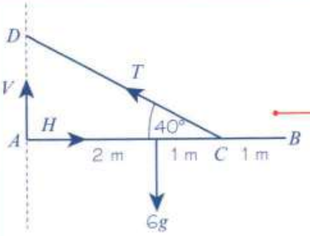
There are other options for points to take moments about. For example, if you were to take moments about the point where the lines of action of R and P meet you would eliminate R and P from your working and simplify your calculation.

Example 9

A uniform rod AB, of mass 6 kg and length 4 m, is smoothly hinged at A. A light inextensible string is attached to the rod at point C, where $AC = 3$ m, and to a point D, vertically above A. If the string keeps the rod in equilibrium in a horizontal position and the angle between the string and the rod is 40°, calculate:

a the tension in the string

b the magnitude and direction of the force exerted on the rod by the wall.



The reaction at the hinge acts at an angle. Write it in terms of its horizontal (H) and vertical (V) components.

Notation The force exerted on the rod by the wall is sometimes called the **reaction at the hinge**.

a Taking moments about A:

$6g \times 2 = T \times 3 \times \sin 40°$

$T = \frac{12g}{3\sin 40°} = 61.0$ N (3 s.f.)

You do not know the magnitude or direction of the reaction at A, so take moments at that point. The perpendicular distance from A to the line of action of T is $3\sin 40°$.

b Consider all of the forces acting on AB:

R(↑), $V + T\sin 40° = 6g \Rightarrow V = 19.6$ N

R(→), $H = T\cos 40° = 46.716...$ N

The sum of the components of the forces in any direction is zero because AB is resting in equilibrium.

So the reaction has magnitude

$\sqrt{19.6^2 + (46.716...)^2} = 50.7$ N (3 s.f.)

and acts at

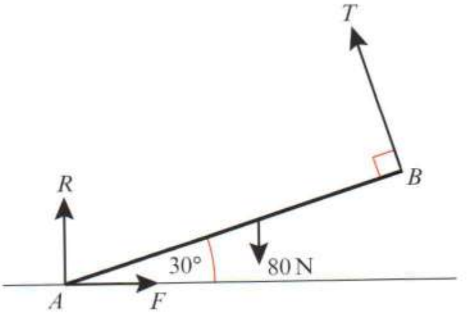
$\arctan \frac{19.6}{46.716...} = 22.8°$ (3 s.f.) to AB.

Use Pythagoras' theorem and trigonometry.

Exercise 7D

Whenever a numerical value of g is required, take $g = 9.8\text{ m s}^{-2}$.

1 A uniform rod AB of weight 80 N rests with its lower end A on a rough horizontal floor. A string attached to end B keeps the rod in equilibrium. The string is held at 90° to the rod. The tension in the string is T. The coefficient of friction between the rod and the ground is μ. R is the normal reaction at A and F is the frictional force at A.
Find the magnitudes of T, R and F, and the least possible value of μ.



2 A uniform ladder of mass 10 kg and length 5 m rests against a smooth vertical wall with its lower end on rough horizontal ground. The ladder rests in equilibrium at an angle of 65° to the horizontal. Find:

a the magnitude of the normal reaction S at the wall

b the magnitude of the normal reaction R at the ground and the frictional force at the ground

c the least possible value of the coefficient of friction between the ladder and the ground.

d State how you have used the assumption that the ladder is uniform in your calculations.

3 A uniform ladder AB of mass 20 kg rests with its top A against a smooth vertical wall and its base B on rough horizontal ground. The coefficient of friction between the ladder and the ground is $\frac{3}{4}$. A mass of 10 kg is attached to the ladder. Given that the ladder is about to slip, find the inclination of the ladder to the horizontal, if the 10 kg mass is attached

a at A **b** at B.

c State how you have used the assumption that the wall is smooth in your calculations.

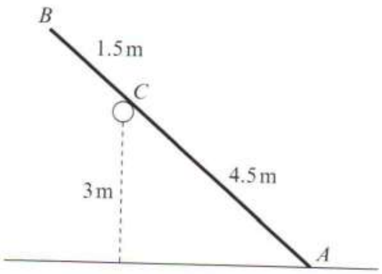
4 A uniform ladder of mass 20 kg and length 8 m rests against a smooth vertical wall with its lower end on rough horizontal ground. The coefficient of friction between the ground and the ladder is 0.3. The ladder is inclined at an angle θ to the horizontal, where $\tan\theta = 2$. A boy of mass 30 kg climbs up the ladder. By modelling the ladder as a uniform rod, the boy as a particle and the wall as smooth and vertical,

a find how far up the ladder the boy can climb before the ladder slips. **(8 marks)**

b Criticise this model with respect to:

i the ladder **ii** the wall. **(2 marks)**

5 A smooth horizontal rail is fixed at a height of 3 m above a rough horizontal surface. A uniform pole AB of weight 4 N and length 6 m is resting with end A on the rough ground and touching the rail at point C. The vertical plane containing the pole is perpendicular to the rail. The distance AC is 4.5 m and the pole is in limiting equilibrium. Calculate:



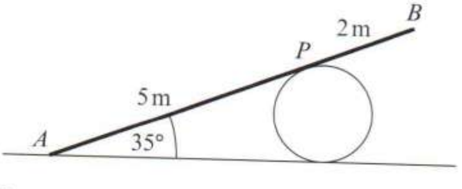
a the magnitude of the force exerted by the rail on the pole

b the coefficient of friction between the pole and the ground.

c State how you have used the assumption that the rail is smooth in your calculations

6 A uniform ladder rests in limiting equilibrium with its top against a smooth vertical wall and its base on a rough horizontal floor. The coefficient of friction between the ladder and the floor is μ. Given that the ladder makes an angle θ with the floor, show that $2\mu\tan\theta = 1$.

7 A uniform ladder AB has length 7 m and mass 20 kg. The ladder is resting against a smooth cylindrical drum at P, where AP is 5 m, with end A in contact with rough horizontal ground. The ladder is inclined at 35° to the horizontal.



Find the normal and frictional components of the contact force at A, and hence find the least possible value of the coefficient of friction between the ladder and the ground.

E/P **8** A uniform ladder rests in limiting equilibrium with one end on rough horizontal ground and the other end against a rough vertical wall. The coefficient of friction between the ladder and the ground is μ_1 and the coefficient of friction between the ladder and the wall is μ_2. Given that the ladder makes an angle θ with the horizontal, show that $\tan\theta = \dfrac{1 - \mu_1\mu_2}{2\mu_1}$ **(8 marks)**

E/P **9** A uniform ladder of weight W rests in equilibrium with one end on rough horizontal ground and the other resting against a smooth vertical wall. The vertical plane containing the ladder is at right angles to the wall and the ladder is inclined at 60° to the horizontal. The coefficient of friction between the ladder and the ground is μ.

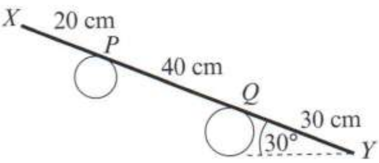
a Find, in terms of W, the magnitude of the force exerted by the wall on the ladder. **(6 marks)**

b Show that $\mu \geqslant \dfrac{\sqrt{3}}{6}$ **(3 marks)**

A load of weight w is attached to the ladder at its upper end (resting against the wall).

c Given that $\mu = \dfrac{\sqrt{3}}{5}$ and that the equilibrium is limiting, find w in terms of W. **(8 marks)**

E/P **10** A uniform rod XY has weight 20 N and length 90 cm. The rod rests on two parallel pegs, with X above Y, in a vertical plane which is perpendicular to the axes of the pegs, as shown in the diagram. The rod makes an angle of 30° to the horizontal and touches the two pegs at P and Q, where $XP = 20$ cm and $XQ = 60$ cm.



a Calculate the normal components of the forces on the rod at P and at Q. **(8 marks)**

The coefficient of friction between the rod and each peg is μ.

b Given that the rod is about to slip, find μ. **(2 marks)**

E **11** A ladder XY, of length l and weight W, has its end X on rough horizontal ground. The coefficient of friction between the ladder and the ground is $\frac{1}{5}$. The end Y of the ladder is resting against a smooth vertical wall. A window cleaner of weight $9W$ stands at the top of the ladder. To stop the ladder from slipping, the window cleaner's assistant applies a horizontal force of magnitude P to the ladder at X, towards the wall. The force acts in a direction which is perpendicular to the wall. The ladder rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle θ with the horizontal ground, where $\tan\theta = \sqrt{3}$. The window cleaner is modelled as a particle and the ladder is modelled as a uniform rod.

a Find, in terms of W, the reaction of the wall on the ladder at Y. **(5 marks)**

b Find, in terms of W, the range of possible values of P for which the ladder remains in equilibrium. **(5 marks)**

c State how you have used the modelling assumption that the ladder is uniform in your calculations. **(1 mark)**

In practice, the ladder is wider and heavier at the bottom. The model is adjusted so the ladder is modelled as a non-uniform rod with its centre of mass closer to the base.

d State, with a reason, the effect this will have on

i the magnitude of the reaction of the wall on the ladder at Y

ii the range of possible values of P for which the ladder remains in equilibrium. **(4 marks)**

E **12** A uniform rod AB of length 4 m and mass 6 kg is smoothly hinged at A. A light inextensible string is attached to the rod at point C, where $AC = 2$ m, and to the point D vertically above A. The string keeps the rod in equilibrium, and the angle between the string and the rod is 40°. Calculate:

a the tension in the string **(4 marks)**

b the magnitude and direction of the force exerted on the rod by the wall. **(6 marks)**

/P **13** A uniform rod AB of length $2a$ m and mass m kg is smoothly hinged at A. It is maintained in equilibrium by a horizontal force of magnitude P acting at B. The rod is inclined at 30° to the horizontal with B below A.

a Show that $P = \frac{\sqrt{3}}{2} mg$ N **(4 marks)**

b Find the magnitude and direction of the force exerted on the rod by the wall. **(6 marks)**

7.5 Dynamics and inclined planes

When a particle is moving along a rough plane, the force of friction is equal to μR, and acts so as to oppose the direction of motion.

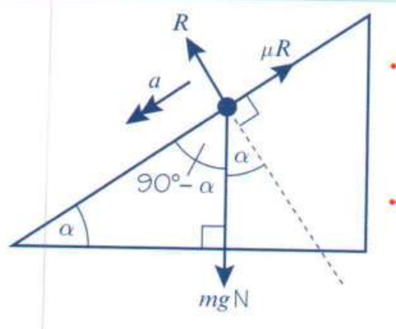
Example 10

A particle is held at rest on a rough plane which is inclined to the horizontal at an angle α, where $\tan \alpha = 0.75$. The coefficient of friction between the particle and the plane is 0.5. The particle is released and slides down the plane. Find:

a the acceleration of the particle

b the distance it slides in the first 2 seconds.

a



Draw a diagram showing all the forces and the acceleration. Note that you are not given the mass of the particle so call it m.

Since the particle slides down the plane friction will be limiting, so $F = \mu R = 0.5R$.

R(↖), $R - mg\cos\alpha = m \times 0 = 0$

$R = mg\cos\alpha$ (1)

Resolve perpendicular to the acceleration.

R(↙), $mg\sin\alpha - \mu R = ma$ (2)

Resolve in the direction of the acceleration and use $F = ma$.

From equation (1),

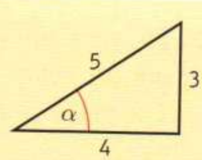
$R = 0.8mg$

If $\tan\alpha = 0.75 = \frac{3}{4}$,

$\cos\alpha = \frac{4}{5} = 0.8$

and $\sin\alpha = \frac{3}{5} = 0.6$

(or use your calculator).



Then equation (2) becomes

$0.6mg - 0.5 \times (0.8mg) = ma$

$0.6g - 0.4g = a$

$0.2g = a$

Substitute for $\sin\alpha$, μ and R and divide through by m.

The acceleration of the particle is $0.2g$ or $2.0\ \text{m s}^{-2}$ (2 s.f.).

b $u = 0$, $a = 0.2g$, $t = 2$, $s = ?$

$s = ut + \frac{1}{2}at^2$

$s = 0 + \frac{1}{2} \times 0.2g \times 2^2$

$= 3.92 = 3.9$ (2 s.f.)

The particle slides 3.9 m (2 s.f.) down the plane.

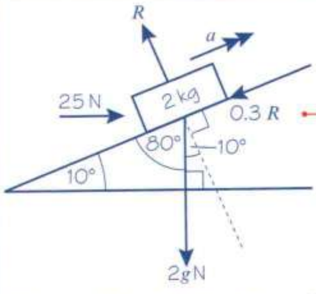
Use the *suvat* equations for motion with constant acceleration. ← Year 1, Chapter 9

Choose the appropriate formula.

Substitute in the values.

Example 11

A box of mass 2 kg is pushed up a rough plane by a horizontal force of magnitude 25 N. The plane is inclined to the horizontal at an angle of 10°. Given that the coefficient of friction between the box and the plane is 0.3, find the acceleration of the box.



Draw a diagram showing all the forces and the acceleration. The box is moving so $F = F_{MAX}$.

R(↖), $R - 2g\cos 10° - 25\cos 80° = 0$

$R = 2g\cos 10° + 25\cos 80°$ (1)

Resolve perpendicular to the slope. The box is moving up the slope so there is no resultant force in this direction.

R(↗), $25\cos 10° - 2g\cos 80° - 0.3R = 2a$

$25\cos 10° - 2g\cos 80°$

$- 0.3(2g\cos 10° + 25\cos 80°) = 2a$

$(25 - 0.6g)\cos 10° - (2g + 7.5)\cos 80°$

$= 2a$

Resolve up the slope and write an equation of motion for the box.

Substitute for R from equation (1) and simplify.

$14.124\ldots = 2a \Rightarrow a = 7.1$ (2 s.f.)

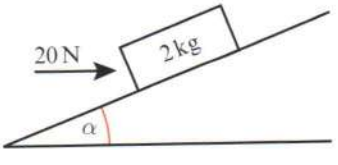
The box accelerates up the plane at 7.1 m s^{-2} (2 s.f.).

Use unrounded values in your calculations, but round your final answer to 2 s.f.

Exercise 7E

1 A particle of mass 0.5 kg is placed on a smooth inclined plane. Given that the plane makes an angle of 20° with the horizontal, find the acceleration of the particle.

2 The diagram shows a box of mass 2 kg being pushed up a smooth plane by a horizontal force of magnitude 20 N. The plane is inclined to the horizontal at an angle α, where $\tan\alpha = \frac{3}{4}$.



Find:

a the normal reaction between the box and the plane

b the acceleration of the box up the plane.

(P) **3** A boy of mass 40 kg slides from rest down a straight slide of length 5 m. The slide is inclined to the horizontal at an angle of 20°. The coefficient of friction between the boy and the slide is 0.1. By modelling the boy as a particle, find:

a the acceleration of the boy

b the speed of the boy at the bottom of the slide.

(P) **4** A block of mass 20 kg is released from rest at the top of a rough slope. The slope is inclined to the horizontal at an angle of 30°. After 6 s the speed of the block is 21 m s^{-1}. Find the coefficient of friction between the block and the slope.

5 A book of mass 2 kg slides down a rough plane inclined at 20° to the horizontal. The acceleration of the book is 1.5 m s^{-2}. Find the coefficient of friction between the book and the plane.

6 A block of mass 4 kg is pulled up a rough slope, inclined at 25° to the horizontal, by means of a rope. The rope lies along the line of the slope. The tension in the rope is 30 N. Given that the acceleration of the block is 2 m s^{-2} find the coefficient of friction between the block and the plane.

(P) **7** A parcel of mass 10 kg is released from rest on a rough plane which is inclined at 25° to the horizontal.

a Find the normal reaction between the parcel and the plane. **(2 marks)**

Two seconds after being released the parcel has moved 4 m down the plane.

b Find the coefficient of friction between the parcel and the plane. **(2 marks)**

(P) **8** A particle P is projected up a rough plane which is inclined at an angle α to the horizontal, where $\tan \alpha = \frac{3}{4}$. The coefficient of friction between the particle and the plane is $\frac{1}{3}$. The particle is projected from the point A with speed 20 m s^{-1} and comes to instantaneous rest at the point B.

a Show that while P is moving up the plane its deceleration is $\frac{13g}{15}$. **(5 marks)**

b Find, to two significant figures, the distance AB. **(2 marks)**

c Find, to two significant figures, the time taken for P to move from A to B. **(2 marks)**

d Find the speed of P when it returns to A. **(7 marks)**

(P) **9** A particle of mass 2 kg is released from rest on a rough slope that is angled at α to the horizontal where $\tan \alpha = \frac{2}{5}$. After 3 seconds the speed of the particle is 6 m s^{-1}. Work out the coefficient of friction μ. **(8 marks)**

(P) **10** A particle of mass m kg is released from rest on a rough slope that is angled at α to the horizontal. The particle begins to accelerate down the slope. Show that the acceleration of the particle is independent of its mass.

(E/P) **11** A particle of mass 5 kg is projected up a rough slope at 16 m s^{-1} and comes to rest at a point P after 5 s. Given that the slope is inclined at 10° to the horizontal,

a work out the coefficient of friction μ. **(7 marks)**

b State, with supporting calculations, whether the particle will remain at rest at P or will begin to slide back down the slope. **(2 marks)**

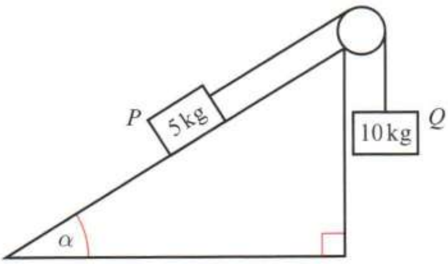
7.6 Connected particles

You need to be able to solve problems about connected particles on inclined and rough surfaces.

Links Unless connected particles are moving in the same direction they must be considered separately.
← **Year 1, Chapter 10**

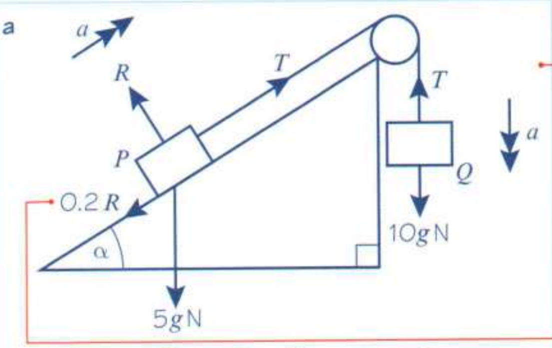
Example 12

Two particles P and Q of masses 5 kg and 10 kg respectively are connected by a light inextensible string. The string passes over a small smooth pulley which is fixed at the top of a rough inclined plane. P rests on the inclined plane and Q hangs on the edge of the plane with the string vertical and taut. The plane is inclined to the horizontal at an angle α where $\tan\alpha = 0.75$, as shown in the diagram. The coefficient of friction between P and the plane is 0.2. The system is released from rest.



a Find the acceleration of the system. **b** Find the tension in the string.

a



Draw a diagram showing all the forces acting on each particle and their accelerations. Friction is limiting.

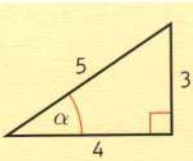
For P: R(↖), $R - 5g\cos\alpha = 0$

$R = 5g \times \frac{4}{5}$

$= 4g$ N

There is no acceleration perpendicular to the plane.

If $\tan\alpha = \frac{3}{4}$, $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$



R(↗), $T - 5g\sin\alpha - 0.2R = 5a$

$T - \cancel{5}g \times \frac{3}{\cancel{5}} - 0.2 \times 4g = 5a$

$T - 3.8g = 5a$ (1)

Resolve in the direction of the acceleration.

Substitute for R and simplify.

For Q: R(↓), $10g - T = 10a$ (2)

$10g - \cancel{T} + \cancel{T} - 3.8g = 10a + 5a$

$6.2g = 15a$

$a = \frac{31g}{75}$ or $4.1\,\text{m s}^{-2}$ (2 s.f.)

b $T - 3.8g = 5 \times \frac{31g}{75}$

$T = 3.8g + \frac{31g}{15} = 57\,\text{N}$ (2 s.f.)

Resolve in the direction of the acceleration.

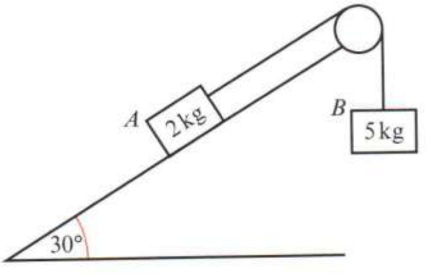
Add equations (1) and (2) to eliminate T.

Either of these answers would be acceptable.

Substitute for a in equation (1), using an unrounded value of a.

Example 13

One end of a light inextensible string is attached to a block A of mass 2 kg. The block A is held at rest on a smooth fixed plane which is inclined to the horizontal at an angle of 30°. The string lies along the line of greatest slope of the plane and passes over a smooth light pulley which is fixed at the top of the plane. The other end of the string is attached to a block B of mass 5 kg. The system is released from rest. By modelling the blocks as particles and ignoring air resistance,



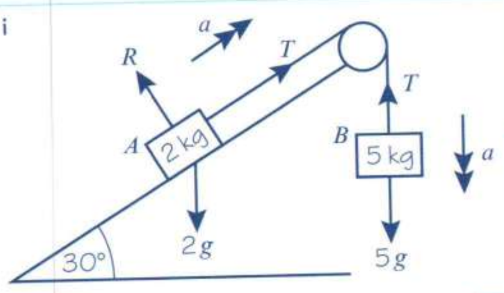
a **i** show that the acceleration of block B is $\frac{4}{7}g$

ii find the tension in the string.

b State how you have used the fact that the string is inextensible in your calculations.

c Calculate the magnitude of the force exerted on the pulley by the string.

a i



Draw a diagram showing all the forces. The pulley is smooth so the tension will be the same on each side.

Consider A:

R(↗), $T - 2g\sin 30° = 2a$

$T = 2a + g$ (1)

Consider B:

R(↓), $5g - T = 5a$

$T = 5g - 5a$ (2)

Write equations of motion for block A and block B separately.

$2a + g = 5g - 5a$

$a = \frac{4}{7}g$

Eliminate T from equations (1) and (2) to find a.

ii $T = 5g - 5a$

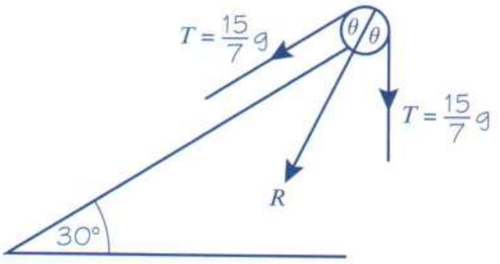
$T = 5g - 5 \times \frac{4}{7}g = \frac{15}{7}g$

Substitute your value of a back into one of the equations to find T.

b The string is inextensible so the acceleration of A and B is the same.

Don't just write that the string does not stretch. You need to state how this fact affects your calculations.

c



$\theta = 30°$

R(↙), $R = 2T\cos\theta$

$R = 2 \times \frac{15}{7}g \times \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{7}g\,\text{N}$

You can leave answers in exact form.

Problem-solving

The force exerted on the pulley by the string is the resultant of the two tensions. The magnitudes are the same so the line of action of the resultant will bisect the lines of actions of the two forces.

Exercise 7F

(P) **1** Two particles P and Q of equal mass are connected by a light inextensible string. The string passes over a small smooth pulley which is fixed at the top of a smooth inclined plane. The plane is inclined to the horizontal at an angle α where $\tan\alpha = 0.75$. Particle P is held at rest on the inclined plane at a distance of 2 m from the pulley and Q hangs freely on the edge of the plane at a distance of 3 m above the ground with the string vertical and taut. Particle P is released. Find the speed with which it hits the pulley.

2 A van of mass 900 kg is towing a trailer of mass 500 kg up a straight road which is inclined to the horizontal at an angle α where $\tan\alpha = 0.75$. The van and the trailer are connected by a light inextensible tow-bar. The engine of the van exerts a driving force of magnitude 12 kN and the van and the trailer experience constant resistances to motion of magnitudes 1600 N and 600 N respectively.

a Find the acceleration of the van.

b Find the tension in the tow-bar.

c Comment on the modelling assumption that the resistances to motion of the van and trailer are constant.

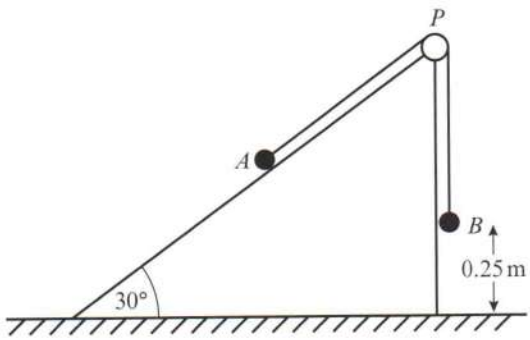
(E/P) **3** Two particles P and Q of mass 2 kg and 3 kg respectively are connected by a light inextensible string. The string passes over a small smooth pulley which is fixed at the top of a rough inclined plane. The plane is inclined to the horizontal at an angle of 30°. Particle P is held at rest on the inclined plane and Q hangs freely with the string vertical and taut. Particle P is released and it accelerates up the plane at $2.5\,\text{m s}^{-2}$. Find:

a the tension in the string **(2 marks)**

b the coefficient of friction between P and the plane **(4 marks)**

c the force exerted by the string on the pulley. **(3 marks)**

E/P **4** Two particles A and B, of mass m kg and 3 kg respectively, are connected by a light inextensible string. The particle A is held resting on a smooth fixed plane inclined at 30° to the horizontal. The string passes over a smooth pulley P fixed at the top of the plane. The portion AP of the string lies along a line of greatest slope of the plane and B hangs freely from the pulley, as shown in the figure. The system is released from rest with B at a height of 0.25 m above horizontal ground.



Immediately after release, B descends with an acceleration of $\frac{2}{5}g$. Given that A does not reach P, calculate:

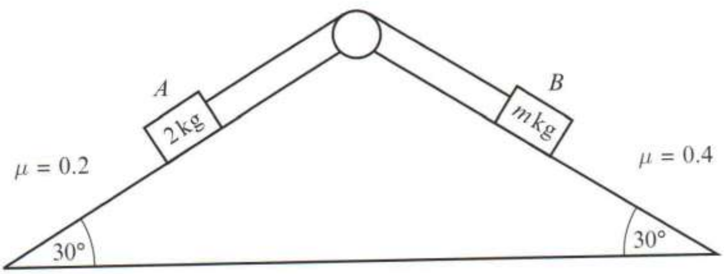
a the tension in the string while B is descending **(2 marks)**

b the value of m. **(4 marks)**

The particle B strikes the ground and does not rebound. Find:

c the time between the instant when B strikes the ground and the instant when A reaches its highest point on the plane. **(6 marks)**

E/P **5** Two particles A and B on back-to-back rough slopes are connected by a light inextensible string that passes over a smooth pulley as shown in the diagram. A has mass 2 kg and B has mass m kg.



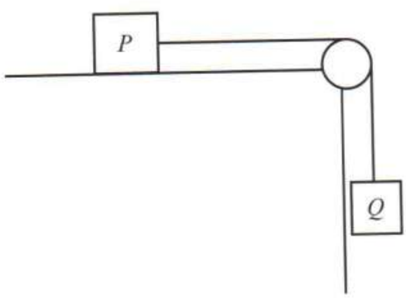
The coefficient of friction between A and the slope is 0.2 and the coefficient of friction between B and the slope is 0.4.

a Show that the maximum value that m can take before the particles begin to move is

$$\frac{10 + 2\sqrt{3}}{5 - 2\sqrt{3}}$$ **(6 marks)**

b Given that $m = 10$, find the acceleration of the particles. **(6 marks)**

E/P **6** A block of metal P of mass 1.5 kg rests on a rough horizontal work bench and is attached to one end of a light inextensible string. The string passes over a small smooth pulley fixed at the edge of the bench. The other end of the string is attached to a box Q of mass 1.6 kg which hangs freely below the pulley, as shown in the diagram. The coefficient of friction between P and the table is μ. The system is released from rest with the string taut. Two seconds after release, Q has velocity 6 m s^{-1}. Modelling P and Q as particles,



a calculate the acceleration of Q **(3 marks)**

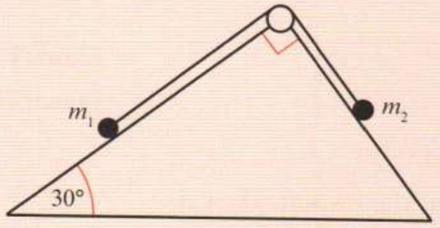
b find the tension in the string **(4 marks)**

c show that μ is 0.434 (3 s.f.). **(5 marks)**

d State how in your calculations you have used the information that the string is inextensible. **(1 mark)**

Challenge

Two particles of mass m_1 and m_2 lie in static equilibrium on a triangular wedge as shown in the diagram. The particles are connected by a light inextensible string that passes over a smooth pulley.



a Given that the wedge is smooth, show that $\frac{m_1}{m_2} = \sqrt{3}$.

b Given instead that the wedge is rough, and that the coefficient of friction between each particle and the wedge is μ, show that

$$\frac{\sqrt{3} - \mu}{1 + \mu\sqrt{3}} \leqslant \frac{m_1}{m_2} \leqslant \frac{\sqrt{3} + \mu}{1 - \mu\sqrt{3}}$$

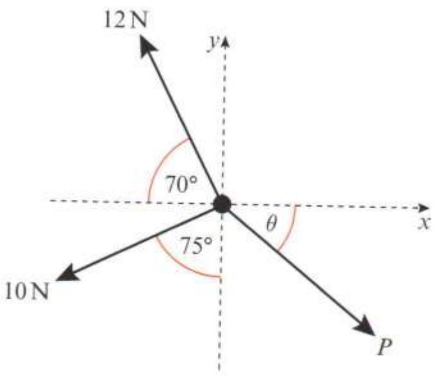
Mixed exercise 7

1 A particle is acted upon by three forces as shown in the diagram.

Given that the particle is in equilibrium, work out:

a the size of angle θ

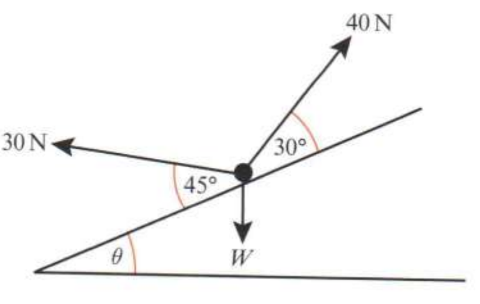
b the magnitude of P.



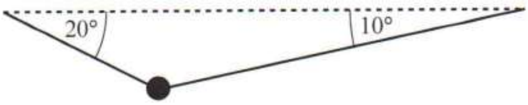
2 A particle is acted upon by three forces as shown in the diagram. Given that it is in equilibrium find:

a the size of angle θ

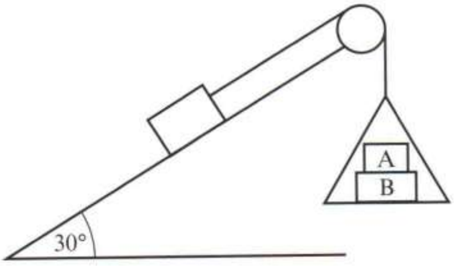
b the magnitude of W.



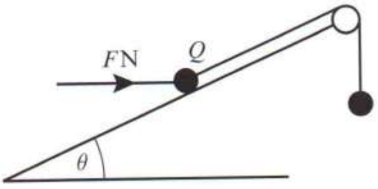
3 An acrobat of mass 55 kg stands on a tightrope. By modelling the acrobat as a particle and the tightrope as two inextensible strings, calculate the tension in the tightrope on each side of the rope.



E/P 4 A box of mass 5 kg sits on a smooth slope that is angled at 30° to the horizontal. It is attached to a light scale-pan by a light inextensible string which passes over a smooth pulley, as shown in the diagram. The scale-pan carries two masses A and B. The mass of A is 2 kg and the mass of B is 5 kg. Work out the force exerted by A on B. **(8 marks)**



E/P 5 A particle Q of mass 5 kg rests in equilibrium on a smooth inclined plane. The plane makes an angle θ with the horizontal, where $\tan\theta = \frac{3}{4}$.



Q is attached to one end of a light inextensible string which passes over a smooth pulley as shown. The other end of the string is attached to a particle of mass 2 kg.

The particle Q is also acted upon by a force of magnitude F N acting horizontally, as shown in the diagram.

Find the magnitude of:

a the force F **(5 marks)**

b the normal reaction between particle Q and the plane. **(3 marks)**

The plane is now assumed to be rough.

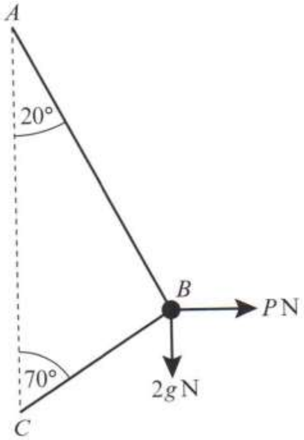
c State, with a reason, which of the following statements is true:

1. F will be larger 2. F will be smaller 3. F could be either larger or smaller. **(2 marks)**

E 6 A smooth bead B of mass 2 kg is threaded on a light inextensible string. The ends of the string are attached to two fixed points A and C where A is vertically above C. The bead is held in equilibrium by a horizontal force of magnitude P N. The sections AB and BC make angles of 20° and 70° with the vertical as shown.

a Show that the tension in the string is 33 N (2 s.f.). **(3 marks)**

b Calculate the value of P. **(3 marks)**

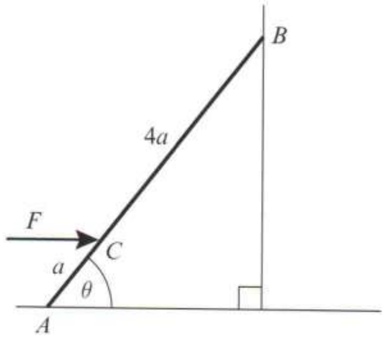


(E) **7** A sledge of mass 50 kg sits on a snowy hill that is angled at 40° to the horizontal. The sledge is held in place by a rope that is angled at 30° above the line of greatest slope of the hill.

a By modelling the sledge as a particle, the hill as a smooth slope and the rope as a light inextensible string, work out the tension in the rope. **(4 marks)**

b Give one criticism of this model. **(1 mark)**

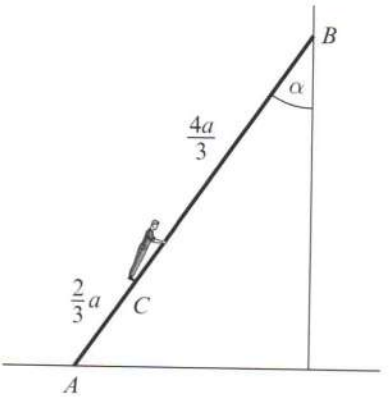
(E/P) **8** A uniform ladder AB has one end A on smooth horizontal ground. The other end B rests against a smooth vertical wall. The ladder is modelled as a uniform rod of mass m and length $5a$. The ladder is kept in equilibrium by a horizontal force F acting at a point C of the ladder where $AC = a$. The force F and the ladder lie in a vertical plane perpendicular to the wall. The ladder is inclined to the horizontal at an angle θ, where $\tan\theta = \frac{9}{5}$, as shown in the diagram.



Show that $F = \frac{25mg}{72}$. **(8 marks)**

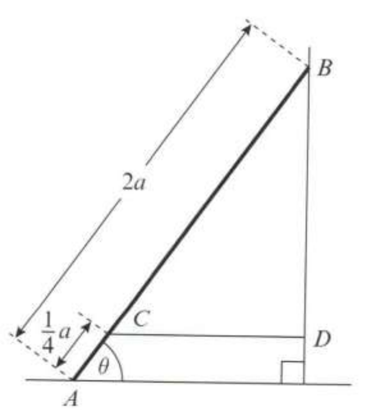
(E/P) **9** A uniform ladder AB, of mass m and length $2a$, has one end A on rough horizontal ground. The other end B rests against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall. The ladder makes an angle α with the vertical, where $\tan\alpha = \frac{3}{4}$. A child of mass $2m$ stands on the ladder at C where $AC = \frac{2}{3}a$, as shown in the diagram. The ladder and the child are in equilibrium.

By modelling the ladder as a rod and the child as a particle, calculate the least possible value of the coefficient of friction between the ladder and the ground.



(8 marks)

(E/P) **10** A uniform ladder, of weight W and length $2a$, rests in equilibrium with one end A on a smooth horizontal floor and the other end B against a rough vertical wall. The ladder is in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the ladder is μ. The ladder makes an angle θ with the floor, where $\tan\theta = \frac{4}{3}$. A horizontal light inextensible string CD is attached to the ladder at the point C, where $AC = \frac{1}{4}a$. The string is attached to the wall at the point D, with BD vertical, as shown in the diagram.



The tension in the string is $\frac{1}{3}W$. By modelling the ladder as a rod,

a find the magnitude of the force of the floor on the ladder **(5 marks)**

b show that $\mu \geqslant \frac{1}{3}$. **(3 marks)**

c State how you have used the modelling assumption that the ladder is a rod. **(1 mark)**

E/P **11** A uniform ladder, of weight W and length 5 m, has one end on rough horizontal ground and the other touching a smooth vertical wall. The coefficient of friction between the ladder and the ground is 0.3.

The top of the ladder touches the wall at a point 4 m vertically above the level of the ground.

a Show that the ladder cannot rest in equilibrium in this position. **(6 marks)**

In order to enable the ladder to rest in equilibrium in the position described above, a brick is attached to the bottom of the ladder.

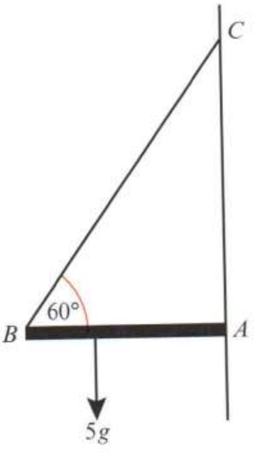
Assuming that this brick is at the lowest point of the ladder, but not touching the ground,

b show that the horizontal frictional force exerted by the ladder on the ground is independent of the mass of the brick **(4 marks)**

c find, in terms of W and g, the smallest mass of the brick for which the ladder will rest in equilibrium. **(3 marks)**

E/P **12** A non-uniform ladder PQ of mass 20 kg and length 4 metres, rests with P on smooth horizontal ground and Q against a rough vertical wall. The coefficient of friction between the ladder and the wall is 0.2. The centre of mass of the ladder is 1 m from P. The ladder is inclined at an angle α to the horizontal, where $\tan \alpha = \frac{5}{2}$. A horizontal force F applied to the base of the ladder can just prevent it from slipping. By modelling the ladder as a rod determine the value of F. **(10 marks)**

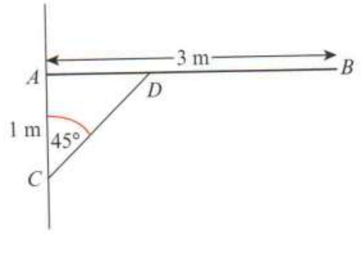
13 The diagram shows a uniform rod AB of length 3 m and of mass 10 kg. The rod is smoothly hinged at A which lies on a vertical wall. A particle of mass 5 kg is suspended 1 m from B. The rod is kept in a horizontal position by a light inextensible string BC, where C lies on the wall directly above A. The plane ABC is perpendicular to the wall and $\angle ABC$ is 60°.



a Calculate the tension in the string.

b Work out the magnitude and direction of the reaction at the hinge.

E/P **14** A uniform pole AB of mass 40 kg and length 3 m, is smoothly hinged to a vertical wall at one end A. The pole is held in equilibrium in a horizontal position by a light rod CD. One end C of the rod is fixed to the wall vertically below A. The other end D is freely jointed to the pole so that angle $ACD = 45°$ and $AC = 1$ m, as shown in the diagram.



Find:

a the thrust in the rod CD **(3 marks)**

b the magnitude of the force exerted by the wall on the pole at A. **(4 marks)**

The rod CD is removed and replaced by a longer light rod CM, where M is the midpoint of AB. The rod is freely jointed to the pole at M. The pole AB remains in equilibrium in a horizontal position.

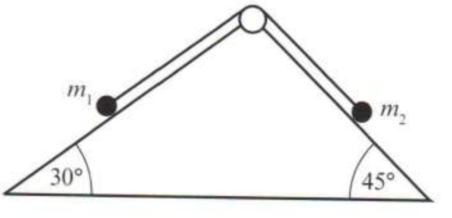
c Show that the force exerted by the wall on the pole at A now acts horizontally. **(2 marks)**

E/P **15** A particle of mass 3 kg is released from rest on a rough slope that is angled at α to the horizontal where $\tan\alpha = \frac{3}{4}$. After 1.5 seconds the particle has travelled 6 m. Work out the coefficient of friction μ. **(6 marks)**

E **16** A particle of mass 5 kg is pushed up a rough slope, inclined at 30° to the horizontal, by a force of 80 N applied at an angle of 10° to the slope. Given that the coefficient of friction of the slope is 0.4, find the acceleration of the particle. **(6 marks)**

E **17** Two particles, A of mass m_1 kg and B of mass m_2 kg are connected by a light inextensible string. The string passes over a smooth pulley, P. A sits on a rough horizontal table, where the coefficient of friction between A and the table is μ, and B lies directly below P. Given that $m_2 > \mu m_1$, show that the acceleration of the system is $\frac{g(m_2 - \mu m_1)}{m_1 + m_2}$. **(5 marks)**

E/P **18** Two particles of masses m_1 and m_2 are connected by a light inextensible string that passes over a smooth pulley. The particles are released from rest on smooth slopes angled at 30° and 45° to the horizontal as shown in the diagram. Given that m_2 is accelerating down the 45° slope at $\frac{1}{2}$ m s^{-2}, show that



$$\frac{m_1}{m_2} = \frac{g\sqrt{2} - 1}{1 + g}.$$

(6 marks)

Summary of key points

1 A particle or rigid body is in static equilibrium if it is at rest and the resultant force acting on the particle is zero.

2 The maximum value of the frictional force $F_{MAX} = \mu R$ is reached when the body you are considering is on the point of moving. The body is then said to be in limiting equilibrium.

3 In general, the force of friction F is such that $F \leqslant \mu R$, and the direction of the frictional force is opposite to the direction in which the body would move if the frictional force were absent.

4 For a rigid body in static equilibrium:

- the body is stationary
- the resultant force in any direction is zero
- the resultant moment is zero.