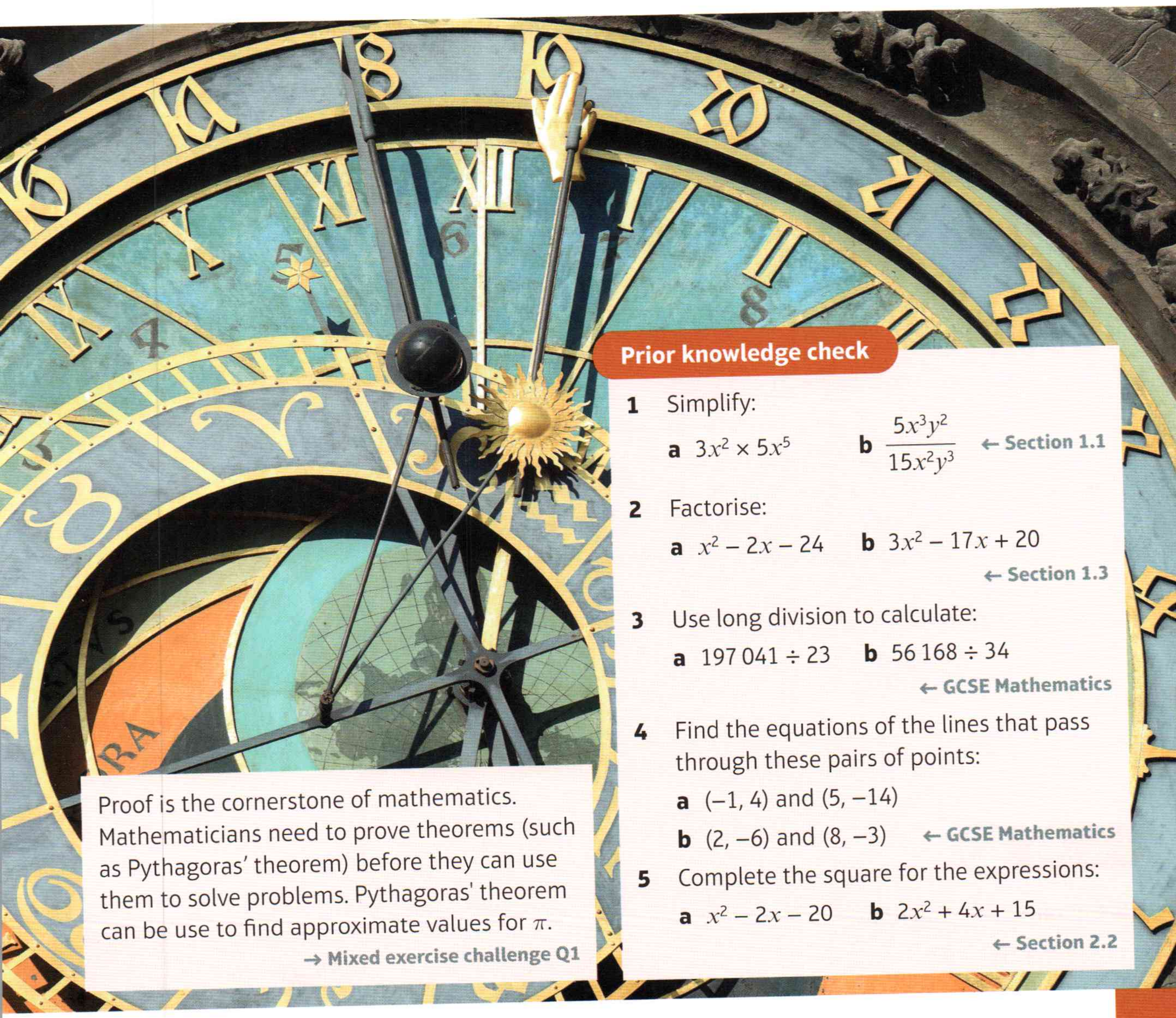
Algebraic methods

7

Objectives

After completing this unit you should be able to:

- Cancel factors in algebraic fractions → pages 138–139
- Divide a polynomial by a linear expression → pages 139–142
- Use the factor theorem to factorise a cubic expression → pages 143–146
- Construct mathematical proofs using algebra → pages 146–150
- Use proof by exhaustion and disproof by counter-example → pages 150–152



Proof is the cornerstone of mathematics. Mathematicians need to prove theorems (such as Pythagoras' theorem) before they can use them to solve problems. Pythagoras' theorem can be use to find approximate values for π.

→ Mixed exercise challenge Q1

Prior knowledge check

1 Simplify:

a $3x^2 \times 5x^5$ **b** $\dfrac{5x^3y^2}{15x^2y^3}$ ← Section 1.1

2 Factorise:

a $x^2 - 2x - 24$ **b** $3x^2 - 17x + 20$ ← Section 1.3

3 Use long division to calculate:

a 197 041 ÷ 23 **b** 56 168 ÷ 34 ← GCSE Mathematics

4 Find the equations of the lines that pass through these pairs of points:

a (−1, 4) and (5, −14)

b (2, −6) and (8, −3) ← GCSE Mathematics

5 Complete the square for the expressions:

a $x^2 - 2x - 20$ **b** $2x^2 + 4x + 15$ ← Section 2.2

7.1 Algebraic fractions

You can simplify algebraic fractions using division.

- **When simplifying an algebraic fraction, where possible factorise the numerator and denominator and then cancel common factors.**

Factorise → Cancel common factor

$$\frac{5x^2-245}{2x^2-15x+7}=\frac{5(x+7)(x-7)}{(2x-1)(x-7)}=\frac{5(x+7)}{2x-1}$$

Example 1

Simplify these fractions:

a $\frac{7x^4-2x^3+6x}{x}$ **b** $\frac{(x+7)(2x-1)}{(2x-1)}$ **c** $\frac{x^2+7x+12}{(x+3)}$ **d** $\frac{x^2+6x+5}{x^2+3x-10}$ **e** $\frac{2x^2+11x+12}{(x+3)(x+4)}$

a $\frac{7x^4-2x^3+6x}{x}$

$=\frac{7x^4}{x}-\frac{2x^3}{x}+\frac{6x}{x}$ — Divide each part of the numerator by x.

$=7x^3-2x^2+6$

b $\frac{(x+7)(2x-1)}{(2x-1)}=x+7$ — Simplify by cancelling the common factor of $(2x-1)$.

c $\frac{x^2+7x+12}{(x+3)}=\frac{(x+3)(x+4)}{(x+3)}$ — Factorise: $x^2+7x+12=(x+3)(x+4)$.

$=x+4$ — Cancel the common factor of $(x+3)$.

d $\frac{x^2+6x+5}{x^2+3x-10}=\frac{(x+5)(x+1)}{(x+5)(x-2)}$ — Factorise: $x^2+6x+5=(x+5)(x+1)$ and $x^2+3x-10=(x+5)(x-2)$.

$=\frac{x+1}{x-2}$ — Cancel the common factor of $(x+5)$.

e $2x^2+11x+12=2x^2+3x+8x+12$

$=x(2x+3)+4(2x+3)$

$=(2x+3)(x+4)$ — Factorise $2x^2+11x+12$

So $\frac{2x^2+11x+12}{(x+3)(x+4)}$

$=\frac{(2x+3)(x+4)}{(x+3)(x+4)}$

$=\frac{2x+3}{x+3}$ — Cancel the common factor of $(x+4)$.

Exercise 7A

1 Simplify these fractions:

a $\frac{4x^4+5x^2-7x}{x}$ **b** $\frac{7x^5-5x^5+9x^3+x^2}{x}$ **c** $\frac{-x^4+4x^2+6}{x}$

d $\dfrac{7x^5 - x^3 - 4}{x}$ **e** $\dfrac{8x^4 - 4x^3 + 6x}{2x}$ **f** $\dfrac{9x^2 - 12x^3 - 3x}{3x}$

g $\dfrac{7x^3 - x^4 - 2}{5x}$ **h** $\dfrac{-4x^2 + 6x^4 - 2x}{-2x}$ **i** $\dfrac{-x^8 + 9x^4 - 4x^3 + 6}{-2x}$

j $\dfrac{-9x^9 - 6x^6 + 4x^4 - 2}{-3x}$

2 Simplify these fractions as far as possible:

a $\dfrac{(x+3)(x-2)}{(x-2)}$ **b** $\dfrac{(x+4)(3x-1)}{(3x-1)}$ **c** $\dfrac{(x+3)^2}{(x+3)}$

d $\dfrac{x^2 + 10x + 21}{(x+3)}$ **e** $\dfrac{x^2 + 9x + 20}{(x+4)}$ **f** $\dfrac{x^2 + x - 12}{(x-3)}$

g $\dfrac{x^2 + x - 20}{x^2 + 2x - 15}$ **h** $\dfrac{x^2 + 3x + 2}{x^2 + 5x + 4}$ **i** $\dfrac{x^2 + x - 12}{x^2 - 9x + 18}$

j $\dfrac{2x^2 + 7x + 6}{(x-5)(x+2)}$ **k** $\dfrac{2x^2 + 9x - 18}{(x+6)(x+1)}$ **l** $\dfrac{3x^2 - 7x + 2}{(3x-1)(x+2)}$

m $\dfrac{2x^2 + 3x + 1}{x^2 - x - 2}$ **n** $\dfrac{x^2 + 6x + 8}{3x^2 + 7x + 2}$ **o** $\dfrac{2x^2 - 5x - 3}{2x^2 - 9x + 9}$

E/P **3** $\dfrac{6x^3 + 3x^2 - 84x}{6x^2 - 33x + 42} = \dfrac{ax(x+b)}{x+c}$, where a, b and c are constants.

Work out the values of a, b and c. **(4 marks)**

7.2 Dividing polynomials

A **polynomial** is a finite expression with positive whole number indices.

- **You can use long division to divide a polynomial by ($x \pm p$), where p is a constant.**

Polynomials	Not polynomials
$2x + 4$	$\sqrt{x}$
$4xy^2 + 3x - 9$	$6x^{-2}$
8	$\dfrac{4}{x}$

Example 2

Divide $x^3 + 2x^2 - 17x + 6$ by $(x - 3)$.

①

$$\begin{array}{r} x^2 \phantom{{}+2x^2-17x+6} \\ x-3\,\overline{)\,x^3 + 2x^2 - 17x + 6} \\ \underline{x^3 - 3x^2} \phantom{{}-17x+6} \\ 5x^2 - 17x \phantom{{}+6} \end{array}$$

Start by dividing the first term of the polynomial by x, so that $x^3 \div x = x^2$.

Next multiply $(x - 3)$ by x^2, so that $x^2 \times (x - 3) = x^3 - 3x^2$.

Now subtract, so that $(x^3 + 2x^2) - (x^3 - 3x^2) = 5x^2$.

Copy $-17x$.

②

$$\begin{array}{r} x^2 + 5x \phantom{{}- 2} \\ x - 3\overline{)x^3 + 2x^2 - 17x + 6} \\ \underline{x^3 - 3x^2} \phantom{{}- 17x + 6} \\ 5x^2 - 17x \phantom{{}+ 6} \\ \underline{5x^2 - 15x} \phantom{{}+ 6} \\ -2x + 6 \end{array}$$

Repeat the method. Divide $5x^2$ by x, so that $5x^2 \div x = 5x$.

Multiply $(x - 3)$ by $5x$, so that $5x \times (x - 3) = 5x^2 - 15x$.

Subtract, so that $(5x^2 - 17x) - (5x^2 - 15x) = -2x$.

Copy +6.

③

$$\begin{array}{r} x^2 + 5x - 2 \\ x - 3\overline{)x^3 + 2x^2 - 17x + 6} \\ \underline{x^3 - 3x^2} \phantom{{}- 17x + 6} \\ 5x^2 - 17x \phantom{{}+ 6} \\ \underline{5x^2 - 15x} \phantom{{}+ 6} \\ -2x + 6 \\ \underline{-2x + 6} \\ 0 \end{array}$$

Repeat the method. Divide $-2x$ by x, so that $-2x \div x = -2$.

Multiply $(x - 3)$ by -2, so that $-2 \times (x - 3) = -2x + 6$.

Subtract, so that $(-2x + 6) - (-2x + 6) = 0$.

No numbers left to copy, so you have finished.

So $\dfrac{x^3 + 2x^2 - 17x + 6}{x - 3} = x^2 + 5x - 2$

This is called the **quotient**.

Example 3

$f(x) = 4x^4 - 17x^2 + 4$

Divide $f(x)$ by $(2x + 1)$, giving your answer in the form $f(x) = (2x + 1)(ax^3 + bx^2 + cx + d)$.

Find $(4x^4 - 17x^2 + 4) \div (2x + 1)$

$$\begin{array}{r} 2x^3 - x^2 - 8x + 4 \\ 2x + 1\overline{)4x^4 + 0x^3 - 17x^2 + 0x + 4} \\ \underline{4x^4 + 2x^3} \phantom{{}- 17x^2 + 0x + 4} \\ -2x^3 - 17x^2 \phantom{{}+ 0x + 4} \\ \underline{-2x^3 - x^2} \phantom{{}+ 0x + 4} \\ -16x^2 + 0x \phantom{{}+ 4} \\ \underline{-16x^2 - 8x} \phantom{{}+ 4} \\ 8x + 4 \\ \underline{8x + 4} \\ 0 \end{array}$$

So

$4x^4 - 17x^2 + 4 = (2x + 1)(2x^3 - x^2 - 8x + 4)$.

Use long division. Include the terms $0x^3$ and $0x$ when you write out $f(x)$.

You need to multiply $(2x + 1)$ by $2x^3$ to get the $4x^4$ term, so write $2x^3$ in the answer, and write $2x^3(2x + 1) = 4x^4 + 2x^3$ below. Subtract and copy the next term.

You need to multiply $(2x + 1)$ by $-x^2$ to get the $-2x^3$ term, so write $-x^2$ in the answer, and write $-x^2(2x + 1) = -2x^3 - x^2$ below. Subtract and copy the next term.

Repeat the method.

$(4x^4 - 17x^2 + 4) \div (2x + 1) = 2x^3 - x^2 - 8x + 4$.

Example 4

Find the remainder when $2x^3 - 5x^2 - 16x + 10$ is divided by $(x - 4)$.

$$
\begin{array}{r}
2x^2 + 3x - 4 \\
x - 4 \overline{) 2x^3 - 5x^2 - 16x + 10} \\
\underline{2x^3 - 8x^2} \\
3x^2 - 16x \\
\underline{3x^2 - 12x} \\
-4x + 10 \\
\underline{-4x + 16} \\
-6
\end{array}
$$

So the remainder is -6.

$(x - 4)$ is not a factor of $2x^3 - 5x^2 - 16x + 10$ as the remainder $\neq 0$.

This means you cannot write the expression in the form $(x - 4)(ax^2 + bx + c)$.

Exercise 7B

1 Write each polynomial in the form $(x \pm p)(ax^2 + bx + c)$ by dividing:

a $x^3 + 6x^2 + 8x + 3$ by $(x + 1)$
b $x^3 + 10x^2 + 25x + 4$ by $(x + 4)$
c $x^3 - x^2 + x + 14$ by $(x + 2)$
d $x^3 + x^2 - 7x - 15$ by $(x - 3)$
e $x^3 - 8x^2 + 13x + 10$ by $(x - 5)$
f $x^3 - 5x^2 - 6x - 56$ by $(x - 7)$

2 Write each polynomial in the form $(x \pm p)(ax^2 + bx + c)$ by dividing:

a $6x^3 + 27x^2 + 14x + 8$ by $(x + 4)$
b $4x^3 + 9x^2 - 3x - 10$ by $(x + 2)$
c $2x^3 + 4x^2 - 9x - 9$ by $(x + 3)$
d $2x^3 - 15x^2 + 14x + 24$ by $(x - 6)$
e $-5x^3 - 27x^2 + 23x + 30$ by $(x + 6)$
f $-4x^3 + 9x^2 - 3x + 2$ by $(x - 2)$

3 Divide:

a $x^4 + 5x^3 + 2x^2 - 7x + 2$ by $(x + 2)$
b $4x^4 + 14x^3 + 3x^2 - 14x - 15$ by $(x + 3)$
c $-3x^4 + 9x^3 - 10x^2 + x + 14$ by $(x - 2)$
d $-5x^5 + 7x^4 + 2x^3 - 7x^2 + 10x - 7$ by $(x - 1)$

4 Divide:

a $3x^4 + 8x^3 - 11x^2 + 2x + 8$ by $(3x + 2)$
b $4x^4 - 3x^3 + 11x^2 - x - 1$ by $(4x + 1)$
c $4x^4 - 6x^3 + 10x^2 - 11x - 6$ by $(2x - 3)$
d $6x^5 + 13x^4 - 4x^3 - 9x^2 + 21x + 18$ by $(2x + 3)$
e $6x^5 - 8x^4 + 11x^3 + 9x^2 - 25x + 7$ by $(3x - 1)$
f $8x^5 - 26x^4 + 11x^3 + 22x^2 - 40x + 25$ by $(2x - 5)$
g $25x^4 + 75x^3 + 6x^2 - 28x - 6$ by $(5x + 3)$
h $21x^5 + 29x^4 - 10x^3 + 42x - 12$ by $(7x - 2)$

5 Divide:

a $x^3 + x + 10$ by $(x + 2)$
b $2x^3 - 17x + 3$ by $(x + 3)$
c $-3x^3 + 50x - 8$ by $(x - 4)$

Hint Include $0x^2$ when you write out $f(x)$.

6 Divide:

a $x^3 + x^2 - 36$ by $(x - 3)$
b $2x^3 + 9x^2 + 25$ by $(x + 5)$
c $-3x^3 + 11x^2 - 20$ by $(x - 2)$

7 Show that $x^3 + 2x^2 - 5x - 10 = (x + 2)(x^2 - 5)$

8 Find the remainder when:

a $x^3 + 4x^2 - 3x + 2$ is divided by $(x + 5)$ **b** $3x^3 - 20x^2 + 10x + 5$ is divided by $(x - 6)$

c $-2x^3 + 3x^2 + 12x + 20$ is divided by $(x - 4)$

9 Show that when $3x^3 - 2x^2 + 4$ is divided by $(x - 1)$ the remainder is 5.

10 Show that when $3x^4 - 8x^3 + 10x^2 - 3x - 25$ is divided by $(x + 1)$ the remainder is -1.

11 Show that $(x + 4)$ is a factor of $5x^3 - 73x + 28$.

12 Simplify $\dfrac{3x^3 - 8x - 8}{x - 2}$

Hint Divide $3x^3 - 8x - 8$ by $(x - 2)$.

13 Divide $x^3 - 1$ by $(x - 1)$.

Hint Write $x^3 - 1$ as $x^3 + 0x^2 + 0x - 1$.

14 Divide $x^4 - 16$ by $(x + 2)$.

(E) 15 $f(x) = 10x^3 + 43x^2 - 2x - 10$

Find the remainder when $f(x)$ is divided by $(5x + 4)$. **(2 marks)**

(E/P) 16 $f(x) = 3x^3 - 14x^2 - 47x - 14$

a Find the remainder when $f(x)$ is divided by $(x - 3)$. **(2 marks)**

b Given that $(x + 2)$ is a factor of $f(x)$, factorise $f(x)$ completely. **(4 marks)**

Problem-solving

Write $f(x)$ in the form $(x + 2)(ax^2 + bx + c)$ then factorise the quadratic factor.

(E/P) 17 **a** Find the remainder when $x^3 + 6x^2 + 5x - 12$ is divided by

i $x - 2$,

ii $x + 3$. **(3 marks)**

b Hence, or otherwise, find all the solutions to the equation $x^3 + 6x^2 + 5x - 12 = 0$. **(4 marks)**

(E/P) 18 $f(x) = 2x^3 + 3x^2 - 8x + 3$

a Show that $f(x) = (2x - 1)(ax^2 + bx + c)$ where a, b and c are constants to be found. **(2 marks)**

b Hence factorise $f(x)$ completely. **(4 marks)**

c Write down all the real roots of the equation $f(x) = 0$. **(2 marks)**

(E/P) 19 $f(x) = 12x^3 + 5x^2 + 2x - 1$

a Show that $(4x - 1)$ is a factor of $f(x)$ and write $f(x)$ in the form $(4x - 1)(ax^2 + bx + c)$. **(6 marks)**

b Hence, show that the equation $12x^3 + 5x^2 + 2x - 1 = 0$ has exactly 1 real solution. **(2 marks)**

7.3 The factor theorem

The factor theorem is a quick way of finding simple linear factors of a polynomial.

- **The factor theorem states that if f(x) is a polynomial then:**
  - **If f(p) = 0, then ($x - p$) is a factor of f(x).**
  - **If ($x - p$) is a factor of f(x), then f(p) = 0.**

Watch out

These two statements are not the same. Here are two similar statements, only one of which is true:

If $x = -2$ then $x^2 = 4$ ✓

If $x^2 = 4$ then $x = -2$ ✗

You can use the factor theorem to quickly factorise a cubic function, g(x):

1 Substitute values into the function until you find a value p such that g(p) = 0.

2 Divide the function by $(x - p)$. — The remainder will be 0 because $(x - p)$ is a factor of g(x).

3 Write $g(x) = (x - p)(ax^2 + bx + c)$ — The other factor will be quadratic.

4 Factorise the quadratic factor, if possible, to write g(x) as a product of three linear factors.

Example 5

Show that $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$ by:

a algebraic division **b** the factor theorem

a

$$\begin{array}{r} x^2 + 3x + 2 \\ x - 2\overline{)x^3 + x^2 - 4x - 4} \\ \underline{x^3 - 2x^2} \\ 3x^2 - 4x \\ \underline{3x^2 - 6x} \\ 2x - 4 \\ \underline{2x - 4} \\ 0 \end{array}$$

So $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Divide $x^3 + x^2 - 4x - 4$ by $(x - 2)$.

The remainder is 0, so $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

b $f(x) = x^3 + x^2 - 4x - 4$

$f(2) = (2)^3 + (2)^2 - 4(2) - 4$

$= 8 + 4 - 8 - 4$

$= 0$

So $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Write the polynomial as a function.

Substitute $x = 2$ into the polynomial.

Use the factor theorem:
If f(p) = 0, then $(x - p)$ is a factor of f(x).
Here $p = 2$, so $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Example 6

a Fully factorise $2x^3 + x^2 - 18x - 9$

b Hence sketch the graph of $y = 2x^3 + x^2 - 18x - 9$

a $f(x) = 2x^3 + x^2 - 18x - 9$

Write the polynomial as a function.

$f(-1) = 2(-1)^3 + (-1)^2 - 18(-1) - 9 = 8$
$f(1) = 2(1)^3 + (1)^2 - 18(1) - 9 = -24$
$f(2) = 2(2)^3 + (2)^2 - 18(2) - 9 = -25$
$f(3) = 2(3)^3 + (3)^2 - 18(3) - 9 = 0$

Try values of x, e.g. $-1, 1, 2, 3, \ldots$ until you find $f(p) = 0$.

$f(p) = 0$.

So $(x - 3)$ is a factor of $2x^3 + x^2 - 18x - 9$.

Use statement 1 from the factor theorem: If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$. Here $p = 3$.

$$
\begin{array}{r}
2x^2 + 7x + 3 \\
x - 3 \overline{)\, 2x^3 + x^2 - 18x - 9} \\
\underline{2x^3 - 6x^2} \qquad\qquad \\
7x^2 - 18x \qquad \\
\underline{7x^2 - 21x} \qquad \\
3x - 9 \\
\underline{3x - 9} \\
0
\end{array}
$$

Use long division to find the quotient when dividing by $(x - 3)$.

You can check your division here: $(x - 3)$ is a factor of $2x^3 + x^2 - 18x - 9$, so the remainder must be 0.

$2x^3 + x^2 - 18x - 9 = (x - 3)(2x^2 + 7x + 3)$
$= (x - 3)(2x + 1)(x + 3)$

$2x^2 + 7x + 3$ can also be factorised.

b $0 = (x - 3)(2x + 1)(x + 3)$

Set $y = 0$ to find the points where the curve crosses the x-axis.

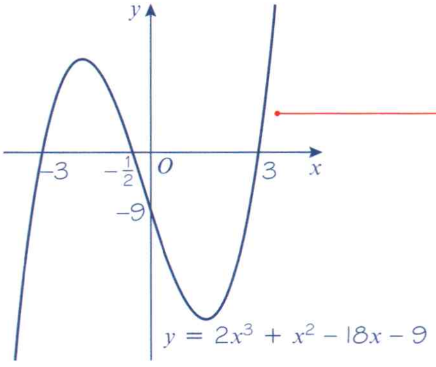
So the curve crosses the x-axis at $(3, 0)$, $(-\frac{1}{2}, 0)$ and $(-3, 0)$.

When $x = 0$, $y = (-3)(1)(3) = -9$

Set $x = 0$ to find the y-intercept.

The curve crosses the y-axis at $(0, -9)$.

$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



This is a cubic graph with a positive coefficient of x^3 and three distinct roots. You should be familiar with its general shape. ← Section 4.1

Example 7

Given that $(x + 1)$ is a factor of $4x^4 - 3x^2 + a$, find the value of a.

$f(x) = 4x^4 - 3x^2 + a$ — Write the polynomial as a function.

$f(-1) = 0$ — Use statement 2 from the factor theorem. $(x - p)$ is a factor of $f(x)$, so $f(p) = 0$. Here $p = -1$.

$4(-1)^4 - 3(-1)^2 + a = 0$

$4 - 3 + a = 0$

$a = -1$

Substitute $x = -1$ and solve the equation for a. Remember $(-1)^4 = 1$.

Exercise 7C

1 Use the factor theorem to show that:

a $(x - 1)$ is a factor of $4x^3 - 3x^2 - 1$

b $(x + 3)$ is a factor of $5x^4 - 45x^2 - 6x - 18$

c $(x - 4)$ is a factor of $-3x^3 + 13x^2 - 6x + 8$.

2 Show that $(x - 1)$ is a factor of $x^3 + 6x^2 + 5x - 12$ and hence factorise the expression completely.

3 Show that $(x + 1)$ is a factor of $x^3 + 3x^2 - 33x - 35$ and hence factorise the expression completely.

4 Show that $(x - 5)$ is a factor of $x^3 - 7x^2 + 2x + 40$ and hence factorise the expression completely.

5 Show that $(x - 2)$ is a factor of $2x^3 + 3x^2 - 18x + 8$ and hence factorise the expression completely.

6 Each of these expressions has a factor $(x \pm p)$. Find a value of p and hence factorise the expression completely.

a $x^3 - 10x^2 + 19x + 30$

b $x^3 + x^2 - 4x - 4$

c $x^3 - 4x^2 - 11x + 30$

7 i Fully factorise the right-hand side of each equation.

ii Sketch the graph of each equation.

a $y = 2x^3 + 5x^2 - 4x - 3$

b $y = 2x^3 - 17x^2 + 38x - 15$

c $y = 3x^3 + 8x^2 + 3x - 2$

d $y = 6x^3 + 11x^2 - 3x - 2$

e $y = 4x^3 - 12x^2 - 7x + 30$

(P) 8 Given that $(x - 1)$ is a factor of $5x^3 - 9x^2 + 2x + a$, find the value of a.

(P) 9 Given that $(x + 3)$ is a factor of $6x^3 - bx^2 + 18$, find the value of b.

(P) 10 Given that $(x - 1)$ and $(x + 1)$ are factors of $px^3 + qx^2 - 3x - 7$, find the values of p and q.

(P) 11 Given that $(x + 1)$ and $(x - 2)$ are factors of $cx^3 + dx^2 - 9x - 10$, find the values of c and d.

Problem-solving

Use the factor theorem to form simultaneous equations.

(P) 12 Given that $(x + 2)$ and $(x - 3)$ are factors of $gx^3 + hx^2 - 14x + 24$, find the values of g and h.

(E) **13** $f(x) = 3x^3 - 12x^2 + 6x - 24$

a Use the factor theorem to show that $(x - 4)$ is a factor of $f(x)$. **(2 marks)**

b Hence, show that 4 is the only real root of the equation $f(x) = 0$. **(4 marks)**

(E) **14** $f(x) = 4x^3 + 4x^2 - 11x - 6$

a Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$. **(2 marks)**

b Factorise $f(x)$ completely. **(4 marks)**

c Write down all the solutions of the equation $4x^3 + 4x^2 - 11x - 6 = 0$. **(1 mark)**

(E) **15** **a** Show that $(x - 2)$ is a factor of $9x^4 - 18x^3 - x^2 + 2x$. **(2 marks)**

b Hence, find four real solutions to the equation $9x^4 - 18x^3 - x^2 + 2x = 0$. **(5 marks)**

Challenge

$f(x) = 2x^4 - 5x^3 - 42x^2 - 9x + 54$

a Show that $f(1) = 0$ and $f(-3) = 0$.

b Hence, solve $f(x) = 0$.

7.4 Mathematical proof

A proof is a logical and structured argument to show that a mathematical statement (or **conjecture**) is always true.
A mathematical proof usually starts with previously established mathematical facts (or **theorems**) and then works through a series of logical steps. The final step in a proof is a **statement** of what has been proven.

Notation A statement that has been proven is called a **theorem**. A statement that has yet to be proven is called a **conjecture**.

Known facts or theorems → Clearly shown logical steps → Statement of proof

A mathematical proof needs to show that something is true in every case.

- **You can prove a mathematical statement is true by deduction. This means starting from known facts or definitions, then using logical steps to reach the desired conclusion.**

Here is an example of proof by deduction:

Statement: The product of two odd numbers is odd.

Demonstration: $5 \times 7 = 35$, which is odd

This is demonstration but it is not a proof. You have only shown one case.

Proof: p and q are integers, so $2p + 1$ and $2q + 1$ are odd numbers.

You can use $2p + 1$ and $2q + 1$ to represent any odd numbers. If you can show that $(2p + 1) \times (2q + 1)$ is always an odd number then you have proved the statement for all cases.

$$(2p + 1) \times (2q + 1) = 4pq + 2p + 2q + 1$$
$$= 2(2pq + p + q) + 1$$

Since p and q are integers, $2pq + p + q$ is also an integer.

So $2(2pq + p + q) + 1$ is one more than an even number.

So the product of two odd numbers is an odd number.

This is the statement of proof.

- **In a mathematical proof you must**
  - **State any information or assumptions you are using**
  - **Show every step of your proof clearly**
  - **Make sure that every step follows logically from the previous step**
  - **Make sure you have covered all possible cases**
  - **Write a statement of proof at the end of your working**

You need to be able to prove results involving identities, such as $(a + b)(a - b) \equiv a^2 - b^2$

- **To prove an identity you should**
  - **Start with the expression on one side of the identity**
  - **Manipulate that expression algebraically until it matches the other side**
  - **Show every step of your algebraic working**

Notation The symbol $\equiv$ means 'is always equal to'. It shows that two expressions are mathematically **identical**.

Watch out Don't try to 'solve' an identity like an equation. Start from one side and manipulate the expression to match the other side.

Example 8

Prove that $(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70$

$(3x + 2)(x - 5)(x + 7)$

$= (3x + 2)(x^2 + 2x - 35)$

$= 3x^3 + 6x^2 - 105x + 2x^2 + 4x - 70$

$= 3x^3 + 8x^2 - 101x - 70$

So

$(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70$

Start with the left-hand side and expand the brackets.

In proof questions you need to show all your working.

Left-hand side = right-hand side.

Example 9

Prove that if $(x - p)$ is a factor of $f(x)$ then $f(p) = 0$.

If $(x - p)$ is a factor of $f(x)$ then

$f(x) = (x - p) \times g(x)$

So $f(p) = (p - p) \times g(p)$

i.e. $f(p) = 0 \times g(p)$

So $f(p) = 0$ as required.

$g(x)$ is a polynomial expression.

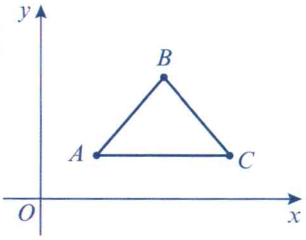
Substitute $x = p$.

$p - p = 0$

Remember $0 \times$ *anything* $= 0$

Example 10

Prove that $A(1, 1)$, $B(3, 3)$ and $C(4, 2)$ are the vertices of a right-angled triangle.



The gradient of line $AB = \frac{3-1}{3-1} = \frac{2}{2} = 1$

The gradient of line $BC = \frac{2-3}{4-3} = \frac{-1}{1} = -1$

The gradient of line $AC = \frac{2-1}{4-1} = \frac{1}{3}$

The gradients are different so the three points are not collinear.

Hence ABC is a triangle.

Gradient of AB × gradient of $BC = 1 \times (-1)$
$= -1$

So AB is perpendicular to BC, and the triangle is a right-angled triangle.

Problem-solving

If you need to prove a geometrical result, it can sometimes help to sketch a diagram as part of your working.

The gradient of a line $= \frac{y_2 - y_1}{x_2 - x_1}$

If the product of two gradients is −1 then the two lines are perpendicular.
Gradient of line AB × gradient of line $BC = -1$

Remember to state what you have proved.

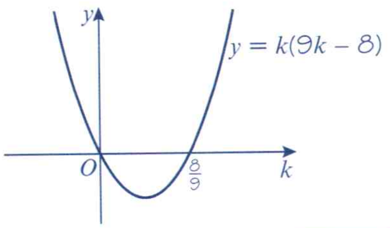
Example 11

The equation $kx^2 + 3kx + 2 = 0$, where k is a constant, has no real roots.

Prove that k satisfies the inequality $0 \leqslant k < \frac{8}{9}$

$kx^2 + 3kx + 2 = 0$ has no real roots, so $b^2 - 4ac < 0$

$(3k)^2 - 4k(2) < 0$
$9k^2 - 8k < 0$
$k(9k - 8) < 0$



State which assumption or information you are using at each stage of your proof.

Use the discriminant. ← Section 2.5

Solve this quadratic inequality by sketching the graph of $y = k(9k - 8)$ ← Section 3.5

The graph shows that when $k(9k - 8) < 0$, $0 < k < \frac{8}{9}$

$0 < k < \frac{8}{9}$

When $k = 0$:

$(0)x^2 + 3(0)x + 2 = 0$

$2 = 0$

Which is impossible, so no real roots

So combining these:

$0 \leqslant k < \frac{8}{9}$ as required

Be really careful to consider all the possible situations. You can't use the discriminant if $k = 0$ so look at this case separately.

Write out all of your conclusions clearly.

$0 < k < \frac{8}{9}$ together with $k = 0$, gives $0 \leqslant k < \frac{8}{9}$

Exercise 7D

Hint The proofs in this exercise are all proofs by deduction.

(P) **1** Prove that $n^2 - n$ is an even number for all values of n.

(P) **2** Prove that $\frac{x}{1+\sqrt{2}} \equiv x\sqrt{2} - x$.

(P) **3** Prove that $(x + \sqrt{y})(x - \sqrt{y}) \equiv x^2 - y$.

(P) **4** Prove that $(2x - 1)(x + 6)(x - 5) \equiv 2x^3 + x^2 - 61x + 30$.

(P) **5** Prove that $x^2 + bx \equiv \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

(P) **6** Prove that the solutions of $x^2 + 2bx + c = 0$ are $x = -b \pm \sqrt{b^2 - c}$.

(P) **7** Prove that $\left(x - \frac{2}{x}\right)^3 \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

(P) **8** Prove that $\left(x^3 - \frac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{-\frac{5}{2}}\right) \equiv x^{\frac{1}{2}}\left(x^4 - \frac{1}{x^4}\right)$

(P) **9** Use completing the square to prove that $3n^2 - 4n + 10$ is positive for all values of n.

(P) **10** Use completing the square to prove that $-n^2 - 2n - 3$ is negative for all values of n.

Problem-solving Any expression that is squared must be $\geqslant 0$.

(P) **11** Prove that $x^2 + 8x + 20 \geqslant 4$ for all values of x. **(3 marks)**

(P) **12** The equation $kx^2 + 5kx + 3 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leqslant k < \frac{12}{25}$ **(4 marks)**

E/P **13** The equation $px^2 - 5x - 6 = 0$, where p is a constant, has two distinct real roots. Prove that p satisfies the inequality $p > -\frac{25}{24}$ **(4 marks)**

P **14** Prove that $A(3, 1)$, $B(1, 2)$ and $C(2, 4)$ are the vertices of a right-angled triangle.

P **15** Prove that quadrilateral $A(1, 1)$, $B(2, 4)$, $C(6, 5)$ and $D(5, 2)$ is a parallelogram.

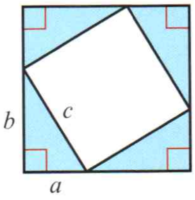
P **16** Prove that quadrilateral $A(2, 1)$, $B(5, 2)$, $C(4, -1)$ and $D(1, -2)$ is a rhombus.

P **17** Prove that $A(-5, 2)$, $B(-3, -4)$ and $C(3, -2)$ are the vertices of an isosceles right-angled triangle.

E/P **18** A circle has equation $(x - 1)^2 + y^2 = k$, where $k > 0$.
The straight line L with equation $y = ax$ cuts the circle at two distinct points.
Prove that $k > \frac{a^2}{1 + a^2}$ **(6 marks)**

E/P **19** Prove that the line $4y - 3x + 26 = 0$ is a tangent to the circle $(x + 4)^2 + (y - 3)^2 = 100$. **(5 marks)**

P **20** The diagram shows a square and four congruent right-angled triangles.
Use the diagram to prove that $a^2 + b^2 = c^2$.



Problem-solving

Find an expression for the area of the large square in terms of a and b.

Challenge

1 Prove that $A(7, 8)$, $B(-1, 8)$, $C(6, 1)$ and $D(0, 9)$ are points on the same circle.

2 Prove that any odd number can be written as the difference of two squares.

7.5 Methods of proof

A mathematical statement can be proved by **exhaustion**. For example, you can prove that the sum of two consecutive square numbers between 100 and 200 is an odd number. The square numbers between 100 and 200 are 121, 144, 169, 196.

$121 + 144 = 265$ which is odd $\quad 144 + 169 = 313$ which is odd $\quad 169 + 196 = 365$ which is odd

So the sum of two consecutive square numbers between 100 and 200 is an odd number.

- **You can prove a mathematical statement is true by exhaustion. This means breaking the statement into smaller cases and proving each case separately.**

This method is better suited to a small number of results. You cannot use one example to prove a statement is true, as one example is only one case.

Example 12

Prove that all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

Problem-solving

Consider the two cases, odd and even numbers, separately.

For odd numbers:

$(2n + 1)^2 = 4n^2 + 4n + 1 = 4n(n + 1) + 1$

$4n(n + 1)$ is a multiple of 4, so $4n(n + 1) + 1$ is 1 more than a multiple of 4.

For even numbers:

$(2n)^2 = 4n^2$

$4n^2$ is a multiple of 4.

All integers are either odd or even, so all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

You can write any odd number in the form $2n + 1$ where n is a positive integer.

You can write any even number in the form $2n$ where n is a positive integer.

A mathematical statement can be disproved using a **counter-example**. For example, to prove that the statement '$3n + 3$ is a multiple of 6 for all values of n' is not true you can use the counter-example when $n = 2$, as $3 \times 2 + 3 = 9$ and 9 is not a multiple of 6.

- **You can prove a mathematical statement is not true by a counter-example. A counter-example is one example that does not work for the statement. You do not need to give more than one example, as one is sufficient to disprove a statement.**

Example 13

Prove that the following statement is **not** true:

'The sum of two consecutive prime numbers is always even.'

2 and 3 are both prime

$2 + 3 = 5$

5 is odd

So the statement is not true.

You only need one counter-example to show that the statement is false.

Example 14

a Prove that for all positive values of x and y:

$$\frac{x}{y} + \frac{y}{x} \geqslant 2$$

b Use a counter-example to show that this is not true when x and y are not both positive.

Watch out You must always start a proof from **known facts**. Never start your proof with the statement you are trying to prove.

a Jottings:

$$\frac{x}{y} + \frac{y}{x} \geqslant 2$$

$$\frac{x^2 + y^2}{xy} \geqslant 2$$

$$x^2 + y^2 - 2xy \geqslant 0$$

$$(x - y)^2 \geqslant 0$$

Problem-solving Use jottings to get some ideas for a good starting point. These don't form part of your proof, but can give you a clue as to what expression you can consider to begin your proof.

Proof:

Consider $(x - y)^2$

*Now you are ready to start your proof. You know that any expression squared is $\geqslant 0$. This is a **known fact** so this is a valid way to begin your proof.*

$$(x - y)^2 \geqslant 0$$

$$x^2 + y^2 - 2xy \geqslant 0$$

$$\frac{x^2 + y^2 - 2xy}{xy} \geqslant 0$$

This step is valid because x and y are both positive so $xy > 0$.

State how you have used the fact that x and y are positive in your proof. If $xy = 0$ you couldn't divide the LHS by xy, and if $xy < 0$, then the direction of the inequality would be reversed.

$$\frac{x}{y} + \frac{y}{x} - 2 \geqslant 0$$

$$\frac{x}{y} + \frac{y}{x} \geqslant 2$$

This was what you wanted to prove so you have finished.

b Try $x = -3$ and $y = 6$

$$\frac{-3}{6} + \frac{6}{-3} = -\frac{1}{2} - 2 = -\frac{5}{2}$$

*Your working for part **a** tells you that the proof fails when $xy < 0$, so try one positive and one negative value.*

This is not $\geqslant 2$ so the statement is not true.

Exercise 7E

(P) **1** Prove that when n is an integer and $1 \leqslant n \leqslant 6$, then $m = n + 2$ is not divisible by 10.

Hint You can try each integer for $1 \leqslant n \leqslant 6$.

(P) **2** Prove that every odd integer between 2 and 26 is either prime or the product of two primes.

(P) **3** Prove that the sum of two consecutive square numbers from 1^2 to 8^2 is an odd number.

(E/P) **4** Prove that all cube numbers are either a multiple of 9 or 1 more or 1 less than a multiple of 9.

(4 marks)

(P) **5** Find a counter-example to disprove each of the following statements:

a If n is a positive integer then $n^4 - n$ is divisible by 4.

b Integers always have an even number of factors.

c $2n^2 - 6n + 1$ is positive for all values of n.

d $2n^2 - 2n - 4$ is a multiple of 3 for all integer values of n.

(P) **6** A student is trying to prove that $x^3 + y^3 < (x + y)^3$.

The student writes:

> $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
> which is more than $x^3 + y^3$ since
> $3x^2y + 3xy^2 > 0$

a Identify the error made in the proof. **(1 mark)**

b Provide a counter-example to show that the statement is not true. **(2 marks)**

Problem-solving

For part **b** you need to write down suitable values of x and y and show that they do not satisfy the inequality.

(P) **7** Prove that for all real values of x

$$(x + 6)^2 \geqslant 2x + 11$$

(3 marks)

(P) **8** Given that a is a positive real number, prove that:

$$a + \frac{1}{a} \geqslant 2$$

(2 marks)

Watch out Remember to state how you use the condition that a is positive.

(P) **9** **a** Prove that for any positive numbers p and q:

$$p + q \geqslant \sqrt{4pq}$$

(3 marks)

b Show, by means of a counter-example, that this inequality does not hold when p and q are both negative. **(2 marks)**

Problem-solving

Use jottings and work backwards to work out what expression to consider.

(P) **10** It is claimed that the following inequality is true for all negative numbers x and y:

$$x + y \geqslant \sqrt{x^2 + y^2}$$

The following proof is offered by a student:

> $x + y \geqslant \sqrt{x^2 + y^2}$
> $(x + y)^2 \geqslant x^2 + y^2$
> $x^2 + y^2 + 2xy \geqslant x^2 + y^2$
> $2xy > 0$ which is true because x and y are both negative, so xy is positive.

a Explain the error made by the student. **(2 marks)**

b By use of a counter-example, verify that the inequality is not satisfied if both x and y are negative. **(1 mark)**

c Prove that this inequality is true if x and y are both positive. **(2 marks)**

Mixed exercise 7

1 Simplify these fractions as far as possible:

a $\dfrac{3x^4 - 21x}{3x}$

b $\dfrac{x^2 - 2x - 24}{x^2 - 7x + 6}$

c $\dfrac{2x^2 + 7x - 4}{2x^2 + 9x + 4}$

2 Divide $3x^3 + 12x^2 + 5x + 20$ by $(x + 4)$.

3 Simplify $\dfrac{2x^3 + 3x + 5}{x + 1}$

E 4 a Show that $(x - 3)$ is a factor of $2x^3 - 2x^2 - 17x + 15$. **(2 marks)**

b Hence express $2x^3 - 2x^2 - 17x + 15$ in the form $(x - 3)(Ax^2 + Bx + C)$, where the values A, B and C are to be found. **(3 marks)**

E 5 a Show that $(x - 2)$ is a factor of $x^3 + 4x^2 - 3x - 18$. **(2 marks)**

b Hence express $x^3 + 4x^2 - 3x - 18$ in the form $(x - 2)(px + q)^2$, where the values p and q are to be found. **(4 marks)**

E 6 Factorise completely $2x^3 + 3x^2 - 18x + 8$. **(6 marks)**

E/P 7 Find the value of k if $(x - 2)$ is a factor of $x^3 - 3x^2 + kx - 10$. **(4 marks)**

E/P 8 $f(x) = 2x^2 + px + q$. Given that $f(-3) = 0$, and $f(4) = 21$:

a find the value of p and q **(6 marks)**

b factorise $f(x)$. **(3 marks)**

E/P 9 $h(x) = x^3 + 4x^2 + rx + s$. Given $h(-1) = 0$, and $h(2) = 30$:

a find the values of r and s **(6 marks)**

b factorise $h(x)$. **(3 marks)**

E 10 $g(x) = 2x^3 + 9x^2 - 6x - 5$.

a Factorise $g(x)$. **(6 marks)**

b Solve $g(x) = 0$. **(2 marks)**

E **11** **a** Show that $(x - 2)$ is a factor of $f(x) = x^3 + x^2 - 5x - 2$. **(2 marks)**

b Hence, or otherwise, find the exact solutions of the equation $f(x) = 0$. **(4 marks)**

E **12** Given that -1 is a root of the equation $2x^3 - 5x^2 - 4x + 3$, find the two positive roots. **(4 marks)**

E **13** $f(x) = x^3 - 2x^2 - 19x + 20$

a Show that $(x + 4)$ is a factor of $f(x)$. **(3 marks)**

b Hence, or otherwise, find all the solutions to the equation $x^3 - 2x^2 - 19x + 20 = 0$. **(4 marks)**

E **14** $f(x) = 6x^3 + 17x^2 - 5x - 6$

a Show that $f(x) = (3x - 2)(ax^2 + bx + c)$, where a, b and c are constants to be found. **(2 marks)**

b Hence factorise $f(x)$ completely. **(4 marks)**

c Write down all the real roots of the equation $f(x) = 0$. **(2 marks)**

15 Prove that $\dfrac{x - y}{\sqrt{x} - \sqrt{y}} \equiv \sqrt{x} + \sqrt{y}$.

16 Use completing the square to prove that $n^2 - 8n + 20$ is positive for all values of n.

17 Prove that the quadrilateral $A(1, 1)$, $B(3, 2)$, $C(4, 0)$ and $D(2, -1)$ is a square.

18 Prove that the sum of two consecutive positive odd numbers less than ten gives an even number.

19 Prove that the statement '$n^2 - n + 3$ is a prime number for all values of n' is untrue.

20 Prove that $\left(x - \dfrac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right) \equiv x^{\frac{1}{3}}\left(x^2 - \dfrac{1}{x^2}\right)$.

21 Prove that $2x^3 + x^2 - 43x - 60 \equiv (x + 4)(x - 5)(2x + 3)$.

22 The equation $x^2 - kx + k = 0$, where k is a positive constant, has two equal roots. Prove that $k = 4$. **(3 marks)**

23 Prove that the distance between opposite edges of a regular hexagon of side length $\sqrt{3}$ is a rational value.

(P) **24** **a** Prove that the difference of the squares of two consecutive even numbers is always divisible by 4.

b Is this statement true for odd numbers? Give a reason for your answer.

(E) **25** A student is trying to prove that $1 + x^2 < (1 + x)^2$.

The student writes:

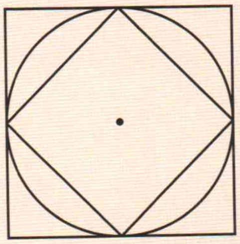
> $(1 + x)^2 = 1 + 2x + x^2$.
> So $1 + x^2 < 1 + 2x + x^2$.

a Identify the error made in the proof. **(1 mark)**

b Provide a counter-example to show that the statement is not true. **(2 marks)**

Challenge

1 The diagram shows two squares and a circle.



a Given that π is defined as the circumference of a circle of diameter 1 unit, prove that $2\sqrt{2} < \pi < 4$.

b By similarly constructing regular hexagons inside and outside a circle, prove that $3 < \pi < 2\sqrt{3}$.

2 Prove that if $f(x) = ax^3 + bx^2 + cx + d$ and $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$.

Summary of key points

1 When simplifying an algebraic fraction, factorise the numerator and denominator where possible and then cancel common factors.

2 You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.

3 The **factor theorem** states that if $f(x)$ is a polynomial then:
- If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$
- If $(x - p)$ is a factor of $f(x)$, then $f(p) = 0$

4 You can prove a mathematical statement is true by **deduction**. This means starting from known facts or definitions, then using logical steps to reach the desired conclusion.

5 In a mathematical proof you must
- State any information or assumptions you are using
- Show every step of your proof clearly
- Make sure that every step follows logically from the previous step
- Make sure you have covered all possible cases
- Write a statement of proof at the end of your working

6 To prove an identity you should
- Start with the expression on one side of the identity
- Manipulate that expression algebraically until it matches the other side
- Show every step of your algebraic working

7 You can prove a mathematical statement is true by **exhaustion**. This means breaking the statement into smaller cases and proving each case separately.

8 You can prove a mathematical statement is not true by a **counter-example**. A counter-example is one example that does not work for the statement. You do not need to give more than one example, as one is sufficient to disprove a statement.