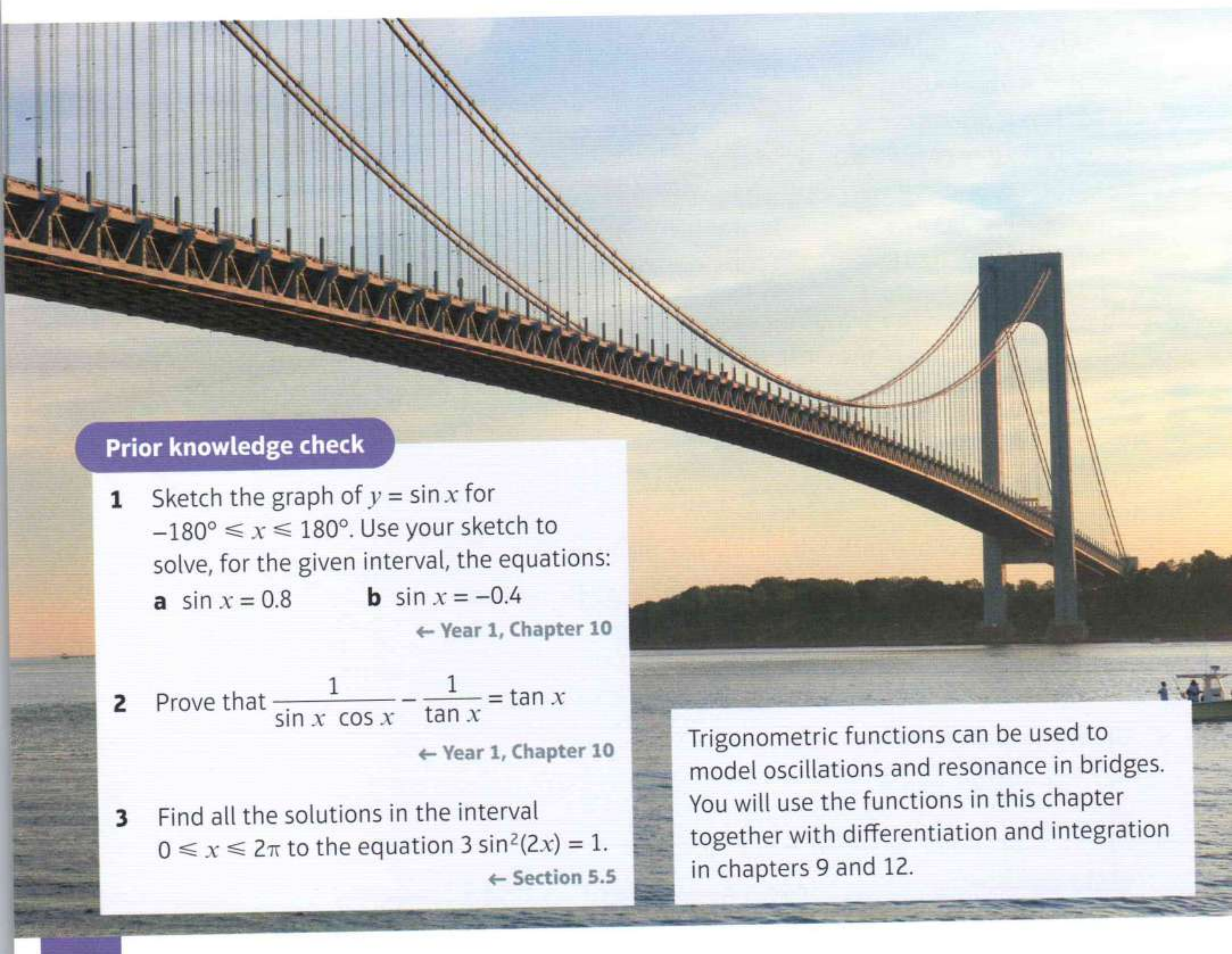
# 6 Trigonometric functions

## Objectives

After completing this chapter you should be able to:

- Understand the definitions of secant, cosecant and cotangent and their relationship to cosine, sine and tangent → pages 143–145
- Understand the graphs of secant, cosecant and cotangent and their domain and range → pages 145–149
- Simplify expressions, prove simple identities and solve equations involving secant, cosecant and cotangent → pages 149–153
- Prove and use $\sec^2 x \equiv 1 + \tan^2 x$ and $\operatorname{cosec}^2 x \equiv 1 + \cot^2 x$ → pages 153–157
- Understand and use inverse trigonometric functions and their domain and ranges. → pages 158–161



## Prior knowledge check

**1** Sketch the graph of $y = \sin x$ for $-180° \leqslant x \leqslant 180°$. Use your sketch to solve, for the given interval, the equations:

**a** $\sin x = 0.8$  **b** $\sin x = -0.4$

← Year 1, Chapter 10

**2** Prove that $\dfrac{1}{\sin x \ \cos x} - \dfrac{1}{\tan x} = \tan x$

← Year 1, Chapter 10

**3** Find all the solutions in the interval $0 \leqslant x \leqslant 2\pi$ to the equation $3\sin^2(2x) = 1$.

← Section 5.5

Trigonometric functions can be used to model oscillations and resonance in bridges. You will use the functions in this chapter together with differentiation and integration in chapters 9 and 12.

## 6.1 Secant, cosecant and cotangent

Secant (sec), cosecant (cosec) and cotangent (cot) are known as the **reciprocal** trigonometric functions.

- **$\sec x = \dfrac{1}{\cos x}$ (undefined for values of $x$ for which $\cos x = 0$)**
- **$\text{cosec}\, x = \dfrac{1}{\sin x}$ (undefined for values of $x$ for which $\sin x = 0$)**
- **$\cot x = \dfrac{1}{\tan x}$ (undefined for values of $x$ for which $\tan x = 0$)**

You can also write $\cot x$ in terms of $\sin x$ and $\cos x$.

- **$\cot x = \dfrac{\cos x}{\sin x}$**

### Example 1

Use your calculator to write down the values of:

**a** sec 280° **b** cot 115°

**a** $\sec 280° = \dfrac{1}{\cos 280°} = 5.76$ (3 s.f.)

**b** $\cot 115° = \dfrac{1}{\tan 115°} = -0.466$ (3 s.f.)

Make sure your calculator is in degrees mode.

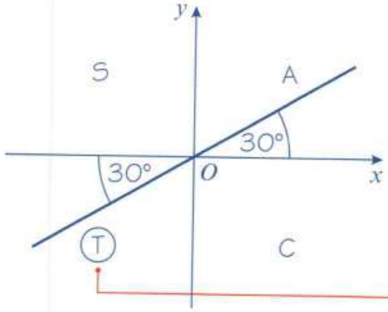
### Example 2

Work out the exact values of:

**a** sec 210° **b** $\text{cosec}\, \dfrac{3\pi}{4}$

*Exact* here means give in surd form.

**a** $\sec 210° = \dfrac{1}{\cos 210°}$




210° is in 3rd quadrant, so cos 210° = −cos 30°.

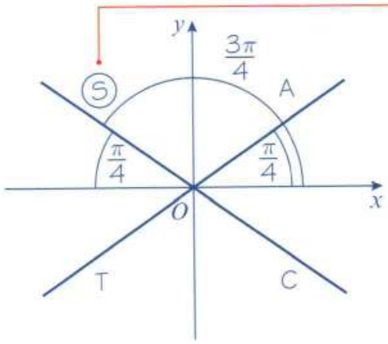
$\cos 30° = \dfrac{\sqrt{3}}{2}$ so $-\cos 30° = -\dfrac{\sqrt{3}}{2}$

So $\sec 210° = -\dfrac{2}{\sqrt{3}}$

Or $\sec 210° = -\dfrac{2\sqrt{3}}{3}$ if you rationalise the denominator.

**b** $\text{cosec}\,\frac{3\pi}{4} = \frac{1}{\sin\left(\frac{3\pi}{4}\right)}$

$\frac{3\pi}{4}$ is in the 2nd quadrant, so $\sin\frac{3\pi}{4} = +\sin\frac{\pi}{4}$




So $\text{cosec}\,\frac{3\pi}{4} = \frac{1}{\sin\left(\frac{\pi}{4}\right)}$

$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

So $\text{cosec}\left(\frac{3\pi}{4}\right) = \sqrt{2}$

## Exercise 6A

**1** Without using your calculator, write down the sign of the following trigonometric ratios.

**a** sec 300° **b** cosec 190° **c** cot 110°
**d** cot 200° **e** sec 95°

**2** Use your calculator to find, to 3 significant figures, the values of:

**a** sec 100° **b** cosec 260° **c** cosec 280°
**d** cot 550° **e** $\cot\frac{4\pi}{3}$ **f** sec 2.4 rad
**g** $\text{cosec}\,\frac{11\pi}{10}$ **h** sec 6 rad

**3** Find the exact values (in surd form where appropriate) of the following:

**a** cosec 90° **b** cot 135° **c** sec 180°
**d** sec 240° **e** cosec 300° **f** cot(−45°)
**g** sec 60° **h** cosec (−210°) **i** sec 225°
**j** $\cot\frac{4\pi}{3}$ **k** $\sec\frac{11\pi}{6}$ **l** $\text{cosec}\left(-\frac{3\pi}{4}\right)$

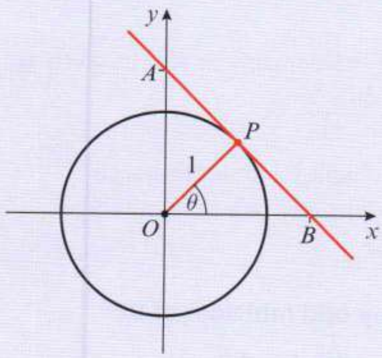
(P) **4** Prove that $\text{cosec}(\pi - x) \equiv \text{cosec}\,x$.

(P) **5** Show that cot 30° sec 30° = 2.

(P) **6** Show that $\text{cosec}\frac{2\pi}{3} + \sec\frac{2\pi}{3} = a + b\sqrt{3}$ where $a$ and $b$ are real numbers to be found.

**Challenge**

The point $P$ lies on the unit circle, centre $O$. The radius $OP$ makes an acute angle of $\theta$ with the positive $x$-axis. The tangent to the circle at $P$ intersects the coordinate axes at points $A$ and $B$.




Prove that

**a** $OB = \sec\theta$

**b** $OA = \text{cosec}\,\theta$

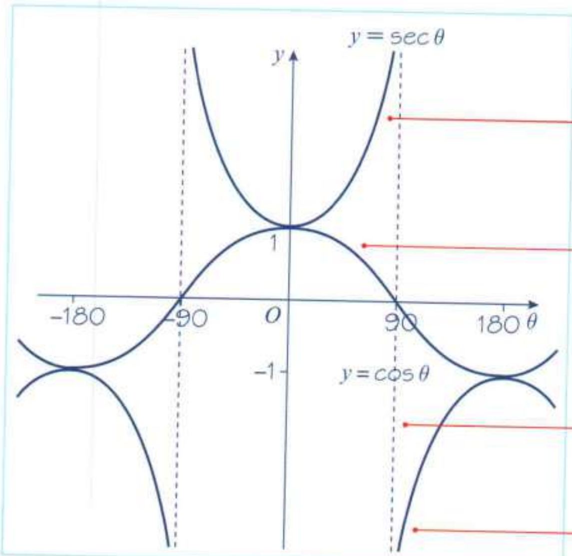
**c** $AP = \cot\theta$

## 6.2 Graphs of sec $x$, cosec $x$ and cot $x$

You can use the graphs of $y = \cos x$, $y = \sin x$ and $y = \tan x$ to sketch the graphs of their reciprocal functions.

**Example 3**

Sketch, in the interval $-180° \leqslant \theta \leqslant 180°$, the graph of $y = \sec\theta$.



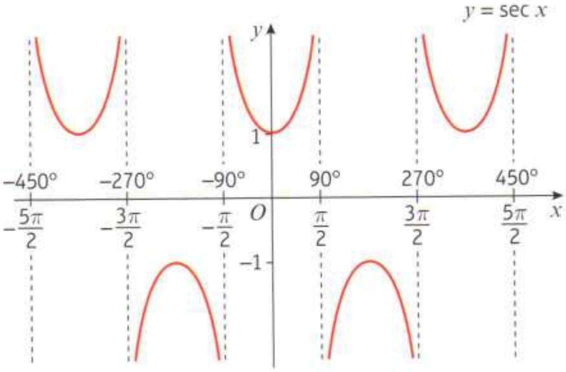

First draw the graph $y = \cos\theta$.
For each value of $\theta$, the value of $\sec\theta$ is the reciprocal of the corresponding value of $\cos\theta$.
In particular: $\cos 0° = 1$, so $\sec 0° = 1$;
and $\cos 180° = -1$, so $\sec 180° = -1$.

As $\theta$ approaches 90° from the left, $\cos\theta$ is +ve but approaches zero, and so $\sec\theta$ is +ve but becomes increasingly large.

At $\theta = 90°$, $\sec\theta$ is undefined and there is a vertical asymptote. This is also true for $\theta = -90°$.

As $\theta$ approaches 90° from the right, $\cos\theta$ is −ve but approaches zero, and so $\sec\theta$ is −ve but becomes increasingly large negative.

- **The graph of $y = \sec x$, $x \in \mathbb{R}$, has symmetry in the $y$-axis and has period 360° or $2\pi$ radians. It has vertical asymptotes at all the values of $x$ for which $\cos x = 0$.**

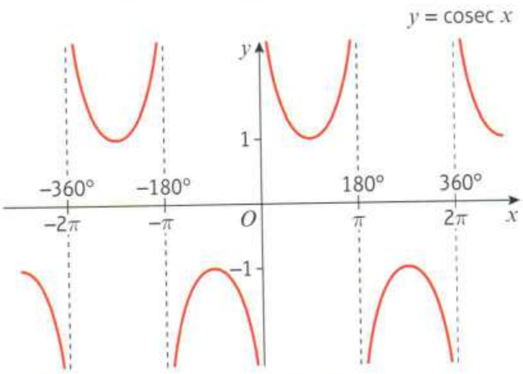



**Notation** The domain can also be given as $x \in \mathbb{R}, x \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$

$\mathbb{Z}$ is the symbol used for **integers**, i.e. positive and negative whole numbers including 0.

- **The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90°, 270°, 450°, \ldots$ or any odd multiple of 90°**
- **In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ or any odd multiple of $\frac{\pi}{2}$**
- **The range of $y = \sec x$ is $y \leqslant -1$ or $y \geqslant 1$**

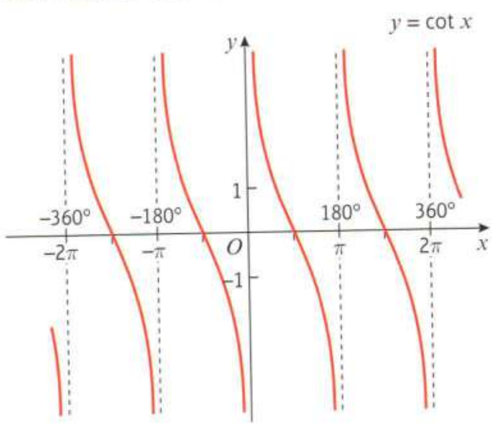
- **The graph of $y = \operatorname{cosec} x$, $x \in \mathbb{R}$, has period 360° or $2\pi$ radians. It has vertical asymptotes at all the values of $x$ for which $\sin x = 0$.**




**Notation** The domain can also be given as $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$.

- **The domain of $y = \operatorname{cosec} x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°**
- **In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of $\pi$**
- **The range of $y = \operatorname{cosec} x$ is $y \leqslant -1$ or $y \geqslant 1$**

- **The graph of $y = \cot x$, $x \in \mathbb{R}$, has period 180° or $\pi$ radians. It has vertical asymptotes at all the values of $x$ for which $\tan x = 0$.**



- **The domain of $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0°$, 180°, 360°,... or any multiple of 180°**
- **In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of $\pi$**
- **The range of $y = \cot x$ is $y \in \mathbb{R}$**

**Notation** The domain can also be given as $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$.

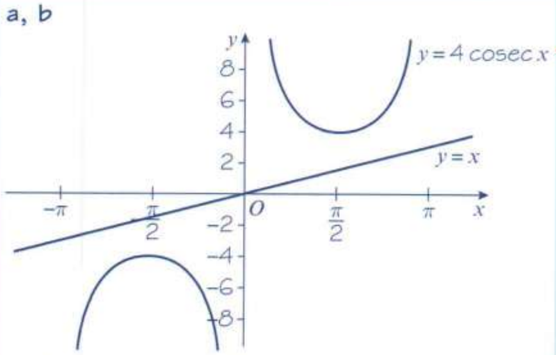
## Example 4

**a** Sketch the graph of $y = 4 \operatorname{cosec} x$, $-\pi \leqslant x \leqslant \pi$.

**b** On the same axes, sketch the line $y = x$.

**c** State the number of solutions to the equation $4 \operatorname{cosec} x - x = 0$, $-\pi \leqslant x \leqslant \pi$.

a, b




$y = 4 \operatorname{cosec} x$ is a stretch of the graph of $y = \operatorname{cosec} x$, scale factor 4 in the $y$-direction. You only need to draw the graph for $-\pi \leqslant x \leqslant \pi$.

**c** $4 \operatorname{cosec} x - x = 0$

$4 \operatorname{cosec} x = x$

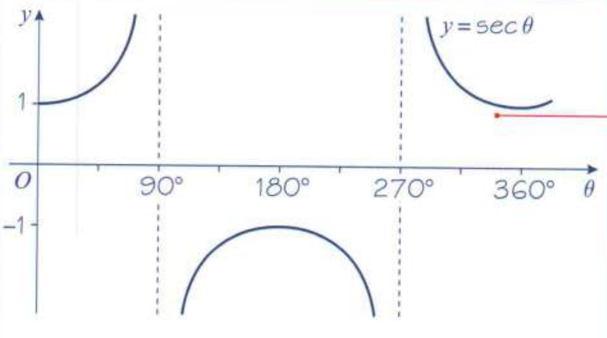
$y = 4 \operatorname{cosec} x$ and $y = x$ do not intersect for $-\pi \leqslant x \leqslant \pi$ so the equation has no solutions in the given range.

**Problem-solving**

The solutions to the equation $f(x) = g(x)$ correspond to the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$.

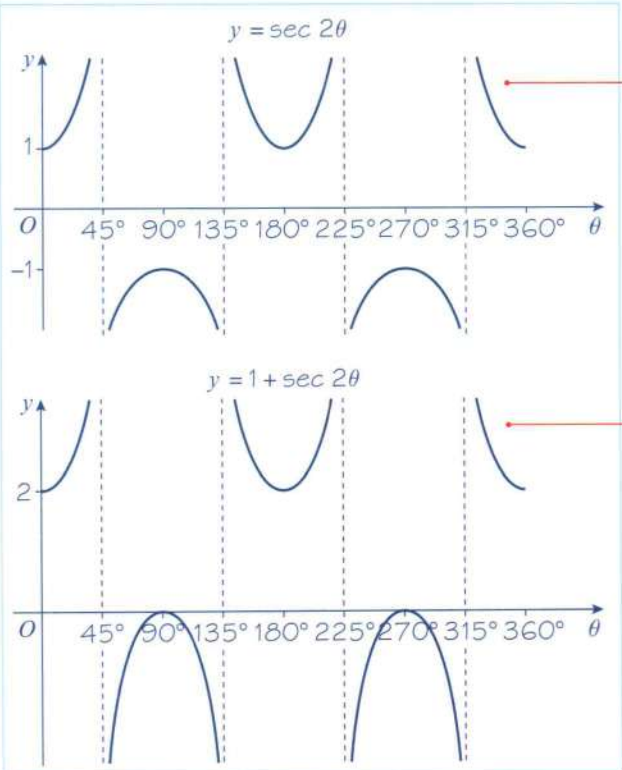
## Example 5

Sketch, in the interval $0 \leqslant \theta \leqslant 360°$, the graph of $y = 1 + \sec 2\theta$.




**Online** Explore transformations of the graphs of reciprocal trigonometric functions using technology.

**Step 1**
Draw the graph of $y = \sec \theta$.




**Step 2**
Stretch in the $\theta$-direction with scale factor $\frac{1}{2}$

**Step 3**
Translate by the vector $\begin{pmatrix}0\\1\end{pmatrix}$.

## Exercise 6B

**1** Sketch, in the interval $-540° \leqslant \theta \leqslant 540°$, the graphs of:

**a** $y = \sec\theta$ **b** $y = \operatorname{cosec}\theta$ **c** $y = \cot\theta$

**2 a** Sketch, on the same set of axes, in the interval $-\pi \leqslant x \leqslant \pi$, the graphs of $y = \cot x$ and $y = -x$.

**b** Deduce the number of solutions of the equation $\cot x + x = 0$ in the interval $-\pi \leqslant x \leqslant \pi$.

**3 a** Sketch, on the same set of axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of $y = \sec\theta$ and $y = -\cos\theta$.

**b** Explain how your graphs show that $\sec\theta = -\cos\theta$ has no solutions.

**4 a** Sketch, on the same set of axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of $y = \cot\theta$ and $y = \sin 2\theta$.

**b** Deduce the number of solutions of the equation $\cot\theta = \sin 2\theta$ in the interval $0 \leqslant \theta \leqslant 360°$.

**5 a** Sketch on separate axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of $y = \tan\theta$ and $y = \cot(\theta + 90°)$.

**b** Hence, state a relationship between $\tan\theta$ and $\cot(\theta + 90°)$.

(P) **6** **a** Describe the relationships between the graphs of:

**i** $y = \tan\left(\theta + \frac{\pi}{2}\right)$ and $y = \tan\theta$ **ii** $y = \cot(-\theta)$ and $y = \cot\theta$

**iii** $y = \text{cosec}\left(\theta + \frac{\pi}{4}\right)$ and $y = \text{cosec}\,\theta$ **iv** $y = \sec\left(\theta - \frac{\pi}{4}\right)$ and $y = \sec\theta$

**b** By considering the graphs of $y = \tan\left(\theta + \frac{\pi}{2}\right)$, $y = \cot(-\theta)$, $y = \text{cosec}\left(\theta + \frac{\pi}{4}\right)$ and $y = \sec\left(\theta - \frac{\pi}{4}\right)$, state which pairs of functions are equal.

(P) **7** Sketch on separate axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of:

**a** $y = \sec 2\theta$ **b** $y = -\text{cosec}\,\theta$ **c** $y = 1 + \sec\theta$

**d** $y = \text{cosec}(\theta - 30°)$ **e** $y = 2\sec(\theta - 60°)$ **f** $y = \text{cosec}(2\theta + 60°)$

**g** $y = -\cot(2\theta)$ **h** $y = 1 - 2\sec\theta$

In each case show the coordinates of any maximum and minimum points, and of any points at which the curve meets the axes.

**8** Write down the periods of the following functions. Give your answers in terms of $\pi$.

**a** $\sec 3\theta$ **b** $\text{cosec}\,\frac{1}{2}\theta$ **c** $2\cot\theta$ **d** $\sec(-\theta)$

(E/P) **9** **a** Sketch, in the interval $-2\pi \leqslant x \leqslant 2\pi$, the graph of $y = 3 + 5\,\text{cosec}\,x$. **(3 marks)**

**b** Hence deduce the range of values of $k$ for which the equation $3 + 5\,\text{cosec}\,x = k$ has no solutions. **(2 marks)**

(E/P) **10** **a** Sketch the graph of $y = 1 + 2\sec\theta$ in the interval $-\pi \leqslant \theta \leqslant 2\pi$. **(3 marks)**

**b** Write down the $\theta$-coordinates of points at which the gradient is zero. **(2 marks)**

**c** Deduce the maximum and minimum values of $\frac{1}{1 + 2\sec\theta}$, and give the smallest positive values of $\theta$ at which they occur. **(4 marks)**

## 6.3 Using sec $x$, cosec $x$ and cot $x$

You need to be able to simplify expressions, prove identities and solve equations involving sec $x$, cosec $x$ and cot $x$.

- **sec $x = k$ and cosec $x = k$ have no solutions for $-1 < k < 1$.**

**Example 6**

Simplify:

**a** $\sin\theta\cot\theta\sec\theta$

**b** $\sin\theta\cos\theta(\sec\theta + \text{cosec}\,\theta)$

**a** $\sin\theta \cot\theta \sec\theta$

$\equiv \sin\theta \times \frac{\cos\theta}{\sin\theta} \times \frac{1}{\cos\theta}$

$\equiv 1$

Write the expression in terms of sin and cos, using $\cot\theta \equiv \frac{\cos\theta}{\sin\theta}$ and $\sec\theta \equiv \frac{1}{\cos\theta}$

**b** $\sec\theta + \operatorname{cosec}\theta \equiv \frac{1}{\cos\theta} + \frac{1}{\sin\theta}$

Write the expression in terms of sin and cos, using $\sec\theta \equiv \frac{1}{\cos\theta}$ and $\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta}$

$\equiv \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}$

Put over common denominator.

So $\sin\theta\cos\theta(\sec\theta + \operatorname{cosec}\theta) = \sin\theta + \cos\theta$

Multiply both sides by $\sin\theta\cos\theta$.

## Example 7

**a** Prove that $\frac{\cot\theta \operatorname{cosec}\theta}{\sec^2\theta + \operatorname{cosec}^2\theta} \equiv \cos^3\theta$.

**b** Hence explain why the equation $\frac{\cot\theta \operatorname{cosec}\theta}{\sec^2\theta + \operatorname{cosec}^2\theta} = 8$ has no solutions.

**a** Consider LHS:

The numerator $\cot\theta \operatorname{cosec}\theta$

$\equiv \frac{\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta} \equiv \frac{\cos\theta}{\sin^2\theta}$

Write the expression in terms of sin and cos, using $\cot\theta \equiv \frac{\cos\theta}{\sin\theta}$ and $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

The denominator $\sec^2\theta + \operatorname{cosec}^2\theta$

$\equiv \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$

Write the expression in terms of sin and cos, using $\sec^2\theta \equiv \left(\frac{1}{\cos\theta}\right)^2 \equiv \frac{1}{\cos^2\theta}$ and $\operatorname{cosec}^2\theta \equiv \frac{1}{\sin^2\theta}$

$\equiv \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$

Remember that $\sin^2\theta + \cos^2\theta \equiv 1$.

$\equiv \frac{1}{\cos^2\theta\sin^2\theta}$

So $\frac{\cot\theta \operatorname{cosec}\theta}{\sec^2\theta + \operatorname{cosec}^2\theta}$

$\equiv \left(\frac{\cos\theta}{\sin^2\theta}\right) \div \left(\frac{1}{\cos^2\theta\sin^2\theta}\right)$

$\equiv \frac{\cos\theta}{\sin^2\theta} \times \frac{\cos^2\theta\sin^2\theta}{1}$

Remember to invert the fraction when changing from ÷ sign to ×.

$\equiv \cos^3\theta$

**b** Since $\frac{\cot\theta \operatorname{cosec}\theta}{\sec^2\theta + \operatorname{cosec}^2\theta} \equiv \cos^3\theta$ we are required to solve the equation $\cos^3\theta = 8$.

$\cos^3\theta = 8 \Rightarrow \cos\theta = 2$ which has no solutions since $-1 \leqslant \cos\theta \leqslant 1$.

**Problem-solving**

Write down the equivalent equation, and state the range of possible values for $\cos\theta$.

**Example 8**

Solve the equations

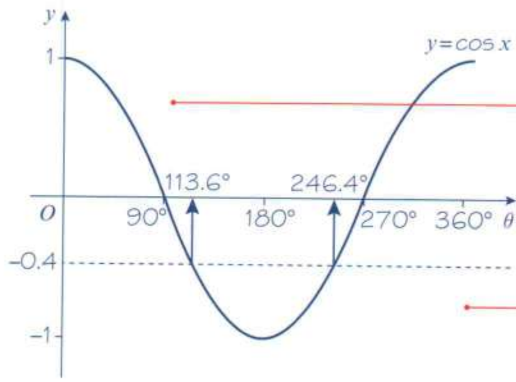
**a** $\sec\theta = -2.5$ **b** $\cot 2\theta = 0.6$

in the interval $0 \leqslant \theta \leqslant 360°$.

**a** $\frac{1}{\cos\theta} = -2.5$

Substitute $\frac{1}{\cos\theta}$ for $\sec\theta$ and then simplify to get an equation in the form $\cos\theta = k$.

$\cos\theta = \frac{1}{-2.5} = -0.4$




Sketch the graph of $y = \cos x$ for the given interval. The graph is symmetrical about $\theta = 180°$. Find the principal value using your calculator then subtract this from 360° to find the second solution.

You could also find all the solutions using a CAST diagram. This method is shown for part **b** below.

$\theta = 113.6°, 246.4° = 114°, 246°$ (3 s.f.)

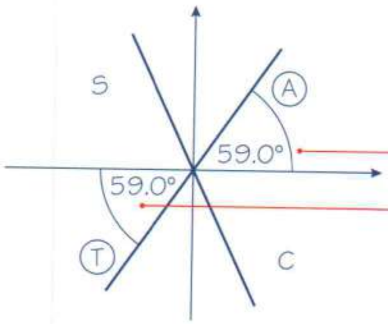
Calculate angles from the diagram.

**b** $\frac{1}{\tan 2\theta} = 0.6$

Substitute $\frac{1}{\tan 2\theta}$ for $\cot 2\theta$ and then simplify to get an equation in the form $\tan 2\theta = k$.

$\tan 2\theta = \frac{1}{0.6} = \frac{5}{3}$

Let $X = 2\theta$, so that you are solving $\tan X = \frac{5}{3}$, in the interval $0 \leqslant X \leqslant 720°$.




Draw the CAST diagram, with the acute angle $X = \tan^{-1}\frac{5}{3}$ drawn to the horizontal in the 1st and 3rd quadrants.

$X = 59.0°, 239.0°, 419.0°, 599.0°$

So $\theta = 29.5°, 120°, 210°, 300°$ (3 s.f.)

Remember that $X = 2\theta$.

## Exercise 6C

1 Rewrite the following as powers of $\sec\theta$, $\operatorname{cosec}\theta$ or $\cot\theta$.

a $\dfrac{1}{\sin^3\theta}$ b $\dfrac{4}{\tan^6\theta}$ c $\dfrac{1}{2\cos^2\theta}$ d $\dfrac{1-\sin^2\theta}{\sin^2\theta}$

e $\dfrac{\sec\theta}{\cos^4\theta}$ f $\sqrt{\operatorname{cosec}^3\theta\cot\theta\sec\theta}$ g $\dfrac{2}{\sqrt{\tan\theta}}$ h $\dfrac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta}$

2 Write down the value(s) of $\cot x$ in each of the following equations.

a $5\sin x = 4\cos x$ b $\tan x = -2$ c $3\dfrac{\sin x}{\cos x} = \dfrac{\cos x}{\sin x}$

3 Using the definitions of **sec**, **cosec**, **cot** and **tan** simplify the following expressions.

a $\sin\theta\cot\theta$ b $\tan\theta\cot\theta$

c $\tan 2\theta\operatorname{cosec} 2\theta$ d $\cos\theta\sin\theta(\cot\theta+\tan\theta)$

e $\sin^3 x\operatorname{cosec} x + \cos^3 x\sec x$ f $\sec A - \sec A\sin^2 A$

g $\sec^2 x\cos^5 x + \cot x\operatorname{cosec} x\sin^4 x$

(P) 4 Prove that:

a $\cos\theta + \sin\theta\tan\theta \equiv \sec\theta$ b $\cot\theta + \tan\theta \equiv \operatorname{cosec}\theta\sec\theta$

c $\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$ d $(1-\cos x)(1+\sec x) \equiv \sin x\tan x$

e $\dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x} \equiv 2\sec x$ f $\dfrac{\cos\theta}{1+\cot\theta} \equiv \dfrac{\sin\theta}{1+\tan\theta}$

(P) 5 Solve, for values of $\theta$ in the interval $0 \leqslant \theta \leqslant 360°$, the following equations. Give your answers to 3 significant figures where necessary.

a $\sec\theta = \sqrt{2}$ b $\operatorname{cosec}\theta = -3$ c $5\cot\theta = -2$ d $\operatorname{cosec}\theta = 2$

e $3\sec^2\theta - 4 = 0$ f $5\cos\theta = 3\cot\theta$ g $\cot^2\theta - 8\tan\theta = 0$ h $2\sin\theta = \operatorname{cosec}\theta$

(P) 6 Solve, for values of $\theta$ in the interval $-180° \leqslant \theta \leqslant 180°$, the following equations:

a $\operatorname{cosec}\theta = 1$ b $\sec\theta = -3$ c $\cot\theta = 3.45$

d $2\operatorname{cosec}^2\theta - 3\operatorname{cosec}\theta = 0$ e $\sec\theta = 2\cos\theta$ f $3\cot\theta = 2\sin\theta$

g $\operatorname{cosec} 2\theta = 4$ h $2\cot^2\theta - \cot\theta - 5 = 0$

(P) 7 Solve the following equations for values of $\theta$ in the interval $0 \leqslant \theta \leqslant 2\pi$. Give your answers in terms of $\pi$.

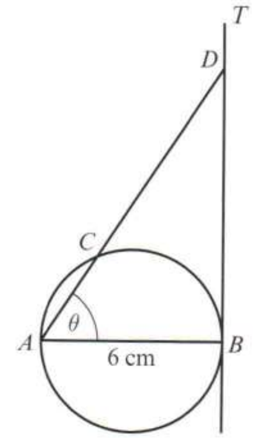
a $\sec\theta = -1$ b $\cot\theta = -\sqrt{3}$

c $\operatorname{cosec}\frac{1}{2}\theta = \dfrac{2\sqrt{3}}{3}$ d $\sec\theta = \sqrt{2}\tan\theta\left(\theta \neq \dfrac{\pi}{2}, \theta \neq \dfrac{3\pi}{2}\right)$

E/P **8** In the diagram $AB = 6$ cm is the diameter of the circle and $BT$ is the tangent to the circle at $B$. The chord $AC$ is extended to meet this tangent at $D$ and $\angle DAB = \theta$.

**a** Show that $CD = 6(\sec\theta - \cos\theta)$ cm. **(4 marks)**

**b** Given that $CD = 16$ cm, calculate the length of the chord $AC$. **(3 marks)**




**Problem-solving**

$AB$ is the diameter of the circle, so $\angle ACB = 90°$.

E/P **9** **a** Prove that $\dfrac{\operatorname{cosec} x - \cot x}{1 - \cos x} \equiv \operatorname{cosec} x$. **(4 marks)**

**b** Hence solve, in the interval $-\pi \leqslant x \leqslant \pi$, the equation $\dfrac{\operatorname{cosec} x - \cot x}{1 - \cos x} = 2$. **(3 marks)**

E/P **10** **a** Prove that $\dfrac{\sin x \tan x}{1 - \cos x} - 1 \equiv \sec x$. **(4 marks)**

**b** Hence explain why the equation $\dfrac{\sin x \tan x}{1 - \cos x} - 1 = -\dfrac{1}{2}$ has no solutions. **(1 mark)**

E/P **11** Solve, in the interval $0 \leqslant x \leqslant 360°$, the equation $\dfrac{1 + \cot x}{1 + \tan x} = 5$. **(8 marks)**

**Problem-solving**

Use the relationship $\cot x = \dfrac{1}{\tan x}$ to form a quadratic equation in $\tan x$. ← Year 1, Section 10.5

## 6.4 Trigonometric identities

You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities.

- $\mathbf{1 + \tan^2 x \equiv \sec^2 x}$
- $\mathbf{1 + \cot^2 x \equiv \operatorname{cosec}^2 x}$

**Link** You can use the unit circle definitions of sin and cos to prove the identity $\sin^2 x + \cos^2 x \equiv 1$. ← Year 1, Section 10.5

**Example 9**

**a** Prove that $1 + \tan^2 x \equiv \sec^2 x$.

**b** Prove that $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$.

**a** $\sin^2 x + \cos^2 x \equiv 1$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \equiv \frac{1}{\cos^2 x}$$

$$\left(\frac{\sin x}{\cos x}\right)^2 + 1 \equiv \left(\frac{1}{\cos x}\right)^2$$

so $1 + \tan^2 x \equiv \sec^2 x$

**b** $\sin^2 x + \cos^2 x \equiv 1$

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \equiv \frac{1}{\sin^2 x}$$

$$1 + \left(\frac{\cos x}{\sin x}\right)^2 \equiv \left(\frac{1}{\sin x}\right)^2$$

so $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

Unless otherwise stated, you can assume the identity $\sin^2 x + \cos^2 x \equiv 1$ in proofs involving cosec, sec and cot in your exam.

Divide both sides of the identity by $\cos^2 x$.

Use $\tan x \equiv \frac{\sin x}{\cos x}$ and $\sec x \equiv \frac{1}{\cos x}$

Divide both sides of the identity by $\sin^2 x$.

Use $\cot x \equiv \frac{\cos x}{\sin x}$ and $\operatorname{cosec} x \equiv \frac{1}{\sin x}$

## Example 10

Given that $\tan A = -\frac{5}{12}$, and that angle $A$ is obtuse, find the exact values of:

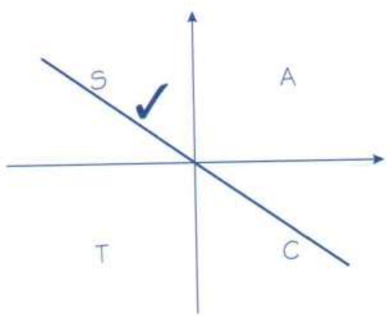
**a** $\sec A$ **b** $\sin A$

**a** Using $1 + \tan^2 A \equiv \sec^2 A$

$$\sec^2 A = 1 + \frac{25}{144} = \frac{169}{144}$$

$$\sec A = \pm\frac{13}{12}$$

$\tan^2 A = \frac{25}{144}$




$$\sec A = -\frac{13}{12}$$

**b** Using $\tan A \equiv \frac{\sin A}{\cos A}$

$$\sin A \equiv \tan A \cos A$$

So $\sin A = \left(-\frac{5}{12}\right) \times \left(-\frac{12}{13}\right)$

$$= \frac{5}{13}$$

**Problem-solving**

You are told that $A$ is obtuse. This means it lies in the second quadrant, so $\cos A$ is negative, and $\sec A$ is also negative.

$\cos A = -\frac{12}{13}$, since $\cos A = \frac{1}{\sec A}$

## Example 11

Prove the identities:

**a** $\operatorname{cosec}^4\theta - \cot^4\theta \equiv \dfrac{1 + \cos^2\theta}{1 - \cos^2\theta}$

**b** $\sec^2\theta - \cos^2\theta \equiv \sin^2\theta(1 + \sec^2\theta)$

**a** LHS $= \operatorname{cosec}^4\theta - \cot^4\theta$

This is the difference of two squares, so factorise.

$\equiv (\operatorname{cosec}^2\theta + \cot^2\theta)(\operatorname{cosec}^2\theta - \cot^2\theta)$

As $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$, so $\operatorname{cosec}^2\theta - \cot^2\theta \equiv 1$.

$\equiv \operatorname{cosec}^2\theta + \cot^2\theta$

$\equiv \dfrac{1}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta}$

Using $\operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$, $\cot\theta \equiv \dfrac{\cos\theta}{\sin\theta}$

$\equiv \dfrac{1 + \cos^2\theta}{\sin^2\theta}$

Using $\sin^2\theta + \cos^2\theta \equiv 1$.

$\equiv \dfrac{1 + \cos^2\theta}{1 - \cos^2\theta} =$ RHS

**b** RHS $= \sin^2\theta + \sin^2\theta\sec^2\theta$

Write in terms of $\sin\theta$ and $\cos\theta$.

Use $\sec\theta \equiv \dfrac{1}{\cos\theta}$.

$\equiv \sin^2\theta + \dfrac{\sin^2\theta}{\cos^2\theta}$

$\equiv \sin^2\theta + \tan^2\theta$

$\dfrac{\sin^2\theta}{\cos^2\theta} \equiv \left(\dfrac{\sin\theta}{\cos\theta}\right)^2 \equiv \tan^2\theta$.

$\equiv (1 - \cos^2\theta) + (\sec^2\theta - 1)$

Look at LHS. It is in terms of $\cos^2\theta$ and $\sec^2\theta$, so use $\sin^2\theta + \cos^2\theta \equiv 1$ and $1 + \tan^2\theta \equiv \sec^2\theta$.

$\equiv \sec^2\theta - \cos^2\theta$

$\equiv$ LHS

**Problem-solving**

You can start from either the LHS or the RHS when proving an identity. Try starting with the LHS using $\cos^2\theta \equiv 1 - \sin^2\theta$ and $\sec^2\theta \equiv 1 + \tan^2\theta$.

## Example 12

Solve the equation $4\operatorname{cosec}^2\theta - 9 = \cot\theta$, in the interval $0 \leqslant \theta \leqslant 360°$.

This is a quadratic equation. You need to write it in terms of one trigonometrical function only, so use $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

The equation can be rewritten as

$$4(1 + \cot^2\theta) - 9 = \cot\theta$$

So $\quad 4\cot^2\theta - \cot\theta - 5 = 0$

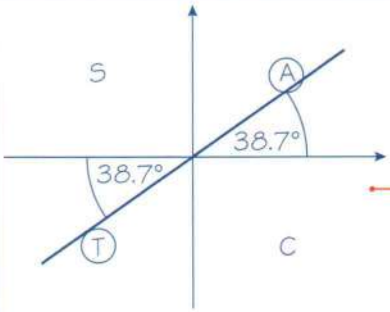
$(4\cot\theta - 5)(\cot\theta + 1) = 0$

Factorise, or solve using the quadratic formula.

So $\quad \cot\theta = \frac{5}{4}$ or $\cot\theta = -1$

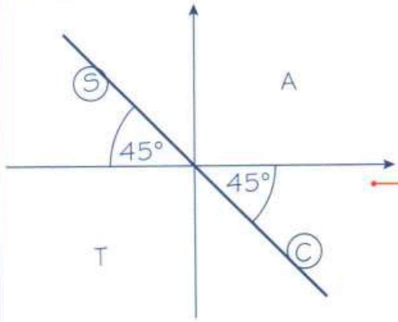
$\therefore \quad \tan\theta = \frac{4}{5}$ or $\tan\theta = -1$

For $\tan\theta = \frac{4}{5}$




$\theta = 38.7°, 219°$ (3 s.f.)

For $\tan\theta = -1$




$\theta = 135°, 315°$

As $\tan\theta$ is +ve, $\theta$ is in the 1st and 3rd quadrants. The acute angle to the horizontal is $\tan^{-1}\frac{4}{5} = 38.7°$.

If $\alpha$ is the value the calculator gives for $\tan^{-1}\frac{4}{5}$, then the solutions are $\alpha$ and $(180° + \alpha)$.

As $\tan\theta$ is −ve, $\theta$ is in the 2nd and 4th quadrants. The acute angle to the horizontal is $\tan^{-1} 1 = 45°$.

If $\alpha$ is the value the calculator gives for $\tan^{-1}(-1)$, then the solutions are $(180° + \alpha)$ and $(360° + \alpha)$, as $\alpha$ is not in the given interval.

**Online** Solve this equation numerically using your calculator.

## Exercise 6D

**Give answers to 3 significant figures where necessary.**

**1** Simplify each of the following expressions.

**a** $1 + \tan^2\frac{1}{2}\theta$ **b** $(\sec\theta - 1)(\sec\theta + 1)$ **c** $\tan^2\theta(\text{cosec}^2\theta - 1)$

**d** $(\sec^2\theta - 1)\cot\theta$ **e** $(\text{cosec}^2\theta - \cot^2\theta)^2$ **f** $2 - \tan^2\theta + \sec^2\theta$

**g** $\dfrac{\tan\theta\sec\theta}{1 + \tan^2\theta}$ **h** $(1 - \sin^2\theta)(1 + \tan^2\theta)$ **i** $\dfrac{\text{cosec}\,\theta\cot\theta}{1 + \cot^2\theta}$

**j** $(\sec^4\theta - 2\sec^2\theta\tan^2\theta + \tan^4\theta)$ **k** $4\,\text{cosec}^2 2\theta + 4\,\text{cosec}^2 2\theta\cot^2 2\theta$

(P) **2** Given that $\text{cosec}\,x = \dfrac{k}{\text{cosec}\,x}$, where $k > 1$, find, in terms of $k$, possible values of $\cot x$.

**3** Given that $\cot\theta = -\sqrt{3}$, and that $90° < \theta < 180°$, find the exact values of:

**a** $\sin\theta$ **b** $\cos\theta$

**4** Given that $\tan\theta = \frac{3}{4}$, and that $180° < \theta < 270°$, find the exact values of:

**a** $\sec\theta$ **b** $\cos\theta$ **c** $\sin\theta$

**5** Given that $\cos\theta = \frac{24}{25}$, and that $\theta$ is a reflex angle, find the exact values of:

**a** $\tan\theta$ **b** $\text{cosec}\,\theta$

(P) **6** Prove the following identities.

**a** $\sec^4\theta - \tan^4\theta \equiv \sec^2\theta + \tan^2\theta$

**b** $\text{cosec}^2 x - \sin^2 x \equiv \cot^2 x + \cos^2 x$

**c** $\sec^2 A(\cot^2 A - \cos^2 A) \equiv \cot^2 A$

**d** $1 - \cos^2\theta \equiv (\sec^2\theta - 1)(1 - \sin^2\theta)$

**e** $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} \equiv 1 - 2\sin^2 A$

**f** $\sec^2\theta + \text{cosec}^2\theta \equiv \sec^2\theta\,\text{cosec}^2\theta$

**g** $\text{cosec}\,A\sec^2 A \equiv \text{cosec}\,A + \tan A\sec A$

**h** $(\sec\theta - \sin\theta)(\sec\theta + \sin\theta) \equiv \tan^2\theta + \cos^2\theta$

(P) **7** Given that $3\tan^2\theta + 4\sec^2\theta = 5$, and that $\theta$ is obtuse, find the exact value of $\sin\theta$.

(P) **8** Solve the following equations in the given intervals.

**a** $\sec^2\theta = 3\tan\theta,\ 0 \leqslant \theta \leqslant 360°$

**b** $\tan^2\theta - 2\sec\theta + 1 = 0,\ -\pi \leqslant \theta \leqslant \pi$

**c** $\text{cosec}^2\theta + 1 = 3\cot\theta,\ -180° \leqslant \theta \leqslant 180°$

**d** $\cot\theta = 1 - \text{cosec}^2\theta,\ 0 \leqslant \theta \leqslant 2\pi$

**e** $3\sec\frac{1}{2}\theta = 2\tan^2\frac{1}{2}\theta,\ 0 \leqslant \theta \leqslant 360°$

**f** $(\sec\theta - \cos\theta)^2 = \tan\theta - \sin^2\theta,\ 0 \leqslant \theta \leqslant \pi$

**g** $\tan^2 2\theta = \sec 2\theta - 1,\ 0 \leqslant \theta \leqslant 180°$

**h** $\sec^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 1,\ 0 \leqslant \theta \leqslant 2\pi$

(E/P) **9** Given that $\tan^2 k = 2\sec k$,

**a** find the value of $\sec k$ **(4 marks)**

**b** deduce that $\cos k = \sqrt{2} - 1$. **(2 marks)**

**c** Hence solve, in the interval $0 \leqslant k \leqslant 360°$, $\tan^2 k = 2\sec k$, giving your answers to 1 decimal place. **(3 marks)**

(E/P) **10** Given that $a = 4\sec x$, $b = \cos x$ and $c = \cot x$,

**a** express $b$ in terms of $a$ **(2 marks)**

**b** show that $c^2 = \dfrac{16}{a^2 - 16}$ **(3 marks)**

(E/P) **11** Given that $x = \sec\theta + \tan\theta$,

**a** show that $\dfrac{1}{x} = \sec\theta - \tan\theta$. **(3 marks)**

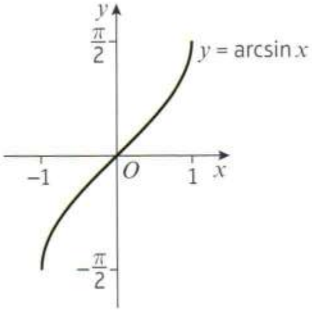
**b** Hence express $x^2 + \dfrac{1}{x^2} + 2$ in terms of $\theta$, in its simplest form. **(5 marks)**

(E/P) **12** Given that $2\sec^2\theta - \tan^2\theta = p$ show that $\text{cosec}^2\theta = \dfrac{p-1}{p-2}$, $p \neq 2$. **(5 marks)**

## 6.5 Inverse trigonometric functions

You need to understand and use the inverse trigonometric functions arcsin $x$, arccos $x$ and arctan $x$ and their graphs.

- **The inverse function of sin $x$ is called arcsin $x$.**



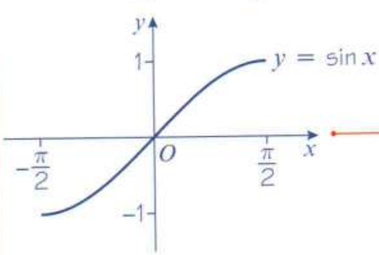

**Hint** The $\sin^{-1}$ function on your calculator will give principal values in the same range as arcsin.

- **The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$.**
- **The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$.**

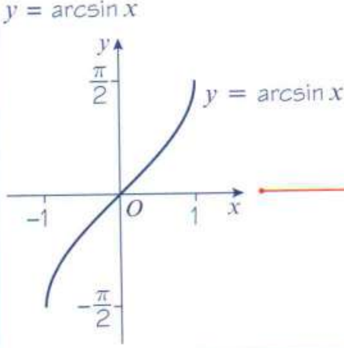
### Example 13

Sketch the graph of $y = \arcsin x$.

$y = \sin x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$




$y = \arcsin x$




**Step 1**
Draw the graph of $y = \sin x$, with the restricted domain of $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Restricting the domain ensures that the inverse function exists since $y = \sin x$ is a **one-to-one** function for the restricted domain. Only one-to-one functions have inverses. ← Section 2.3

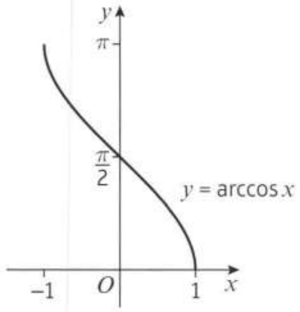
**Step 2**
Reflect in the line $y = x$.

The domain of arcsin $x$ is $-1 \leqslant x \leqslant 1$; the range is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$

Remember that the $x$ and $y$ coordinates of points interchange when reflecting in $y = x$. For example:

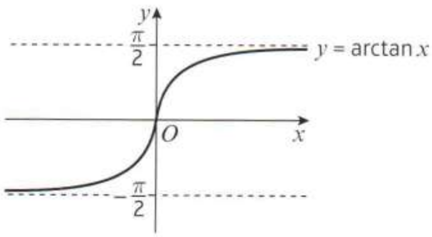
$$\left(\frac{\pi}{2}, 1\right) \rightarrow \left(1, \frac{\pi}{2}\right)$$

- **The inverse function of cos $x$ is called arccos $x$.**




- **The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$.**
- **The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$.**

- **The inverse function of tan $x$ is called arctan $x$.**




**Watch out** Unlike arcsin $x$ and arccos $x$, the function arctan $x$ is defined for all real values of $x$.

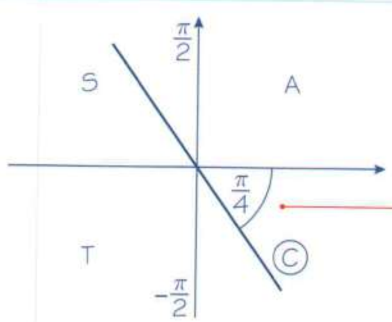
- **The domain of $y = \arctan x$ is $x \in \mathbb{R}$.**
- **The range of $y = \arctan x$ is $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$ or $-90° < \arctan x < 90°$.**

## Example 14

Work out, in radians, the values of:

**a** $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ **b** $\arccos(-1)$ **c** $\arctan(\sqrt{3})$

**a**



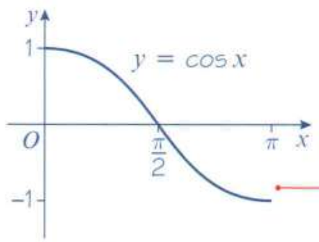

$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

You need to solve, in the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, the equation $\sin x = -\frac{\sqrt{2}}{2}$.

The angle to the horizontal is $\frac{\pi}{4}$ and, as sin is –ve, it is in the 4th quadrant.

**Online** Use your calculator to evaluate inverse trigonometric functions in radians.

b

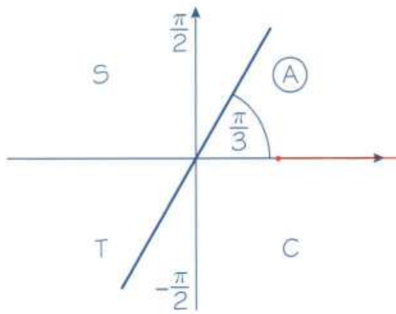



$\arccos(-1) = \pi$

You need to solve, in the interval $0 \leqslant x \leqslant \pi$, the equation $\cos x = -1$.

Draw the graph of $y = \cos x$.

c




$\arctan(\sqrt{3}) = \frac{\pi}{3}$

You need to solve, in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the equation $\tan x = \sqrt{3}$.

The angle to the horizontal is $\frac{\pi}{3}$ and, as tan is +ve, it is in the 1st quadrant.

You can verify these results using the $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ functions on your calculator.

## Exercise 6E

**In this exercise, all angles are given in radians.**

**1** Without using a calculator, work out, giving your answer in terms of $\pi$:

**a** $\arccos(0)$ **b** $\arcsin(1)$ **c** $\arctan(-1)$ **d** $\arcsin\left(-\frac{1}{2}\right)$

**e** $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ **f** $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ **g** $\arcsin\left(\sin\frac{\pi}{3}\right)$ **h** $\arcsin\left(\sin\frac{2\pi}{3}\right)$

**2** Find:

**a** $\arcsin\left(\frac{1}{2}\right) + \arcsin\left(-\frac{1}{2}\right)$ **b** $\arccos\left(\frac{1}{2}\right) - \arccos\left(-\frac{1}{2}\right)$ **c** $\arctan(1) - \arctan(-1)$

(P) **3** Without using a calculator, work out the values of:

**a** $\sin\left(\arcsin\frac{1}{2}\right)$ **b** $\sin\left(\arcsin\left(-\frac{1}{2}\right)\right)$

**c** $\tan(\arctan(-1))$ **d** $\cos(\arccos 0)$

(P) **4** Without using a calculator, work out the exact values of:

**a** $\sin\left(\arccos\left(\frac{1}{2}\right)\right)$ **b** $\cos\left(\arcsin\left(-\frac{1}{2}\right)\right)$ **c** $\tan\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$

**d** $\sec(\arctan(\sqrt{3}))$ **e** $\text{cosec}(\arcsin(-1))$ **f** $\sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)$

(P) **5** Given that $\arcsin k = \alpha$, where $0 < k < 1$ and $\alpha$ is in radians, write down, in terms of $\alpha$, the first two positive values of $x$ satisfying the equation $\sin x = k$.

(E/P) **6** Given that $x$ satisfies $\arcsin x = k$, where $0 < k < \frac{\pi}{2}$,

**a** state the range of possible values of $x$ **(1 mark)**

**b** express, in terms of $x$,

**i** $\cos k$ **ii** $\tan k$ **(4 marks)**

Given, instead, that $-\frac{\pi}{2} < k < 0$,

**c** how, if at all, are your answers to part **b** affected? **(2 marks)**

(P) **7** Sketch the graphs of:

**a** $y = \frac{\pi}{2} + 2\arcsin x$ **b** $y = \pi - \arctan x$

**c** $y = \arccos(2x + 1)$ **d** $y = -2\arcsin(-x)$

(E/P) **8** The function f is defined as $\mathrm{f}: x \mapsto \arcsin x$, $-1 \leqslant x \leqslant 1$, and the function g is such that $\mathrm{g}(x) = \mathrm{f}(2x)$.

**a** Sketch the graph of $y = \mathrm{f}(x)$ and state the range of f. **(3 marks)**

**b** Sketch the graph of $y = \mathrm{g}(x)$. **(2 marks)**

**c** Define g in the form $\mathrm{g}: x \mapsto \ldots$ and give the domain of g. **(3 marks)**

**d** Define $\mathrm{g}^{-1}$ in the form $\mathrm{g}^{-1}: x \mapsto \ldots$ **(2 marks)**

(E/P) **9 a** Prove that for $0 \leqslant x \leqslant 1$, $\arccos x = \arcsin\sqrt{1 - x^2}$ **(4 marks)**

**b** Give a reason why this result is not true for $-1 \leqslant x \leqslant 0$. **(2 marks)**

## Challenge

**a** Sketch the graph of $y = \sec x$, with the restricted domain $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{2}$

**b** Given that $\operatorname{arcsec} x$ is the inverse function of $\sec x$, $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{2}$, sketch the graph of $y = \operatorname{arcsec} x$ and state the range of $\operatorname{arcsec} x$.

## Mixed exercise 6

**Give any non-exact answers to equations to 1 decimal place.**

E/P **1** Solve $\tan x = 2\cot x$, in the interval $-180° \leqslant x \leqslant 90°$. **(4 marks)**

E/P **2** Given that $p = 2\sec\theta$ and $q = 4\cos\theta$, express $p$ in terms of $q$. **(4 marks)**

E/P **3** Given that $p = \sin\theta$ and $q = 4\cot\theta$, show that $p^2q^2 = 16(1 - p^2)$. **(4 marks)**

P **4** **a** Solve, in the interval $0 < \theta < 180°$,

**i** $\operatorname{cosec}\theta = 2\cot\theta$ **ii** $2\cot^2\theta = 7\operatorname{cosec}\theta - 8$

**b** Solve, in the interval $0 \leqslant \theta \leqslant 360°$,

**i** $\sec(2\theta - 15°) = \operatorname{cosec} 135°$ **ii** $\sec^2\theta + \tan\theta = 3$

**c** Solve, in the interval $0 \leqslant x \leqslant 2\pi$,

**i** $\operatorname{cosec}\left(x + \frac{\pi}{15}\right) = -\sqrt{2}$ **ii** $\sec^2 x = \frac{4}{3}$

E/P **5** Given that $5\sin x\cos y + 4\cos x\sin y = 0$, and that $\cot x = 2$, find the value of $\cot y$. **(5 marks)**

P **6** Prove that:

**a** $(\tan\theta + \cot\theta)(\sin\theta + \cos\theta) \equiv \sec\theta + \operatorname{cosec}\theta$

**b** $\dfrac{\operatorname{cosec} x}{\operatorname{cosec} x - \sin x} \equiv \sec^2 x$

**c** $(1 - \sin x)(1 + \operatorname{cosec} x) \equiv \cos x \cot x$

**d** $\dfrac{\cot x}{\operatorname{cosec} x - 1} - \dfrac{\cos x}{1 + \sin x} \equiv 2\tan x$

**e** $\dfrac{1}{\operatorname{cosec}\theta - 1} + \dfrac{1}{\operatorname{cosec}\theta + 1} \equiv 2\sec\theta\tan\theta$

**f** $\dfrac{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}{1 + \tan^2\theta} \equiv \cos^2\theta$

E/P **7** **a** Prove that $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} \equiv 2\operatorname{cosec} x$. **(4 marks)**

**b** Hence solve, in the interval $-2\pi \leqslant x \leqslant 2\pi$, $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = -\dfrac{4}{\sqrt{3}}$ **(4 marks)**

E/P **8** Prove that $\dfrac{1 + \cos\theta}{1 - \cos\theta} \equiv (\operatorname{cosec}\theta + \cot\theta)^2$ **(4 marks)**

E **9** Given that $\sec A = -3$, where $\frac{\pi}{2} < A < \pi$,

**a** calculate the exact value of $\tan A$ **(3 marks)**

**b** show that $\operatorname{cosec} A = \dfrac{3\sqrt{2}}{4}$ **(3 marks)**

**10** Given that $\sec\theta = k$, $|k| \geqslant 1$, and that $\theta$ is obtuse, express in terms of $k$:

**a** $\cos\theta$ **b** $\tan^2\theta$ **c** $\cot\theta$ **d** $\operatorname{cosec}\theta$

(E) **11** Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec\left(x + \frac{\pi}{4}\right) = 2$, giving your answers in terms of $\pi$. **(5 marks)**

(E/P) **12** Find, in terms of $\pi$, the value of $\arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)$. **(4 marks)**

(E/P) **13** Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec^2 x - \frac{2\sqrt{3}}{3}\tan x - 2 = 0$, giving your answers in terms of $\pi$. **(5 marks)**

(E/P) **14** **a** Factorise $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2$. **(2 marks)**

**b** Hence solve $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2 = 0$, in the interval $0 \leqslant x \leqslant 360°$. **(4 marks)**

(E/P) **15** Given that $\arctan(x - 2) = -\frac{\pi}{3}$, find the value of $x$. **(3 marks)**

(E) **16** On the same set of axes sketch the graphs of $y = \cos x$, $0 \leqslant x \leqslant \pi$, and $y = \arccos x$, $-1 \leqslant x \leqslant 1$, showing the coordinates of points at which the curves meet the axes. **(4 marks)**

(E/P) **17** **a** Given that $\sec x + \tan x = -3$, use the identity $1 + \tan^2 x \equiv \sec^2 x$ to find the value of $\sec x - \tan x$. **(3 marks)**

**b** Deduce the values of:

**i** $\sec x$ **ii** $\tan x$ **(3 marks)**

**c** Hence solve, in the interval $-180° \leqslant x \leqslant 180°$, $\sec x + \tan x = -3$. **(3 marks)**

(E/P) **18** Given that $p = \sec\theta - \tan\theta$ and $q = \sec\theta + \tan\theta$, show that $p = \frac{1}{q}$ **(4 marks)**

(E/P) **19** **a** Prove that $\sec^4\theta - \tan^4\theta = \sec^2\theta + \tan^2\theta$. **(3 marks)**

**b** Hence solve, in the interval $-180° \leqslant \theta \leqslant 180°$, $\sec^4\theta = \tan^4\theta + 3\tan\theta$. **(4 marks)**

(P) **20** **a** Sketch the graph of $y = \sin x$ and shade in the area representing $\int_0^{\frac{\pi}{2}} \sin x \, dx$.

**b** Sketch the graph of $y = \arcsin x$ and shade in the area representing $\int_0^1 \arcsin x \, dx$.

**c** By considering the shaded areas explain why $\int_0^{\frac{\pi}{2}} \sin x \, dx + \int_0^1 \arcsin x \, dx = \frac{\pi}{2}$

(P) **21** Show that $\cot 60° \sec 60° = \frac{2\sqrt{3}}{3}$

(E/P) **22** **a** Sketch, in the interval $-2\pi \leqslant x \leqslant 2\pi$, the graph of $y = 2 - 3\sec x$. **(3 marks)**

**b** Hence deduce the range of values of $k$ for which the equation $2 - 3\sec x = k$ has no solutions. **(2 marks)**

(P) **23** **a** Sketch the graph of $y = 3\arcsin x - \frac{\pi}{2}$, showing clearly the exact coordinates of the end-points of the curve. **(4 marks)**

**b** Find the exact coordinates of the point where the curve crosses the $x$-axis. **(3 marks)**

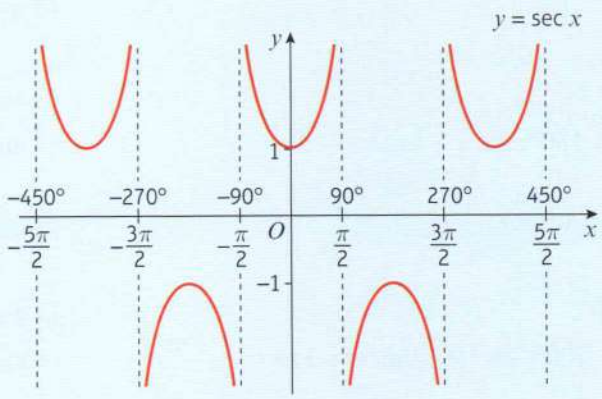
**24** **a** Prove that for $0 < x \leqslant 1$, $\arccos x = \arctan \dfrac{\sqrt{1-x^2}}{x}$

**b** Prove that for $-1 \leqslant x < 0$, $\arccos x = k + \arctan \dfrac{\sqrt{1-x^2}}{x}$, where $k$ is a constant to be found.

## Summary of key points

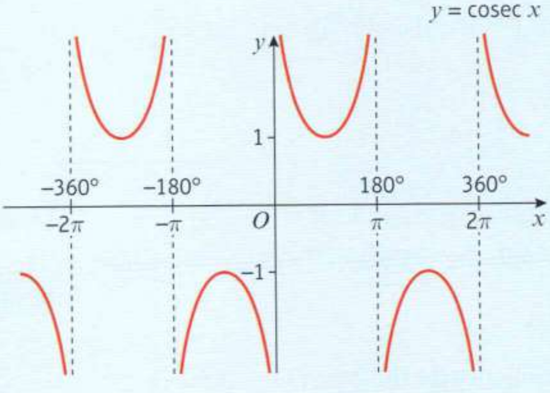
**1**
- $\sec x = \dfrac{1}{\cos x}$ (undefined for values of $x$ for which $\cos x = 0$)
- $\text{cosec}\, x = \dfrac{1}{\sin x}$ (undefined for values of $x$ for which $\sin x = 0$)
- $\cot x = \dfrac{1}{\tan x}$ (undefined for values of $x$ for which $\tan x = 0$)
- $\cot x = \dfrac{\cos x}{\sin x}$

**2** The graph of $\boldsymbol{y = \sec x}$, $x \in \mathbb{R}$, has symmetry in the $y$-axis and has period 360° or $2\pi$ radians. It has vertical asymptotes at all the values of $x$ for which $\cos x = 0$.



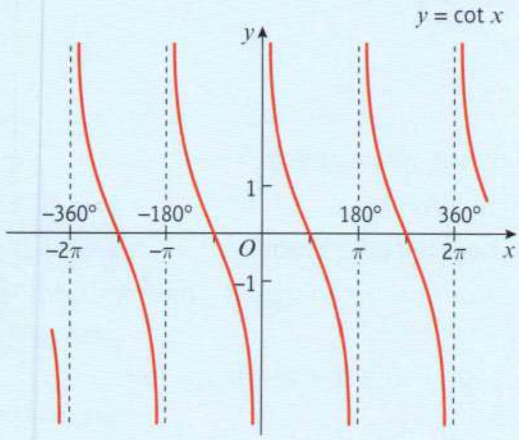

- The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90°, 270°, 450°, \ldots$ or any odd multiple of 90°.
- In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ or any odd multiple of $\frac{\pi}{2}$
- The range of $y = \sec x$ is $y \leqslant -1$ or $y \geqslant 1$.

**3** The graph of $\boldsymbol{y = \text{cosec}\, x}$, $x \in \mathbb{R}$, has period 360° or $2\pi$ radians. It has vertical asymptotes at all the values of $x$ for which $\sin x = 0$.




- The domain of $y = \text{cosec}\, x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°.
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of $\pi$
- The range of $y = \text{cosec}\, x$ is $y \leqslant -1$ or $y \geqslant 1$.

**4** The graph of $\boldsymbol{y = \cot x}$, $x \in \mathbb{R}$, has period 180° or $\pi$ radians. It has vertical asymptotes at all the values of $x$ for which $\tan x = 0$.




- The domain of $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0°$, 180°, 360°, … or any multiple of 180°.
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi$, … or any multiple of $\pi$.
- The range of $y = \cot x$ is $y \in \mathbb{R}$.

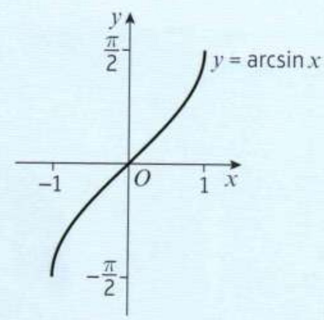
**5** $\sec x = k$ and $\operatorname{cosec} x = k$ have no solutions for $-1 < k < 1$.

**6** You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities:

- $1 + \tan^2 x \equiv \sec^2 x$
- $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

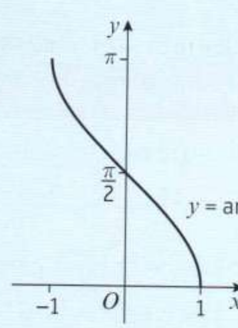
**7** The inverse function of $\sin x$ is called **arcsin** $\boldsymbol{x}$.

- The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$




**8** The inverse function of $\cos x$ is called **arccos** $\boldsymbol{x}$.

- The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$




**9** The inverse function of $\tan x$ is called **arctan** $\boldsymbol{x}$.

- The domain of $y = \arctan x$ is $x \in \mathbb{R}$
- The range of $y = \arctan x$ is $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$ or $-90° < \arctan x < 90°$