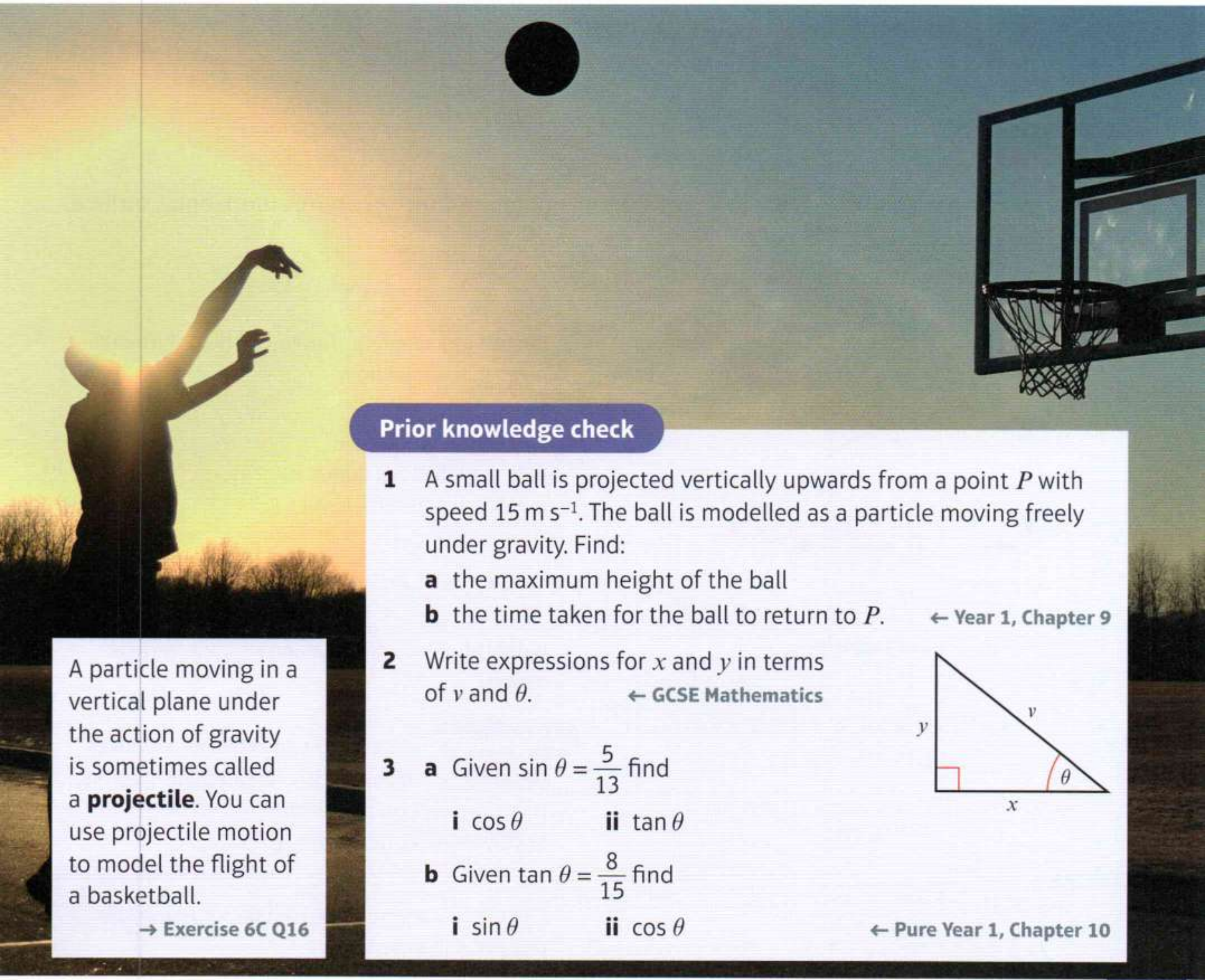
Projectiles

6

Objectives

After completing this chapter you should be able to:

- Model motion under gravity for an object projected horizontally → pages 108–111
- Resolve velocity into components → pages 111–113
- Solve problems involving particles projected at an angle → pages 113–120
- Derive the formulae for time of flight, range and greatest height, and the equation of the path of a projectile → pages 120–125



A particle moving in a vertical plane under the action of gravity is sometimes called a **projectile**. You can use projectile motion to model the flight of a basketball.

→ Exercise 6C Q16

Prior knowledge check

1 A small ball is projected vertically upwards from a point P with speed 15 m s^{-1}. The ball is modelled as a particle moving freely under gravity. Find:

a the maximum height of the ball

b the time taken for the ball to return to P. ← Year 1, Chapter 9

2 Write expressions for x and y in terms of v and θ. ← GCSE Mathematics

3 **a** Given $\sin\theta = \frac{5}{13}$ find

i $\cos\theta$ **ii** $\tan\theta$

b Given $\tan\theta = \frac{8}{15}$ find

i $\sin\theta$ **ii** $\cos\theta$ ← Pure Year 1, Chapter 10

6.1 Horizontal projection

You can model the motion of a projectile as a particle being acted on by a single force, gravity. In this model you ignore the effects of air resistance and any rotational movement on the particle.

You can analyse the motion of a projectile by considering its horizontal and vertical motion separately. Because gravity acts vertically downwards, there is **no force** acting on the particle in the horizontal direction.

- **The horizontal motion of a projectile is modelled as having constant velocity ($a = 0$). You can use the formula $s = vt$.**

The force due to gravity is modelled as being constant, so the vertical acceleration is constant.

- **The vertical motion of a projectile is modelled as having constant acceleration due to gravity ($a = g$).**

Use $g = 9.8\text{ m s}^{-2}$ unless the question specifies a different value.

Links You can use the constant acceleration formulae for the vertical motion of a projectile:

$v = u + at$

$s = \left(\frac{u + v}{2}\right)t$

$v^2 = u^2 + 2as$

$s = ut + \frac{1}{2}at^2$

$s = vt - \frac{1}{2}at^2$

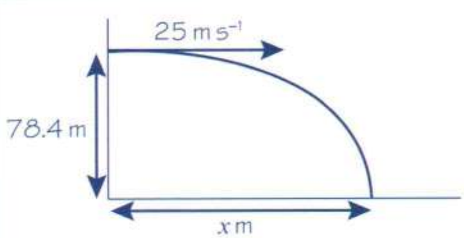
← Year 1, Chapter 9

Example 1

A particle is projected horizontally at 25 m s^{-1} from a point 78.4 metres above a horizontal surface. Find:

a the time taken by the particle to reach the surface

b the horizontal distance travelled in that time.



First draw a diagram showing all the information given in the question.

Notation u_x is the initial horizontal velocity. u_y is the initial vertical velocity.

Projected horizontally, R(→), $u_x = 25$

Taking the downwards direction as positive, R(↓), $u_y = 0$

The particle is projected horizontally so $u_y = 0$.

a R(↓), $u = 0,\ s = 78.4,\ a = 9.8,\ t = ?$

Write the values of u, s, a and t for the vertical motion.

Watch out The sign of g (positive or negative) depends on which direction is chosen as positive. Positive direction downwards: $g = 9.8\text{ m s}^{-2}$ Positive direction upwards: $g = -9.8\text{ m s}^{-2}$

$s = ut + \frac{1}{2}at^2$

$78.4 = 0 + \frac{1}{2} \times 9.8 \times t^2$

$78.4 = 4.9t^2$

$\frac{78.4}{4.9} = t^2$

$t^2 = 16$ so $t = 4\text{ s}$

The time taken must be positive so choose the positive square root.

b R(→), $u = 25$, $s = x$, $t = 4$

$s = vt$

$x = 25 \times 4$ so $x = 100$ m

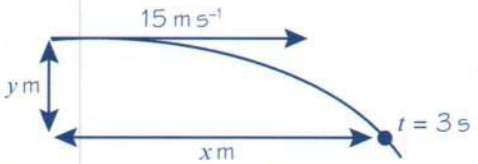
Your answer to part **a** tells you the time taken for the particle to hit the surface. The horizontal motion has constant velocity so you can use: distance = speed × time.

Example 2

A particle is projected horizontally with a velocity of 15 m s^{-1}. Find:

a the horizontal and vertical components of the displacement of the particle from the point of projection after 3 seconds

b the distance of the particle from the point of projection after 3 seconds.



Draw a diagram based on the information in the question.

Projected horizontally, R(→), $u_x = 15$

Taking the downwards direction as positive, R(↓), $u_y = 0$

a R(↓), $u = 0$, $s = y$, $a = 9.8$, $t = 3$

$s = ut + \frac{1}{2}at^2$

$y = 0 + \frac{1}{2} \times 9.8 \times 3^2$

$y = 44$ m (2 s.f.)

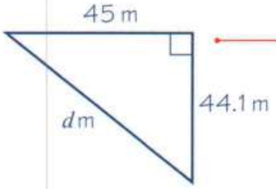
R(→), $u = 15$, $s = x$, $t = 3$

$s = vt$

$x = 15 \times 3 = 45$ m

Use $s = ut + \frac{1}{2}at^2$ to find the vertical distance. This is the same distance as the particle would travel in 3 seconds if it was dropped and fell under the action of gravity.

b



$d = \sqrt{45^2 + 44.1^2}$

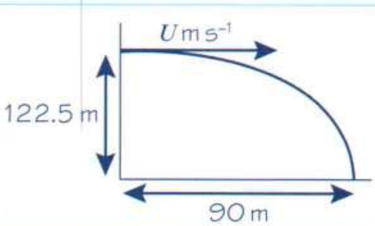
$= \sqrt{3969.81}$

$d = 63$ m (2 s.f.)

The distance travelled is the magnitude of the displacement vector. Sketch a right-angled triangle showing the components and use Pythagoras' Theorem.

Example 3

A particle is projected horizontally with a speed of U m s^{-1} from a point 122.5 m above a horizontal plane. The particle hits the plane at a point which is at a horizontal distance of 90 m away from the starting point. Find the initial speed of the particle.



Projected horizontally, R(→), $u_x = U$
Taking the downwards direction as positive, R(↓), $u_y = 0$
R(↓), $u = 0$, $s = 122.5$, $a = 9.8$, $t = ?$

$s = ut + \frac{1}{2}at^2$

$122.5 = 0 + \frac{1}{2} \times 9.8 \times t^2$

$122.5 = 4.9t^2$

$t^2 = 25$ so $t = 5\,\text{s}$

R(→), $v = U$, $s = 90$, $t = 5$

$s = vt$

$90 = U \times 5$

$U = 90 \div 5$ so $U = 18\,\text{m}\,\text{s}^{-1}$

Problem-solving

Many projectile problems can be solved by first using the **vertical motion** to find the total time taken.

Substitute $t = 5$ into the equation for horizontal motion to find U.

Exercise 6A

1 A particle is projected horizontally at $20\,\text{m}\,\text{s}^{-1}$ from a point h metres above horizontal ground. It lands on the ground 5 seconds later. Find:

a the value of h

b the horizontal distance travelled between the time the particle is projected and the time it hits the ground.

2 A particle is projected horizontally with a velocity of $18\,\text{m}\,\text{s}^{-1}$. Find:

a the horizontal and vertical components of the displacement of the particle from the point of projection after 2 seconds

b the distance of the particle from the point of projection after 2 seconds.

3 A particle is projected horizontally with a speed of $U\,\text{m}\,\text{s}^{-1}$ from a point 160 m above a horizontal plane. The particle hits the plane at a point which is at a horizontal distance of 95 m away from the point of projection. Find the initial speed of the particle.

4 A particle is projected horizontally from a point A which is 16 m above horizontal ground. The projectile strikes the ground at a point B which is at a horizontal distance of 140 m from A. Find the speed of projection of the particle.

(P) **5** A particle is projected horizontally with velocity $20\,\text{m}\,\text{s}^{-1}$ along a flat smooth table-top from a point 2 m from the table edge. The particle then leaves the table-top which is at a height of 1.2 m from the floor. Work out the total time taken for the particle to travel from the point of projection until it lands on the floor.

(E) **6** A darts player throws darts at a dart board which hangs vertically. The motion of a dart is modelled as that of a particle moving freely under gravity. The darts move in a vertical plane which is perpendicular to the plane of the dart board. A dart is thrown horizontally with an initial velocity of $14\,\text{m}\,\text{s}^{-1}$. It hits the board at a point which is 9 cm below the level from which it was thrown.

Find the horizontal distance from the point where the dart was thrown to the dart board. **(4 marks)**

E/P 7 A particle of mass 2.5 kg is projected along a horizontal rough surface with a velocity of $5\,\text{m}\,\text{s}^{-1}$. After travelling a distance of 2 m the ball leaves the rough surface as a projectile and lands on the ground which is 1.2 m vertically below. Given that the total time taken for the ball to travel from the initial point of projection to the point when it lands is 1.0 seconds, find:

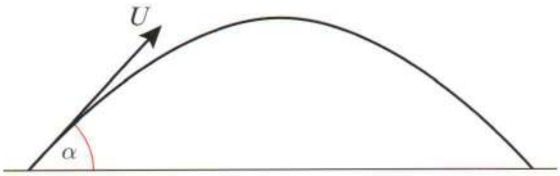
a the time for which the particle is in contact with the surface **(4 marks)**

b the coefficient of friction between the particle and the surface **(6 marks)**

c the horizontal distance travelled from the point of projection to the point where the particle hits the ground. **(3 marks)**

6.2 Horizontal and vertical components

Suppose a particle is projected with initial velocity U, at an angle α above the horizontal. The angle α is called the **angle of projection**.

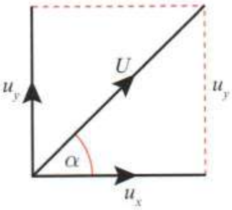


You can **resolve** the velocity into **components** that act horizontally and vertically:

Links This is the same technique as you use to resolve forces into components. ← Section 5.1

$\cos\alpha = \frac{u_x}{U}$ so $u_x = U\cos\alpha$

$\sin\alpha = \frac{u_y}{U}$ so $u_y = U\sin\alpha$



- **When a particle is projected with initial velocity U, at an angle α above the horizontal:**
  - **The horizontal component of the initial velocity is $U\cos\alpha$**
  - **The vertical component of the initial velocity is $U\sin\alpha$**

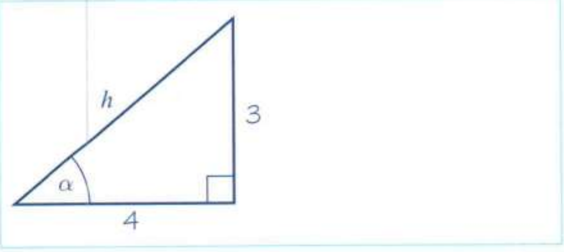
Example 4

A particle is projected from a point on a horizontal plane with an initial velocity of $40\,\text{m}\,\text{s}^{-1}$ at an angle α above the horizontal, where $\tan\alpha = \frac{3}{4}$.

a Find the horizontal and vertical components of the initial velocity.

Given that the vectors **i** and **j** are unit vectors acting in a vertical plane, horizontally and vertically respectively,

b express the initial velocity as a vector in terms of **i** and **j**.



Problem-solving

When you are given a value for $\tan\alpha$ you can find the values of $\cos\alpha$ and $\sin\alpha$ without working out the value of α. Here $\tan\alpha = \frac{3}{4} = \frac{\text{opp}}{\text{adj}}$, so sketch a right-angled triangle with opposite 3 and adjacent 4.

a $\tan\alpha = \frac{3}{4}$ so $h = \sqrt{3^2 + 4^2} = 5$

$\sin\alpha = \frac{3}{5}$ $\cos\alpha = \frac{4}{5}$

R(→), $u_x = u\cos\alpha = 40 \times \frac{4}{5} = 32\,\text{m s}^{-1}$

R(↑), $u_y = u\sin\alpha = 40 \times \frac{3}{5} = 24\,\text{m s}^{-1}$

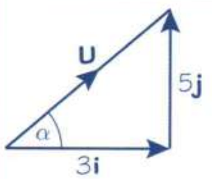
b $\mathbf{U} = (32\mathbf{i} + 24\mathbf{j})\,\text{m s}^{-1}$

Online Find cos α and sin α using your calculator.

You can write velocity as a vector using **i–j** notation. Remember to include units.

Example 5

A particle is projected with velocity $\mathbf{U} = (3\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$, where **i** and **j** are the unit vectors in the horizontal and vertical directions respectively. Find the initial speed of the particle and its angle of projection.



Speed, $|\mathbf{U}| = \sqrt{3^2 + 5^2} = \sqrt{34}\,\text{m s}^{-1}$

$\tan\alpha = \frac{5}{3}$

$\alpha = 59°$ (2 s.f.)

Initial speed is $\sqrt{34}\,\text{m s}^{-1}$ and the particle is projected at an angle of 59° above the horizontal.

Speed is the magnitude of the velocity vector. If the initial velocity is $p\mathbf{i} + q\mathbf{j}\,\text{m s}^{-1}$, the initial speed is $\sqrt{p^2 + q^2}$.

When an initial velocity is given in the form $p\mathbf{i} + q\mathbf{j}\,\text{m s}^{-1}$, the values of p and q are the horizontal and vertical components of the velocity respectively.

Exercise 6B

In this exercise **i** and **j** are unit vectors acting in a vertical plane, horizontally and vertically respectively.

1 A particle is projected from a point on a horizontal plane with an initial velocity of $25\,\text{m s}^{-1}$ at an angle of 40° above the horizontal.

a Find the horizontal and vertical components of the initial velocity.

b Express the initial velocity as a vector in the form $p\mathbf{i} + q\mathbf{j}\,\text{m s}^{-1}$.

2 A particle is projected from a cliff top with an initial velocity of $18\,\text{m s}^{-1}$ at an angle of 20° below the horizontal.

a Find the horizontal and vertical components of the initial velocity.

b Express the initial velocity as a vector in the form $p\mathbf{i} + q\mathbf{j}\,\text{m s}^{-1}$.

3 A particle is projected from a point on level ground with an initial velocity of $35\,\text{m s}^{-1}$ at an angle α above the horizontal, where $\tan\alpha = \frac{5}{12}$.

a Find the horizontal and vertical components of the initial velocity.

b Express the initial velocity as a vector in terms of **i** and **j**.

4 A particle is projected from the top of a building with an initial velocity of $28\,\text{m s}^{-1}$ at an angle θ below the horizontal, where $\tan\theta = \frac{7}{24}$.

a Find the horizontal and vertical components of the initial velocity.

b Express the initial velocity as a vector in terms of **i** and **j**.

5 A particle is projected with initial velocity $\mathbf{U} = (6\mathbf{i} + 9\mathbf{j})\,\text{m s}^{-1}$. Find the initial speed of the particle and its angle of projection.

6 A particle is projected with initial velocity $\mathbf{U} = (4\mathbf{i} - 5\mathbf{j})\,\text{m s}^{-1}$. Find the initial speed of the particle and its angle of projection.

(P) **7** A particle is projected with initial velocity $\mathbf{U} = 3k\mathbf{i} + 2k\mathbf{j}\,\text{m s}^{-1}$.

a Find the angle of projection.

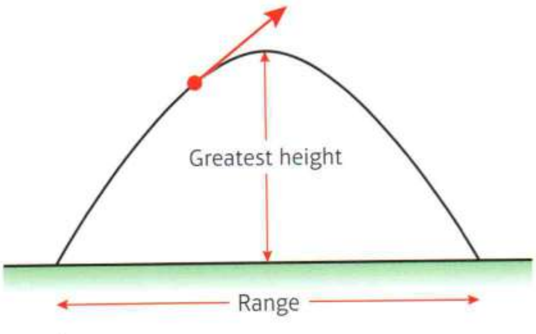
Given the initial speed is $3\sqrt{13}\,\text{m s}^{-1}$,

b find the value of k.

6.3 Projection at any angle

You can solve problems involving particles projected at any angle by resolving the initial velocity into horizontal and vertical components.

The distance from the point from which the particle was projected to the point where it strikes the horizontal plane is called the **range**.



The time the particle takes to move from its point of projection to the point where it strikes the horizontal plane is called the **time of flight** of the projectile.

- **A projectile reaches its point of greatest height when the vertical component of its velocity is equal to 0.**

Example 6

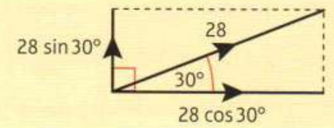
A particle P is projected from a point O on a horizontal plane with speed $28\,\text{m s}^{-1}$ and with angle of elevation 30°. After projection, the particle moves freely under gravity until it strikes the plane at a point A. Find:

a the greatest height above the plane reached by P **b** the time of flight of P **c** the distance OA.

Resolving the velocity of projection horizontally and vertically:

R(→), $u_x = 28\cos 30° = 24.248...$

R(↑), $u_y = 28\sin 30° = 14$



a Taking the upwards direction as positive:

R(↑), $u = 14$, $v = 0$, $a = -9.8$, $s = ?$

$v^2 = u^2 + 2as$

$0^2 = 14^2 - 2 \times 9.8 \times s$

$s = \dfrac{14^2}{2 \times 9.8} = 10$

The greatest height above the plane reached by P is 10 m.

At the highest point the vertical component of the velocity is zero.

The vertical motion is motion with constant acceleration.

b R(↑), $s = 0$, $u = 14$, $a = -9.8$, $t = ?$

$s = ut + \frac{1}{2}at^2$

$0 = 14t - 4.9t^2$

$= t(14 - 4.9t)$

$t = 0$ or $t = \dfrac{14}{4.9} = 2.857...$

The time of flight is 2.9 s (2 s.f.)

When the particle strikes the plane, it is at the same height (zero) as when it started.

$t = 0$ corresponds to the point from which P was projected and can be ignored.

Watch out In this example the value for g was given to 2 significant figures so your answers should be given to 2 significant figures.

c R(→), distance = speed × time

$= 28\cos 30° \times 2.857...$

$= 69.282...$

$OA = 69$ m (2 s.f.)

There is no horizontal acceleration.

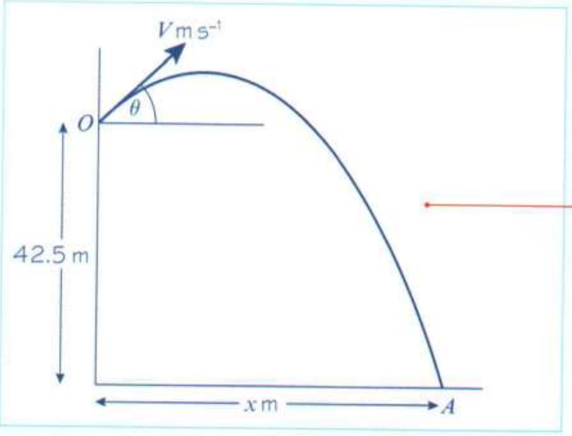
Use the unrounded value for the time of flight.

Example 7

A particle is projected from a point O with speed V m s^{-1} and at an angle of elevation of θ, where $\tan\theta = \frac{4}{3}$. The point O is 42.5 m above a horizontal plane. The particle strikes the plane at a point A, 5 s after it is projected.

a Show that $V = 20$.

b Find the distance between O and A.



Start by drawing a diagram.

Resolving the velocity of projection horizontally and vertically:

R(→), $u_x = V\cos\theta = \frac{3}{5}V$

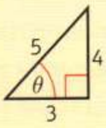
R(↑), $u_y = V\sin\theta = \frac{4}{5}V$

You will need $\sin\theta$ and $\cos\theta$ to resolve the initial velocity.
When you know $\tan\theta$ you can draw a triangle to find $\cos\theta$ and $\sin\theta$.

$\tan\theta = \frac{4}{3}$

$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$



a Taking the upwards direction as positive:

R(↑), $s = -42.5$, $u = \frac{4}{5}V$, $g = -9.8$, $t = 5$

$s = ut + \frac{1}{2}at^2$

$-42.5 = \frac{4}{5}V \times 5 - 4.9 \times 25$

$4V = 4.9 \times 25 - 42.5 = 80$

$V = \frac{80}{4} = 20$, as required.

Use the formula $s = ut + \frac{1}{2}at^2$ to obtain an equation in V.

b Let the horizontal distance moved be x m:

R(→), distance = speed × time

$x = \frac{3}{5}V \times 5 = 3V = 60$

Use the value of V found in part **a** to find the horizontal distance moved by the particle.

Using Pythagoras' Theorem:

$OA^2 = 42.5^2 + 60^2 = 5406.25$

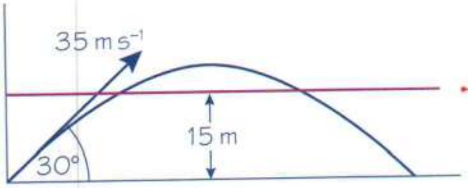
$OA = \sqrt{5406.25} = 73.527...$

The distance between O and A is 74 m, to 2 significant figures.

Example 8

A particle is projected from a point O with speed $35\,\text{m}\,\text{s}^{-1}$ at an angle of elevation of 30°.
The particle moves freely under gravity.

Find the length of time for which the particle is 15 m or more above O.



The particle is 15 m above O twice. First on the way up and then on the way down.

Resolving the initial velocity vertically:

In this example the horizontal component of the initial velocity is not used.

R(↑), $u_y = 35\sin 30° = 17.5$

$s = 15$, $u = 17.5$, $a = -9.8$, $t = ?$

$s = ut + \frac{1}{2}at^2$

$15 = 17.5t - 4.9t^2$

$4.9t^2 - 17.5t + 15 = 0$

Form a quadratic equation in t to find the two times when the particle is 15 m above O. Between these two times, the particle will be more than 15 m above O.

Multiplying by 10:

$49t^2 - 175t + 150 = 0$

$(7t - 10)(7t - 15) = 0$

$t = \frac{10}{7}, \frac{15}{7}$

Online Use your calculator to solve a quadratic equation.

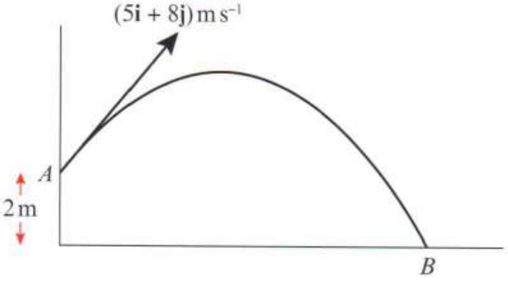
$\frac{15}{7} - \frac{10}{7} = \frac{5}{7}$

The particle is 15 m or more above O for $\frac{5}{7}$ s.

You could also give this answer as a decimal to 2 significant figures, 0.71 s.

Example 9

A ball is struck by a racket at a point A which is 2 m above horizontal ground. Immediately after being struck, the ball has velocity $(5\mathbf{i} + 8\mathbf{j})\,\text{m s}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors horizontally and vertically respectively. After being struck, the ball travels freely under gravity until it strikes the ground at the point B, as shown in the diagram above. Find:



a the greatest height above the ground reached by the ball

b the speed of the ball as it reaches B

c the angle the velocity of the ball makes with the ground as the ball reaches B.

a Taking the upwards direction as positive:

R(↑), $u = 8$, $v = 0$, $a = -9.8$, $s = ?$

$v^2 = u^2 + 2as$

$0^2 = 8^2 - 2 \times 9.8 \times s$

$s = \frac{64}{19.6} = 3.265...$

The greatest height above the ground reached by the ball is $2 + 3.265... = 5.3$ m, to 2 significant figures.

> The velocity of projection has been given as a vector in terms of **i** and **j**. The horizontal component is 5 and the vertical component is 8.

> This is the greatest height above the point of projection. You need to add 2 m to find the height above the ground.

b The horizontal component of the velocity of the ball at B is $5\,\text{m s}^{-1}$.

The vertical component of the velocity of the ball at B is given by:

R(↑), $s = -2$, $u = 8$, $a = -9.8$, $v = ?$

$v^2 = u^2 + 2as$

$= 8^2 + 2 \times (-9.8) \times (-2) = 103.2$

The speed at B is given by:

$v^2 = 5^2 + 103.2 = 128.2$

$v = \sqrt{128.2}$

The speed of the ball as it reaches B is $11\,\text{m s}^{-1}$, to 2 significant figures.

> The horizontal motion is motion with constant speed, so the horizontal component of the velocity never changes.

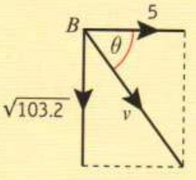
> There is no need to find the square root of 103.2 at this point, as you need v^2 in the next stage of the calculation.

c The angle is given by:

$\tan\theta = \frac{\sqrt{103.2}}{5} \Rightarrow \theta = 64°$ (2 s.f.)

The angle the velocity of the ball makes with the ground as the ball reaches B is 64°, to the nearest degree.

> As the ball reaches B, its velocity has two components as shown below.



> The magnitude (speed) and direction of the velocity are found using trigonometry and Pythagoras' Theorem.

Exercise 6C

In this exercise **i** and **j** are unit vectors acting in a vertical plane, horizontally and vertically respectively.

Whenever a numerical value of g is required, take $g = 9.8\,\text{m s}^{-2}$ unless otherwise stated.

1 A particle is projected with speed $35\,\text{m s}^{-1}$ at an angle of elevation of 60°. Find the time the particle takes to reach its greatest height.

2 A ball is projected from a point 5 m above horizontal ground with speed $18\,\text{m s}^{-1}$ at an angle of elevation of 40°. Find the height of the ball above the ground 2 s after projection.

3 A stone is projected from a point above horizontal ground with speed $32\,\text{m s}^{-1}$, at an angle of 10° below the horizontal. The stone takes 2.5 s to reach the ground. Find:

a the height of the point of projection above the ground

b the distance from the point on the ground vertically below the point of projection to the point where the stone reaches the ground.

4 A projectile is launched from a point on horizontal ground with speed $150\,\text{m s}^{-1}$ at an angle of 10° above the horizontal. Find:

a the time the projectile takes to reach its highest point above the ground

b the range of the projectile.

5 A particle is projected from a point O on a horizontal plane with speed $20\,\text{m s}^{-1}$ at an angle of elevation of 45°. The particle moves freely under gravity until it strikes the ground at a point X. Find:

a the greatest height above the plane reached by the particle

b the distance OX.

(P) **6** A ball is projected from a point A on level ground with speed $24\,\text{m s}^{-1}$. The ball is projected at an angle θ to the horizontal where $\sin\theta = \frac{4}{5}$. The ball moves freely under gravity until it strikes the ground at a point B. Find:

a the time of flight of the ball

b the distance from A to B.

(P) **7** A particle is projected with speed $21\,\text{m s}^{-1}$ at an angle of elevation α. Given that the greatest height reached above the point of projection is 15 m, find the value of α, giving your answer to the nearest degree.

8 A particle P is projected from the origin with velocity $(12\mathbf{i} + 24\mathbf{j})\,\text{m s}^{-1}$, where **i** and **j** are horizontal and vertical unit vectors respectively. The particle moves freely under gravity. Find:

a the position vector of P after 3 s

b the speed of P after 3 s.

(P) **9** A stone is thrown with speed 30 m s^{-1} from a window which is 20 m above horizontal ground. The stone hits the ground 3.5 s later. Find:

a the angle of projection of the stone

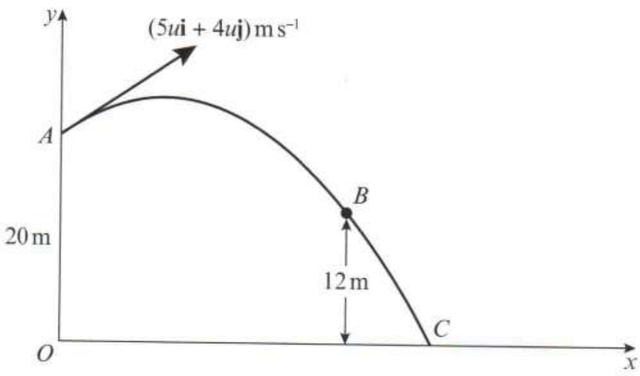
b the horizontal distance from the window to the point where the stone hits the ground.

(E/P) **10** A ball is thrown from a point O on horizontal ground with speed $U\text{ m s}^{-1}$ at an angle of elevation of θ, where $\tan\theta = \frac{3}{4}$. The ball strikes a vertical wall which is 20 m from O at a point which is 3 m above the ground. Find:

a the value of U **(6 marks)**

b the time from the instant the ball is thrown to the instant that it strikes the wall. **(2 marks)**

(E/P) **11** A particle P is projected from a point A with position vector $20\mathbf{j}$ m with respect to a fixed origin O. The velocity of projection is $(5u\mathbf{i} + 4u\mathbf{j})\text{ m s}^{-1}$. The particle moves freely under gravity, passing through a point B, which has position vector $(k\mathbf{i} + 12\mathbf{j})$ m, where k is a constant, before reaching the point C on the x-axis, as shown in the diagram. The particle takes 4 s to move from A to B. Find:



a the value of u **(4 marks)**

b the value of k **(2 marks)**

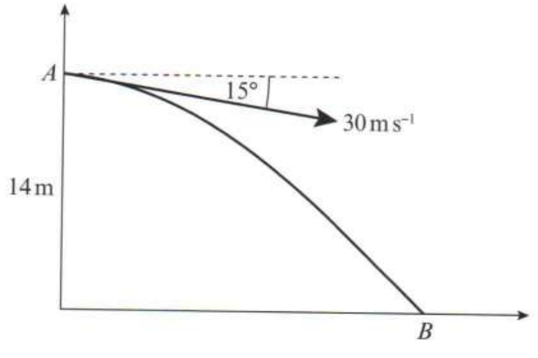
c the angle the velocity of P makes with the x-axis as it reaches C. **(6 marks)**

Watch out When finding a square root involving use of $g = 9.8\text{ m s}^{-2}$ to work out an answer, an exact surd answer is **not** acceptable.

(E) **12** A stone is thrown from a point A with speed 30 m s^{-1} at an angle of 15° below the horizontal. The point A is 14 m above horizontal ground. The stone strikes the ground at the point B, as shown in the diagram. Find:

a the time the stone takes to travel from A to B **(6 marks)**

b the distance AB. **(2 marks)**

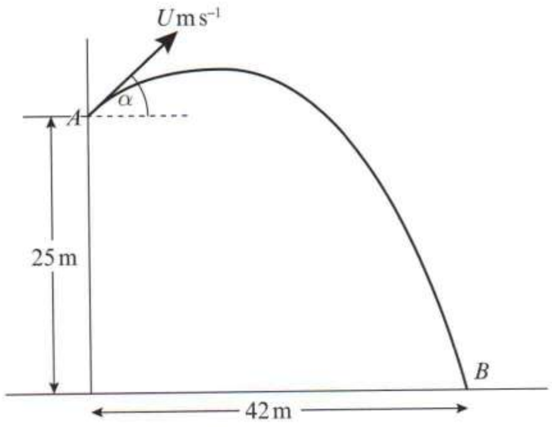


E/P **13** A particle is projected from a point on level ground with speed U m s^{-1} and angle of elevation α. The maximum height reached by the particle is 42 m above the ground and the particle hits the ground 196 m from its point of projection.

Find the value of α and the value of U. **(9 marks)**

E/P **14** In this question use $g = 10$ m s^{-2}.

An object is projected with speed U m s^{-1} from a point A at the top of a vertical building. The point A is 25 m above the ground. The object is projected at an angle α above the horizontal, where $\tan\alpha = \frac{5}{12}$. The object hits the ground at the point B, which is at a horizontal distance of 42 m from the foot of the building, as shown in the diagram. The object is modelled as a particle moving freely under gravity.



Find:

a the value of U **(6 marks)**

b the time taken by the object to travel from A to B **(2 marks)**

c the speed of the object when it is 12.4 m above the ground, giving your answer to 2 significant figures. **(5 marks)**

E/P **15** An object is projected from a fixed origin O with velocity $(4\mathbf{i} + 5\mathbf{j})$ m s^{-1}. The particle moves freely under gravity and passes through the point P with position vector $k(\mathbf{i} - \mathbf{j})$ m, where k is a positive constant.

a Find the value of k. **(6 marks)**

b Find:

i the speed of the object at the instant when it passes through P

ii the direction of motion of the object at the instant when it passes through P. **(7 marks)**

E/P **16** A basketball player is standing on the floor 10 m from the basket. The height of the basket is 3.05 m, and he shoots the ball from a height of 2 m, at an angle of 40° above the horizontal.

The basketball can be modelled as a particle moving in a vertical plane. Given that the ball passes through the basket,

a find the speed with which the basketball is thrown. **(6 marks)**

b State two factors that can be ignored by modelling the basketball as a particle. **(2 marks)**

Challenge

A vertical tower is 85 m high. A stone is projected at a speed of 20 m s^{-1} from the top of a tower at an angle of α below the horizontal. At the same time, a second stone is projected horizontally at a speed of 12 m s^{-1} from a window in the tower 45 m above the ground.

Given that the two stones move freely under gravity in the same vertical plane, and that they collide in mid-air, show that the time that elapses between the moment they are projected and the moment they collide is 2.5 s.

6.4 Projectile motion formulae

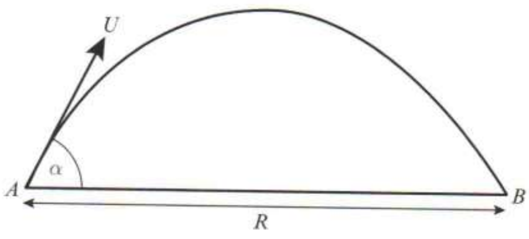
You need to be able to derive general formulae related to the motion of a particle which is projected from a point on a horizontal plane and moves freely under gravity.

Example 10

A particle is projected from a point on a horizontal plane with an initial velocity U at an angle α above the horizontal and moves freely under gravity until it hits the plane at point B. Given that the acceleration due to gravity is g, find expressions for:

a the time of flight, T

b the range, R, on the horizontal plane.



Taking the upwards direction as positive and resolving the velocity of projection:

R(↑), $u_y = U\sin\alpha$

R(→), $u_x = U\cos\alpha$

a Considering vertical motion:

R(↑), $u = U\sin\alpha,\ s = 0,\ a = -g,\ t = T$

$s = ut + \frac{1}{2}at^2$

$0 = (U\sin\alpha)T - \frac{1}{2} \times g \times T^2$

$0 = T\left(U\sin\alpha - \frac{gT}{2}\right)$

either $T = 0$ (at A) or $U\sin\alpha - \frac{gT}{2} = 0$

so $T = \frac{2U\sin\alpha}{g}$

Online Explore the parametric equations for the path of the particle and their Cartesian form, both algebraically and graphically using technology.

When the particle reaches the horizontal plane, the vertical displacement is 0.

Taking out the factor T, one solution is $T = 0$ which is at the start of the motion.

Problem-solving

Follow the same steps as you would if you were given values of U and α and asked to find the time of flight and the range. The answer will be an algebraic expression in terms of U and α instead of a numerical value.

b Considering horizontal motion:

R(→), $v_x = U\cos\alpha$, $s = R$, $t = T$

$s = v_x t$

$R = U\cos\alpha \times T$ using $T = \frac{2U\sin\alpha}{g}$

$R = U\cos\alpha \times \frac{2U\sin\alpha}{g} = \frac{2U^2\sin\alpha\cos\alpha}{g}$

Using $2\sin\alpha\cos\alpha \equiv \sin 2\alpha$:

$R = \frac{U^2\sin 2\alpha}{g}$

Substitute for T in the equation $R = U\cos\alpha \times T$

$U\cos\alpha \times \frac{2U\sin\alpha}{g} = \frac{U\cos\alpha}{1} \times \frac{2U\sin\alpha}{g}$

Use the double-angle formula for $\sin 2\alpha$.

← **Pure Year 2, Section 7.3**

Notation g is usually left as a letter in the formulae for projectile motion.

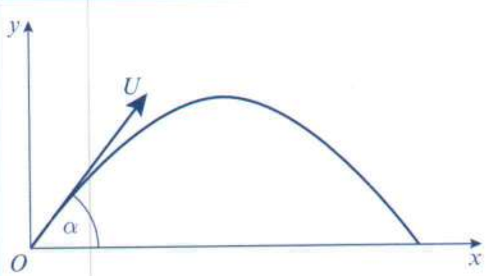
Example 11

A particle is projected from a point with speed U at an angle of elevation α and moves freely under gravity. When the particle has moved a horizontal distance x, its height above the point of projection is y.

a Show that $y = x\tan\alpha - \frac{gx^2}{2U^2}(1 + \tan^2\alpha)$.

A particle is projected from a point O on a horizontal plane, with speed 28 m s^{-1} at an angle of elevation α. The particle passes through a point B, which is at a horizontal distance of 32 m from O and at a height of 8 m above the plane.

b Find the two possible values of α, giving your answers to the nearest degree.



a R(→), $u_x = U\cos\alpha$

R(↑), $u_y = U\sin\alpha$

Resolve the velocity of projection horizontally and vertically.

For the horizontal motion:

R(→), $s = vt$

$x = U\cos\alpha \times t$ (1)

For the vertical motion, taking upwards as positive:

R(↑), $s = ut + \frac{1}{2}at^2$

$y = U\sin\alpha \times t - \frac{1}{2}gt^2$ (2)

You have obtained two equations, labelled (1) and (2). Both equations contain t and the result you have been asked to show has no t in it. You must eliminate t using substitution.

If the upwards direction is taken as positive, the vertical acceleration is $-g$.

Rearranging (1) to make t the subject of the formula:

$$t = \frac{x}{U\cos\alpha} \qquad (3)$$

Substituting (3) into (2):

$$y = U\sin\alpha \times \frac{x}{U\cos\alpha} - \frac{1}{2}g\left(\frac{x}{U\cos\alpha}\right)^2$$

Using $\tan\alpha \equiv \frac{\sin\alpha}{\cos\alpha}$ and $\frac{1}{\cos\alpha} \equiv \sec\alpha$,

$$y = x\tan\alpha - \frac{gx^2}{2U^2}\sec^2\alpha$$

Using $\sec^2\alpha \equiv 1 + \tan^2\alpha$,

$$y = x\tan\alpha - \frac{gx^2}{2U^2}(1 + \tan^2\alpha)\text{, as required.}$$

b Using the result in **a** with $U = 28$, $x = 32$, $y = 8$ and $g = 9.8$

$$8 = 32\tan\alpha - 6.4(1 + \tan^2\alpha)$$

Rearranging as a quadratic in $\tan\alpha$:

$$6.4\tan^2\alpha - 32\tan\alpha + 14.4 = 0$$
$$4\tan^2\alpha - 20\tan\alpha + 9 = 0$$
$$(2\tan\alpha - 1)(2\tan\alpha - 9) = 0$$
$$\tan\alpha = \tfrac{1}{2}, \tfrac{9}{2}$$

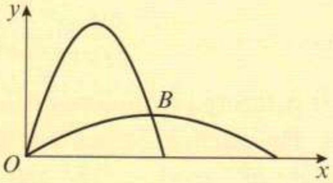
$\alpha = 27°$ and $77°$, to the nearest degree

(1) and (2) are parametric equations describing the path of the particle. You can eliminate the parameter, t, to find the Cartesian form of the path. ← Pure Year 2, Section 8.5

To obtain a quadratic expression in $\tan\alpha$, you need to use the identity $\sec^2\alpha \equiv 1 + \tan^2\alpha$. ← Pure Year 2, Section 6.4

You could use your calculator to solve this equation.

There are two possible angles of elevation for which the particle will pass through B. This sketch illustrates the two paths.



- **For a particle which is projected from a point on a horizontal plane with an initial velocity U at an angle α above the horizontal, and that moves freely under gravity:**
  - **Time of flight $= \frac{2U\sin\alpha}{g}$**
  - **Time to reach greatest height $= \frac{U\sin\alpha}{g}$**
  - **Range on horizontal plane $= \frac{U^2\sin 2\alpha}{g}$**
  - **Equation of trajectory: $y = x\tan\alpha - gx^2\frac{(1+\tan^2\alpha)}{2U^2}$**

  where y is the vertical height of the particle, x is the horizontal distance from the point of projection, and g is the acceleration due to gravity.

Watch out You need to know how to derive the equations. But be careful of using them in projectile problems. They are hard to memorise, and it is usually safer to answer projectile problems using the techniques covered in Section 16.3.

Hint The equation for the trajectory of the particle is a **quadratic** equation for y in x. This proves that the path of a projectile moving freely under gravity is a quadratic curve, or **parabola**.

Exercise 6D

Whenever a numerical value of g is required, take $g = 9.8\,\text{m}\,\text{s}^{-2}$ unless otherwise stated.

(P) **1** A particle is launched from a point on a horizontal plane with initial velocity $U\,\text{m}\,\text{s}^{-1}$ at an angle of elevation α. The particle moves freely under gravity until it strikes the plane.
The greatest height of the particle is h m.

Show that $h = \dfrac{U^2 \sin^2 \alpha}{2g}$

(P) **2** A particle is projected from a point with speed $21\,\text{m}\,\text{s}^{-1}$ at an angle of elevation α and moves freely under gravity. When the particle has moved a horizontal distance x m, its height above the point of projection is y m.

a Show that $y = x \tan \alpha - \dfrac{x^2}{90 \cos^2 \alpha}$

b Given that $y = 8.1$ when $x = 36$, find the value of $\tan \alpha$.

(P) **3** A projectile is launched from a point on a horizontal plane with initial speed $U\,\text{m}\,\text{s}^{-1}$ at an angle of elevation α. The particle moves freely under gravity until it strikes the plane. The range of the projectile is R m.

a Show that the time of flight of the particle is $\dfrac{2U \sin \alpha}{g}$ seconds.

b Show that $R = \dfrac{U^2 \sin 2\alpha}{g}$.

c Deduce that, for a fixed u, the greatest possible range is when $\alpha = 45°$.

d Given that $R = \dfrac{2U^2}{5g}$, find the two possible values of the angle of elevation at which the projectile could have been launched.

(P) **4** A firework is launched vertically with a speed of $v\,\text{m}\,\text{s}^{-1}$. When it reaches its maximum height, the firework explodes into two parts, which are projected horizontally in opposite directions, each with speed $2v\,\text{m}\,\text{s}^{-1}$.
Show that the two parts of the firework land a distance $\dfrac{4v^2}{g}$ m apart.

(E/P) **5** A particle is projected from a point O with speed U at an angle of elevation α above the horizontal and moves freely under gravity. When the particle has moved a horizontal distance x, its height above O is y.

a Show that $y = x \tan \alpha - \dfrac{gx^2}{2U^2 \cos^2 \alpha}$ **(4 marks)**

A boy throws a stone from a point P at the end of a pier. The point P is 15 m above sea level. The stone is projected with a speed of $8\,\text{m}\,\text{s}^{-1}$ at an angle of elevation of 40°. By modelling the ball as a particle moving freely under gravity,

b find the horizontal distance of the stone from P when the ball is 2 m above sea level. **(5 marks)**

E/P **6** A particle is projected from a point with speed U at an angle of elevation α above the horizontal and moves freely under gravity. When it has moved a horizontal distance x, its height above the point of projection is y.

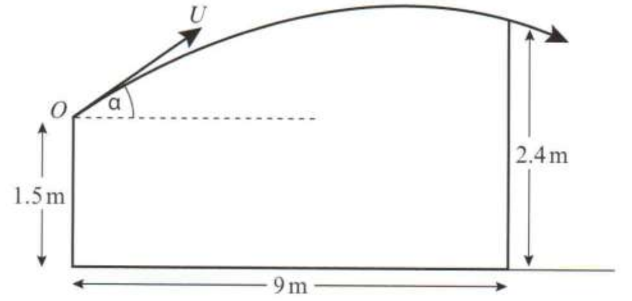
a Show that $y = x\tan\alpha - \frac{gx^2}{2U^2}(1 + \tan^2\alpha)$ **(5 marks)**

An athlete throws a javelin from a point P at a height of 2 m above horizontal ground. The javelin is projected at an angle of elevation of 45° with a speed of 30 m s^{-1}. By modelling the javelin as a particle moving freely under gravity,

b find, to 3 significant figures, the horizontal distance of the javelin from P when it hits the ground **(5 marks)**

c find, to 2 significant figures, the time elapsed from the point the javelin is thrown to the point it hits the ground. **(2 marks)**

E/P **7** A girl playing volleyball on horizontal ground hits the ball towards the net 9 m away from a point 1.5 m above the ground. The ball moves in a vertical plane which is perpendicular to the net. The ball just passes over the top of the net, which is 2.4 m above the ground, as shown in the diagram.



The ball is modelled as a particle projected with initial speed U m s^{-1} from point O, 1.5 m above the ground at an angle α to the horizontal.

a By writing down expressions for the horizontal and vertical distances from O to the ball, t seconds after it was hit, show that when the ball passes over the net

$$0.9 = 9\tan\alpha - \frac{81g}{2U^2\cos^2\alpha}$$ **(6 marks)**

Given that $\alpha = 30°$,

b find the speed of the ball as it passes over the net. **(6 marks)**

E/P **8** In this question **i** and **j** are unit vectors in a horizontal and upward vertical direction respectively. An object is projected from a fixed point A on horizontal ground with velocity $(k\mathbf{i} + 2k\mathbf{j})$ m s^{-1}, where k is a positive constant. The object moves freely under gravity until it strikes the ground at B, where it immediately comes to rest. Relative to O, the position vector of a point on the path of the object is $(x\mathbf{i} + y\mathbf{j})$ m.

a Show that $y = 2x - \frac{gx^2}{2k^2}$ **(5 marks)**

Given that $AB = R$ m and the maximum vertical height of the object above the ground is H m,

b using the result in part **a**, or otherwise, find, in terms of k and g,

i R **ii** H **(6 marks)**

Challenge

A stone is projected from a point on a straight sloping hill. Given that the hill slopes downwards at an angle of 45°, and that the stone is projected at an angle of 45° above the horizontal with speed $U\,\text{m s}^{-1}$.

Show that the stone lands a distance $\frac{2\sqrt{2}\,U^2}{g}$ m down the hill.

Mixed exercise 6

Whenever a numerical value of g is required, take $g = 9.8\,\text{m s}^{-2}$ unless otherwise stated.

1 A particle P is projected from a point O on a horizontal plane with speed $42\,\text{m s}^{-1}$ and with angle of elevation 45°. After projection, the particle moves freely under gravity until it strikes the plane. Find:

a the greatest height above the plane reached by P

b the time of flight of P.

2 A stone is thrown horizontally with speed $21\,\text{m s}^{-1}$ from a point P on the edge of a cliff h metres above sea level. The stone lands in the sea at a point Q, where the horizontal distance of Q from the cliff is 56 m.

Calculate the value of h.

(E) **3** A ball is thrown from a window above a horizontal lawn. The velocity of projection is $15\,\text{m s}^{-1}$ and the angle of elevation is α, where $\tan\alpha = \frac{4}{3}$. The ball takes 4 s to reach the lawn. Find:

a the horizontal distance between the point of projection and the point where the ball hits the lawn **(3 marks)**

b the vertical height above the lawn from which the ball was thrown. **(3 marks)**

(E) **4** A projectile is fired with velocity $40\,\text{m s}^{-1}$ at an angle of elevation of 30° from a point A on horizontal ground. The projectile moves freely under gravity until it reaches the ground at the point B. Find:

a the distance AB **(5 marks)**

b the speed of the projectile at the first instant when it is 15 m above the ground. **(5 marks)**

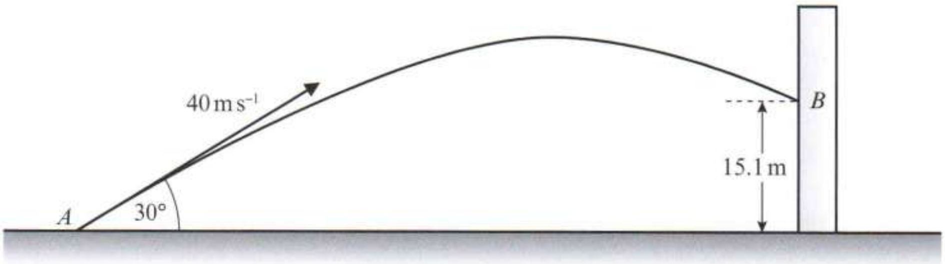
(E/P) **5** A particle P is projected from a point on a horizontal plane with speed U at an angle of elevation θ.

a Show that the range of the projectile is $\frac{U^2 \sin 2\theta}{g}$. **(6 marks)**

b Hence find, as θ varies, the maximum range of the projectile. **(2 marks)**

c Given that the range of the projectile is $\frac{2U^2}{3g}$, find the two possible value of θ.
Give your answers to the nearest 0.1°. **(3 marks)**

E **6**



A golf ball is driven from a point A with a speed of $40\,\text{m}\,\text{s}^{-1}$ at an angle of elevation of $30°$. On its downward flight, the ball hits an advertising hoarding at a height 15.1 m above the level of A, as shown in the diagram above. Find:

a the time taken by the ball to reach its greatest height above A **(3 marks)**

b the time taken by the ball to travel from A to B **(6 marks)**

c the speed with which the ball hits the hoarding. **(5 marks)**

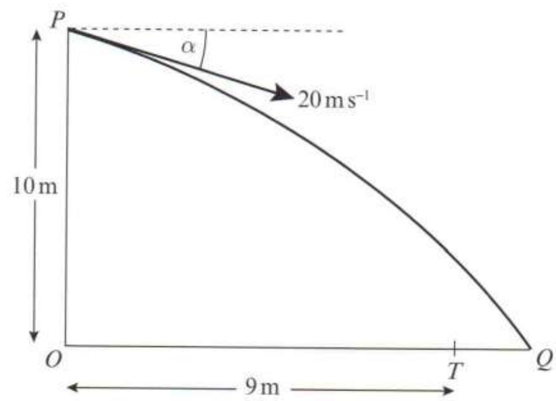
E/P **7** In this question use $g = 10\,\text{m}\,\text{s}^{-2}$.
A boy plays a game at a fairground. He needs to throw a ball through a hole in a vertical target to win a prize. The motion of the ball is modelled as that of a particle moving freely under gravity. The ball moves in a vertical plane which is perpendicular to the plane of the target. The boy throws the ball horizontally at the same height as the hole with a speed of $10\,\text{m}\,\text{s}^{-1}$. It hits the target at a point 20 cm below the hole.

a Find the horizontal distance from the point where the ball was thrown to the target. **(4 marks)**

The boy throws the ball again with the same speed and at the same distance from the target.

b Work out the possible angles above the horizontal the boy could throw the ball so that it passes through the hole. **(6 marks)**

E/P **8** In this question use $g = 10\,\text{m}\,\text{s}^{-2}$.
A stone is thrown from a point P at a target, which is on horizontal ground. The point P is 10 m above the point O on the ground. The stone is thrown from P with speed $20\,\text{m}\,\text{s}^{-1}$ at an angle of α below the horizontal, where $\tan\alpha = \frac{3}{4}$.
The stone is modelled as a particle and the target as a point T. The distance OT is 9 m. The stone misses the target and hits the ground at the point Q, where OTQ is a straight line, as shown in the diagram. Find:



a the time taken by the ball to travel from P to Q **(5 marks)**

b the distance TQ. **(4 marks)**

The point A is on the path of the ball vertically above T.

c Find the speed of the ball at A. **(5 marks)**

E/P **9** A vertical mast is 32 m high. Two balls P and Q are projected simultaneously. Ball P is projected horizontally from the top of the mast with speed $18\ \text{m s}^{-1}$. Ball Q is projected from the bottom of the mast with speed $30\ \text{m s}^{-1}$ at an angle α above the horizontal. The balls move freely under gravity in the same vertical plane and collide in mid-air. By considering the horizontal motion of each ball,

a prove that $\cos\alpha = \frac{3}{5}$ **(4 marks)**

b Find the time which elapses between the instant when the balls are projected and the instant when they collide. **(4 marks)**

Challenge

A cruise ship is 250 m long, and is accelerating forwards in a straight line at a constant rate of $1.5\ \text{m s}^{-2}$. A golfer stands at the stern (back) of the cruise ship and hits a golf ball towards the bow (front). Given that the golfer hits the golf ball at an angle of elevation of 60°, and that the ball lands directly on the bow of the cruise ship, find the speed, v, with which the golfer hits the ball.

Summary of key points

1 The **horizontal** motion of a projectile is modelled as having **constant velocity** ($a = 0$). You can use the formula $s = vt$.

2 The **vertical** motion of a projectile is modelled as having **constant acceleration** due to gravity ($a = g$).

3 When a particle is projected with initial velocity U, at an angle α above the horizontal:

- The **horizontal component** of the initial velocity is $U\cos\alpha$
- The **vertical component** of the initial velocity is $U\sin\alpha$

4 A projectile reaches its point of greatest height when the vertical component of its velocity is equal to 0.

5 For a particle which is projected from a point on a horizontal plane with an initial velocity U at an angle α above the horizontal, and that moves freely under gravity:

- Time of flight $= \frac{2U\sin\alpha}{g}$
- Time to reach greatest height $= \frac{U\sin\alpha}{g}$
- Range on horizontal plane $= \frac{U^2\sin 2\alpha}{g}$
- Equation of trajectory: $y = x\tan\alpha - gx^2\frac{(1+\tan^2\alpha)}{2U^2}$

where y is the vertical height of the particle, x is the horizontal distance from the point of projection, and g is the acceleration due to gravity.