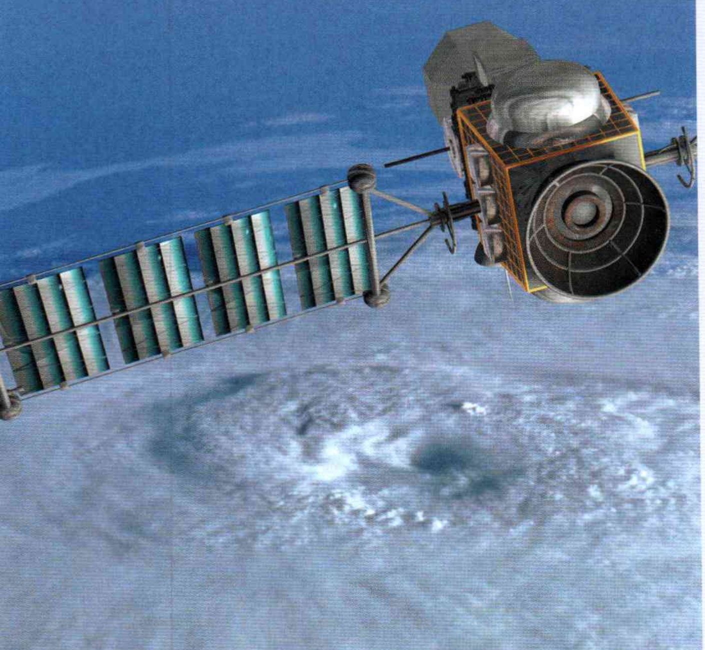
Circles

6

Objectives

After completing this unit you should be able to:

- Find the mid point of a line segment → pages 114 – 115
- Find the equation of the perpendicular bisector to a line segment → pages 116 – 117
- Know how to find the equation of a circle → pages 117 – 120
- Solve geometric problems involving straight lines and circles → pages 121 – 122
- Use circle properties to solve problems on coordinate grids → pages 123 – 128
- Find the angle in a semicircle and solve other problems involving circles and triangles → pages 128 – 132



Geostationary orbits are circular orbits around the Earth. Meteorologists use geostationary satellites to provide information about the Earth's surface and atmosphere.

Prior knowledge check

1 Write each of the following in the form $(x + p)^2 + q$:

a $x^2 + 10x + 28$ **b** $x^2 - 6x + 1$

c $x^2 - 12x$ **d** $x^2 + 7x$ ← Section 2.2

2 Find the equation of the line passing through each of the following pairs of points:

a $A(0, -6)$ and $B(4, 3)$

b $P(7, -5)$ and $Q(-9, 3)$

c $R(-4, -2)$ and $T(5, 10)$ ← Section 5.2

3 Use the discriminant to determine whether the following have two real solutions, one real solution or no real solutions.

a $x^2 - 7x + 14 = 0$

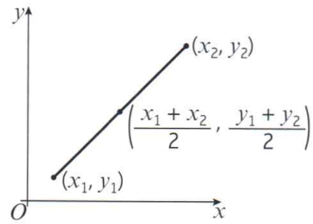
b $x^2 + 11x + 8 = 0$

c $4x^2 + 12x + 9 = 0$ ← Section 2.5

4 Find the equation of the line that passes through the point $(3, -4)$ and is perpendicular to the line with equation $6x - 5y - 1 = 0$ ← Section 5.3

6.1 Midpoints and perpendicular bisectors

You can find the **midpoint** of a line segment by averaging the x- and y-coordinates of its endpoints.

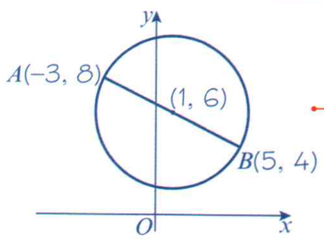


- **The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.**

Notation A **line segment** is a finite part of a straight line with two distinct endpoints.

Example 1

The line segment AB is a diameter of a circle, where A and B are $(-3, 8)$ and $(5, 4)$ respectively. Find the coordinates of the centre of the circle.



Draw a sketch. Remember the centre of a circle is the midpoint of a diameter.

The centre of the circle is $\left(\frac{-3 + 5}{2}, \frac{8 + 4}{2}\right)$

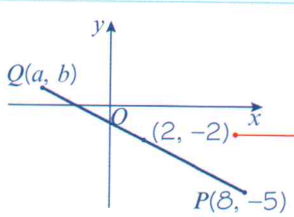
$= \left(\frac{2}{2}, \frac{12}{2}\right) = (1, 6)$

Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
Here $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (5, 4)$.

Example 2

The line segment PQ is a diameter of the circle centre $(2, -2)$. Given that P is $(8, -5)$, find the coordinates of Q.

Problem-solving In coordinate geometry problems, it is often helpful to draw a sketch showing the information given in the question.



$(2, -2)$ is the mid-point of (a, b) and $(8, -5)$.

Let Q have coordinates (a, b).

$\left(\frac{8 + a}{2}, \frac{-5 + b}{2}\right) = (2, -2)$

Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
Here $(x_1, y_1) = (8, -5)$ and $(x_2, y_2) = (a, b)$.

So $\frac{8 + a}{2} = 2$ $\frac{-5 + b}{2} = -2$

Compare the x- and y-coordinates separately.

$8 + a = 4$ $-5 + b = -4$

$a = -4$ $b = 1$

Rearrange the equations to find a and b.

So, Q is $(-4, 1)$.

Exercise 6A

1 Find the midpoint of the line segment joining each pair of points:

a $(4, 2), (6, 8)$ b $(0, 6), (12, 2)$ c $(2, 2), (-4, 6)$

d $(-6, 4), (6, -4)$ e $(7, -4), (-3, 6)$ f $(-5, -5), (-11, 8)$

g $(6a, 4b), (2a, -4b)$ h $(-4u, 0), (3u, -2v)$ i $(a + b, 2a - b), (3a - b, -b)$

j $(4\sqrt{2}, 1)\ (2\sqrt{2}, 7)$ k $(\sqrt{2} - \sqrt{3}, 3\sqrt{2} + 4\sqrt{3}), (3\sqrt{2} + \sqrt{3}, -\sqrt{2} + 2\sqrt{3})$

(P) 2 The line segment AB has endpoints $A(-2, 5)$ and $B(a, b)$. The midpoint of AB is $M(4, 3)$. Find the values of a and b.

3 The line segment PQ is a diameter of a circle, where P and Q are $(-4, 6)$ and $(7, 8)$ respectively. Find the coordinates of the centre of the circle.

(P) 4 The line segment RS is a diameter of a circle, where R and S are $\left(\frac{4a}{5}, -\frac{3b}{4}\right)$ and $\left(\frac{2a}{5}, \frac{5b}{4}\right)$ respectively. Find the coordinates of the centre of the circle.

Problem-solving

Your answer will be in terms of a and b.

5 The line segment AB is a diameter of a circle, where A and B are $(-3, -4)$ and $(6, 10)$ respectively.

a Find the coordinates of the centre of the circle.

b Show the centre of the circle lies on the line $y = 2x$.

(P) 6 The line segment JK is a diameter of a circle, where J and K are $\left(\frac{3}{4}, \frac{4}{3}\right)$ and $\left(-\frac{1}{2}, 2\right)$ respectively. The centre of the circle lies on the line segment with equation $y = 8x + b$. Find the value of b.

(P) 7 The line segment AB is a diameter of a circle, where A and B are $(0, -2)$ and $(6, -5)$ respectively. Show that the centre of the circle lies on the line $x - 2y - 10 = 0$.

(P) 8 The line segment FG is a diameter of the circle centre $(6, 1)$. Given F is $(2, -3)$, find the coordinates of G.

(P) 9 The line segment CD is a diameter of the circle centre $(-2a, 5a)$. Given D has coordinates $(3a, -7a)$, find the coordinates of C.

(P) 10 The points $M(3, p)$ and $N(q, 4)$ lie on the circle centre $(5, 6)$. The line segment MN is a diameter of the circle. Find the values of p and q.

Problem-solving

Use the formula for finding the midpoint:
$\left(\frac{3 + q}{2}, \frac{p + 4}{2}\right) = (5, 6)$

(P) 11 The points $V(-4, 2a)$ and $W(3b, -4)$ lie on the circle centre $(b, 2a)$. The line segment VW is a diameter of the circle. Find the values of a and b.

Challenge

A triangle has vertices at $A(3, 5)$, $B(7, 11)$ and $C(p, q)$. The midpoint of side BC is $M(8, 5)$.

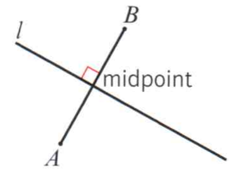
a Find the values of p and q.

b Find the equation of the straight line joining the midpoint of AB to the point M.

c Show that the line in part **b** is parallel to the line AC.

Links You can also prove results like this using vectors.
→ Section 11.5

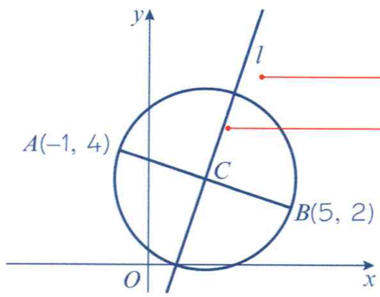
- **The perpendicular bisector of a line segment AB is the straight line that is perpendicular to AB and passes through the midpoint of AB.**



If the gradient of AB is m then the gradient of its perpendicular bisector, l, will be $-\frac{1}{m}$

Example 3

The line segment AB is a diameter of the circle centre C, where A and B are $(-1, 4)$ and $(5, 2)$ respectively. The line l passes through C and is perpendicular to AB. Find the equation of l.



Draw a sketch.

l is the perpendicular bisector of AB.

The centre of the circle is $\left(\frac{-1+5}{2}, \frac{4+2}{2}\right)$
$= (2, 3)$

Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Here $(x_1, y_1) = (-1, 4)$ and $(x_2, y_2) = (5, 2)$.

The gradient of the line segment AB is $\frac{2-4}{5-(-1)} = -\frac{1}{3}$

Use $m = \frac{y_2 - y_1}{x_2 - x_1}$. Here $(x_1, y_1) = (-1, 4)$ and $(x_2, y_2) = (5, 2)$.

Gradient of $l = 3$.

Remember the product of the gradients of two perpendicular lines is $= -1$, so $-\frac{1}{3} \times 3 = -1$.

The equation of l is

$y - 3 = 3(x - 2)$

$y - 3 = 3x - 6$

So $y = 3x - 3$

The perpendicular line l passes through the point $(2, 3)$ and has gradient 3, so use $y - y_1 = m(x - x_1)$ with $m = 3$ and $(x_1, y_1) = (2, 3)$.

Rearrange the equation into the form $y = mx + c$.

Exercise 6B

1 Find the perpendicular bisector of the line segment joining each pair of points:

  a $A(-5, 8)$ and $B(7, 2)$ b $C(-4, 7)$ and $D(2, 25)$ c $E(3, -3)$ and $F(13, -7)$

  d $P(-4, 7)$ and $Q(-4, -1)$ e $S(4, 11)$ and $T(-5, -1)$ f $X(13, 11)$ and $Y(5, 11)$

E/P 2 The line FG is a diameter of the circle centre C, where F and G are $(-2, 5)$ and $(2, 9)$ respectively. The line l passes through C and is perpendicular to FG. Find the equation of l. **(7 marks)**

P 3 The line JK is a diameter of the circle centre P, where J and K are $(0, -3)$ and $(4, -5)$ respectively. The line l passes through P and is perpendicular to JK. Find the equation of l. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

4 Points A, B, C and D have coordinates $A(-4, -9)$, $B(6, -3)$, $C(11, 5)$ and $D(-1, 9)$.

  a Find the equation of the perpendicular bisector of line segment AB.

  b Find the equation of the perpendicular bisector of line segment CD.

  c Find the coordinates of the point of intersection of the two perpendicular bisectors.

(P) **5** Point X has coordinates $(7, -2)$ and point Y has coordinates $(4, q)$. The perpendicular bisector of XY has equation $y = 4x + b$. Find the value of q and the value of b.

Problem-solving

It is often easier to find unknown values in the order they are given in the question. Find q first, then find b.

Challenge

Triangle PQR has vertices at $P(6, 9)$, $Q(3, -3)$ and $R(-9, 3)$.

a Find the perpendicular bisectors of each side of the triangle.

b Show that all three perpendicular bisectors meet at a single point, and find the coordinates of that point.

Links This point of intersection is called the **circumcentre** of the triangle. → Section 6.5

6.2 Equation of a circle

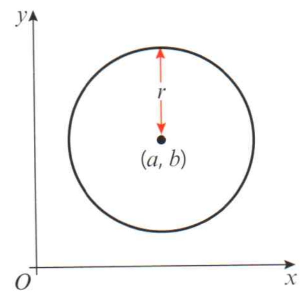
A circle is the set of points that are equidistant from a fixed point. You can use Pythagoras' theorem to derive equations of circles on a coordinate grid.

For any point (x, y) on the circumference of a circle, you can use Pythagoras' theorem to show the relationship between x, y and the radius r.

- **The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.**

When a circle has a centre (a, b) and radius r, you can use the following general form of the equation of a circle.

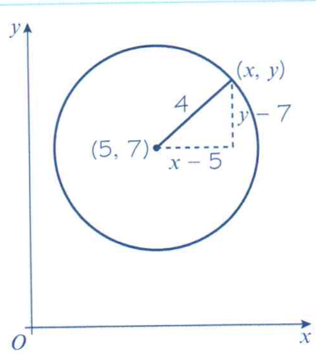
- **The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.**



Links This circle is a translation of the circle $x^2 + y^2 = r^2$ by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$. ← Section 4.5

Example 4

Write down the equation of the circle with centre (5, 7) and radius 4.



Online Explore the general form of the equation of a circle using technology.

$(x - 5)^2 + (y - 7)^2 = 4^2$ — Substitute $a = 5$, $b = 7$ and $r = 4$ into the equation.

$(x - 5)^2 + (y - 7)^2 = 16$ — Simplify by calculating $4^2 = 16$.

Example 5

A circle has equation $(x-3)^2 + (y+4)^2 = 20$.

a Write down the centre and radius of the circle.

b Show that the circle passes through $(5, -8)$.

a Centre $(3, -4)$, radius $\sqrt{20} = 2\sqrt{5}$

b $(x-3)^2 + (y+4)^2 = 20$

Substitute $(5, -8)$

$(5-3)^2 + (-8+4)^2 = 2^2 + (-4)^2$

$= 4 + 16$

$= 20$

So the circle passes through the point $(5, -8)$.

$r^2 = 20$ so $r = \sqrt{20}$

Substitute $x = 5$ and $y = -8$ into the equation of the circle.

$(5, -8)$ satisfies the equation of the circle.

Example 6

The line segment AB is a diameter of a circle, where A and B are $(4, 7)$ and $(-8, 3)$ respectively. Find the equation of the circle.

Length of $AB = \sqrt{(4-(-8))^2 + (7-3)^2}$

$= \sqrt{12^2 + 4^2}$

$= \sqrt{160}$

$= \sqrt{16} \times \sqrt{10}$

$= 4\sqrt{10}$

So the radius is $2\sqrt{10}$.

The centre is $\left(\frac{4+(-8)}{2}, \frac{7+3}{2}\right) = (-2, 5)$.

The equation of the circle is

$(x+2)^2 + (y-5)^2 = (2\sqrt{10})^2$

Or $(x+2)^2 + (y-5)^2 = 40$.

Use $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here $(x_1, y_1) = (-8, 3)$ and $(x_2, y_2) = (4, 7)$

Problem-solving

You need to work out the steps of this problem yourself:

- Find the radius of the circle by finding the length of the diameter and dividing by 2.
- Find the centre of the circle by finding the midpoint of AB.
- Write down the equation of the circle.

Remember the centre of a circle is at the midpoint of a diameter. Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

You can multiply out the brackets in the equation of a circle to find it in an alternate form:

$$(x-a)^2 + (y-b)^2 = r^2$$

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$

$$x^2 + y^2 - 2ax - 2by + b^2 + a^2 - r^2 = 0$$

Compare the constant terms with the equation given in the key point: $b^2 + a^2 - r^2 = c$ so $r = \sqrt{f^2 + g^2 - c}$

- **The equation of a circle can be given in the form:**

  $$\mathbf{x^2 + y^2 + 2fx + 2gy + c = 0}$$

- **This circle has centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$**

If you need to find the centre and radius of a circle with an equation given in expanded form it is usually safest to **complete the square** for the x and y terms.

Example 7

Find the centre and the radius of the circle with the equation $x^2 + y^2 - 14x + 16y - 12 = 0$.

Rearrange into the form $(x - a)^2 + (y - b)^2 = r^2$.

$x^2 + y^2 - 14x + 16y - 12 = 0$

$x^2 - 14x + y^2 + 16y - 12 = 0$ (1)

Completing the square for x terms and y terms.

$x^2 - 14x = (x - 7)^2 - 49$

$y^2 + 16y = (y + 8)^2 - 64$

Substituting back into (1)

$(x - 7)^2 - 49 + (y + 8)^2 - 64 - 12 = 0$

$(x - 7)^2 + (y + 8)^2 = 125$

$(x - 7)^2 + (y + 8)^2 = (\sqrt{125})^2$

$\sqrt{125} = \sqrt{25} \times \sqrt{5} = 5\sqrt{5}$

The equation of the circle is

$(x - 7)^2 + (y + 8)^2 = (5\sqrt{5})^2$

The circle has centre $(7, -8)$ and radius $= 5\sqrt{5}$.

Links You need to **complete the square** for the terms in x and for the terms in y. ← Section 2.2

Group the x terms and y terms together.

Move the number terms to the right-hand side of the equation.

Write the equation in the form $(x - a)^2 + (y - b)^2 = r^2$.

Simplify $\sqrt{125}$.

You could also compare the original equation with:
$x^2 + y^2 + 2fx + 2gy + c = 0$
$f = -7$, $g = 8$ and $c = -12$ so the circle has centre $(7, -8)$ and radius $\sqrt{(-7)^2 + 8^2 - (-12)} = 5\sqrt{5}$.

Exercise 6C

1 Write down the equation of each circle:

a Centre (3, 2), radius 4 **b** Centre (−4, 5), radius 6 **c** Centre (5, −6), radius $2\sqrt{3}$

d Centre $(2a, 7a)$, radius $5a$ **e** Centre $(-2\sqrt{2}, -3\sqrt{2})$, radius 1

2 Write down the coordinates of the centre and the radius of each circle:

a $(x + 5)^2 + (y - 4)^2 = 9^2$ **b** $(x - 7)^2 + (y - 1)^2 = 16$ **c** $(x + 4)^2 + y^2 = 25$

d $(x + 4a)^2 + (y + a)^2 = 144a^2$ **e** $(x - 3\sqrt{5})^2 + (y + \sqrt{5})^2 = 27$

3 In each case, show that the circle passes through the given point:

a $(x - 2)^2 + (y - 5)^2 = 13$, point (4, 8) **b** $(x + 7)^2 + (y - 2)^2 = 65$, point (0, −2)

c $x^2 + y^2 = 25^2$, point (7, −24) **d** $(x - 2a)^2 + (y + 5a)^2 = 20a^2$, point $(6a, -3a)$

e $(x - 3\sqrt{5})^2 + (y - \sqrt{5})^2 = (2\sqrt{10})^2$ point, $(\sqrt{5}, -\sqrt{5})$

(P) **4** The point (4, −2) lies on the circle centre (8, 1).
Find the equation of the circle.

Hint First find the radius of the circle.

E/P **5** The line PQ is the diameter of the circle, where P and Q are (5, 6) and (−2, 2) respectively. Find the equation of the circle. **(5 marks)**

E/P **6** The point (1, −3) lies on the circle $(x-3)^2 + (y+4)^2 = r^2$. Find the value of r. **(3 marks)**

E/P **7** The points $P(2, 2)$, $Q(2+\sqrt{3}, 5)$ and $R(2-\sqrt{3}, 5)$ lie on the circle $(x-2)^2 + (y-4)^2 = r^2$.

a Find the value of r. **(2 marks)**

b Show that $\triangle PQR$ is equilateral. **(3 marks)**

E/P **8** **a** Show that $x^2 + y^2 - 4x - 11 = 0$ can be written in the form $(x-a)^2 + y^2 = r^2$, where a and r are numbers to be found. **(2 marks)**

b Hence write down the centre and radius of the circle with equation $x^2 + y^2 - 4x - 11 = 0$ **(2 marks)**

Problem-solving

Start by writing $(x^2 - 4x)$ in the form $(x-a)^2 - b$.

E/P **9** **a** Show that $x^2 + y^2 - 10x + 4y - 20 = 0$ can be written in the form $(x-a)^2 + (y-b)^2 = r^2$, where a, b and r are numbers to be found. **(2 marks)**

b Hence write down the centre and radius of the circle with equation $x^2 + y^2 - 10x + 4y - 20 = 0$. **(2 marks)**

10 Find the centre and radius of the circle with each of the following equations.

a $x^2 + y^2 - 2x + 8y - 8 = 0$

b $x^2 + y^2 + 12x - 4y = 9$

c $x^2 + y^2 - 6y = 22x - 40$

d $x^2 + y^2 + 5x - y + 4 = 2y + 8$

e $2x^2 + 2y^2 - 6x + 5y = 2x - 3y - 3$

Hint Start by writing the equation in one of the following forms:
$(x-a)^2 + (y-b)^2 = r^2$
$x^2 + y^2 + 2fx + 2gy + c = 0$

E/P **11** A circle C has equation $x^2 + y^2 + 12x + 2y = k$, where k is a constant.

a Find the coordinates of the centre of C. **(2 marks)**

b State the range of possible values of k. **(2 marks)**

Problem-solving

A circle must have a positive radius.

E/P **12** The point $P(7, -14)$ lies on the circle with equation $x^2 + y^2 + 6x - 14y = 483$. The point Q also lies on the circle such that PQ is a diameter. Find the coordinates of point Q. **(4 marks)**

E/P **13** The circle with equation $(x-k)^2 + y^2 = 41$ passes through the point (3, 4). Find the two possible values of k. **(5 marks)**

Challenge

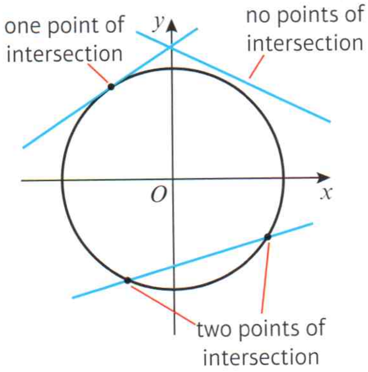
1 A circle with equation $(x-k)^2 + (y-2)^2 = 50$ passes through the point (4, −5). Find the possible values of k and the equation of each circle.

2 By completing the square for x and y, show that the equation $x^2 + y^2 + 2fx + 2gy + c = 0$ describes a circle with centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$.

6.3 Intersections of straight lines and circles

You can use algebra to find the coordinates of intersection of a straight line and a circle.

- **A straight line can intersect a circle once, by just touching the circle, or twice.**
  Not all straight lines will intersect a given circle.



Example 8

Find the coordinates of the points where the line $y = x + 5$ meets the circle $x^2 + (y - 2)^2 = 29$.

Online Explore intersections of straight lines and circles using GeoGebra.

$$x^2 + (y - 2)^2 = 29$$
$$x^2 + (x + 5 - 2)^2 = 29$$

Solve the equations simultaneously, so substitute $y = x + 5$ into the equation of the circle. ← Section 3.2

$$x^2 + (x + 3)^2 = 29$$
$$x^2 + x^2 + 6x + 9 = 29$$
$$2x^2 + 6x - 20 = 0$$
$$x^2 + 3x - 10 = 0$$
$$(x + 5)(x - 2) = 0$$

Simplify the equation to form a quadratic equation.

So $x = -5$ and $x = 2$.

The resulting quadratic equation has two distinct solutions, so the line intersects the circle at two distinct points.

$x = -5$: $y = -5 + 5 = 0$

$x = 2$: $y = 2 + 5 = 7$

Now find the y-coordinates, so substitute the values of x into the equation of the line.

The line meets the circle at $(-5, 0)$ and $(2, 7)$.

Remember to write the answers as coordinates.

Example 9

Show that the line $y = x - 7$ does not meet the circle $(x + 2)^2 + y^2 = 33$.

$$(x + 2)^2 + y^2 = 33$$
$$(x + 2)^2 + (x - 7)^2 = 33$$

Try to solve the equations simultaneously, so substitute $y = x - 7$ into the equation of the circle.

$$x^2 + 4x + 4 + x^2 - 14x + 49 = 33$$
$$2x^2 - 10x + 20 = 0$$
$$x^2 - 5x + 10 = 0$$

Now $b^2 - 4ac = (-5)^2 - 4 \times 1 \times 10$

$= 25 - 40$

$= -15$

Use the discriminant $b^2 - 4ac$ to test for roots of the quadratic equation. ← Section 2.5

$b^2 - 4ac < 0$, so the line does not meet the circle.

Problem-solving

If $b^2 - 4ac > 0$ there are two distinct roots.
If $b^2 - 4ac = 0$ there is a repeated root.
If $b^2 - 4ac < 0$ there are no real roots.

Exercise 6D

1 Find the coordinates of the points where the circle $(x-1)^2 + (y-3)^2 = 45$ meets the x-axis.

Hint Substitute $y = 0$ into the equation.

2 Find the coordinates of the points where the circle $(x-2)^2 + (y+3)^2 = 29$ meets the y-axis.

3 The line $y = x + 4$ meets the circle $(x-3)^2 + (y-5)^2 = 34$ at A and B. Find the coordinates of A and B.

4 Find the coordinates of the points where the line $x + y + 5 = 0$ meets the circle $x^2 + 6x + y^2 + 10y - 31 = 0$.

(P) 5 Show that the line $x - y - 10 = 0$ does not meet the circle $x^2 - 4x + y^2 = 21$.

Problem-solving
Attempt to solve the equations simultaneously. Use the discriminant to show that the resulting quadratic equation has no solutions.

(E/P) 6 a Show that the line $x + y = 11$ meets the circle with equation $x^2 + (y-3)^2 = 32$ at only one point. **(4 marks)**

b Find the coordinates of the point of intersection. **(1 mark)**

(E/P) 7 The line $y = 2x - 2$ meets the circle $(x-2)^2 + (y-2)^2 = 20$ at A and B.

a Find the coordinates of A and B. **(5 marks)**

b Show that AB is a diameter of the circle. **(2 marks)**

(E/P) 8 The line $x + y = a$ meets the circle $(x-p)^2 + (y-6)^2 = 20$ at $(3, 10)$, where a and p are constants.

a Work out the value of a. **(1 mark)**

b Work out the two possible values of p. **(5 marks)**

(E/P) 9 The circle with equation $(x-4)^2 + (y+7)^2 = 50$ meets the straight line with equation $x - y - 5 = 0$ at points A and B.

a Find the coordinates of the points A and B. **(5 marks)**

b Find the equation of the perpendicular bisector of line segment AB. **(3 marks)**

c Show that the perpendicular bisector of AB passes through the centre of the circle. **(1 mark)**

d Find the area of triangle OAB. **(2 marks)**

(E/P) 10 The line with equation $y = kx$ intersects the circle with equation $x^2 - 10x + y^2 - 12y + 57 = 0$ at two distinct points.

a Show that $21k^2 - 60k + 32 < 0$. **(5 marks)**

b Determine the range of possible values for k. Round your answer to 2 decimal places. **(3 marks)**

(E/P) 11 The line with equation $y = 4x - 1$ does not intersect the circle with equation $x^2 + 2x + y^2 = k$. Find the range of possible values of k.

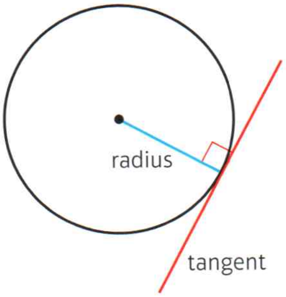
(E/P) 12 The line with equation $y = 2x + 5$ meets the circle with equation $x^2 + kx + y^2 = 4$ at exactly one point. Find two possible values of k. **(7 marks)**

Problem-solving
If you are solving a problem where there are 0, 1 or 2 solutions (or points of intersection), you might be able to use the discriminant.

6.4 Use tangent and chord properties

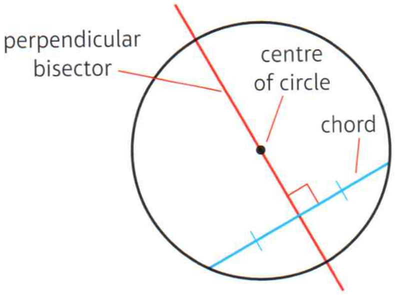
You can use the properties of tangents and chords within circles to solve geometric problems. A tangent to a circle is a straight line that intersects the circle at only one point.

- **A tangent to a circle is perpendicular to the radius of the circle at the point of intersection.**



A chord is a line segment that joins two points on the circumference of a circle.

- **The perpendicular bisector of a chord will go through the centre of a circle.**



Online Explore the circle theorems using GeoGebra.

Example 10

The circle C has equation $(x - 2)^2 + (y - 6)^2 = 100$.

a Verify that the point $P(10, 0)$ lies on C.

b Find an equation of the tangent to C at the point $(10, 0)$, giving your answer in the form $ax + by + c = 0$.

a $(x - 2)^2 + (y - 6)^2 = (10 - 2)^2 + (0 - 6)^2$

$= 8^2 + (-6)^2$ — Substitute $(x, y) = (10, 0)$ into the equation for the circle.

$= 64 + 36$

$= 100$ ✓ — The point $P(10, 0)$ satisfies the equation, so P lies on C.

b The centre of circle C is $(2, 6)$. Find the gradient of the line between $(2, 6)$ and P. — A circle with equation $(x - a)^2 + (y - b)^2 = r^2$ has centre (a, b).

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 0}{2 - 10} = \frac{6}{-8} = -\frac{3}{4}$ — Use the gradient formula with $(x_1, y_1) = (10, 0)$ and $(x_2, y_2) = (2, 6)$

The gradient of the tangent is $\frac{4}{3}$ — The tangent is perpendicular to the radius at that point. If the gradient of the radius is m then the gradient of the tangent will be $-\frac{1}{m}$

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{4}{3}(x - 10)$ — Substitute $(x_1, y_1) = (10, 0)$ and $m = \frac{4}{3}$ into the equation for a straight line.

$3y = 4x - 40$ — Simplify.

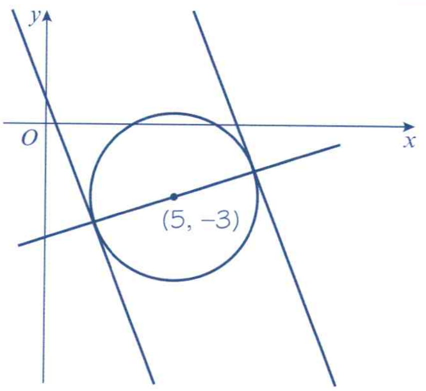
$4x - 3y - 40 = 0$ — Leave the answer in the correct form.

Example 11

A circle C has equation $(x - 5)^2 + (y + 3)^2 = 10$.

The line l is a tangent to the circle and has gradient -3.

Find two possible equations for l, giving your answers in the form $y = mx + c$.



Problem-solving

Draw a sketch showing the circle and the two possible tangents with gradient −3. If you are solving a problem involving tangents and circles there is a good chance you will need to use the radius at the point of intersection, so draw this on your sketch.

Find a line that passes through the centre of the circle that is perpendicular to the tangents.

This line will intersect the circle at the same points where the tangent intersects the circle.

The gradient of this line is $\frac{1}{3}$

The gradient of the tangents is −3, so the gradient of a perpendicular line will be $\frac{-1}{-3} = \frac{1}{3}$

The coordinates of the centre of circle are $(5, -3)$

A circle with equation $(x - a)^2 + (y - b)^2 = r^2$ has centre (a, b).

$$y - y_1 = m(x - x_1)$$
$$y + 3 = \frac{1}{3}(x - 5)$$
$$y + 3 = \frac{1}{3}x - \frac{5}{3}$$

Substitute $(x_1, y_1) = (5, -3)$ and $m = \frac{1}{3}$ into the equation for a straight line.

$$y = \frac{1}{3}x - \frac{14}{3}$$

This is the equation of the line passing through the circle.

$$(x - 5)^2 + (y + 3)^2 = 10$$
$$(x - 5)^2 + \left(\frac{1}{3}x - \frac{14}{3} + 3\right)^2 = 10$$

Substitute $y = \frac{1}{3}x - \frac{14}{3}$ into the equation for a circle to find the points of intersection.

$$(x - 5)^2 + \left(\frac{1}{3}x - \frac{5}{3}\right)^2 = 10$$
$$x^2 - 10x + 25 + \frac{1}{9}x^2 - \frac{10}{9}x + \frac{25}{9} = 10$$

Simplify the expression.

$$\frac{10}{9}x^2 - \frac{100}{9}x + \frac{250}{9} = 10$$
$$10x^2 - 100x + 250 = 90$$
$$10x^2 - 100x + 160 = 0$$
$$x^2 - 10x + 16 = 0$$
$$(x - 8)(x - 2) = 0$$

Factorise to find the values of x.

$$x = 8 \text{ or } x = 2$$
$$y = -\frac{6}{3} = -2 \text{ or } y = -4$$

Substitute $x = 8$ and $x = 2$ into $y = \frac{1}{3}x - \frac{14}{3}$.

So the tangents will intersect the circle at (8, −2) and (2, −4)

$y - y_1 = m(x - x_1)$

$y + 2 = -3(x - 8)$ — Substitute $(x_1, y_1) = (8, -2)$ and $m = -3$ into the equation for a straight line.

$y = -3x + 22$ — This is one possible equation for the tangent.

$y - y_1 = m(x - x_1)$

$y + 4 = -3(x - 2)$ — Substitute $(x_1, y_1) = (2, -4)$ and $m = -3$ into the equation for a straight line.

$y = -3x + 2$ — This is the other possible equation for the tangent.

Example 12

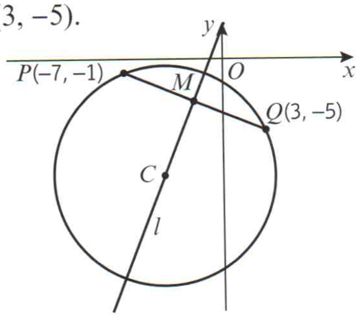
The points P and Q lie on a circle with centre C, as shown in the diagram.
The point P has coordinates (−7, −1) and the point Q has coordinates (3, −5).
M is the midpoint of the line segment PQ.
The line l passes through the points M and C.

a Find an equation for l.

Given that the y-coordinate of C is −8,

b show that the x-coordinate of C is −4

c find an equation of the circle.



a The midpoint M of line segment PQ is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-7 + 3}{2}, \frac{-1 + (-5)}{2}\right)$$

Use the midpoint formula with $(x_1, y_1) = (-7, -1)$ and $(x_2, y_2) = (3, -5)$.

$$= (-2, -3)$$

Gradient of $PQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - (-1)}{3 - (-7)}$

Use the gradient formula with $(x_1, y_1) = (-7, -1)$ and $(x_2, y_2) = (3, -5)$.

$$= \frac{-4}{10} = -\frac{2}{5}$$

The gradient of a line perpendicular to PQ is $\frac{5}{2}$

Problem-solving

If a gradient is given as a fraction, you can find the perpendicular gradient quickly by turning the fraction upside down and changing the sign.

$y - y_1 = m(x - x_1)$

$y + 3 = \frac{5}{2}(x + 2)$ — Substitute $(x_1, y_1) = (-2, -3)$ and $m = \frac{5}{2}$ into the equation of a straight line.

$y + 3 = \frac{5}{2}x + 5$

$y = \frac{5}{2}x + 2$ — Simplify and leave in the form $y = mx + c$.

b $y = \frac{5}{2}x + 2$

$-8 = \frac{5}{2}x + 2$ — The perpendicular bisector of any chord passes through the centre of the circle. Substitute $y = -8$ into the equation of the straight line to find the corresponding x-coordinate.

$\frac{5}{2}x = -10$

$x = -4$ — Solve the equation to find x.

c The centre of the circle is (−4, −8).
To find the radius of the circle:

$CQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3 - (-4))^2 + (-5 - (-8))^2}$

$= \sqrt{49 + 9} = \sqrt{58}$

So the circle has a radius of $\sqrt{58}$.
The equation of the circle is:

$(x - a)^2 + (y - b)^2 = r^2$

$(x + 4)^2 + (y + 8)^2 = 58$

The radius is the length of the line segment CP or CQ.

Substitute $(x_1, y_1) = (-4, -8)$ and $(x_2, y_2) = (3, -5)$.

Substitute $(a, b) = (-4, -8)$ and $r = \sqrt{58}$ into the equation of a circle.

Exercise 6E

1 The line $x + 3y - 11 = 0$ touches the circle $(x + 1)^2 + (y + 6)^2 = r^2$ at (2, 3).

a Find the radius of the circle.

b Show that the radius at (2, 3) is perpendicular to the line.

2 The point $P(1, -2)$ lies on the circle centre (4, 6).

a Find the equation of the circle.

b Find the equation of the tangent to the circle at P.

3 The points A and B with coordinates (−1, −9) and (7, −5) lie on the circle C with equation $(x - 1)^2 + (y + 3)^2 = 40$.

a Find the equation of the perpendicular bisector of the line segment AB.

b Show that the perpendicular bisector of AB passes through the centre of the circle C.

(P) **4** The points P and Q with coordinates (3, 1) and (5, −3) lie on the circle C with equation $x^2 - 4x + y^2 + 4y = 2$.

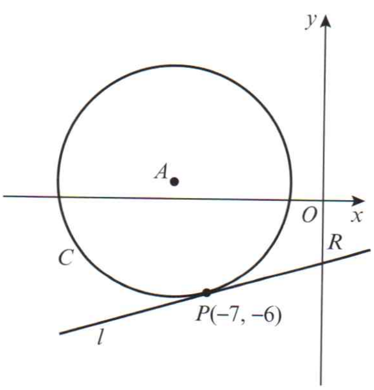
a Find the equation of the perpendicular bisector of the line segment PQ.

b Show that the perpendicular bisector of PQ passes through the centre of the circle C.

(E) **5** The circle C has equation $x^2 + 18x + y^2 - 2y + 29 = 0$.

a Verify the point $P(-7, -6)$ lies on C. **(2 marks)**

b Find an equation for the tangent to C at the point P, giving your answer in the form $y = mx + b$. **(4 marks)**



c Find the coordinates of R, the point of intersection of the tangent and the y-axis. **(2 marks)**

d Find the area of the triangle APR. **(2 marks)**

E/P **6** The tangent to the circle $(x + 4)^2 + (y - 1)^2 = 242$ at $(7, -10)$ meets the y-axis at S and the x-axis at T.

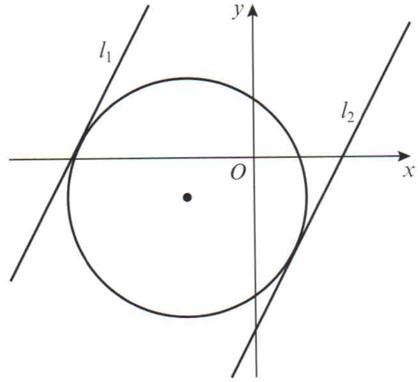
a Find the coordinates of S and T. **(5 marks)**

b Hence, find the area of $\triangle OST$, where O is the origin. **(3 marks)**

E/P **7** The circle C has equation $(x + 5)^2 + (y + 3)^2 = 80$.

The line l is a tangent to the circle and has gradient 2.

Find two possible equations for l giving your answers in the form $y = mx + c$. **(8 marks)**



E/P **8** The line with equation $2x + y - 5 = 0$ is a tangent to the circle with equation

$(x - 3)^2 + (y - p)^2 = 5$

a Find the two possible values of p. **(8 marks)**

b Write down the coordinates of the centre of the circle in each case. **(2 marks)**

Problem-solving

The line is a tangent to the circle so it must intersect at exactly one point. You can use the discriminant to determine the values of p for which this occurs.

E/P **9** The circle C has centre $P(11, -5)$ and passes through the point $Q(5, 3)$.

a Find an equation for C. **(3 marks)**

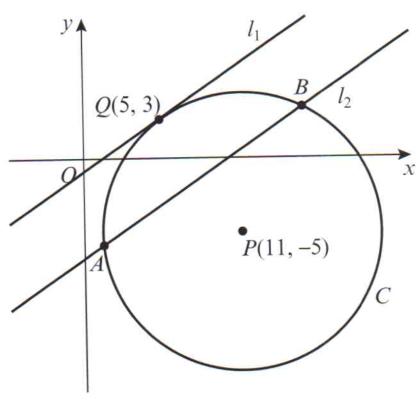
The line l_1 is a tangent to C at the point Q.

b Find an equation for l_1. **(4 marks)**

The line l_2 is parallel to l_1 and passes through the midpoint of PQ. Given that l_2 intersects C at A and B

c find the coordinates of points A and B **(4 marks)**

d find the length of the line segment AB, leaving your answer in its simplest surd form. **(3 marks)**



E/P **10** The points R and S lie on a circle with centre $C(a, -2)$, as shown in the diagram.

The point R has coordinates $(2, 3)$ and the point S has coordinates $(10, 1)$.

M is the midpoint of the line segment RS.

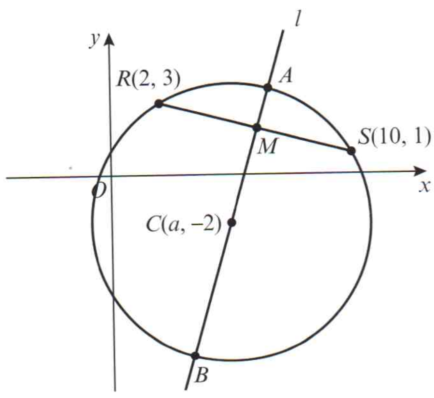
The line l passes through M and C.

a Find an equation for l. **(4 marks)**

b Find the value of a. **(2 marks)**

c Find the equation of the circle. **(3 marks)**

d Find the points of intersection, A and B, of the line l and the circle. **(5 marks)**



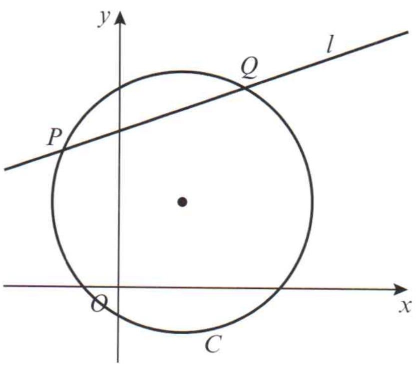
E/P **11** The circle C has equation $x^2 - 4x + y^2 - 6y = 7$.

The line l with equation $x - 3y + 17 = 0$ intersects the circle at the points P and Q.

a Find the coordinates of the point P and the point Q. **(4 marks)**

b Find the equation of the tangent at the point P and the point Q. **(4 marks)**

c Find the equation of the perpendicular bisector of the chord PQ. **(3 marks)**



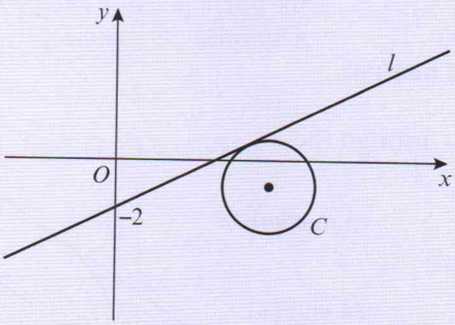
d Show that the two tangents and the perpendicular bisector intersect at a single point and find the coordinates of the point of intersection. **(2 marks)**

Challenge

1 The circle C has equation $(x - 7)^2 + (y + 1)^2 = 5$.

The line l with positive gradient passes through $(0, -2)$ and is a tangent to the circle.

Find an equation of l, giving your answer in the form $y = mx + c$.



2 The circle with centre C has equation $(x - 2)^2 + (y - 1)^2 = 10$.

The tangents to the circle at points P and Q meet at the point R with coordinates $(6, -1)$.

a Show that $CPRQ$ is a square.

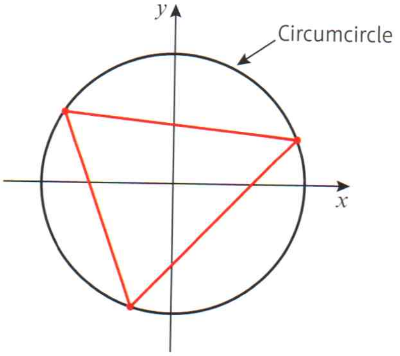
b Hence find the equations of both tangents.

Problem-solving

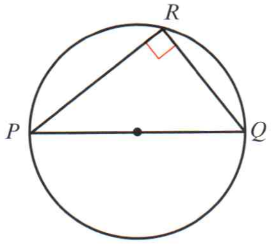
Use the point $(0, -2)$ to write an equation for the tangent in terms of m. Substitute this equation into the equation for the circle.

6.5 Circles and triangles

A triangle consists of three points, called vertices.
It is always possible to draw a unique circle through the three vertices of any triangle. This circle is called the **circumcircle** of the triangle. The centre of the circle is called the **circumcentre** of the triangle and is the point where the perpendicular bisectors of each side intersect.



For a right-angled triangle, the hypotenuse of the triangle is a diameter of the circumcircle.

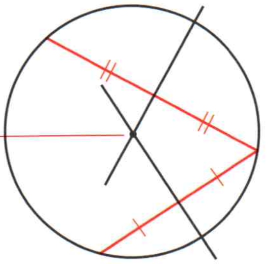


You can state this result in two other ways:

- **If $\angle PRQ = 90°$ then R lies on the circle with diameter PQ.**
- **The angle in a semicircle is always a right angle.**

To find the centre of a circle given any three points on the circumference:

- **Find the equations of the perpendicular bisectors of two different chords.**
- **Find the coordinates of the point of intersection of the perpendicular bisectors.**

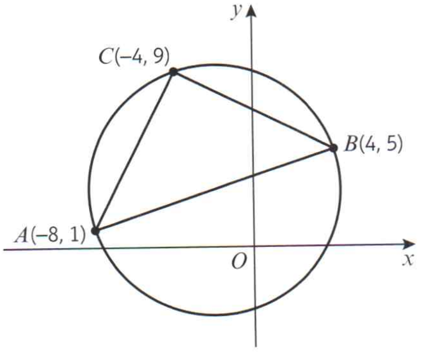


Example 13

The points $A(-8, 1)$, $B(4, 5)$ and $C(-4, 9)$ lie on the circle, as shown in the diagram.

a Show that AB is a diameter of the circle.

b Find an equation of the circle.



a Test triangle ABC to see if it is a right-angled triangle.

$AB^2 = (4-(-8))^2 + (5-1)^2$
$= 12^2 + 4^2 = 160$

$AC^2 = (-4-(-8))^2 + (9-1)^2$
$= 4^2 + 8^2 = 80$

$BC^2 = (-4-4)^2 + (9-5)^2$
$= (-8)^2 + 4^2 = 80$

Use $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ to determine the length of each side of the triangle ABC.

Now, $80 + 80 = 160$ so $AC^2 + BC^2 = AB^2$

Use Pythagoras' theorem to test if triangle ABC is a right-angled triangle.

So triangle ABC is a right-angled triangle and AB is the diameter of the circle.

If ABC is a right-angled triangle, its longest side must be a diameter of the circle that passes through all three points.

b Find the midpoint M of AB.

The centre of the circle is the midpoint of AB.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-8+4}{2}, \frac{1+5}{2}\right)$$
$$= (-2, 3)$$

Substitute $(x_1, y_1) = (-8, 1)$ and $(x_2, y_2) = (4, 5)$.

The diameter is $\sqrt{160} = 4\sqrt{10}$

From part **a**, $AB^2 = 160$.

The radius is $2\sqrt{10}$

The radius is half the diameter.

$(x-a)^2 + (y-b)^2 = r^2$

$(x+2)^2 + (y-3)^2 = (2\sqrt{10})^2$

$(x+2)^2 + (y-3)^2 = 40$

Substitute $(a, b) = (-2, 3)$ and $r = 2\sqrt{10}$ into the equation for a circle.

Example 14

The points $P(3, 16)$, $Q(11, 12)$ and $R(-7, 6)$ lie on the circumference of a circle. The equation of the perpendicular bisector of PQ is $y = 2x$.

a Find the equation of the perpendicular bisector of PR.

b Find the centre of the circle.

c Work out the equation of the circle.

Online Explore triangles and their circumcircles using GeoGebra.

a The midpoint of PR is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{3 + (-7)}{2}, \frac{16 + 6}{2}\right) = (-2, 11)$

> The perpendicular bisector of PR passes through the midpoint of PR.
> Substitute $(x_1, y_1) = (3, 16)$ and $(x_2, y_2) = (-7, 6)$ into the midpoint formula.

The gradient of PR is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 16}{-7 - 3}$

$= \frac{-10}{-10} = 1$

> Substitute $(x_1, y_1) = (3, 16)$ and $(x_2, y_2) = (-7, 6)$ into the gradient formula.

The gradient of a line perpendicular to PR is -1.

$y - y_1 = m(x - x_1)$

$y - 11 = -1(x - (-2))$

> Substitute $m = -1$ and $(x_1, y_1) = (-2, 11)$ into the equation for a straight line.

$y - 11 = -x - 2$

$y = -x + 9$

> Simplify and leave in the form $y = mx + c$.

b Equation of perpendicular bisector to PQ: $y = 2x$

Equation of perpendicular bisector to PR: $y = -x + 9$

> Solve these two equations simultaneously to find the point of intersection. The two perpendicular bisectors intersect at the centre of the circle.

$2x = -x + 9$

$3x = 9$

$x = 3$

> This is the x-coordinate of the centre of the circle.

$y = 2x$

$y = 2(3) = 6$

> Substitute $x = 3$ to find the y-coordinate of the centre of the circle.

The centre of the circle is at $(3, 6)$.

c Find the distance between $(3, 6)$ and $Q(11, 12)$.

> The radius of the circle is the distance from the centre to a point on the circumference of the circle.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(11 - 3)^2 + (12 - 6)^2}$

> Substitute $(x_1, y_1) = (3, 6)$ and $(x_2, y_2) = (11, 12)$ into the distance formula.

$d = \sqrt{64 + 36}$

$d = \sqrt{100} = 10$

> Simplify to find the radius of the circle.

The circle through the points P, Q and R has a radius of 10.

The centre of the circle is $(3, 6)$.

The equation for the circle is

$(x - 3)^2 + (y - 6)^2 = 100$

> Substitute $(a, b) = (3, 6)$ and $r = 10$ into $(x - a)^2 + (y - b)^2 = r^2$

Exercise 6F

1 The points $U(-2, 8)$, $V(7, 7)$ and $W(-3, -1)$ lie on a circle.

a Show that triangle UVW has a right angle.

b Find the coordinates of the centre of the circle.

c Write down an equation for the circle.

2 The points $A(2, 6)$, $B(5, 7)$ and $C(8, -2)$ lie on a circle.

a Show that AC is a diameter of the circle.

b Write down an equation for the circle.

c Find the area of the triangle ABC.

3 The points $A(-3, 19)$, $B(9, 11)$ and $C(-15, 1)$ lie on the circumference of a circle.

a Find the equation of the perpendicular bisector of

i AB **ii** AC

b Find the coordinates of the centre of the circle.

c Write down an equation for the circle.

4 The points $P(-11, 8)$, $Q(-6, -7)$ and $R(4, -7)$ lie on the circumference of a circle.

a Find the equation of the perpendicular bisector of

i PQ **ii** QR

b Find an equation for the circle.

(P) 5 The points $R(-2, 1)$, $S(4, 3)$ and $T(10, -5)$ lie on the circumference of a circle C. Find an equation for the circle.

Problem-solving

Use headings in your working to keep track of what you are working out at each stage.

(P) 6 Consider the points $A(3, 15)$, $B(-13, 3)$, $C(-7, -5)$ and $D(8, 0)$.

a Show that ABC is a right-angled triangle.

b Find the equation of the circumcircle of triangle ABC.

c Hence show that A, B, C and D all lie on the circumference of this circle.

(P) 7 The points $A(-1, 9)$, $B(6, 10)$, $C(7, 3)$ and $D(0, 2)$ lie on a circle.

a Show that $ABCD$ is a square.

b Find the area of $ABCD$.

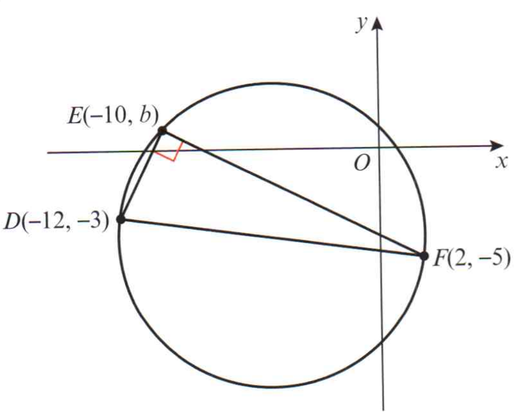
c Find the centre of the circle.

(E/P) 8 The points $D(-12, -3)$, $E(-10, b)$ and $F(2, -5)$ lie on the circle C as shown in the diagram.

Given that $\angle DEF = 90°$ and $b > 0$

a show that $b = 1$ **(5 marks)**

b find an equation for C. **(4 marks)**



E/P **9** A circle has equation $x^2 + 2x + y^2 - 24y - 24 = 0$

a Find the centre and radius of the circle. **(3 marks)**

b The points $A(-13, 17)$ and $B(11, 7)$ both lie on the circumference of the circle. Show that AB is a diameter of the circle. **(3 marks)**

c The point C lies on the negative x-axis and the angle $ACB = 90°$. Find the coordinates of C. **(3 marks)**

Mixed exercise 6

P **1** The line segment QR is a diameter of the circle centre C, where Q and R have coordinates (11, 12) and (−5, 0) respectively. The point P has coordinates (13, 6).

a Find the coordinates of C.

b Find the radius of the circle.

c Write down the equation of the circle.

d Show that P lies on the circle.

P **2** Show that (0, 0) lies inside the circle $(x - 5)^2 + (y + 2)^2 = 30$.

E/P **3** The circle C has equation $x^2 + 3x + y^2 + 6y = 3x - 2y - 7$.

a Find the centre and radius of the circle. **(4 marks)**

b Find the points of intersection of the circle and the y-axis. **(3 marks)**

c Show that the circle does not intersect the x-axis. **(2 marks)**

4 The centres of the circles $(x - 8)^2 + (y - 8)^2 = 117$ and $(x + 1)^2 + (y - 3)^2 = 106$ are P and Q respectively.

a Show that P lies on $(x + 1)^2 + (y - 3)^2 = 106$.

b Find the length of PQ.

P **5** The points $A(-1, 0)$, $B\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $C\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ are the vertices of a triangle.

a Show that the circle $x^2 + y^2 = 1$ passes through the vertices of the triangle.

b Show that $\triangle ABC$ is equilateral.

E/P **6** A circle with equation $(x - k)^2 + (y - 3k)^2 = 13$ passes through the point (3, 0).

a Find two possible values of k. **(6 marks)**

b Given that $k > 0$, write down the equation of the circle. **(1 mark)**

E/P **7** The line with $3x - y - 9 = 0$ does not intersect the circle with equation $x^2 + px + y^2 + 4y = 20$. Show that $42 - 10\sqrt{10} < p < 42 + 10\sqrt{10}$. **(6 marks)**

P **8** The line $y = 2x - 8$ meets the coordinate axes at A and B. The line segment AB is a diameter of the circle. Find the equation of the circle.

P **9** The circle centre (8, 10) meets the x-axis at (4, 0) and $(a, 0)$.

a Find the radius of the circle.

b Find the value of a.

10 The circle $(x - 5)^2 + y^2 = 36$ meets the x-axis at P and Q. Find the coordinates of P and Q.

11 The circle $(x + 4)^2 + (y - 7)^2 = 121$ meets the y-axis at $(0, m)$ and $(0, n)$. Find the values of m and n.

(E) **12** The circle C with equation $(x + 5)^2 + (y + 2)^2 = 125$ meets the positive coordinate axes at $A(a, 0)$ and $B(0, b)$.

a Find the values of a and b. **(2 marks)**

b Find the equation of the line AB. **(2 marks)**

c Find the area of the triangle OAB, where O is the origin. **(2 marks)**

(P) **13** The circle, centre (p, q) radius 25, meets the x-axis at $(-7, 0)$ and $(7, 0)$, where $q > 0$.

a Find the values of p and q.

b Find the coordinates of the points where the circle meets the y-axis.

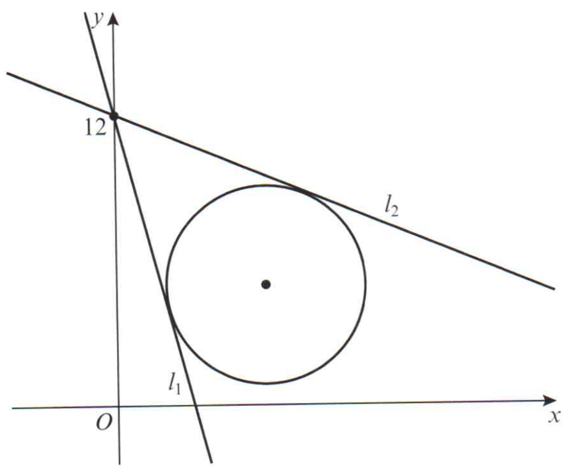
(P) **14** The point $A(-3, -7)$ lies on the circle centre $(5, 1)$. Find the equation of the tangent to the circle at A.

(P) **15** The line segment AB is a chord of a circle centre $(2, -1)$, where A and B are $(3, 7)$ and $(-5, 3)$ respectively. AC is a diameter of the circle. Find the area of $\triangle ABC$.

(E/P) **16** The circle C has equation $(x - 6)^2 + (y - 5)^2 = 17$.

The lines l_1 and l_2 are each a tangent to the circle and intersect at the point $(0, 12)$.

Find the equations of l_1 and l_2, giving your answers in the form $y = mx + c$. **(8 marks)**



(E/P) **17** The points A and B lie on a circle with centre C, as shown in the diagram.

The point A has coordinates $(3, 7)$ and the point B has coordinates $(5, 1)$.

M is the midpoint of the line segment AB.

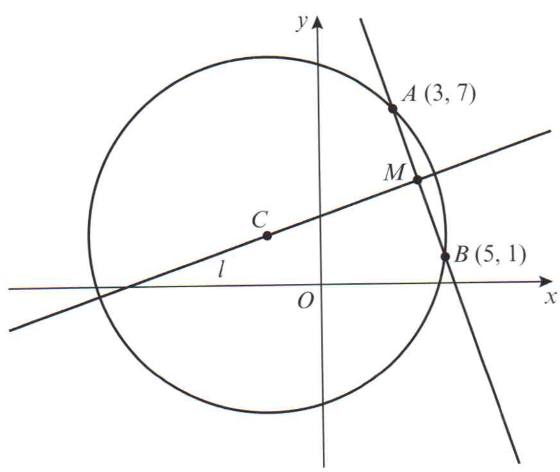
The line l passes through the points M and C.

a Find an equation for l. **(4 marks)**

Given that the x-coordinate of C is -2:

b find an equation of the circle **(4 marks)**

c find the area of the triangle ABC. **(3 marks)**

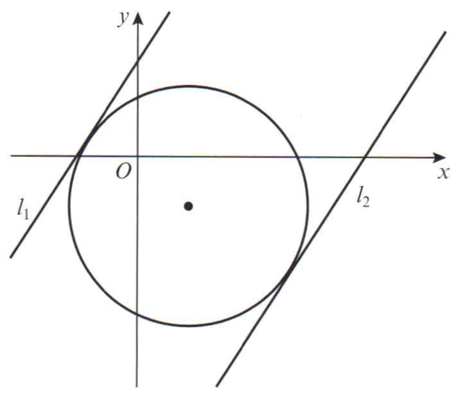


E/P **18** The circle C has equation $(x-3)^2 + (y+3)^2 = 52$.

The baselines l_1 and l_2 are tangents to the circle and have gradient $\frac{3}{2}$

a Find the points of intersection, P and Q, of the tangents and the circle. **(6 marks)**

b Find the equations of lines l_1 and l_2, giving your answers in the form $ax + by + c = 0$. **(2 marks)**



E/P **19** The circle C has equation $x^2 + 6x + y^2 - 2y = 7$.

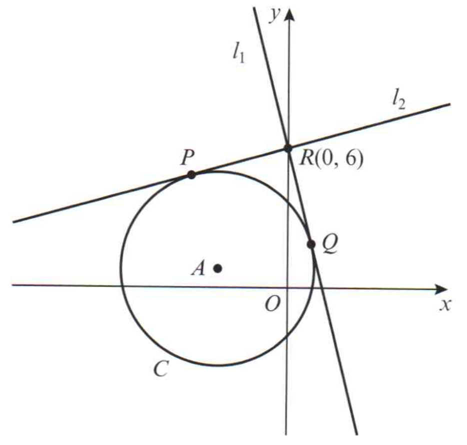
The lines l_1 and l_2 are tangents to the circle.

They intersect at the point $R(0, 6)$.

a Find the equations of lines l_1 and l_2, giving your answers in the form $y = mx + b$. **(6 marks)**

b Find the points of intersection, P and Q, of the tangents and the circle. **(4 marks)**

c Find the area of quadrilateral $APRQ$. **(2 marks)**



E/P **20** The circle C has a centre at $(6, 9)$ and a radius of $\sqrt{50}$.

The line l_1 with equation $x + y - 21 = 0$ intersects the circle at the points P and Q.

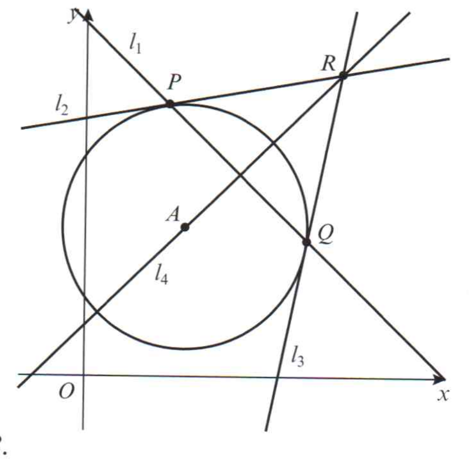
a Find the coordinates of the point P and the point Q. **(5 marks)**

b Find the equations of l_2 and l_3, the tangents at the points P and Q respectively. **(4 marks)**

c Find the equation of l_4, the perpendicular bisector of the chord PQ. **(4 marks)**

d Show that the two tangents and the perpendicular bisector intersect and find the coordinates of R, the point of intersection. **(2 marks)**

e Calculate the area of the kite $APRQ$. **(3 marks)**



P **21** The line $y = -3x + 12$ meets the coordinate axes at A and B.

a Find the coordinates of A and B.

b Find the coordinates of the midpoint of AB.

c Find the equation of the circle that passes through A, B and O, where O is the origin.

E/P **22** The points $A(-3, -2)$, $B(-6, 0)$ and $C(1, q)$ lie on the circumference of a circle such that $\angle BAC = 90°$.

a Find the value of q. **(4 marks)**

b Find the equation of the circle. **(4 marks)**

E/P **23** The points $R(-4, 3)$, $S(7, 4)$ and $T(8, -7)$ lie on the circumference of a circle.

a Show that RT is the diameter of the circle. **(4 marks)**

b Find the equation of the circle. **(4 marks)**

P **24** The points $A(-4, 0)$, $B(4, 8)$ and $C(6, 0)$ lie on the circumference of circle C.

Find the equation of the circle.

P **25** The points $A(-7, 7)$, $B(1, 9)$, $C(3, 1)$ and $D(-7, 1)$ lie on a circle.

a Find the equation of the perpendicular bisector of:

i AB **ii** CD

b Find the equation of the circle.

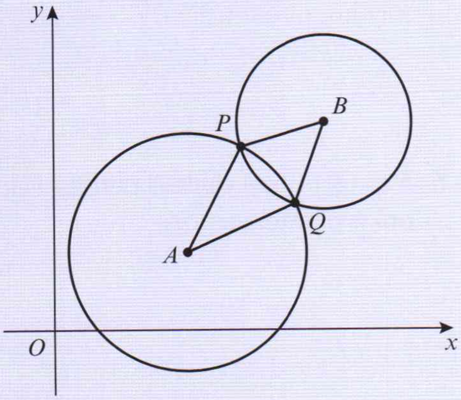
Challenge

The circle with equation $(x - 5)^2 + (y - 3)^2 = 20$ with centre A intersects the circle with equation $(x - 10)^2 + (y - 8)^2 = 10$ with centre B at the points P and Q.

a Find the equation of the line containing the points P and Q in the form $ax + by + c = 0$.

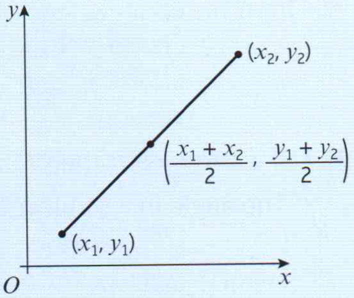
b Find the coordinates of the points P and Q.

c Find the area of the kite $APBQ$.

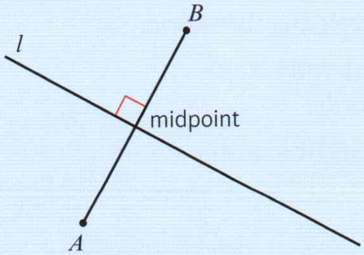


Summary of key points

1 The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



2 The perpendicular bisector of a line segment AB is the straight line that is perpendicular to AB and passes through the midpoint of AB.



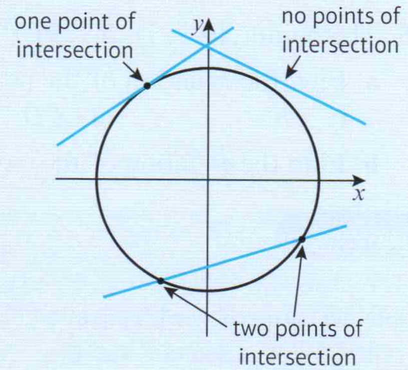
If the gradient of AB is m then the gradient of its perpendicular bisector, l, will be $-\frac{1}{m}$

3 The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

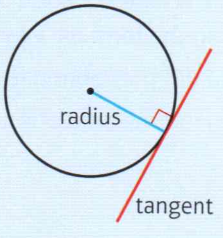
4 The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

5 The equation of a circle can be given in the form: $x^2 + y^2 + 2fx + 2gy + c = 0$
This circle has centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$

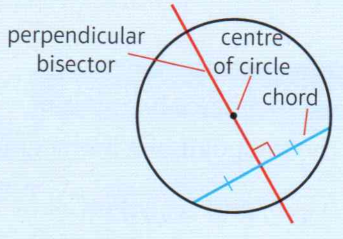
6 A straight line can intersect a circle once, by just touching the circle, or twice. Not all straight lines will intersect a given circle.



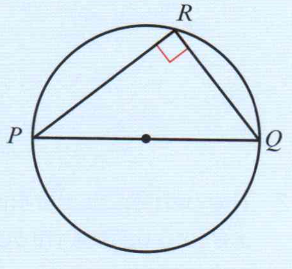
7 A tangent to a circle is perpendicular to the radius of the circle at the point of intersection.



8 The perpendicular bisector of a chord will go through the centre of a circle.



9
- If $\angle PRQ = 90°$ then R lies on the circle with diameter PQ.
- The angle in a semicircle is always a right angle.



10 To find the centre of a circle given any three points:
- Find the equations of the perpendicular bisectors of two different chords.
- Find the coordinates of intersection of the perpendicular bisectors.