Radians

5

Objectives

After completing this unit you should be able to:

- Convert between degrees and radians and apply this to trigonometric graphs and their transformations → pages 114–116
- Know exact values of angles measured in radians → pages 117–118
- Find an arc length using radians → pages 118–122
- Find areas of sectors and segments using radians → pages 122–128
- Solve trigonometric equations in radians → pages 128–132
- Use approximate trigonometric values when θ is small → pages 133–135

Radians are units for measuring angles. They are used in mechanics to describe circular motion, and can be used to work out the distances between the pods around the edge of a Ferris wheel. → Exercise 5B Q13

Prior knowledge check

1 Write down the exact values of the following trigonometric ratios.

a $\cos 120°$ **b** $\sin 225°$ **c** $\tan(-300°)$

d $\sin(-480°)$ ← Year 1, Chapter 10

2 Simplify each of the following expressions.

a $(\tan\theta\cos\theta)^2 + \cos^2\theta$

b $1 - \dfrac{1}{\cos^2\theta}$ **c** $\sqrt{1 - \dfrac{\sin\theta\cos\theta}{\tan\theta}}$

← Year 1, Chapter 10

3 Show that

a $(\sin 2\theta + \cos 2\theta)^2 \equiv 1 + 2\sin 2\theta\cos 2\theta$

b $\dfrac{2}{\sin\theta} - 2\sin\theta \equiv \dfrac{2\cos^2\theta}{\sin\theta}$

← Year 1, Chapter 10

4 Solve the following equations for θ in the interval $0 \leqslant \theta \leqslant 360°$, giving your answers to 3 significant figures where they are not exact.

a $4\cos\theta + 2 = 3$ **b** $2\sin 2\theta = 1$

c $6\tan^2\theta + 10\tan\theta - 4 = \tan\theta$

d $10 + 5\cos\theta = 12\sin^2\theta$

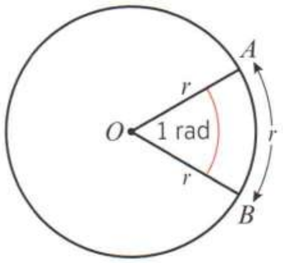
← Year 1, Chapter 10

5.1 Radian measure

So far you have probably only measured angles in degrees, with one degree representing $\frac{1}{360}$ of a complete revolution or circle.

You can also measure angles in units called **radians**. 1 radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.

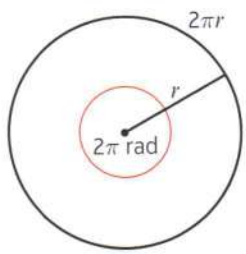
If the arc AB has length r, then $\angle AOB$ is 1 radian.



Links You always use radian measure when you are differentiating or integrating trigonometric functions. → Sections 9.1, 11.1

Notation You can write 1 radian as 1 rad.

The circumference of a circle of radius r is an arc of length $2\pi r$, so it subtends an angle of 2π radians at the centre of the circle.



- **2π radians = 360°**
- **π radians = 180°**
- **1 radian = $\frac{180°}{\pi}$**

Hint This means that 1 radian = 57.295...°

Example 1

Convert the following angles into degrees.

a $\frac{7\pi}{8}$ rad **b** $\frac{4\pi}{15}$ rad

a $\frac{7\pi}{8}$ rad
$= \frac{7}{8} \times 180°$
$= 157.5°$

b $\frac{4\pi}{15}$ rad
$= 4 \times \frac{180°}{15}$
$= 48°$

1 radian = $\frac{180°}{\pi}$, so multiply by $\frac{180°}{\pi}$
$\frac{7\pi}{8} \times \frac{180°}{\pi} = \frac{7}{8} \times 180°$

Example 2

Convert the following angles into radians. Leave your answers in terms of π.

a 150° **b** 110°

a $150° = 150 \times \frac{\pi}{180}$ rad
$= \frac{5\pi}{6}$ rad

b $110° = 110 \times \frac{\pi}{180}$ rad
$= \frac{11\pi}{18}$ rad

$1° = \frac{\pi}{180}$ radians, so multiply by $\frac{\pi}{180}$

Your calculator will often give you exact answers in terms of π.

You should learn these important angles in radians:

- **$30° = \frac{\pi}{6}$ radians**
- **$60° = \frac{\pi}{3}$ radians**
- **$180° = \pi$ radians**
- **$45° = \frac{\pi}{4}$ radians**
- **$90° = \frac{\pi}{2}$ radians**
- **$360° = 2\pi$ radians**

Example 3

Find: **a** $\sin(0.3 \text{ rad})$ **b** $\cos(\pi \text{ rad})$ **c** $\tan(2 \text{ rad})$

Give your answers correct to 2 decimal places where appropriate.

a $\sin(0.3 \text{ rad}) = 0.30$ (2 d.p.)
b $\cos(\pi \text{ rad}) = -1$
c $\tan(2 \text{ rad}) = -2.19$ (2 d.p.)

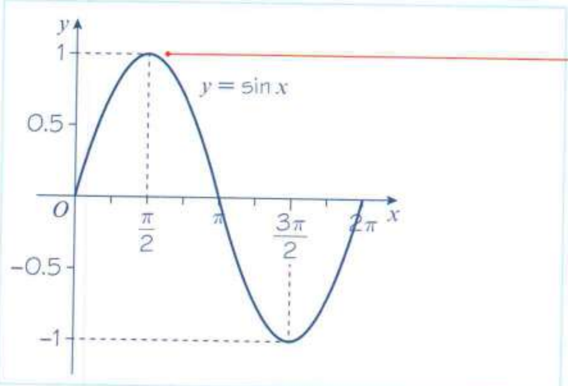
Online Use your calculator to evaluate trigonometric functions in radians.

Watch out You need to make sure your calculator is in radians mode.

Example 4

Sketch the graph of $y = \sin x$ for $0 \leqslant x \leqslant 2\pi$.

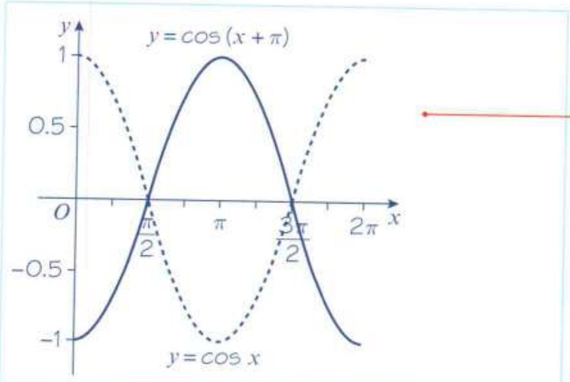
If the range includes values given in terms of π, you can assume that the angle has been given in radians.



$\sin\left(\frac{\pi}{2}\right) = \sin 90° = 1$

Example 5

Sketch the graph of $y = \cos(x + \pi)$ for $0 \leqslant x \leqslant 2\pi$.



The graph of $y = \cos(x + a)$ is a translation of the graph $y = \cos x$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

Exercise 5A

1 Convert the following angles in radians to degrees.

a $\frac{\pi}{20}$ **b** $\frac{\pi}{15}$ **c** $\frac{5\pi}{12}$ **d** $\frac{5\pi}{4}$ **e** $\frac{3\pi}{2}$ **f** 3π

2 Convert the following angles to degrees, giving your answer to 1 d.p.

a 0.46 rad **b** 1 rad **c** 1.135 rad **d** $\sqrt{3}$ rad

3 Evaluate the following, giving your answers to 3 significant figures.

a $\sin(0.5\text{ rad})$ **b** $\cos(\sqrt{2}\text{rad})$ **c** $\tan(1.05\text{ rad})$ **d** $\sin(2\text{ rad})$ **e** $\sin(3.6\text{ rad})$

4 Convert the following angles to radians, giving your answers as multiples of π.

a 8° **b** 10° **c** 22.5° **d** 30° **e** 112.5°
f 240° **g** 270° **h** 315° **i** 330°

5 Convert the following angles to radians, giving your answers to 3 significant figures.

a 50° **b** 75° **c** 100° **d** 160° **e** 230° **f** 320°

6 Sketch the graphs of:

a $y = \tan x$ for $0 \leqslant x \leqslant 2\pi$ **b** $y = \cos x$ for $-\pi \leqslant x \leqslant \pi$

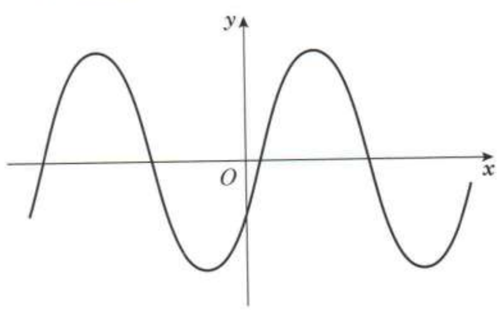
Mark any points where the graphs cut the coordinate axes.

7 Sketch the following graphs for the given ranges, marking any points where the graphs cut the coordinate axes.

a $y = \sin(x - \pi)$ for $-\pi \leqslant x \leqslant \pi$ **b** $y = \cos 2x$ for $0 \leqslant x \leqslant 2\pi$

c $y = \tan\left(x + \frac{\pi}{2}\right)$ for $-\pi \leqslant x \leqslant \pi$ **d** $y = \sin\frac{1}{3}x + 1$ for $0 \leqslant x \leqslant 6\pi$

E/P **8** The diagram shows the curve with equation $y = \cos\left(x - \frac{2\pi}{3}\right)$, $-2\pi \leqslant x \leqslant 2\pi$.



Problem-solving

Make sure you write down the coordinates of all four points of intersection with the x-axis and the coordinates of the y-intercept.

Write down the coordinates of the points at which the curve meets the coordinate axes. **(3 marks)**

Challenge

Describe all the angles, θ, in radians, that satisfy:

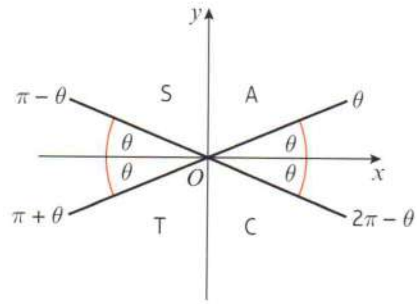
a $\cos\theta = 1$
b $\sin\theta = -1$
c $\tan\theta$ is undefined.

Hint You can use $n\pi$, where n is an integer, to describe any integer multiple of π.

You need to learn the exact values of the trigonometric ratios of these angles measured in radians:

- $\sin\frac{\pi}{6} = \frac{1}{2}$
- $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
- $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
- $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
- $\cos\frac{\pi}{3} = \frac{1}{2}$
- $\tan\frac{\pi}{3} = \sqrt{3}$
- $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\tan\frac{\pi}{4} = 1$

You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angle made with the x-axis, θ.



- $\sin(\pi - \theta) = \sin\theta$
- $\sin(\pi + \theta) = -\sin\theta$
- $\sin(2\pi - \theta) = -\sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(\pi + \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$

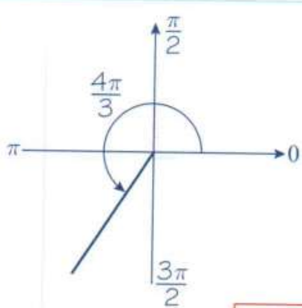
Links The CAST diagram shows you which trigonometric ratios are positive in which quadrant. You can also use the symmetry properties and periods of the graphs of sin, cos and tan to find these results. ← Year 1, Chapter 10

Example 6

Find the exact values of:

a $\cos\frac{4\pi}{3}$ **b** $\sin\left(\frac{-7\pi}{6}\right)$

a



$\cos\frac{4\pi}{3} = \cos\left(\pi + \frac{\pi}{3}\right) = -\cos\frac{\pi}{3}$

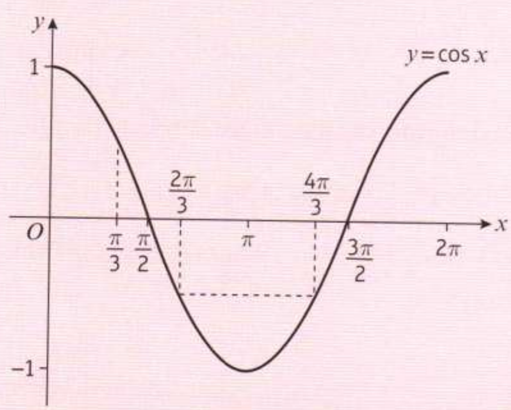
So $\cos\frac{4\pi}{3} = -\frac{1}{2}$

$\frac{4\pi}{3}$ is $\frac{\pi}{3}$ bigger than π.

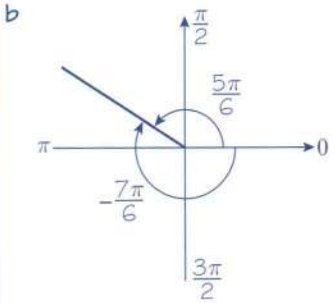
Use $\cos(\pi + \theta) = -\cos\theta$.

Problem-solving

You can also use the symmetry properties of $y = \cos x$:



b



$\sin\left(-\frac{7\pi}{6}\right) = \sin\frac{5\pi}{6} = \sin\frac{\pi}{6}$

Use $\sin(\pi - \theta) = \sin\theta$.

So $\sin\left(-\frac{7\pi}{6}\right) = \frac{1}{2}$

Exercise 5B

1 Express the following as trigonometric ratios of either $\frac{\pi}{6}$, $\frac{\pi}{4}$ or $\frac{\pi}{3}$, and hence find their exact values.

a $\sin\frac{3\pi}{4}$ **b** $\sin\left(-\frac{\pi}{3}\right)$ **c** $\sin\frac{11\pi}{6}$

d $\cos\frac{2\pi}{3}$ **e** $\cos\frac{5\pi}{3}$ **f** $\cos\frac{5\pi}{4}$

g $\tan\frac{3\pi}{4}$ **h** $\tan\left(-\frac{5\pi}{4}\right)$ **i** $\tan\frac{7\pi}{6}$

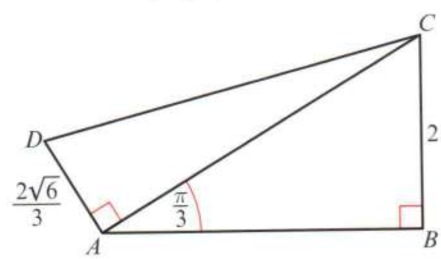
2 Without using a calculator, find the exact values of the following trigonometric ratios.

a $\sin\frac{7\pi}{3}$ **b** $\sin\left(-\frac{5\pi}{3}\right)$ **c** $\cos\left(-\frac{7\pi}{6}\right)$

d $\cos\frac{11\pi}{4}$ **e** $\tan\frac{5\pi}{3}$ **f** $\tan\left(-\frac{2\pi}{3}\right)$

(P) 3 The diagram shows a right-angled triangle ACD on another right-angled triangle ABC with $AD = \frac{2\sqrt{6}}{3}$ and $BC = 2$.

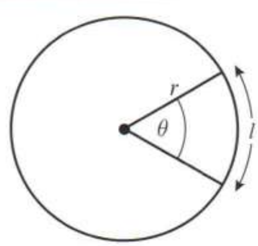
Show that $DC = k\sqrt{2}$, where k is a constant to be determined.



5.2 Arc length

Using radians greatly simplifies the formula for **arc length**.

- **To find the arc length l of a sector of a circle use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.**



Example 7

Find the length of the arc of a circle of radius 5.2 cm, given that the arc subtends an angle of 0.8 rad at the centre of the circle.

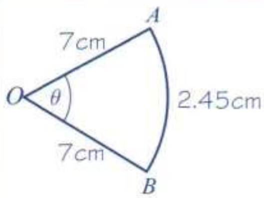
Online Explore the arc length of a sector using GeoGebra.

Arc length = 5.2 × 0.8 = 4.16 cm

Use $l = r\theta$, with $r = 5.2$ and $\theta = 0.8$.

Example 8

An arc AB of a circle with radius 7 cm and centre O has a length of 2.45 cm. Find the angle $\angle AOB$ subtended by the arc at the centre of the circle.



$l = r\theta$

$2.45 = 7\theta$

$\frac{2.45}{7} = \theta$

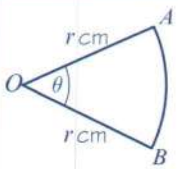
$\theta = 0.35$ rad

Use $l = r\theta$, with $l = 2.45$ and $r = 7$.

Using this formula gives the angle in radians.

Example 9

An arc AB of a circle, with centre O and radius r cm, subtends an angle of θ radians at O. The perimeter of the sector AOB is P cm. Express r in terms of P and θ.



$P = r\theta + 2r$

$= r(2 + \theta)$

So $r = \frac{P}{(2 + \theta)}$

Problem-solving

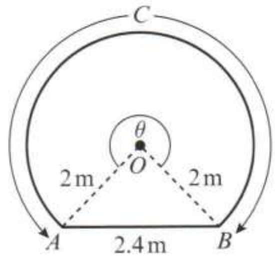
When given a problem in words, it is often a good idea to sketch and label a diagram to help you to visualise the information you have and what you need to find.

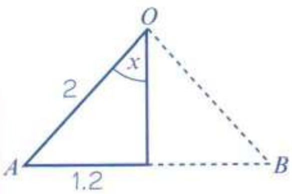
The perimeter = arc $AB + OA + OB$, where arc $AB = r\theta$.

Factorise.

Example 10

The border of a garden pond consists of a straight edge AB of length 2.4 m, and a curved part C, as shown in the diagram.
The curved part is an arc of a circle, centre O and radius 2 m.
Find the length of C.





$\sin x = \frac{1.2}{2}$

$x = 0.6435...$ rad

Acute $\angle AOB = 2x$ rad
$= 2 \times 0.6435...$
$= 1.2870...$ rad

So $\theta = (2\pi - 1.2870...)$ rad
$= 4.9961...$ rad

So length of $C = 9.99$ m (3 s.f.)

Online Explore the area of a sector using GeoGebra.

Problem-solving

Look for opportunities to use the basic trigonometric ratios rather than the more complicated cosine rule or sine rule. AOB is an isosceles triangle, so you can divide it into congruent right-angled triangles. Make sure your calculator is in radians mode.

C subtends the reflex angle θ at O, so length of $C = 2\theta$.

θ + acute $\angle AOB = 2\pi$ rad

$C = 2\theta$

Exercise 5C

1 An arc AB of a circle, centre O and radius r cm, subtends an angle θ radians at O.
The length of AB is l cm.

a Find l when: **i** $r = 6, \theta = 0.45$ **ii** $r = 4.5, \theta = 0.45$ **iii** $r = 20, \theta = \frac{3}{8}\pi$

b Find r when: **i** $l = 10, \theta = 0.6$ **ii** $l = 1.26, \theta = 0.7$ **iii** $l = 1.5\pi, \theta = \frac{5}{12}\pi$

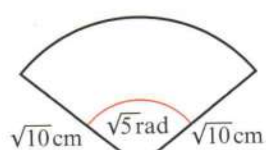
c Find θ when: **i** $l = 10, r = 7.5$ **ii** $l = 4.5, r = 5.625$ **iii** $l = \sqrt{12}, r = \sqrt{3}$

(P) 2 A minor arc AB of a circle, centre O and radius 10 cm, subtends an angle x at O. The major arc AB subtends an angle $5x$ at O. Find, in terms of π, the length of the minor arc AB.

Notation The **minor arc** AB is the shorter arc between points A and B on a circle.

3 An arc AB of a circle, centre O and radius 6 cm, has length l cm. Given that the chord AB has length 6 cm, find the value of l, giving your answer in terms of π.

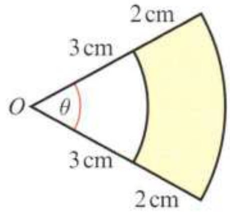
4 The sector of a circle of radius $\sqrt{10}$ cm contains an angle of $\sqrt{5}$ radians, as shown in the diagram. Find the length of the arc, giving your answer in the form $p\sqrt{q}$ cm, where p and q are integers.



(P) **5** Referring to the diagram, find:

a the perimeter of the shaded region when $\theta = 0.8$ radians.

b the value of θ when the perimeter of the shaded region is 14 cm.



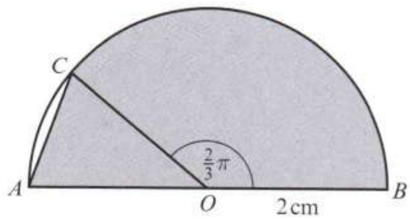
Problem-solving

The radius of the larger arc is $3 + 2 = 5$ cm.

(P) **6** A sector of a circle of radius r cm contains an angle of 1.2 radians. Given that the sector has the same perimeter as a square of area $36\,\text{cm}^2$, find the value of r.

(P) **7** A sector of a circle of radius 15 cm contains an angle of θ radians. Given that the perimeter of the sector is 42 cm, find the value of θ.

(E/P) **8** In the diagram AB is the diameter of a circle, centre O and radius 2 cm.
The point C is on the circumference such that $\angle COB = \frac{2}{3}\pi$ radians.



a State the value, in radians, of $\angle COA$. **(1 mark)**

The shaded region enclosed by the chord AC, arc CB and AB is the template for a brooch.

b Find the exact value of the perimeter of the brooch. **(5 marks)**

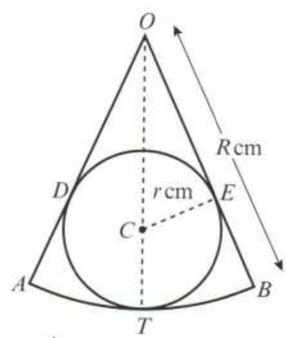
(P) **9** The points A and B lie on the circumference of a circle with centre O and radius 8.5 cm.
The point C lies on the major arc AB. Given that $\angle ACB = 0.4$ radians, calculate the length of the minor arc AB.

(E/P) **10** In the diagram OAB is a sector of a circle, centre O and radius R cm, and $\angle AOB = 2\theta$ radians. A circle, centre C and radius r cm, touches the arc AB at T, and touches OA and OB at D and E respectively, as shown.

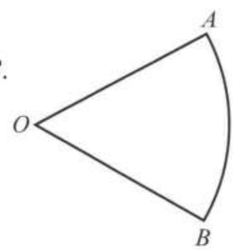
a Write down, in terms of R and r, the length of OC. **(1 mark)**

b Using $\triangle OCE$, show that $R\sin\theta = r(1 + \sin\theta)$. **(3 marks)**

c Given that $\sin\theta = \frac{3}{4}$ and that the perimeter of the sector OAB is 21 cm, find r, giving your answer to 3 significant figures. **(7 marks)**



(P) **11** The diagram shows a sector AOB.
The perimeter of the sector is twice the length of the arc AB.
Find the size of angle AOB.



(P) **12** A circular Ferris wheel has 24 pods equally spaced on its circumference.

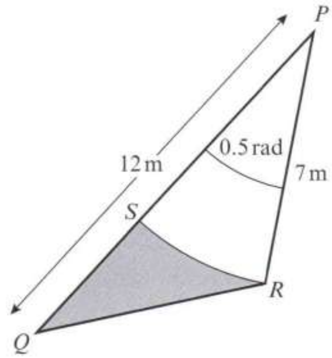
Given the arc length between each pod is $\frac{3\pi}{2}$ m, and modelling each pod as a particle,

a calculate the diameter of the Ferris wheel.

Given that it takes approximately 30 seconds for a pod to complete one revolution,

b estimate the speed of the pod in km/h.

(E/P) **13** The diagram shows a triangular garden, PQR, with $PQ = 12\,\text{m}$, $PR = 7\,\text{m}$ and $\angle QPR = 0.5$ radians. The curve SR is a small path separating the shaded patio area and the lawn, and is an arc of a circle with centre at P and radius 7 m.

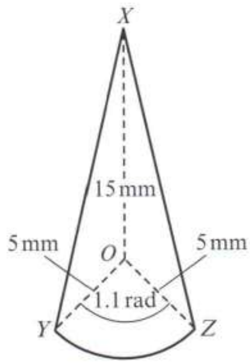


Find:

a the length of the path SR **(2 marks)**

b the perimeter of the shaded patio, giving your answer to 3 significant figures. **(4 marks)**

(E/P) **14** The shape XYZ shown is a design for an earring.



The straight lines XY and XZ are equal in length. The curve YZ is an arc of a circle with centre O and radius 5 mm. The size of $\angle YOZ$ is 1.1 radians and $XO = 15\,\text{mm}$.

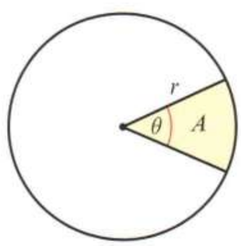
a Find the size of $\angle XOZ$, in radians, to 3 significant figures. **(2 marks)**

b Find the total perimeter of the earring, to the nearest mm. **(6 marks)**

5.3 Areas of sectors and segments

Using radians also greatly simplifies the formula for the area of a **sector**.

- **To find the area A of a sector of a circle use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.**



Notation A sector of a circle is the portion of a circle enclosed by two radii and an arc. The smaller area is known as the **minor** sector and the larger is known as the **major** sector.

Example 11

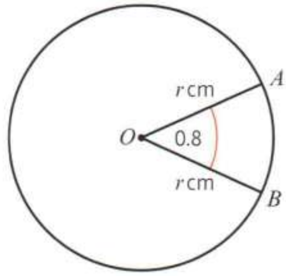
Find the area of the sector of a circle of radius 2.44 cm, given that the sector subtends an angle of 1.4 radians at the centre of the circle.

Area of sector $= \frac{1}{2} \times 2.44^2 \times 1.4$
$= 4.17\,\text{cm}^2$ (3 s.f.)

Use $A = \frac{1}{2}r^2\theta$ with $r = 2.44$ and $\theta = 1.4$.

Example 12

In the diagram, the area of the minor sector AOB is $28.9\,\text{cm}^2$.
Given that $\angle AOB = 0.8$ radians, calculate the value of r.



$28.9 = \frac{1}{2}r^2 \times 0.8 = 0.4r^2$

So $r^2 = \frac{28.9}{0.4} = 72.25$

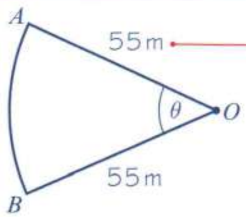
$r = \sqrt{72.25} = 8.5$

Let area of sector be $A\,\text{cm}^2$, and use $A = \frac{1}{2}r^2\theta$.

Use the positive square root in this case as a length cannot be negative.

Example 13

A plot of land is in the shape of a sector of a circle of radius 55 m. The length of fencing that is erected along the edge of the plot to enclose the land is 176 m. Calculate the area of the plot of land.



Arc $AB = 176 - (55 + 55)$

$= 66\text{ m}$

$66 = 55\theta$

So $\theta = 1.2$ radians

Area of plot $= \frac{1}{2} \times 55^2 \times 1.2$

$= 1815\text{ m}^2$

Online Explore the area of a segment using GeoGebra.

Draw a diagram including all the data and let the angle of the sector be θ.

Problem-solving

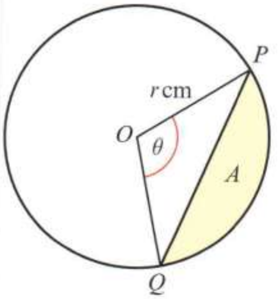
In order to find the area of the sector, you need to know θ. Use the information about the perimeter to find the arc length AB.

As the perimeter is given, first find length of arc AB.

Use the formula for arc length, $l = r\theta$.

Use the formula for area of a sector, $A = \frac{1}{2}r^2\theta$.

You can find the area of a **segment** by subtracting the area of triangle OPQ from the area of sector OPQ.



Using $\frac{1}{2}r^2\theta$ for the area of the sector and $\frac{1}{2}ab\sin\theta$ for the area of a triangle:

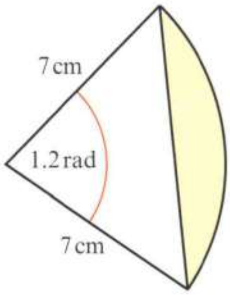
$A = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$= \frac{1}{2}r^2(\theta - \sin\theta)$

- **The area of a segment in a circle of radius r is $A = \frac{1}{2}r^2(\theta - \sin\theta)$**

Example 14

The diagram shows a sector of a circle. Find the area of the shaded segment.



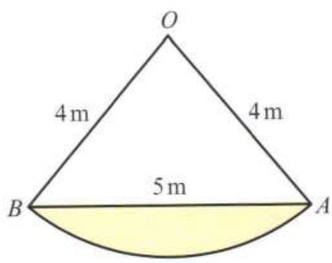
Area of segment $= \frac{1}{2} \times 7^2(1.2 - \sin 1.2)$

$= \frac{1}{2} \times 49 \times 0.26796...$

$= 6.57\ \text{cm}^2$ (3 s.f.)

Use $A = \frac{1}{2}r^2(\theta - \sin\theta)$ with $r = 7$ and $\theta = 1.2$ radians. Make sure your calculator is in radians mode when calculating $\sin\theta$.

Example 15



In the diagram above, OAB is a sector of a circle, radius 4 m. The chord AB is 5 m long. Find the area of the shaded segment.

Problem-solving

In order to find the area of the segment you need to know angle AOB. You can use the cosine rule in triangle AOB, or divide the triangle into two right-angled triangles and use the trigonometric ratios.

Calculate angle AOB first:

$\cos\angle AOB = \frac{4^2 + 4^2 - 5^2}{2 \times 4 \times 4}$

$= \frac{7}{32}$

So $\angle AOB = 1.3502...$

Area of shaded segment

$= \frac{1}{2} \times 4^2(1.3502... - \sin 1.3502...)$

$= \frac{1}{2} \times 16 \times 0.37448...$

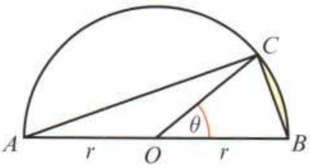
$= 3.00\ \text{m}^2$ (3 s.f.)

Use the cosine rule for a non-right-angled triangle.

Watch out Use unrounded values in your calculations wherever possible to avoid rounding errors. You can use the memory function or answer button on your calculator.

Example 16

In the diagram, AB is the diameter of a circle of radius r cm, and $\angle BOC = \theta$ radians. Given that the area of $\triangle AOC$ is three times that of the shaded segment, show that $3\theta - 4\sin\theta = 0$.



Area of segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

Area of $\triangle AOC = \frac{1}{2}r^2\sin(\pi - \theta)$

$= \frac{1}{2}r^2\sin\theta$

So $\frac{1}{2}r^2\sin\theta = 3 \times \frac{1}{2}r^2(\theta - \sin\theta)$

$\sin\theta = 3(\theta - \sin\theta)$

So $3\theta - 4\sin\theta = 0$

Area of segment = area of sector – area of triangle.

$\angle AOB = \pi$ radians.

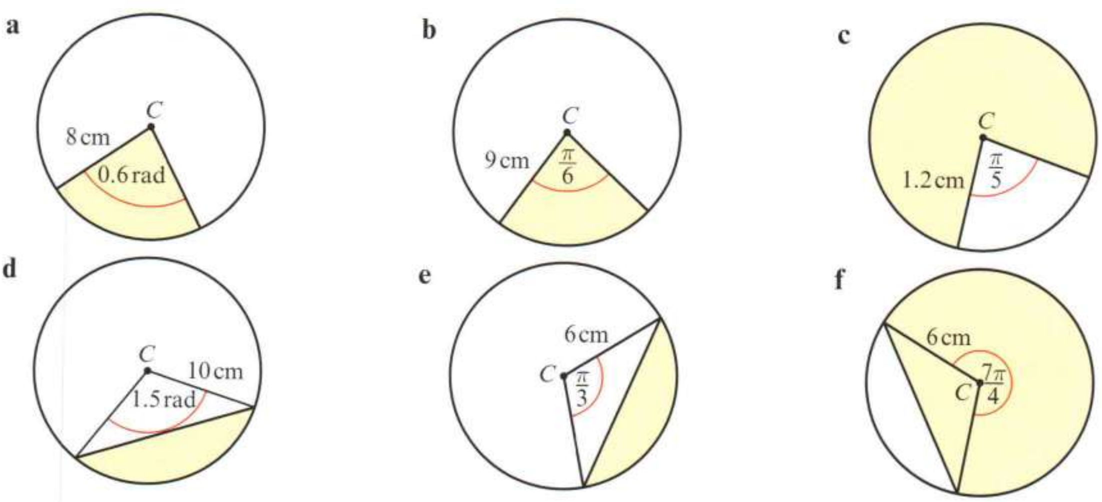
Area of $\triangle AOC = 3 \times$ area of shaded segment.

Problem-solving

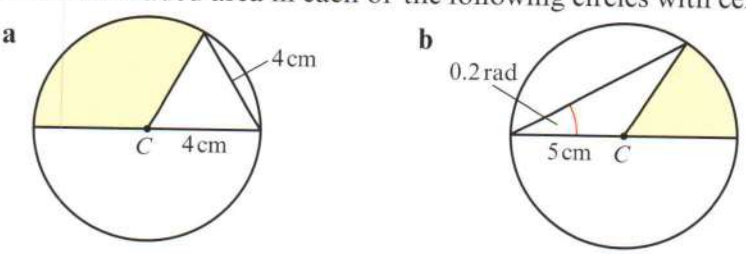
You might need to use circle theorems or properties when solving problems. The angle in a semicircle is a right angle so $\angle ACB = \frac{\pi}{2}$

Exercise 5D

1 Find the shaded area in each of the following circles. Leave your answers in terms of π where appropriate.

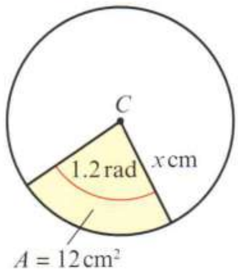


2 Find the shaded area in each of the following circles with centre C.

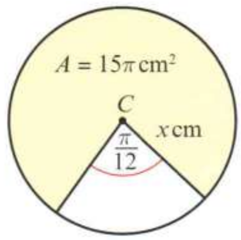


3 For the following circles with centre C, the area A of the shaded sector is given. Find the value of x in each case.

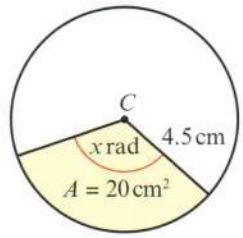
a



b



c



4 The arc AB of a circle, centre O and radius 6 cm, has length 4 cm.
Find the area of the minor sector AOB.

5 The chord AB of a circle, centre O and radius 10 cm, has length 18.65 cm and subtends an angle of θ radians at O.

a Show that $\cos\theta = -0.739$ (to 3 significant figures).

b Find the area of the minor sector AOB.

(P) **6** The area of a sector of a circle of radius 12 cm is 100 cm^2. Find the perimeter of the sector.

7 The arc AB of a circle, centre O and radius r cm, is such that $\angle AOB = 0.5$ radians.
Given that the perimeter of the minor sector AOB is 30 cm,

a calculate the value of r

b show that the area of the minor sector AOB is 36 cm^2

c calculate the area of the segment enclosed by the chord AB and the minor arc AB.

(P) **8** The arc AB of a circle, centre O and radius x cm, is such that angle $AOB = \frac{\pi}{12}$ radians.
Given that the arc length AB is l cm,

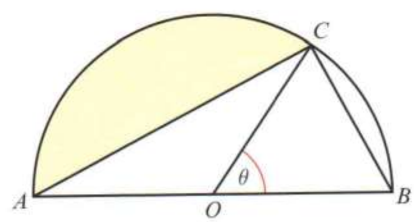
a show that the area of the sector can be written as $\frac{6l^2}{\pi}$

The area of the full circle is 3600π cm^2.

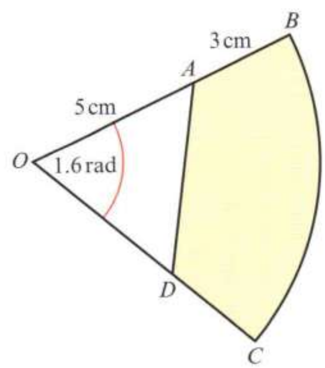
b Find the arc length of AB.

c Calculate the value of x.

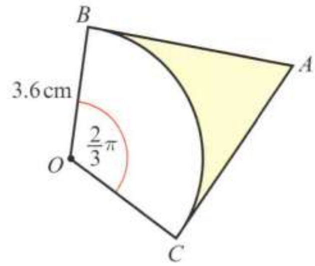
(P) **9** In the diagram, AB is the diameter of a circle of radius r cm and $\angle BOC = \theta$ radians.
Given that the area of ΔCOB is equal to that of the shaded segment, show that $\theta + 2\sin\theta = \pi$.



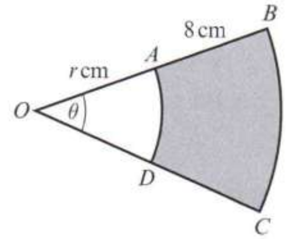
(P) **10** In the diagram, BC is the arc of a circle, centre O and radius 8 cm. The points A and D are such that $OA = OD = 5$ cm. Given that $\angle BOC = 1.6$ radians, calculate the area of the shaded region.



11 In the diagram, AB and AC are tangents to a circle, centre O and radius 3.6 cm. Calculate the area of the shaded region, given that $\angle BOC = \frac{2\pi}{3}$ radians.



12 In the diagram, AD and BC are arcs of circles with centre O, such that $OA = OD = r$ cm, $AB = DC = 8$ cm and $\angle BOC = \theta$ radians.

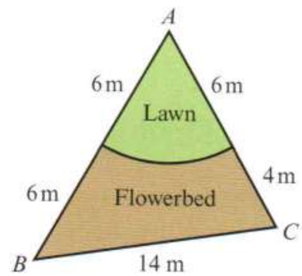


a Given that the area of the shaded region is 48 cm², show that $r = \frac{6}{\theta} - 4$ **(4 marks)**

b Given also that $r = 10\theta$, calculate the perimeter of the shaded region. **(6 marks)**

13 A sector of a circle of radius 28 cm has perimeter P cm and area A cm². Given that $A = 4P$, find the value of P.

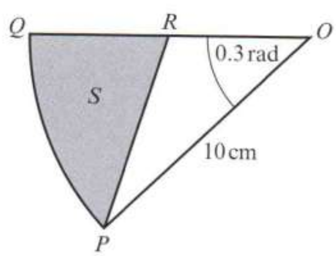
14 The diagram shows a triangular plot of land. The sides AB, BC and CA have lengths 12 m, 14 m and 10 m respectively. The lawn is a sector of a circle, centre A and radius 6 m.



a Show that $\angle BAC = 1.37$ radians, correct to 3 significant figures.

b Calculate the area of the flowerbed.

15 The diagram shows OPQ, a sector of a circle with centre O, radius 10 cm, with $\angle POQ = 0.3$ radians.



The point R is on OQ such that the ratio $OR : RQ$ is $1 : 3$. The region S, shown shaded in the diagram, is bounded by QR, RP and the arc PQ.

Find:

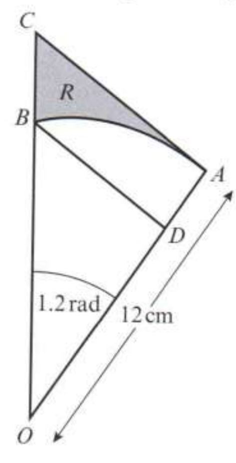
a the perimeter of S, giving your answer to 3 significant figures **(6 marks)**

b the area of S, giving your answer to 3 significant figures. **(6 marks)**

16 The diagram shows the sector OAB of a circle with centre O, radius 12 cm and angle 1.2 radians.

The line AC is a tangent to the circle with centre O, and OBC is a straight line.

The region R is bounded by the arc AB and the lines AC and CB.

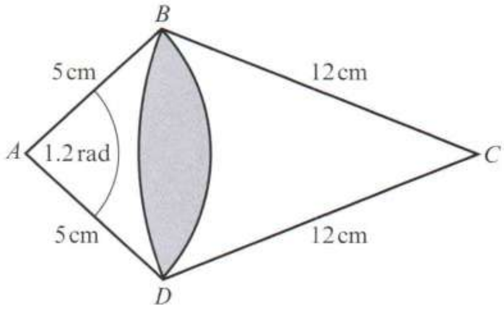


a Find the area of R, giving your answer to 2 decimal places. **(8 marks)**

The line BD is parallel to AC.

b Find the perimeter of DAB. **(5 marks)**

(P) 17



The diagram shows two intersecting sectors: ABD, with radius 5 cm and angle 1.2 radians, and CBD, with radius 12 cm.
Find the area of the overlapping section.

Challenge

Find an expression for the area of a sector of a circle with radius r and arc length l.

5.4 Solving trigonometric equations

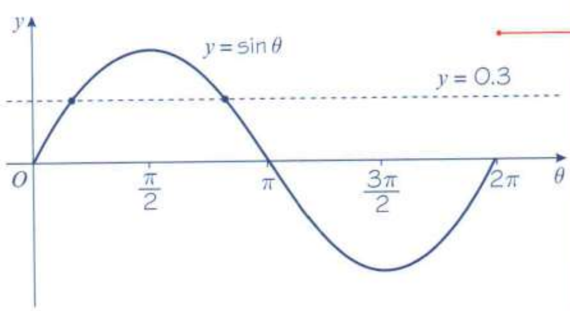
In Year 1, you learned how to solve trigonometric equations in degrees. You can solve trigonometric equations in radians in the same way.

Example 17

Find the solutions of these equations in the interval $0 \leqslant \theta \leqslant 2\pi$:

a $\sin\theta = 0.3$ **b** $4\cos\theta = 2$ **c** $5\tan\theta + 3 = 1$

a $\sin\theta = 0.3$
So $\theta = 0.304692654...$



Draw the graph of $y = \sin\theta$ for the given interval.

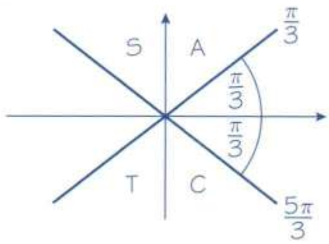
$\sin\theta = 0.3$ where the line $y = 0.3$ cuts the curve.
Hence $\theta = 0.305$ rad or 2.84 rad (3 s.f.)

Find the first value using your calculator in radians mode.
Since the sine curve is symmetrical in the interval $0 < \theta < \pi$, the second value is obtained by $\pi - 0.30469...$

b $4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

So $\theta = \frac{\pi}{3}$

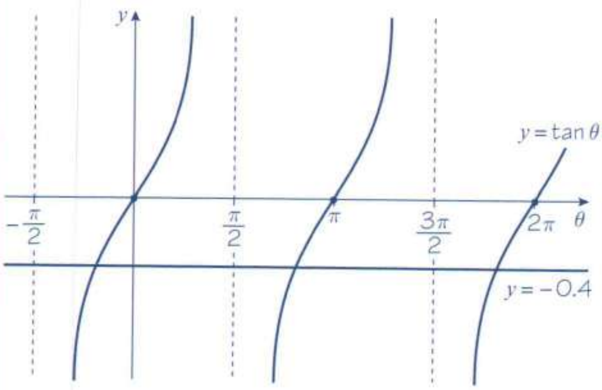


So $\theta = \frac{\pi}{3}$ or $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

c $5\tan\theta + 3 = 1$

$5\tan\theta = -2$

$\tan\theta = -0.4$



$\tan\theta = -0.4$ where the line $y = -0.4$ cuts the curve.

$\tan^{-1}(-0.4) = -0.3805\ldots$ rad

So $\theta = 2.76108\ldots$ rad (2.76 rad to 3 s.f.)

or $\theta = 5.90267\ldots$ rad (5.90 rad to 3 s.f.)

Watch out When the interval is given in radians, make sure you answer in radians.

First rewrite in the form $\cos\theta = \ldots$

Use exact values where possible.

Putting $\frac{\pi}{3}$ in the four positions shown gives the angles $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$ but cosine is only positive in the 1st and 4th quadrants.

For the 2nd value, since we are working in radians, we use $2\pi - \theta$ instead of $360° - \theta$.

Draw the graph of $y = \tan\theta$ for the interval $-\frac{\pi}{2} < \theta < 2\pi$ since the principal value given by $\tan^{-1}(-0.4)$ is negative.

Use the symmetry and period of the tangent graph to find the required values.

Watch out Always check that your final values are within the given range; in this case $0 < \theta < 2\pi$ (remember $2\pi \approx 6.283\ldots$)

Example 18

Solve the equation $17\cos\theta + 3\sin^2\theta = 13$ in the interval $0 \leqslant \theta \leqslant 2\pi$.

$17\cos\theta + 3\sin^2\theta = 13$
$17\cos\theta + 3(1 - \cos^2\theta) = 13$
$17\cos\theta + 3 - 3\cos^2\theta = 13$

Problem-solving

Use the trigonometric identity $\sin^2\theta + \cos^2\theta \equiv 1$. Trigonometric identities work the same in radians as in degrees.

$0 = 3\cos^2\theta - 17\cos\theta + 10$

This is a quadratic so rearrange to make one side 0.

$0 = 3Y^2 - 17Y + 10$

If $Y = \cos\theta$, then $Y^2 = \cos^2\theta$.

$0 = (3Y - 2)(Y - 5)$
So $3Y - 2 = 0$ or $Y - 5 = 0$
$Y = \frac{2}{3}$ or $Y = 5$

Solve the quadratic equation.

i.e. $\cos\theta = \frac{2}{3}$ or $\cos\theta = 5$

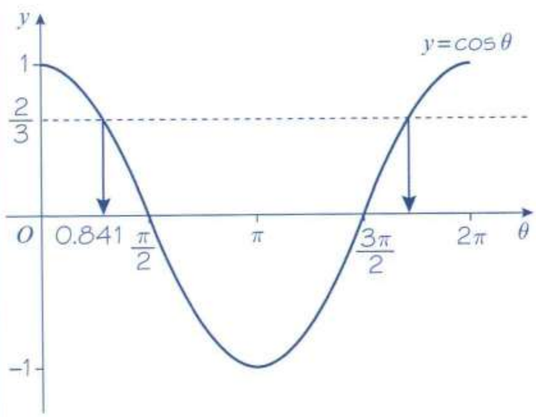
The value of $\cos\theta$ is between −1 and 1, so reject $\cos\theta = 5$.

$\cos\theta = \frac{2}{3}$

Solve this equation to find θ.

So $\theta = 0.841068...$ rad

Since the interval is given in radians, answer in radians.



Second solution is $2\pi - 0.841068...$
$= 5.442116...$
$\theta = 0.841$ or 5.44 (3 s.f.)

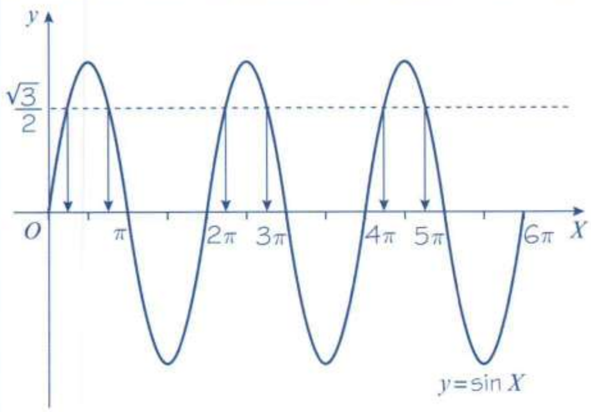
Example 19

Solve the equation $\sin 3\theta = \frac{\sqrt{3}}{2}$, in the interval $0 \leqslant \theta \leqslant 2\pi$.

Let $X = 3\theta$

So $\sin X = \frac{\sqrt{3}}{2}$

As $X = 3\theta$, then the interval for X is $0 \leqslant X \leqslant 6\pi$



$X = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}$

i.e. $3\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}$

So $\theta = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}$

Replace 3θ by X and solve as normal.

Remember to transform the interval: $0 \leqslant \theta \leqslant 2\pi$ becomes $0 \leqslant 3\theta \leqslant 6\pi$

Remember $X = 3\theta$ so divide each value by 3.

Always check that your solutions for θ are in the given interval for θ, in this case $0 \leqslant \theta \leqslant 2\pi$.

Exercise 5E

1 Solve the following equations for θ, in the interval $0 \leqslant \theta \leqslant 2\pi$, giving your answers to 3 significant figures where they are not exact.

a $\cos\theta = 0.7$ **b** $\sin\theta = -0.2$ **c** $\tan\theta = 5$ **d** $\cos\theta = -1$

2 Solve the following equations for θ, in the interval $0 \leqslant \theta \leqslant 2\pi$, giving your answers to 3 significant figures where they are not exact.

a $4\sin\theta = 3$ **b** $7\tan\theta = 1$ **c** $8\tan\theta = 15$ **d** $\sqrt{5}\cos\theta = \sqrt{2}$

3 Solve the following equations for θ, in the interval $0 \leqslant \theta \leqslant 2\pi$, giving your answers to 3 significant figures where they are not exact.

a $5\cos\theta + 1 = 3$ **b** $\sqrt{5}\sin\theta + 2 = 1$ **c** $8\tan\theta - 5 = 5$ **d** $\sqrt{7}\cos\theta - 1 = \sqrt{2}$

4 Solve the following equations for θ, giving your answers to 3 significant figures where appropriate, in the intervals indicated:

a $\sqrt{3}\tan\theta - 1 = 0, -\pi \leqslant \theta \leqslant \pi$ **b** $5\sin\theta = 1, -\pi \leqslant \theta \leqslant 2\pi$

c $8\cos\theta = 5, -2\pi \leqslant \theta \leqslant 2\pi$ **d** $3\cos\theta - 1 = 0.02, -\pi \leqslant \theta \leqslant 3\pi$

e $0.4\tan\theta - 5 = -7, 0 \leqslant \theta \leqslant 4\pi$ **f** $\cos\theta - 1 = -0.82, \frac{\pi}{2} \leqslant \theta \leqslant \frac{7\pi}{3}$

5 Solve the following equations for θ, in the interval $0 \leqslant \theta \leqslant 2\pi$, giving your answers to 3 significant figures where they are not exact.

a $5\cos 2\theta = 4$ **b** $5\sin 3\theta + 3 = 1$

c $\sqrt{3}\tan 4\theta - 5 = -4$ **d** $\sqrt{10}\cos 2\theta + \sqrt{2} = 3\sqrt{2}$

6 Solve the following equations for θ, giving your answers to 3 significant figures where appropriate, in the intervals indicated.

a $\sqrt{2}\sin 3\theta - 1 = 0, \quad -\pi \leqslant \theta \leqslant \pi$ **b** $2\cos 4\theta = -1, \quad -\pi \leqslant \theta \leqslant 2\pi$

c $8\tan 2\theta = 7, \quad -2\pi \leqslant \theta \leqslant 2\pi$ **d** $6\cos 2\theta - 1 = 0.2, \quad -\pi \leqslant \theta \leqslant 3\pi$

(P) **7** Solve the following equations for θ, in the interval $0 \leqslant \theta \leqslant 2\pi$, giving your answers to 3 significant figures where they are not exact.

a $4\cos^2\theta = 2$ **b** $3\tan^2\theta + \tan\theta = 0$

c $\cos^2\theta - 2\cos\theta = 3$ **d** $2\sin^2 2\theta - 5\cos 2\theta = -2$

(P) **8** Solve the following equations for θ, in the interval $0 \leqslant \theta \leqslant 2\pi$, giving your answers to 3 significant figures where they are not exact.

a $\cos\theta + 2\sin^2\theta + 1 = 0$ **b** $10\sin^2\theta = 3\cos^2\theta$

c $4\cos^2\theta + 8\sin^2\theta = 2\sin^2\theta - 2\cos^2\theta$ **d** $2\sin^2\theta - 7 + 12\cos\theta = 0$

9 Solve, for $0 \leqslant x < 2\pi$,

a $\cos\left(x - \frac{\pi}{12}\right) = \frac{1}{\sqrt{2}}$ **b** $\sin 3x = -\frac{1}{2}$

c $\cos(2\theta + 0.2) = -0.2, -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$ **d** $\tan\left(2\theta + \frac{\pi}{4}\right) = 1, 0 \leqslant \theta \leqslant 2\pi$

(E/P) **10 a** Solve, for $-\pi \leqslant \theta < \pi$, $(1 + \tan\theta)(5\sin\theta - 2) = 0$. **(4 marks)**

b Solve, for $0 \leqslant x < 2\pi$, $4\tan x = 5\sin x$. **(6 marks)**

(E) **11** Find all the solutions, in the interval $0 \leqslant x \leqslant 2\pi$, to the equation $8\cos^2 x + 6\sin x - 6 = 3$ giving each solution to one decimal place. **(6 marks)**

(E/P) **12** Find, for $0 \leqslant x \leqslant 2\pi$, all the solutions of $\cos^2 x - 1 = \frac{7}{2}\sin^2 x - 2$ giving each solution to one decimal place. **(6 marks)**

(E/P) **13** Show that the equation $8\sin^2 x + 4\sin x - 20 = 4$ has no solutions. **(3 marks)**

(E/P) **14 a** Show that the equation $\tan^2 x - 2\tan x - 6 = 0$ can be written as $\tan x = p \pm \sqrt{q}$ where p and q are numbers to be found. **(3 marks)**

b Hence solve, for $0 \leqslant x \leqslant 3\pi$, the equation $\tan^2 x - 2\tan x - 6 = 0$ giving your answers to 1 decimal place where appropriate. **(5 marks)**

(E/P) **15** In the triangle ABC, $AB = 5$ cm, $AC = 4$ cm, $\angle ABC = 0.5$ radians and $\angle ACB = x$ radians.

a Use the sine rule to find the value of $\sin x$, giving your answer to 3 decimal places. **(3 marks)**

Given that there are two possible values of x,

b find these values of x, giving your answers to 2 decimal places. **(3 marks)**

5.5 Small angle approximations

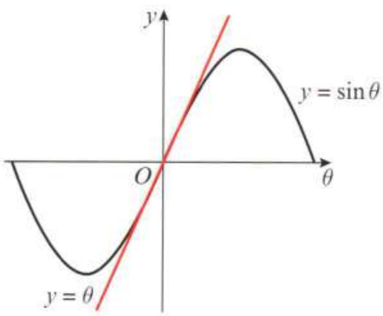
You can use radians to find **approximations** for the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

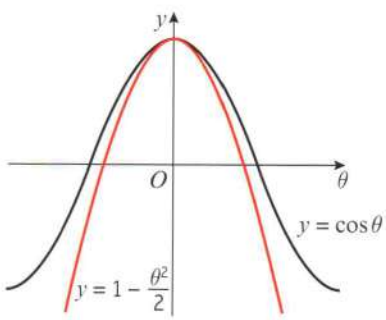
- **When θ is small and measured in radians:**
  - $\sin\theta \approx \theta$
  - $\tan\theta \approx \theta$
  - $\cos\theta \approx 1 - \frac{\theta^2}{2}$

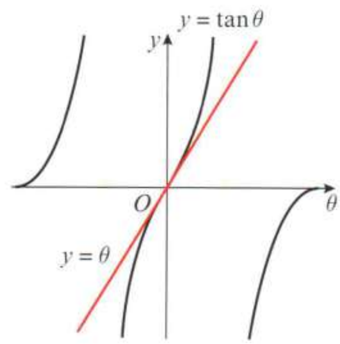
You can see why these approximations work by looking at the graphs of $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ for values of θ close to 0.

Notation In mathematics 'small' is a relative concept. Consequently, there is not a fixed set of numbers which are small and a fixed set which are not. In this case, it is useful to think of small as being really close to 0.

Online Use technology to explore approximate values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for values of θ close to 0.







Example 20

When θ is small, find the approximate value of:

a $\dfrac{\sin 2\theta + \tan\theta}{2\theta}$ **b** $\dfrac{\cos 4\theta - 1}{\theta \sin 2\theta}$

a $\dfrac{\sin 2\theta + \tan\theta}{2\theta} \approx \dfrac{2\theta + \theta}{2\theta}$

If $\sin\theta \approx \theta$ then $\sin 2\theta \approx 2\theta$

$= \dfrac{3\theta}{2\theta} = \dfrac{3}{2}$

When θ is small, $\dfrac{\sin 2\theta + \tan\theta}{2\theta} \approx \dfrac{3}{2}$

Note that this approximation is only valid when θ is small and measured in radians.

b $\dfrac{\cos 4\theta - 1}{\theta \sin 2\theta} \approx \dfrac{1 - \frac{(4\theta)^2}{2} - 1}{\theta \times 2\theta}$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$ so $\cos 4\theta \approx 1 - \frac{(4\theta)^2}{2}$

$= \dfrac{1 - \frac{16\theta^2}{2} - 1}{2\theta^2}$

$= \dfrac{\frac{-16\theta^2}{2}}{2\theta^2} = -\dfrac{8\theta^2}{2\theta^2}$

$= -4$

Example 21

a Show that, when θ is small, $\sin 5\theta + \tan 2\theta - \cos 2\theta \approx 2\theta^2 + 7\theta - 1$.

b Hence state the approximate value of $\sin 5\theta + \tan 2\theta - \cos 2\theta$ for small values of θ.

a For small values of θ:

$$\sin 5\theta + \tan 2\theta - \cos 2\theta \approx 5\theta + 2\theta - \left(1 - \frac{(2\theta)^2}{2}\right)$$

Use the small angle approximations for sin, cos and tan.

$$= 7\theta - 1 + \frac{4\theta^2}{2}$$

$$= 7\theta - 1 + 2\theta^2$$

When θ is small,

$\sin 5\theta + \tan 2\theta - \cos 2\theta \approx 2\theta^2 + 7\theta - 1$

b So, for small θ, $\sin 5\theta + \tan 2\theta - \cos 2\theta \approx -1$

When θ is small, terms in θ^2 and θ will also be small, so you can disregard the terms $2\theta^2$ and 7θ.

Exercise 5F

1 When θ is small, find the approximate values of:

a $\dfrac{\sin 4\theta - \tan 2\theta}{3\theta}$ **b** $\dfrac{1 - \cos 2\theta}{\tan 2\theta \sin\theta}$ **c** $\dfrac{3\tan\theta - \theta}{\sin 2\theta}$

2 When θ is small, show that:

a $\dfrac{\sin 3\theta}{\theta \sin 4\theta} = \dfrac{3}{4\theta}$ **b** $\dfrac{\cos\theta - 1}{\tan 2\theta} = -\dfrac{\theta}{4}$ **c** $\dfrac{\tan 4\theta + \theta^2}{3\theta - \sin 2\theta} = 4 + \theta$

3 **a** Find $\cos(0.244 \text{ rad})$ correct to 6 decimal places.

b Use the approximation for $\cos\theta$ to find an approximate value for $\cos(0.244 \text{ rad})$.

c Calculate the percentage error in your approximation.

d Calculate the percentage error in the approximation for $\cos 0.75$ rad.

e Explain the difference between your answers to parts **c** and **d**.

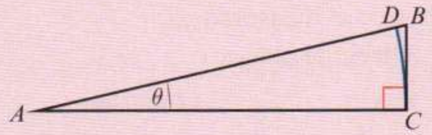
(P) **4** The percentage error for $\sin\theta$ for a given value of θ is 1%. Show that $100\theta = 101\sin\theta$.

(E/P) **5** **a** When θ is small, show that the expression $\dfrac{4\cos 3\theta - 2 + 5\sin\theta}{1 - \sin 2\theta}$ can be written as $9\theta + 2$. **(3 marks)**

b Hence write down the value of $\dfrac{4\cos 3\theta - 2 + 5\sin\theta}{1 - \sin 2\theta}$ when θ is small. **(1 mark)**

Challenge

1 The diagram shows a right-angled triangle ABC. $\angle BAC = \theta$. An arc, CD, of the circle with centre A and radius AC has been drawn on the diagram in blue.



a Write an expression for the arc length CD in terms of AC and θ.

Given that θ is small so that, $AC = AD \approx AB$ and $CD \approx BC$,

b deduce that $\sin\theta \approx \theta$ and $\tan\theta \approx \theta$.

2 a Using the binomial expansion and ignoring terms in x^4 and higher powers of x, find an approximation for $\sqrt{1 - x^2}$, $|x| < 1$.

b Hence show that for small θ, $\cos\theta \approx 1 - \frac{\theta^2}{2}$. You may assume that $\sin\theta \approx \theta$.

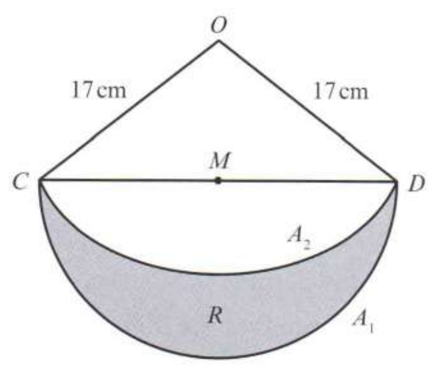
Mixed exercise 5

(P) **1** Triangle ABC is such that $AB = 5$ cm, $AC = 10$ cm and $\angle ABC = 90°$. An arc of a circle, centre A and radius 5 cm, cuts AC at D.

a State, in radians, the value of $\angle BAC$.

b Calculate the area of the region enclosed by BC, DC and the arc BD.

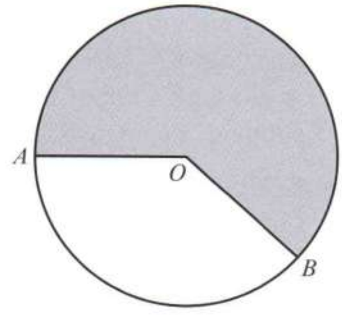
(E/P) **2** The diagram shows the triangle OCD with $OC = OD = 17$ cm and $CD = 30$ cm. The midpoint of CD is M. A semicircular arc A_1, with centre M is drawn, with CD as diameter. A circular arc A_2 with centre O and radius 17 cm, is drawn from C to D. The shaded region R is bounded by the arcs A_1 and A_2. Calculate, giving answers to 2 decimal places:



a the area of the triangle OCD **(4 marks)**

b the area of the shaded region R. **(5 marks)**

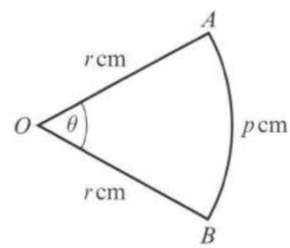
(E/P) **3** The diagram shows a circle, centre O, of radius 6 cm. The points A and B are on the circumference of the circle. The area of the shaded major sector is 80 cm^2. Given that $\angle AOB = \theta$ radians, where $0 < \theta < \pi$, calculate:



a the value, to 3 decimal places, of θ **(3 marks)**

b the length in cm, to 2 decimal places, of the minor arc AB. **(2 marks)**

E/P **4** The diagram shows a sector OAB of a circle, centre O and radius r cm. The length of the arc AB is p cm and $\angle AOB$ is θ radians.



a Find θ in terms of p and r. **(2 marks)**

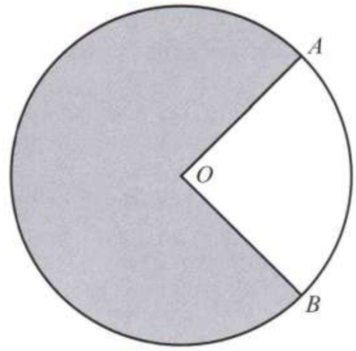
b Deduce that the area of the sector is $\frac{1}{2}pr\,\text{cm}^2$. **(2 marks)**

Given that $r = 4.7$ and $p = 5.3$, where each has been measured to 1 decimal place, find, giving your answer to 3 decimal places:

c the least possible value of the area of the sector **(2 marks)**

d the range of possible values of θ. **(3 marks)**

E **5** The diagram shows a circle centre O and radius 5 cm. The length of the minor arc AB is 6.4 cm.



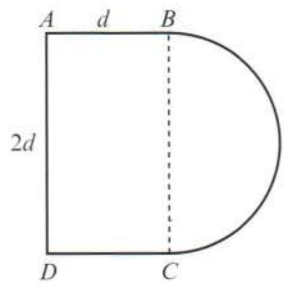
a Calculate, in radians, the size of the acute angle AOB. **(2 marks)**

The area of the minor sector AOB is $R_1\,\text{cm}^2$ and the area of the shaded major sector is $R_2\,\text{cm}^2$.

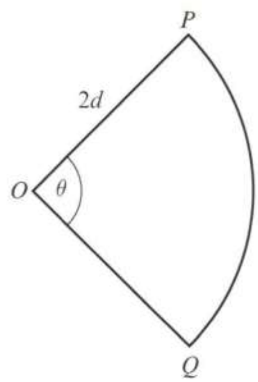
b Calculate the value of R_1. **(2 marks)**

c Calculate $R_1 : R_2$ in the form $1 : p$, giving the value of p to 3 significant figures. **(3 marks)**

E/P **6** The diagrams show the cross-sections of two drawer handles. Shape X is a rectangle $ABCD$ joined to a semicircle with BC as diameter. The length $AB = d$ cm and $BC = 2d$ cm. Shape Y is a sector OPQ of a circle with centre O and radius $2d$ cm. Angle POQ is θ radians.



Shape X



Shape Y

Given that the areas of shapes X and Y are equal,

a prove that $\theta = 1 + \frac{\pi}{4}$ **(5 marks)**

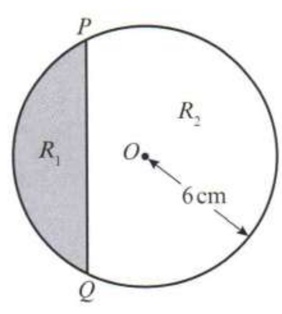
Using this value of θ, and given that $d = 3$, find in terms of π:

b the perimeter of shape X **(3 marks)**

c the perimeter of shape Y. **(3 marks)**

d Hence find the difference, in mm, between the perimeters of shapes X and Y. **(1 mark)**

E/P **7** The diagram shows a circle centre O and radius 6 cm. The chord PQ divides the circle into a minor segment R_1 of area $A_1\,\text{cm}^2$ and a major segment R_2 of area $A_2\,\text{cm}^2$. The chord PQ subtends an angle θ radians at O.

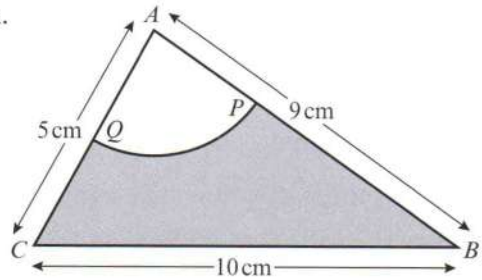


a Show that $A_1 = 18(\theta - \sin\theta)$. **(2 marks)**

Given that $A_2 = 3A_1$,

b show that $\sin\theta = \theta - \frac{\pi}{2}$ **(4 marks)**

E/P **8** Triangle ABC has $AB = 9$ cm, $BC = 10$ cm and $CA = 5$ cm. A circle, centre A and radius 3 cm, intersects AB and AC at P and Q respectively, as shown in the diagram.



a Show that, to 3 decimal places, $\angle BAC = 1.504$ radians. **(2 marks)**

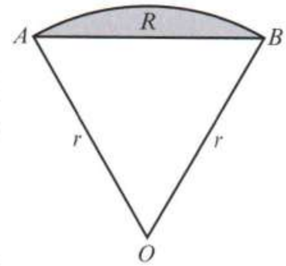
b Calculate:

i the area, in cm², of the sector APQ

ii the area, in cm², of the shaded region $BPQC$

iii the perimeter, in cm, of the shaded region $BPQC$. **(8 marks)**

E/P **9** The diagram shows the sector OAB of a circle of radius r cm. The area of the sector is 15 cm² and $\angle AOB = 1.5$ radians.



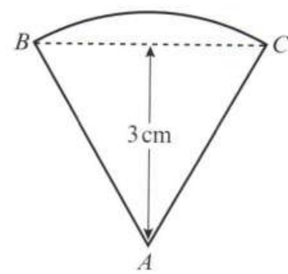
a Prove that $r = 2\sqrt{5}$. **(2 marks)**

b Find, in cm, the perimeter of the sector OAB. **(3 marks)**

The segment R, shaded in the diagram, is enclosed by the arc AB and the straight line AB.

c Calculate, to 3 decimal places, the area of R. **(2 marks)**

E/P **10** The shape of a badge is a sector ABC of a circle with centre A and radius AB, as shown in the diagram. The triangle ABC is equilateral and has perpendicular height 3 cm.

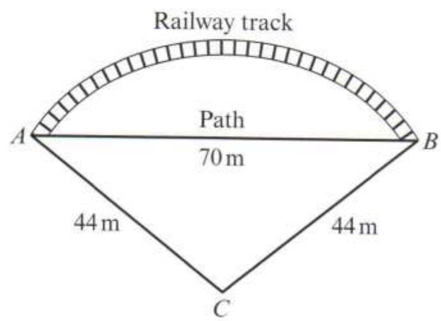


a Find, in surd form, the length of AB. **(2 marks)**

b Find, in terms of π, the area of the badge. **(2 marks)**

c Prove that the perimeter of the badge is $\frac{2\sqrt{3}}{3}(\pi + 6)$ cm. **(4 marks)**

E **11** There is a straight path of length 70 m from the point A to the point B. The points are joined also by a railway track in the form of an arc of the circle whose centre is C and whose radius is 44 m, as shown in the diagram.



a Show that the size, to 2 decimal places, of $\angle ACB$ is 1.84 radians. **(2 marks)**

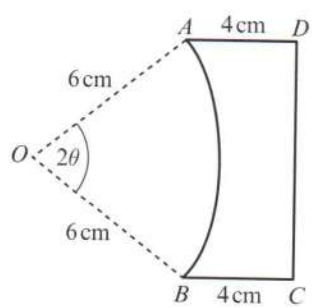
b Calculate:

i the length of the railway track

ii the shortest distance from C to the path

iii the area of the region bounded by the railway track and the path. **(6 marks)**

P **12** The diagram shows the cross-section $ABCD$ of a glass prism. $AD = BC = 4$ cm and both are at right angles to DC. AB is the arc of a circle, centre O and radius 6 cm. Given that $\angle AOB = 2\theta$ radians, and that the perimeter of the cross-section is $2(7 + \pi)$ cm,



a show that $(2\theta + 2\sin\theta - 1) = \frac{\pi}{3}$

b verify that $\theta = \frac{\pi}{6}$

c find the area of the cross-section.

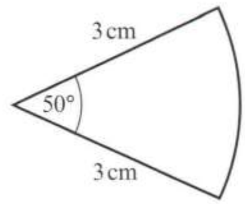
(P) **13** Two circles C_1 and C_2, both of radius 12 cm have centres O_1 and O_2 respectively. O_1 lies on the circumference of C_2; O_2 lies on the circumference of C_1. The circles intersect at A and B, and enclose the region R.

a Show that $\angle AO_1B = \frac{2\pi}{3}$

b Hence write down, in terms of π, the perimeter of R.

c Find the area of R, giving your answer to 3 significant figures.

(E/P) **14** A teacher asks a student to find the area of the following sector. The attempt is shown below.



Area $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 3^2 \times 50$
$= 225\text{ cm}^2$

a Identify the mistake made by the student. **(1 mark)**

b Calculate the correct area of the sector. **(2 marks)**

15 When θ is small, find the approximate values of:

a $\frac{\cos\theta - 1}{\theta\tan 2\theta}$

b $\frac{2(1 - \cos\theta) - 1}{\tan\theta - 1}$

16 a When θ is small, show that the expression $\frac{7 + 2\cos 2\theta}{\tan 2\theta + 3}$ can be written as $3 - 2\theta$. **(3 marks)**

b Hence write down the value of $\frac{7 + 2\cos 2\theta}{\tan 2\theta + 3}$ when θ is small. **(1 mark)**

(E/P) **17 a** When θ is small, show that the equation

$$32\cos 5\theta + 203\tan 10\theta = 182$$

can be written as

$$40\theta^2 - 203\theta + 15 = 0$$ **(4 marks)**

b Hence, find the solutions of the equation

$$32\cos 5\theta + 203\tan 10\theta = 182$$ **(3 marks)**

c Comment on the validity of your solutions. **(1 mark)**

(P) **18** When θ is small, find the approximate value of $\cos^4\theta - \sin^4\theta$.

19 Solve the following equations for θ, giving your answers to 3 significant figures where appropriate, in the intervals indicated.

a $3\sin\theta = 2,\ 0 \leqslant \theta \leqslant \pi$

b $\sin\theta = -\cos\theta,\ -\pi \leqslant \theta \leqslant \pi$

c $\tan\theta + \frac{1}{\tan\theta} = 2,\ 0 \leqslant \theta \leqslant 2\pi$

d $2\sin^2\theta - \sin\theta - 1 = \sin^2\theta,\ -\pi \leqslant \theta \leqslant \pi$

20 **a** Sketch the graphs of $y = 5\sin x$ and $y = 3\cos x$ on the same axes $(0 \leqslant x \leqslant 2\pi)$, marking on all the points where the graphs cross the axes.

b Write down how many solutions there are in the given range for the equation $5\sin x = 3\cos x$.

c Solve the equation $5\sin x = 3\cos x$ algebraically, giving your answers to 3 significant figures.

(E) **21** **a** Express $4\sin\theta - \cos\left(\frac{\pi}{2} - \theta\right)$ as a single trigonometric function. **(1 mark)**

b Hence solve $4\sin\theta - \cos\left(\frac{\pi}{2} - \theta\right) = 1$ in the interval $0 \leqslant \theta \leqslant 2\pi$. Give your answers to 3 significant figures. **(3 marks)**

(E/P) **22** Find the values of x in the interval $0 < x < \frac{3\pi}{2}$ which satisfy the equation

$$\frac{\sin 2x + 0.5}{1 - \sin 2x} = 2$$

(6 marks)

(E/P) **23** A teacher asks two students to solve the equation $2\cos^2 x = 1$ for $-\pi \leqslant x \leqslant \pi$.
The attempts are shown below.

Student A:

$\cos x = \pm\frac{1}{\sqrt{2}}$

Reject $-\frac{1}{\sqrt{2}}$ as cosine cannot be negative

$x = \frac{\pi}{4}$ or $x = -\frac{\pi}{4}$

Student B:

$2\cos^2 x = 1$

$\cos x = \pm\frac{1}{2}$

$x = \frac{\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}$

a Identify the mistake made by Student A. **(1 mark)**

b Identify the mistake made by Student B. **(1 mark)**

c Calculate the correct solutions to the equation. **(4 marks)**

(E/P) **24** A teacher asks a student to solve the equation $2\tan 2x = 5$ for $0 \leqslant x \leqslant 2\pi$.
The attempt is shown below.

$2\tan 2x = 5$
$\tan 2x = 2.5$
$2x = 1.19, 4.33$
$x = 0.595$ rad or 2.17 rad (3 s.f.)

Problem-solving

Solve the equation yourself then compare your working with the student's answer.

a Identify the mistake made by the student. **(1 mark)**

b Calculate the correct solutions to the equation. **(4 marks)**

(E/P) **25** **a** Show that the equation

$$5\sin x = 1 + 2\cos^2 x$$

can be written in the form

$$2\sin^2 x + 5\sin x - 3 = 0$$

(2 marks)

b Solve, for $0 \leqslant x < 2\pi$,

$$2\sin^2 x + 5\sin x - 3 = 0$$

(4 marks)

E/P **26** **a** Show that the equation

$$4\sin^2 x + 9\cos x - 6 = 0$$

can be written as

$$4\cos^2 x - 9\cos x + 2 = 0$$ **(2 marks)**

b Hence solve, for $0 \leqslant x < 4\pi$,

$$4\sin^2 x + 9\cos x - 6 = 0$$

giving your answers to 1 decimal place. **(6 marks)**

E/P **27** **a** Show that the equation

$$\tan 2x = 5\sin 2x$$

can be written in the form

$$(1 - 5\cos 2x)\sin 2x = 0$$ **(2 marks)**

b Hence solve, for $0 \leqslant x \leqslant \pi$,

$$\tan 2x = 5\sin 2x$$

giving your answers to 1 decimal place where appropriate. You must show clearly how you obtained your answers. **(5 marks)**

E **28** **a** Sketch, for $0 \leqslant x \leqslant 2\pi$, the graph of $y = \cos\left(x + \frac{\pi}{6}\right)$. **(2 marks)**

b Write down the exact coordinates of the points where the graph meets the coordinate axes. **(3 marks)**

c Solve, for $0 \leqslant x \leqslant 2\pi$, the equation

$$y = \cos\left(x + \frac{\pi}{6}\right) = 0.65,$$

giving your answers in radians to 2 decimal places. **(5 marks)**

E **29** Solve, for $0 \leqslant x \leqslant \pi$, the equation

$$\sin\left(3x + \frac{\pi}{3}\right) = 0.45$$

giving your answers in radians to two decimal places. **(5 marks)**

Challenge

Use the small angle approximations to determine whether the following equations have any solutions close to $\theta = 0$. In each case, state whether each root of the resulting quadratic equation is likely to correspond to a solution of the original equation.

a $9\sin\theta\tan\theta + 25\tan\theta = 6$

b $2\tan\theta + 3 = 5\cos 4\theta$

c $\sin 4\theta = 37 - 2\cos 2\theta$

Summary of key points

1 • 2π radians = 360° • π radians = 180° • 1 radian = $\frac{180°}{\pi}$

2 • $30° = \frac{\pi}{6}$ radians • $45° = \frac{\pi}{4}$ radians • $60° = \frac{\pi}{3}$ radians

• $90° = \frac{\pi}{2}$ radians • $180° = \pi$ radians • $360° = 2\pi$ radians

3 You need to learn the exact values of the trigonometric ratios of these angles measured in radians.

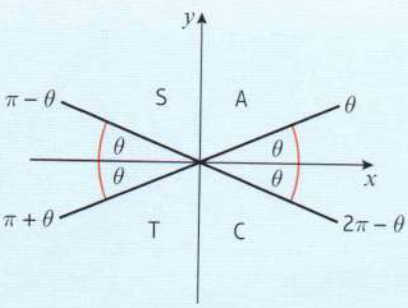
• $\sin\frac{\pi}{6} = \frac{1}{2}$ • $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ • $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

• $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ • $\cos\frac{\pi}{3} = \frac{1}{2}$ • $\tan\frac{\pi}{3} = \sqrt{3}$

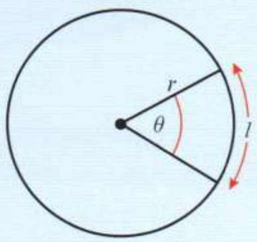
• $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ • $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ • $\tan\frac{\pi}{4} = 1$

4 You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angle made with the x-axis, θ.

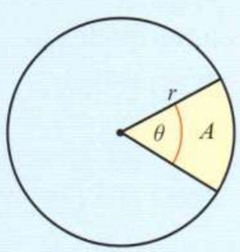
- $\sin(\pi - \theta) = \sin\theta$
- $\sin(\pi + \theta) = -\sin\theta$
- $\sin(2\pi - \theta) = -\sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(\pi + \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$



5 To find the arc length l of a sector of a circle use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



6 To find the area A of a sector of a circle use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



7 The area of a segment in a circle of radius r is

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

8 When θ is small and measured in radians:

• $\sin\theta \approx \theta$ • $\tan\theta \approx \theta$ • $\cos\theta \approx 1 - \frac{\theta^2}{2}$