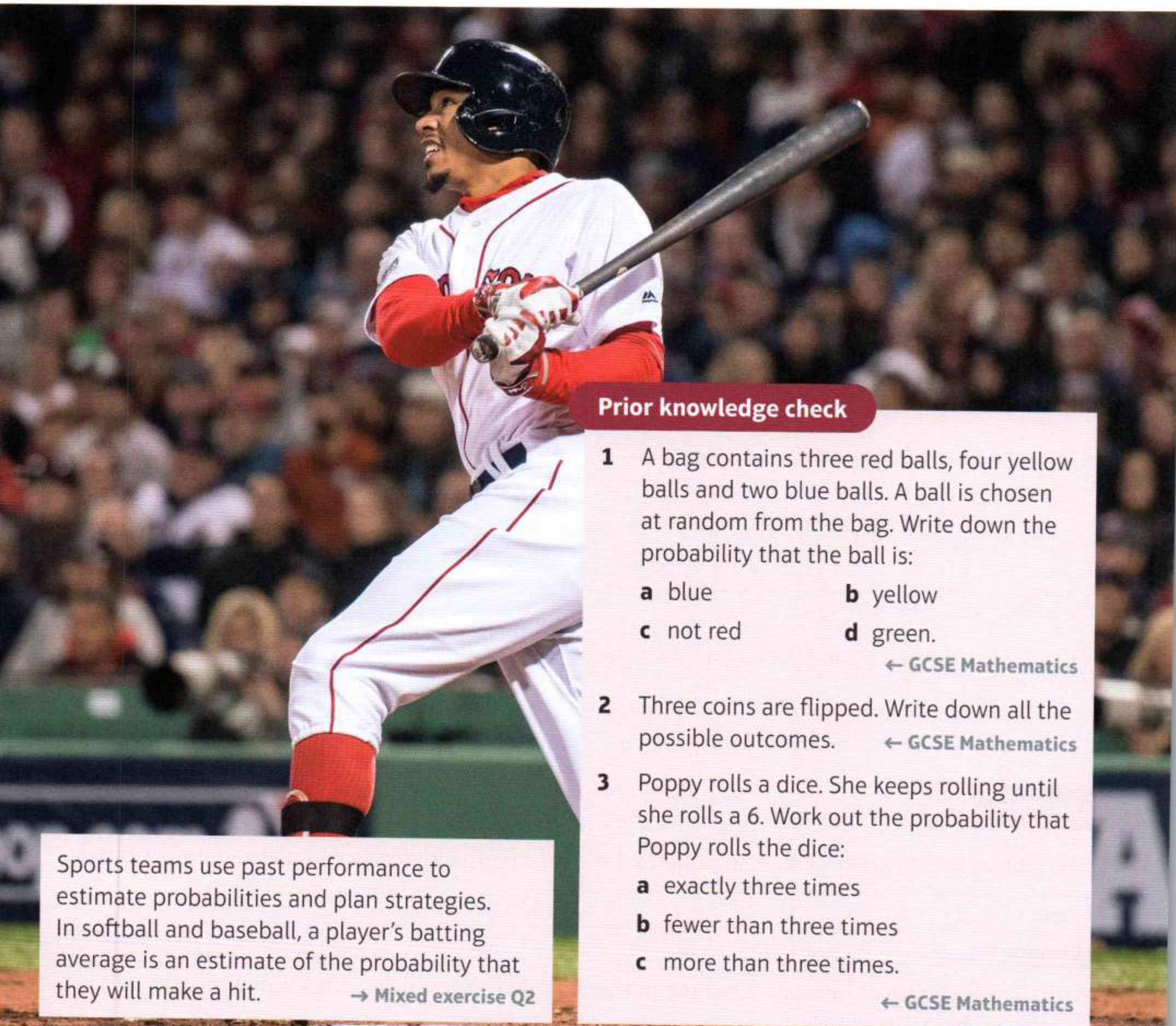
Probability

5

Objectives

After completing this chapter you should be able to:

- Calculate probabilities for single events → pages 70–72
- Draw and interpret Venn diagrams → pages 72–75
- Understand mutually exclusive and independent events, and determine whether two events are independent → pages 75–78
- Use and understand tree diagrams → pages 78–80



Sports teams use past performance to estimate probabilities and plan strategies. In softball and baseball, a player's batting average is an estimate of the probability that they will make a hit. → Mixed exercise Q2

Prior knowledge check

1 A bag contains three red balls, four yellow balls and two blue balls. A ball is chosen at random from the bag. Write down the probability that the ball is:

a blue **b** yellow

c not red **d** green.

← GCSE Mathematics

2 Three coins are flipped. Write down all the possible outcomes. ← GCSE Mathematics

3 Poppy rolls a dice. She keeps rolling until she rolls a 6. Work out the probability that Poppy rolls the dice:

a exactly three times

b fewer than three times

c more than three times.

← GCSE Mathematics

5.1 Calculating probabilities

If you want to predict the chance of something happening, you use probability.

An **experiment** is a repeatable process that gives rise to a number of **outcomes**.

An **event** is a collection of one or more outcomes.

A **sample space** is the set of all possible outcomes.

Where outcomes are **equally likely** the probability of an event is the number of outcomes in the event divided by the total number of possible outcomes.

All events have probability between 0 (impossible) and 1 (certain). Probabilities are usually written as fractions or decimals.

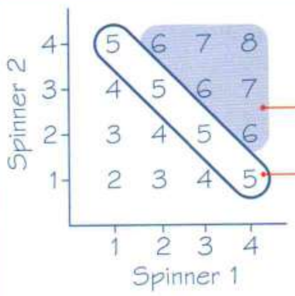
Example 1

Two fair spinners each have four sectors numbered 1 to 4. The two spinners are spun together and the sum of the numbers indicated on each spinner is recorded.

Find the probability of the spinners indicating a sum of:

a exactly 5 **b** more than 5.

Draw a sample space diagram showing all possible outcomes.



There are 4 × 4 = 16 points. Each of these points is equally likely as the spinners are fair.

There are 4 outcomes for part **a**.

a $P(5) = \frac{4}{16} = \frac{1}{4}$

There are four 5s and 16 outcomes altogether.
P( ) is short for 'probability of'. The answer P(5) can also be written as 0.25.

b $P(\text{more than } 5) = \frac{6}{16} = \frac{3}{8}$

There are six sums more than 5 for this part (shaded blue). They form the top right corner of the diagram.
The answer can also be written as 0.375.

Example 2

The table shows the times taken, in minutes, for a group of students to complete a number puzzle.

Time, t (min)	$5 \leq t < 7$	$7 \leq t < 9$	$9 \leq t < 11$	$11 \leq t < 13$	$13 \leq t < 15$
Frequency	6	13	12	5	4

A student is chosen at random. Find the probability that they finished the number puzzle:

a in under 9 minutes **b** in over 10.5 minutes.

a P(finished in under 9 minutes) = $\frac{19}{40}$

b $3 + 5 + 4 = 12$

P(finished in over 10.5 minutes) = $\frac{12}{40} = \frac{3}{10}$

There are 40 students overall.

$6 + 13 = 19$ finished in under 9 minutes.

Problem-solving

Use interpolation: 10.5 minutes lies $\frac{3}{4}$ of the way through the $9 \leqslant t < 11$ class, so approximately $\frac{1}{4}$ of the 12 students finished in more than 10.5 minutes. Your answer is an estimate because you don't know the exact number of students who took longer than 10.5 minutes.

Exercise 5A

1 Two coins are tossed. Find the probability of both coins showing the same outcome.

2 Two six-sided dice are thrown and their product, X, is recorded.

a Draw a sample space diagram showing all the possible outcomes of this experiment.

b Find the probability of each event:

i $X = 24$ **ii** $X < 5$ **iii** X is even.

(P) **3** The masses of 140 adult Bullmastiffs are recorded in a table. One dog is chosen at random.

Mass, m (kg)	Frequency
$45 \leqslant m < 48$	17
$48 \leqslant m < 51$	25
$51 \leqslant m < 54$	42
$54 \leqslant m < 57$	33
$57 \leqslant m < 60$	21
$60 \leqslant m < 63$	2

a Find the probability that the dog has a mass of 54 kg or more.

b Find the probability that the dog has a mass between 48 kg and 57 kg.

The probability that a Rottweiler chosen at random has a mass under 53 kg is 0.54.

c Is it more or less likely than this that a Bullmastiff chosen at random has a mass under 53 kg? State one assumption that you have made in making your decision.

Hint Use interpolation.

(P) **4** The lengths, in cm, of 240 koalas are recorded in a table.

One koala is chosen at random.

Length, l (cm)	Frequency (male)	Frequency (female)
$65 \leqslant l < 70$	4	14
$70 \leqslant l < 75$	20	15
$75 \leqslant l < 80$	24	32
$80 \leqslant l < 85$	47	27
$85 \leqslant l < 90$	31	26

a Find the probability that the koala is female.

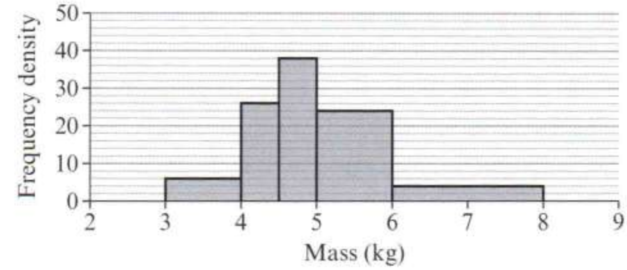
b Find the probability that the koala is less than 80 cm long.

c Find the probability that the koala is a male between 75 cm and 85 cm long.

Koalas under 72 cm long are called juvenile.

d Estimate the probability that a koala chosen at random is juvenile. State one assumption you have made in making your estimate.

E/P **5** The histogram shows the distribution of masses, in kg, of 70 adult cats.

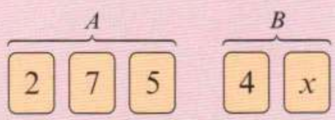


a Find the probability that a cat chosen at random has a mass more than 5 kg. **(2 marks)**

b Estimate the probability that a cat chosen at random has a mass less than 6.5 kg. **(3 marks)**

Challenge

Samira picks one card at random from group A and one card at random from group B.

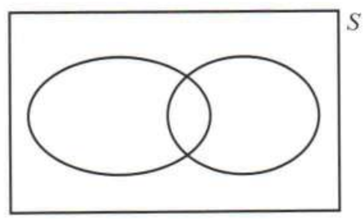


She records the product, Y, of the two cards as the result of her experiment. Given that x is an integer and that P(Y is even) = P($Y \geqslant 20$), find the possible values of x.

5.2 Venn diagrams

- **A Venn diagram can be used to represent events graphically. Frequencies or probabilities can be placed in the regions of the Venn diagram.**

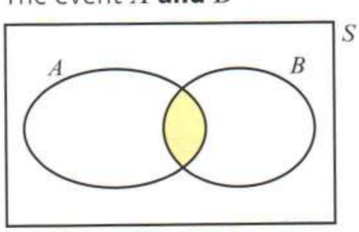
Venn diagrams are named after the English mathematician John Venn (1834–1923).



A rectangle represents the sample space, S, and it contains closed curves that represent events.

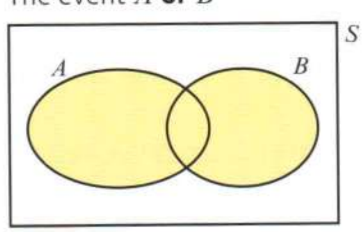
For events A and B in a sample space S:

1 The event A **and** B



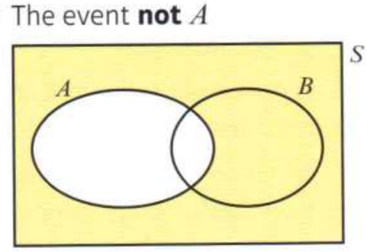
This event is also called the **intersection** of A and B. It represents the event that both A and B occur.

2 The event A **or** B



This event is also called the **union** of A and B. It represents the event that either A or B, or both, occur.

3 The event **not** A



This event is also called the **complement** of A. It represents the event that A does not occur.

P(not A) = 1 − P(A)

You can write numbers of outcomes (frequencies) or the probability of the events in a Venn diagram to help solve problems.

Example 3

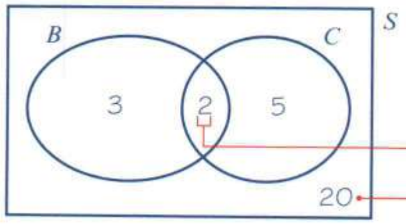
In a class of 30 students, 7 are in the choir, 5 are in the school band, and 2 are in the choir and the band. A student is chosen at random from the class.

a Draw a Venn diagram to represent this information.

b Find the probability that:

i the student is not in the band **ii** the student is not in the choir or the band.

a



Put the number in both the choir and the band in the intersection of B and C.

This region represents the events in the sample space that are not in C or B:

$30 - (3 + 2 + 5) = 20$.

b **i** A student not in the band is not B.

$P(\text{not } B) = \frac{25}{30} = \frac{5}{6}$

There are $5 + 20 = 25$ outcomes not in B, out of 30 equally likely outcomes.

ii P(student is not in the choir or the band)

$= \frac{20}{30} = \frac{2}{3}$

20 outcomes are in neither event.

Example 4

A vet surveys 100 of her clients. She finds that:

- 25 own dogs
- 15 own dogs and cats
- 11 own dogs and fish
- 53 own cats
- 10 own cats and fish
- 7 own dogs, cats and fish
- 40 own fish

A client is chosen at random.

Find the probability that the client:

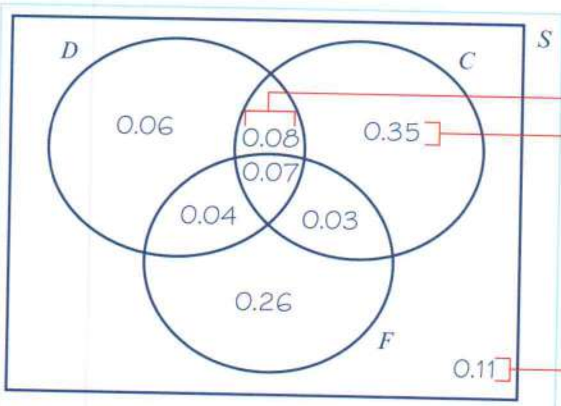
a owns dogs only

b does not own fish

c does not own dogs, cats or fish.

Problem-solving

You can use a Venn diagram with probabilities to solve this problem, but it could also be solved using the number of outcomes. There are 7 clients who own all three pets. Start with 0.07 in the intersection of all three events.



Work outwards to the intersections.

$0.15 - 0.07 = 0.08$

Each of 'dogs only', 'cats only' and 'fish only' can be worked out by further subtractions:

$0.53 - (0.08 + 0.07 + 0.03) = 0.35$ for 'cats only'

As the probability of the whole sample space is 1, the final area is $1 - (0.26 + 0.04 + 0.07 + 0.03 + 0.06 + 0.08 + 0.35) = 0.11$

a P(owns dogs only) = 0.06

b P(does not own fish) = 1 − 0.4 = 0.6

c P(does not own dogs, cats or fish) = 0.11

This is the value on the Venn diagram outside D, C and F.

Exercise 5B

1 There are 25 students in a certain tutor group at Philips College. There are 16 students in the tutor group studying German, 14 studying French and 6 students studying both French and German.

a Draw a Venn diagram to represent this information.

b Find the probability that a randomly chosen student in the tutor group:

i studies French
ii studies French and German
iii studies French but not German
iv does not study French or German.

2 There are 125 diners in a restaurant who were surveyed to find out if they had ordered garlic bread, beer or cheesecake:

15 diners had ordered all three items
43 diners had ordered garlic bread
40 diners had ordered beer
44 diners had ordered cheesecake
20 had ordered beer and cheesecake
26 had ordered garlic bread and cheesecake
25 had ordered garlic bread and beer

a Draw a Venn diagram to represent this information.

A diner is chosen at random. Find the probability that the diner ordered:

b i all three items
ii beer but not cheesecake and not garlic bread
iii garlic bread and beer but not cheesecake
iv none of these items.

3 A group of 275 people at a music festival were asked if they play guitar, piano or drums:

one person plays all three instruments
65 people play guitar and piano
10 people play piano and drums
30 people play guitar and drums
15 people play piano only
20 people play guitar only
35 people play drums only

a Draw a Venn diagram to represent this information.

b A festival goer is chosen at random from the group.
Find the probability that the person chosen:

i plays the piano
ii plays at least two of guitar, piano and drums
iii plays exactly one of the instruments
iv plays none of the instruments.

(P) 4 The probability that a child in a school has blue eyes is 0.27 and the probability that they have blonde hair is 0.35. The probability that the child will have blonde hair or blue eyes or both is 0.45. A child is chosen at random from the school. Find the probability that the child has:

a blonde hair and blue eyes

b blonde hair but not blue eyes

c neither feature.

Hint Draw a Venn diagram to help you.

(E/P) 5 A patient going into a doctor's waiting room reads *Hiya* magazine with probability 0.6 and *Dakor* magazine with probability 0.4. The probability that the patient reads either one or both of the magazines is 0.7. Find the probability that the patient reads:

a both magazines **(2 marks)**

b *Hiya* magazine only. **(2 marks)**

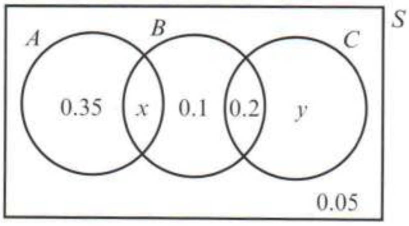
6 The Venn diagram shows the probabilities of members of a sports club taking part in various activities.

A represents the event that the member takes part in archery.

B represents the event that the member takes part in badminton.

C represents the event that the member takes part in croquet.

Given that $P(B) = 0.45$:



a find x **(1 mark)**

b find y. **(2 marks)**

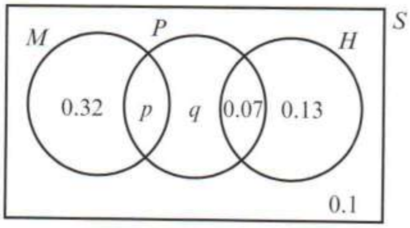
7 The Venn diagram shows the probabilities that students at a sixth-form college study certain subjects.

M represents the event that the student studies mathematics.

P represents the event that the student studies physics.

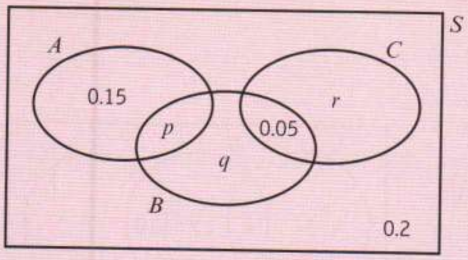
H represents the event that the student studies history.

Given that $P(M) = P(P)$, find the values of p and q. **(4 marks)**



Challenge

The Venn diagram shows the probabilities of a group of children liking three types of sweet.

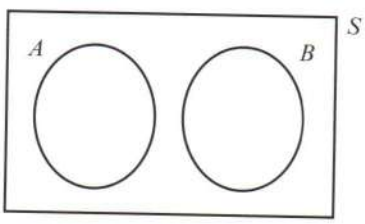


Given that $P(B) = 2P(A)$ and that $P(\text{not } C) = 0.83$, find the values of p, q and r.

5.3 Mutually exclusive and independent events

When events have no outcomes in common they are called **mutually exclusive**.

In a Venn diagram, the closed curves do not overlap and you can use a simple addition rule to work out combined probabilities:



- **For mutually exclusive events, P(A or B) = P(A) + P(B).**

When one event has no effect on another, they are **independent**. Therefore if A and B are independent, the probability of A happening is the same whether or not B happens.

- **For independent events, P(A and B) = P(A) × P(B).**

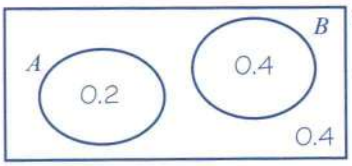
You can use this **multiplication rule** to determine whether events are independent.

Example 5

Events A and B are mutually exclusive and $P(A) = 0.2$ and $P(B) = 0.4$.

Find: **a** $P(A \text{ or } B)$ **b** $P(A \text{ but not } B)$ **c** $P(\text{neither } A \text{ nor } B)$

a



$P(A \text{ or } B) = P(A) + P(B)$
$= 0.2 + 0.4 = 0.6$

b $P(A \text{ but not } B) = P(A) = 0.2$

c $P(\text{neither } A \text{ nor } B) = 0.4$

> A and B are mutually exclusive so the closed curves do not intersect.

> Use the simple addition rule.

> Everything in A is 'not B'.

> This is everything outside of both circles: $1 - P(A \text{ or } B)$.

Example 6

Events A and B are independent and $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{5}$.

Find $P(A \text{ and } B)$.

$P(A \text{ and } B) = P(A) \times P(B) = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$

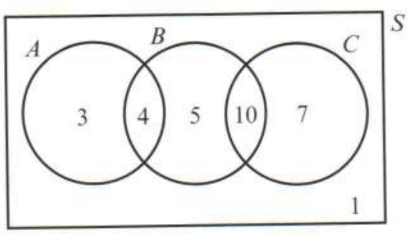
> A and B are independent so you can use the multiplication rule for independent events.

Example 7

The Venn diagram shows the number of students in a particular class who watch any of three popular TV programmes.

a Find the probability that a student chosen at random watches B or C or both.

b Determine whether watching A and watching B are statistically independent.



a $4 + 5 + 10 + 7 = 26$

$P(\text{watches } B \text{ or } C \text{ or both}) = \frac{26}{30} = \frac{13}{15}$

b $P(A) = \frac{3 + 4}{30} = \frac{7}{30}$

$P(B) = \frac{4 + 5 + 10}{30} = \frac{19}{30}$

$P(A \text{ and } B) = \frac{4}{30} = \frac{2}{15}$

$P(A) \times P(B) = \frac{7}{30} \times \frac{19}{30} = \frac{133}{900}$

So $P(A \text{ and } B) \neq P(A) \times P(B)$

Therefore watching A and watching B are *not* independent.

> Take the probabilities from the Venn diagram.

> Multiply the two probabilities and check whether they give the same answer as $P(A \text{ and } B)$.

Problem-solving

Show your calculations and then write down a conclusion stating whether or not the events are independent.

Exercise 5C

1 Events A and B are mutually exclusive. $P(A) = 0.2$ and $P(B) = 0.5$.

a Draw a Venn diagram to represent these two events.

b Find P(A or B).

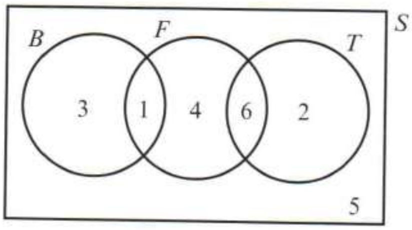
c Find P(neither A nor B).

2 Two fair dice are rolled and the result on each die is recorded. Show that the events 'the sum of the scores on the dice is 4' and 'both dice land on the same number' are not mutually exclusive.

3 $P(A) = 0.5$ and $P(B) = 0.3$. Given that events A and B are independent, find P(A and B).

4 $P(A) = 0.15$ and P(A and B) $= 0.045$. Given that events A and B are independent, find $P(B)$.

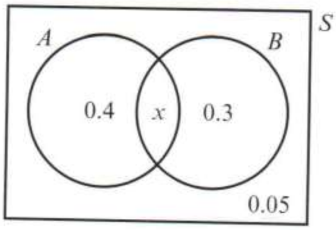
5 The Venn diagram shows the number of children in a play group who like playing with bricks (B), action figures (F) or trains (T).



a State, with a reason, which two types of toy are mutually exclusive.

b Determine whether the events 'plays with bricks' and 'plays with action figures' are independent.

(E) **6** The Venn diagram shows the probabilities that a group of students like pasta (A) or pizza (B).



a Write down the value of x. **(1 mark)**

b Determine whether the events 'like pasta' and 'like pizza' are independent. **(3 marks)**

(P) **7** S and T are two events such that $P(S) = 0.3$, $P(T) = 0.4$ and P(S but not T) $= 0.18$.

a Show that S and T are independent.

b Find:

i P(S and T) **ii** P(neither S nor T).

(P) **8** W and X are two events such that $P(W) = 0.5$, P(W and not X) $= 0.25$ and P(neither W nor X) $= 0.3$. State, with a reason, whether W and X are independent events.

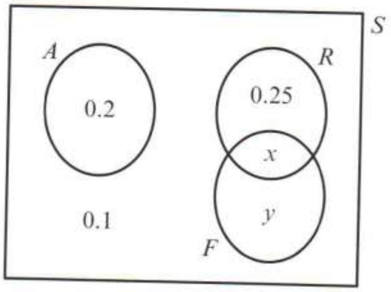
(3 marks)

(P) **9** The Venn diagram shows the probabilities of members of a social club taking part in charitable activities.

A represents taking part in an archery competition.

R represents taking part in a raffle.

F represents taking part in a fun run.

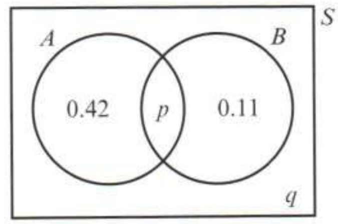


The probability that a member takes part in the archery competition or the raffle is 0.6.

a Find the value of x and the value of y. **(2 marks)**

b Show that events R and F are not independent. **(3 marks)**

(P) **10** Given that A and B are independent, find the two possible values for p and q.



Challenge

A and B are independent events in a sample space S. Given that A and B are independent, prove that:

a A and 'not B' are independent

b 'not A' and 'not B' are independent.

5.4 Tree diagrams

- **A tree diagram can be used to show the outcomes of two (or more) events happening in succession.**

Example 8

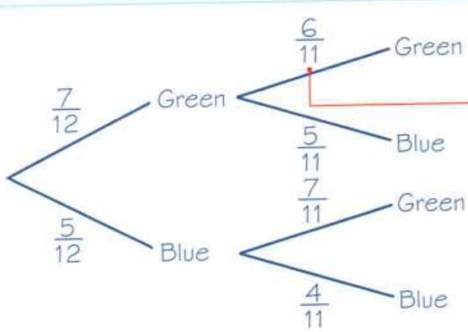
A bag contains seven green beads and five blue beads. A bead is taken from the bag at random and not replaced. A second bead is then taken from the bag.

Find the probability that:

a both beads are green

b the beads are different colours.

a



Draw a tree diagram to show the events.

There are now only 6 green beads and 11 beads in total.

P(green and green) $= \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$

Multiply along the branch of the tree diagram.

b P(different colours)

= P(green then blue) + P(blue then green)

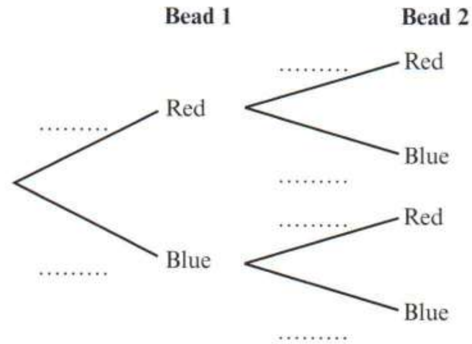
$= \frac{7}{12} \times \frac{5}{11} + \frac{5}{12} \times \frac{7}{11} = \frac{35}{66}$

Multiply along each branch and add the two probabilities.

Exercise 5D

1 A bag contains three red beads and five blue beads. A bead is chosen at random from the bag, the colour is recorded and the bead is replaced. A second bead is chosen and the colour recorded.

a Copy and complete this tree diagram to show the outcomes of the experiment.



b Find the probability that both beads are blue.

c Find the probability that the second bead is blue.

2 A box contains nine cards numbered 1 to 9. A card is drawn at random and not replaced. It is noted whether the number is odd or even. A second card is drawn and it is also noted whether this number is odd or even.

a Draw a tree diagram to represent this experiment.

Hint The first card is not replaced.

b Find the probability that both cards are even.

c Find the probability that one card is odd and the other card is even.

3 The probability that Charlie takes the bus to school is 0.4. If he doesn't take the bus, he walks. The probability that Charlie is late to school if he takes the bus is 0.2. The probability he is late to school if he walks is 0.3.

a Draw a tree diagram to represent this information.

b Find the probability that Charlie is late to school.

(E) 4 Mr Dixon plays golf. The probability that he scores par or under on the first hole is 0.7. If he scores par or under on the first hole, the probability he scores par or under on the second hole is 0.8. If he doesn't score par or under on the first hole, the probability that he scores par or under on the second hole is 0.4.

a Draw a tree diagram to represent this information. **(3 marks)**

b State whether the events 'scores par or under on the first hole' and 'scores par or under par on the second hole' are independent. **(1 mark)**

c Find the probability that Mr Dixon scores par or under on only one hole. **(3 marks)**

(/P) 5 A biased coin is tossed three times and it is recorded whether it falls heads or tails. P(heads) = $\frac{1}{3}$

a Draw a tree diagram to represent this experiment. **(3 marks)**

b Find the probability that the coin lands on heads all three times. **(1 mark)**

c Find the probability that the coin lands on heads only once. **(2 marks)**

The whole experiment is repeated for a second trial.

d Find the probability of obtaining either 3 heads or 3 tails in both trials. **(3 marks)**

E/P 6 A bag contains 13 tokens, 4 coloured blue, 3 coloured red and 6 coloured yellow. Two tokens are drawn from the bag without replacement.

a Find the probability that both tokens are yellow. **(2 marks)**

A third token is drawn from the bag.

b Write down the probability that the third token is yellow, given that the first two are yellow. **(1 mark)**

c Find the probability that all three tokens are different colours. **(4 marks)**

Mixed exercise 5

E/P 1 There are 15 coloured beads in a bag; seven beads are red, three are blue and five are green. Three beads are selected at random from the bag and replaced. Find the probability that:

a the first and second beads chosen are red and the third bead is blue or green **(3 marks)**

b one red, one blue and one green bead are chosen. **(3 marks)**

2 A baseball player has a batting average of 0.341. This means her probability of making a hit when she bats is 0.341. She bats three times in one game. Estimate the probability that:

a she makes three hits

b she makes no hits

c she makes at least one hit.

P 3 The scores of 250 students in a test are recorded in a table.

Score, s	Frequency (male)	Frequency (female)
$20 \leqslant s < 25$	7	8
$25 \leqslant s < 30$	15	13
$30 \leqslant s < 35$	18	19
$35 \leqslant s < 40$	25	30
$40 \leqslant s < 45$	30	26
$45 \leqslant s < 50$	27	32

One student is chosen at random.

a Find the probability that the student is female.

b Find the probability that the student scored less than 35.

c Find the probability that the student is male with a score s such that $25 \leq s < 35$.

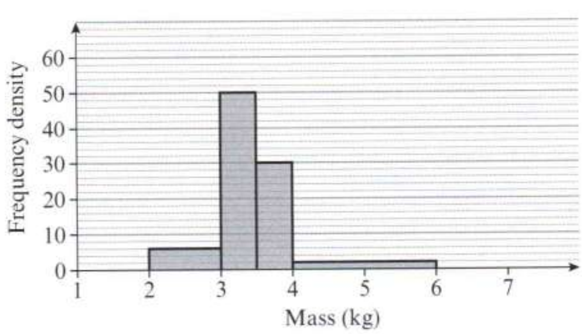
In order to pass the test, students must score 37 or more.

d Estimate the probability that a student chosen at random passes the test. State one assumption you have made in making your estimate.

E/P 4 The histogram shows the distribution of masses, in kg, of 50 newborn babies.

a Find the probability that a baby chosen at random has a mass greater than 3 kg. **(2 marks)**

b Estimate the probability that a baby chosen at random has a mass less than 3.75 kg. **(3 marks)**



(E) **5** A study was made of a group of 150 children to determine which of three cartoons they watch on television. The following results were obtained:

35 watch Toontime
54 watch Porky
62 watch Skellingtons
9 watch Toontime and Porky
14 watch Porky and Skellingtons
12 watch Toontime and Skellingtons
4 watch Toontime, Porky and Skellingtons

a Draw a Venn diagram to represent this data. **(4 marks)**

b Find the probability that a randomly selected child from the study watches:

i none of the three cartoons **(2 marks)**

ii no more than one of the cartoons. **(2 marks)**

(P) **6** The events A and B are such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$. $P(A \text{ or } B) = \frac{1}{2}$.

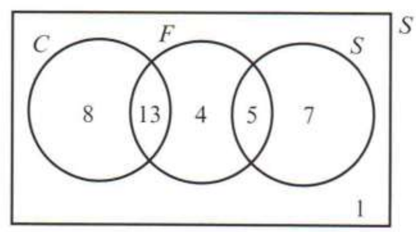
a Represent these probabilities on a Venn diagram.

b Show that A and B are independent.

(E) **7** The Venn diagram shows the number of students who like cricket (C), football (F) or swimming (S).

a Which two sports are mutually exclusive? **(1 mark)**

b Determine whether the events 'likes cricket' and 'likes football' are independent. **(3 marks)**



(E/P) **8** For events J and K, $P(J \text{ or } K) = 0.5$, $P(K \text{ but not } J) = 0.2$ and $P(J \text{ but not } K) = 0.25$.

a Draw a Venn diagram to represent events J and K and the sample space S. **(3 marks)**

b Determine whether events J and K are independent. **(3 marks)**

(E) **9** A survey of a group of students revealed that 85% have a mobile phone, 60% have an MP3 player and 5% have neither phone nor MP3 player.

a Find the proportion of students who have both gadgets. **(2 marks)**

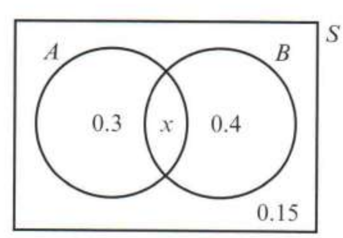
b Draw a Venn diagram to represent this information. **(3 marks)**

c A student is chosen at random. Find the probability that they only own a mobile phone. **(2 marks)**

d Are the events 'own a mobile phone' and 'own an MP3 player' independent? Justify your answer. **(3 marks)**

(E/P) **10** The Venn diagram shows the probabilities that a group of children like cake (A) or crisps (B).

Determine whether the events 'like cake' and 'like crisps' are independent. **(3 marks)**



(E/P) **11** A computer game has three levels and one of the objectives of every level is to collect a diamond. The probability that Becca collects a diamond on the first level is $\frac{4}{5}$, on the second level is $\frac{2}{3}$ and on the third level is $\frac{1}{2}$. The events are independent.

a Draw a tree diagram to represent Becca collecting diamonds on the three levels of the game. **(4 marks)**

b Find the probability that Becca:

i collects all three diamonds **(2 marks)**

ii collects only one diamond. **(3 marks)**

c Find the probability that she collects at least two diamonds each time she plays. **(3 marks)**

(P) **12** In a factory, machines A, B and C produce electronic components. Machine A produces 16% of the components, machine B produces 50% of the components and machine C produces the rest. Some of the components are defective. 4% of the components produced by machine A are defective, as are 3% of those prduced by machine B and 7% of those produced by machine C.

a Draw a tree diagram to represent this information.

b Find the probability that a randomly selected component is:

i produced by machine B and is defective **ii** defective.

Challenge

The members of a cycling club are married couples. For any married couple in the club, the probability that the husband is retired is 0.7 and the probability that the wife is retired 0.4. Given that the wife is retired, the probability that the husband is retired is 0.8.

Two married couples are chosen at random.

Find the probability that only one of the two husbands and only one of the two wives is retired.

Summary of key points

1. A **Venn diagram** can be used to represent events graphically. Frequencies or probabilities can be placed in the regions of the Venn diagram.
2. For **mutually exclusive** events, P(A or B) = P(A) + P(B).
3. For **independent** events, P(A and B) = P(A) × P(B).
4. A **tree diagram** can be used to show the outcomes of two (or more) events happening in succession.