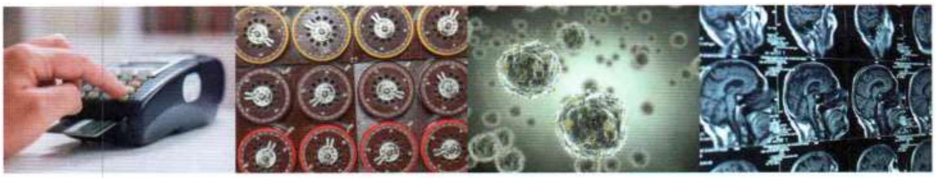
Review exercise 1



(E/P) **1** Prove by contradiction that there are infinitely many prime numbers. **(4)**

← Section 1.1

(E/P) **2** Prove that the equation $x^2 - 2 = 0$ has no rational solutions.
You may assume that if n^2 is an even integer then n is also an even integer. **(4)**

← Section 1.1

(E) **3** Express $\dfrac{4x}{x^2 - 2x - 3} + \dfrac{1}{x^2 + x}$ as a single fraction in its simplest form. **(4)**

← Section 1.2

(P) **4** $f(x) = 1 - \dfrac{3}{x+2} + \dfrac{3}{(x+2)^2}, x \neq -2$

a Show that $f(x) = \dfrac{x^2 + x + 1}{(x+2)^2}, x \neq -2$.

b Show that $x^2 + x + 1 > 0$ for all values of $x, x \neq -2$.

c Show that $f(x) > 0$ for all values of x, $x \neq -2$.

← Section 1.2

(E) **5** Show that $\dfrac{2x - 1}{(x-1)(2x-3)}$ can be written in the form $\dfrac{A}{x-1} + \dfrac{B}{2x-3}$ where A and B are constants to be found. **(3)**

← Section 1.3

(E) **6** Given that

$$\frac{3x + 7}{(x+1)(x+2)(x+3)} \equiv \frac{P}{x+1} + \frac{Q}{x+2} + \frac{R}{x+3}$$

where P, Q and R are constants, find the values of P, Q and R. **(4)**

← Section 1.3

(E) **7** $f(x) = \dfrac{2}{(2-x)(1+x)^2}, x \neq -1, x \neq 2$.

Find the values of A, B and C such that

$$f(x) = \frac{A}{2-x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$ **(4)**

← Section 1.4

8 $$\frac{14x^2 + 13x + 2}{(x+1)(2x+1)^2} \equiv \frac{A}{x+1} + \frac{B}{2x+1} + \frac{C}{(2x+1)^2}$$

Find the values of the constants A, B and C.

← Section 1.4

(E) **9** Given that $\dfrac{3x^2 + 6x - 2}{x^2 + 4} \equiv d + \dfrac{ex + f}{x^2 + 4}$

find the values of d, e and f. **(4)**

← Section 1.5

(E) **10** $p(x) = \dfrac{9 - 3x - 12x^2}{(1-x)(1+2x)}$

Show that $p(x)$ can be written in the form $A + \dfrac{B}{1-x} + \dfrac{C}{1+2x}$, where A, B and C are constants to be found. **(4)**

← Sections 1.3, 1.5

(E) **11** Solve the inequality $|4x + 3| > 7 - 2x$. **(3)**

← Section 2.1

(E/P) **12** The function $p(x)$ is defined by

$$p: x \mapsto \begin{cases} 4x + 5, & x < -2 \\ -x^2 + 4, & x \geqslant -2 \end{cases}$$

a Sketch $p(x)$, stating its range. **(3)**

b Find the exact values of a such that $p(a) = -20$. **(4)**

← Section 2.2

E/P **13** The functions p and q are defined by

$p(x) = \frac{1}{x+4}, x \in \mathbb{R}, x \neq -4$

$q(x) = 2x - 5, x \in \mathbb{R}$

a Find an expression for $qp(x)$ in the form $\frac{ax+b}{cx+d}$ **(3)**

b Solve $qp(x) = 15$. **(3)**

Let $r(x) = qp(x)$.

c Find $r^{-1}(x)$, stating its domain. **(3)**

← Section 2.3

E/P **14** The functions f and g are defined by:

$f: x \mapsto \frac{x+2}{x}, x \in \mathbb{R}, x \neq 0$

$g: x \mapsto \ln(2x-5), x \in \mathbb{R}, x > \frac{5}{2}$

a Sketch the graph of f. **(3)**

b Show that $f^2(x) = \frac{3x+2}{x+2}$ **(3)**

c Find the exact value of $gf\left(\frac{1}{4}\right)$.

(2)

d Find $g^{-1}(x)$, stating its domain. **(3)**

← Section 2.3, 2.4

E/P **15** The functions p and q are defined by:

$p(x) = 3x + b, x \in \mathbb{R}$

$q(x) = 1 - 2x, x \in \mathbb{R}$

Given that $pq(x) = qp(x)$,

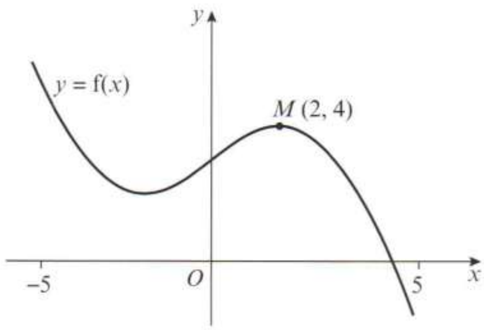
a show that $b = -\frac{2}{3}$ **(3)**

b find $p^{-1}(x)$ and $q^{-1}(x)$ **(3)**

c show that $p^{-1}q^{-1}(x) = q^{-1}p^{-1}(x) = \frac{ax+b}{c}$, where a, b and c are integers to be found. **(4)**

← Section 2.3, 2.4

E **16**



The figure shows the graph of $y = f(x), -5 \leqslant x \leqslant 5$

The point $M(2, 4)$ is the maximum turning point of the graph.

Sketch, on separate diagrams, the graphs of:

a $y = f(x) + 3$ **(2)**

b $y = |f(x)|$ **(2)**

c $y = f(|x|)$ **(2)**

Show on each graph the coordinates of any maximum turning points.

← Sections 2.5, 2.6

E/P **17** The function h is defined by

$h: x \mapsto 2(x+3)^2 - 8, x \in \mathbb{R}$

a Draw a sketch of $y = h(x)$, labelling the turning points and the x- and y-intercepts. **(4)**

b Write down the coordinates of the turning points on the graphs with equations:

i $y = 3h(x+2)$ **(2)**

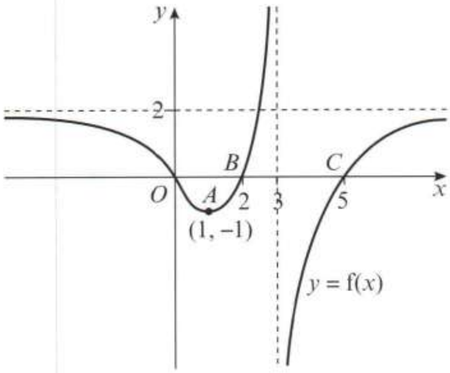
ii $y = h(-x)$ **(2)**

iii $y = |h(x)|$ **(2)**

c Sketch the curve with equation $y = h(-|x|)$. On your sketch show the coordinates of all turning points and all x- and y-intercepts. **(4)**

← Sections 2.5, 2.6

E **18**



The diagram shows a sketch of the graph of $y = \mathrm{f}(x)$.

The curve has a minimum at the point $A\,(1, -1)$, passes through x-axis at the origin, and the points $B\,(2, 0)$ and $C\,(5, 0)$; the asymptotes have equations $x = 3$ and $y = 2$.

a Sketch, on separate axes, the graphs of:

i $y = |\mathrm{f}(x)|$ **(2)**

ii $y = -\mathrm{f}(x + 1)$ **(2)**

iii $y = \mathrm{f}(-2x)$ **(2)**

in each case, showing the images of the points A, B and C.

b State the number of solutions to each equation.

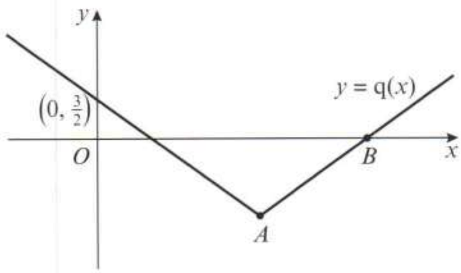
i $3|\mathrm{f}(x)| = 2$ **(2)**

ii $2|\mathrm{f}(x)| = 3.$ **(2)**

← Sections 2.6, 2.7

E/P **19** The diagram shows a sketch of part of the graph $y = \mathrm{q}(x)$, where

$\mathrm{q}(x) = \frac{1}{2}|x + b| - 3,\ b < 0$



The graph cuts the y-axis at $\left(0, \frac{3}{2}\right)$.

a Find the value of b. **(2)**

b Find the coordinates of A and B. **(3)**

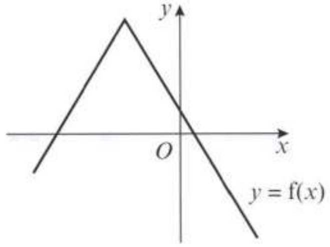
c Solve $\mathrm{q}(x) = -\frac{1}{3}x + 5.$ **(5)**

← Section 2.7

E/P **20** The function f is defined by

$\mathrm{f}(x) = -\frac{5}{3}|x + 4| + 8,\ x \in \mathbb{R}$

The diagram shows a sketch of the graph $y = \mathrm{f}(x)$.



a State the range of f. **(1)**

b Give a reason why $\mathrm{f}^{-1}(x)$ does not exist. **(1)**

c Solve the inequality $\mathrm{f}(x) > \frac{2}{3}x + 4.$ **(5)**

d State the range of values of k for which the equation $\mathrm{f}(x) = \frac{5}{3}x + k$ has no solutions. **(2)**

← Section 2.7

E/P **21** The 4th, 5th and 6th terms in an arithmetic sequence are:

$12 - 7k,\ 3k^2,\ k^2 - 10k$

a Find two possible values of k. **(3)**

Given that the sequence contains only integer terms,

b find the first term and the common difference. **(2)**

← Section 3.1

E **22** The 4th term of an arithmetic sequence is 72. The 11th term is 51. The sum of the first n terms is 1125.

a Show that $3n^2 - 165n + 2250 = 0.$ **(4)**

b Find the two possible values for n. **(2)**

← Section 3.2

E/P **23** **a** Find, in terms of p, the 30th term of the arithmetic sequence

$(19p - 18), (17p - 8), (15p + 2), \ldots$

giving your answer in its simplest form. **(2)**

b Given $S_{31} = 0$, find the value of p. **(3)**

← Sections 3.1, 3.2

E/P **24** The second term of a geometric sequence is 256. The eighth term of the same sequence is 900. The common ratio is r, $r > 0$.

a Show that r satisfies the equation

$6\ln r + \ln\left(\frac{64}{225}\right) = 0$ **(3)**

b Find the value of r correct to 3 significant figures. **(3)**

← Section 3.3

E/P **25** The first three terms of a geometric sequence are 10, $\frac{50}{6}$ and $\frac{250}{36}$.

a Find the sum to infinity of the series. **(3)**

Given that the sum to k terms of the series is greater than 55,

b show that $k > \dfrac{\log\left(\frac{1}{12}\right)}{\log\left(\frac{5}{6}\right)}$ **(4)**

c find the smallest possible value of k. **(1)**

← Sections 3.4, 3.5

E/P **26** A geometric series has first term 4 and common ratio r. The sum of the first three terms of the series is 7.

a Show that $4r^2 + 4r - 3 = 0$. **(3)**

b Find the two possible values of r. **(2)**

Given that r is positive,

c find the sum to infinity of the series. **(2)**

← Sections 3.4, 3.5

E/P **27** The fourth, fifth and sixth terms of a geometric series are x, 3 and $x + 8$.

a Find the two possible values of x and the corresponding values of the common ratio. **(4)**

Given that the sum to infinity of the series exists,

b find the first term **(1)**

c find the sum to infinity of the series. **(2)**

← Sections 3.3, 3.5

E/P **28** A sequence $a_1, a_2, a_3, \ldots$ is defined by

$a_1 = k$,

$a_{n+1} = 3a_n + 5, n \geqslant 1$

where k is a positive integer.

a Write down an expression for a_2 in terms of k. **(1)**

b Show that $a_3 = 9k + 20$. **(2)**

c **i** Find $\sum_{r=1}^{4} a_r$ in terms of k. **(2)**

ii Show that $\sum_{r=1}^{4} a_r$ is divisible by 10. **(2)**

← Sections 3.6, 3.7

E/P **29** At the end of year 1, a company employs 2400 people. A model predicts that the number of employees will increase by 6% each year, forming a geometric sequence.

a Find the predicted number of employees after 4 years, giving your answer to the nearest 10. **(3)**

The company expects to expand in this way until the total number of employees first exceeds 6000 at the end of a year, N.

b Show that $(N - 1)\log 1.06 > \log 2.5$ **(3)**

c Find the value of N. **(2)**

The company has a charity scheme whereby they match any employee charity contribution exactly.

d Given that the average employee charity contribution is £5 each year, find the total charity donation over the 10-year period from the end of year 1 to the end of year 10. Give your answer to the nearest £1000. **(3)**

← Section 3.8

E/P **30** A geometric series is given by

$6 - 24x + 96x^2 - \ldots$

The series is convergent.

a Write down a condition on x. **(1)**

Given that $\sum_{r=1}^{\infty} 6 \times (-4x)^{r-1} = \frac{24}{5}$

b Calculate the value of x. **(5)**

← Sections 3.5, 3.6

E **31** $g(x) = \frac{1}{\sqrt{1-x}}$

a Show that the series expansion of $g(x)$ up to and including the x^3 term is

$1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16}$ **(5)**

b State the range of values of x for which the expansion is valid. **(1)**

← Section 4.1

P **32** When $(1 + ax)^n$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 45 respectively.

a Find the value of a and the value of n.

b Find the coefficient of x^3.

c Find the set of values of x for which the expansion is valid.

← Section 4.1

E **33** **a** Find the binomial expansion of $(1 + 4x)^{\frac{3}{2}}$ in ascending powers of x up to and including the x^3 term, simplifying each term. **(4)**

b Show that, when $x = \frac{3}{100}$, the exact value of $(1 + 4x)^{\frac{3}{2}}$ is $\frac{112\sqrt{112}}{1000}$ **(2)**

c Substitute $x = \frac{3}{100}$ into the binomial expansion in part **a** and hence obtain an approximation to $\sqrt{112}$. Give your answer to 5 decimal places. **(3)**

d Calculate the percentage error in your estimate to 5 decimal places. **(2)**

← Section 4.1

E **34** $f(x) = (1 + x)(3 + 2x)^{-3}$, $|x| < \frac{3}{2}$

Find the binomial expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction. **(5)**

← Section 4.2

E **35** $h(x) = \sqrt{4 - 9x}$, $|x| < \frac{4}{9}$

a Find the series expansion of $h(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(4)**

b Show that, when $x = \frac{1}{100}$, the exact value of $h(x)$ is $\frac{\sqrt{391}}{10}$ **(2)**

c Use the series expansion in part **a** to estimate the value of $h\left(\frac{1}{100}\right)$ and state the degree of accuracy of your approximation. **(3)**

← Section 4.2

E/P **36** Given that $(a + bx)^{-2}$ has binomial expansion $\frac{1}{4} + \frac{1}{4}x + cx^2 + \ldots$

a Find the values of the constants a, b and c. **(4)**

b Find the coefficient of the x^3 term in the expansion. **(2)**

← Section 4.2

E/P **37** $g(x) = \frac{3 + 5x}{(1 + 3x)(1 - x)}$, $|x| < \frac{1}{3}$

Given that $g(x)$ can be expressed in the form $g(x) = \frac{A}{1 + 3x} + \frac{B}{1 - x}$

a find the values of A and B. **(3)**

b Hence, or otherwise, find the series expansion of $f(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(6)**

← Sections 4.1, 4.3

E/P **38** $\frac{3x-1}{(1-2x)^2} \equiv \frac{A}{1-2x} + \frac{B}{(1-2x)^2}$, $|x| < \frac{1}{2}$

a Find the values of A and B. **(3)**

b Hence, or otherwise, expand $\frac{3x-1}{(1-2x)^2}$ in ascending powers of x, as far as the term in x^3. Give each coefficient as a simplified fraction. **(6)**

← Sections 4.1, 4.3

E/P **39** $f(x) = \frac{25}{(3+2x)^2(1-x)}$, $|x| < 1$

f(x) can be expressed in the form

$\frac{A}{3+2x} + \frac{B}{(3+2x)^2} + \frac{C}{1-x}$

a Find the values of A, B and C. **(4)**

b Hence, or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the term in x^2. Simplify each term. **(6)**

← Sections 4.1, 4.2, 4.3

E/P **40** $\frac{4x^2+30x+31}{(x+4)(2x+3)} = A + \frac{B}{x+4} + \frac{C}{2x+3}$

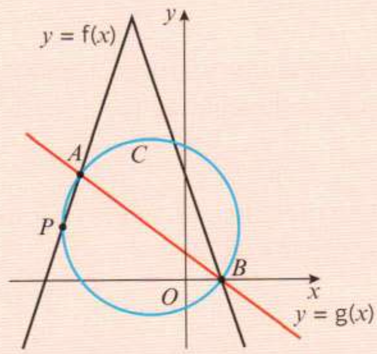
a Find the values of the constants A, B and C. **(4)**

b Hence, or otherwise, expand $\frac{4x^2+31x+30}{(x+4)(2x+3)}$ in ascending powers of x, as far as the term in x^2. Give each coefficient as a simplified fraction. **(7)**

← Sections 4.1, 4.2, 4.3

Challenge

1 The functions f and g are defined by
$f(x) = -3|x+3| + 15, x \in \mathbb{R}$
$g(x) = -\frac{3}{4}x + \frac{3}{2}, x \in \mathbb{R}$
The diagram shows a sketch of the graphs $y = f(x)$ and $y = g(x)$, which intersect at points A and B. M is the midpoint of AB.
The circle C, with centre M, passes through points A and B, and meets $y = f(x)$ at point P as shown in the diagram.



a Find the equation of the circle.

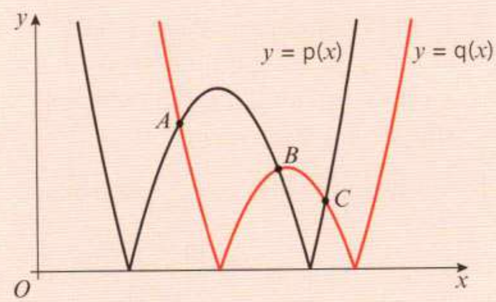
b Find the area of the triangle APB.

← Section 2.6

2 Given that $a_{n+1} = a_n + k$, $a_1 = m$ and $\sum_{i=6}^{11} a_i = \sum_{i=12}^{15} a_i$ show that $m = \frac{5}{2}k$.

← Section 3.6

3 The diagram shows a sketch of the functions $p(x) = |x^2 - 8x + 12|$ and $q(x) = |x^2 - 11x + 28|$.



Find the exact values of the x-coordinates of the points A, B and C.

← Section 2.5