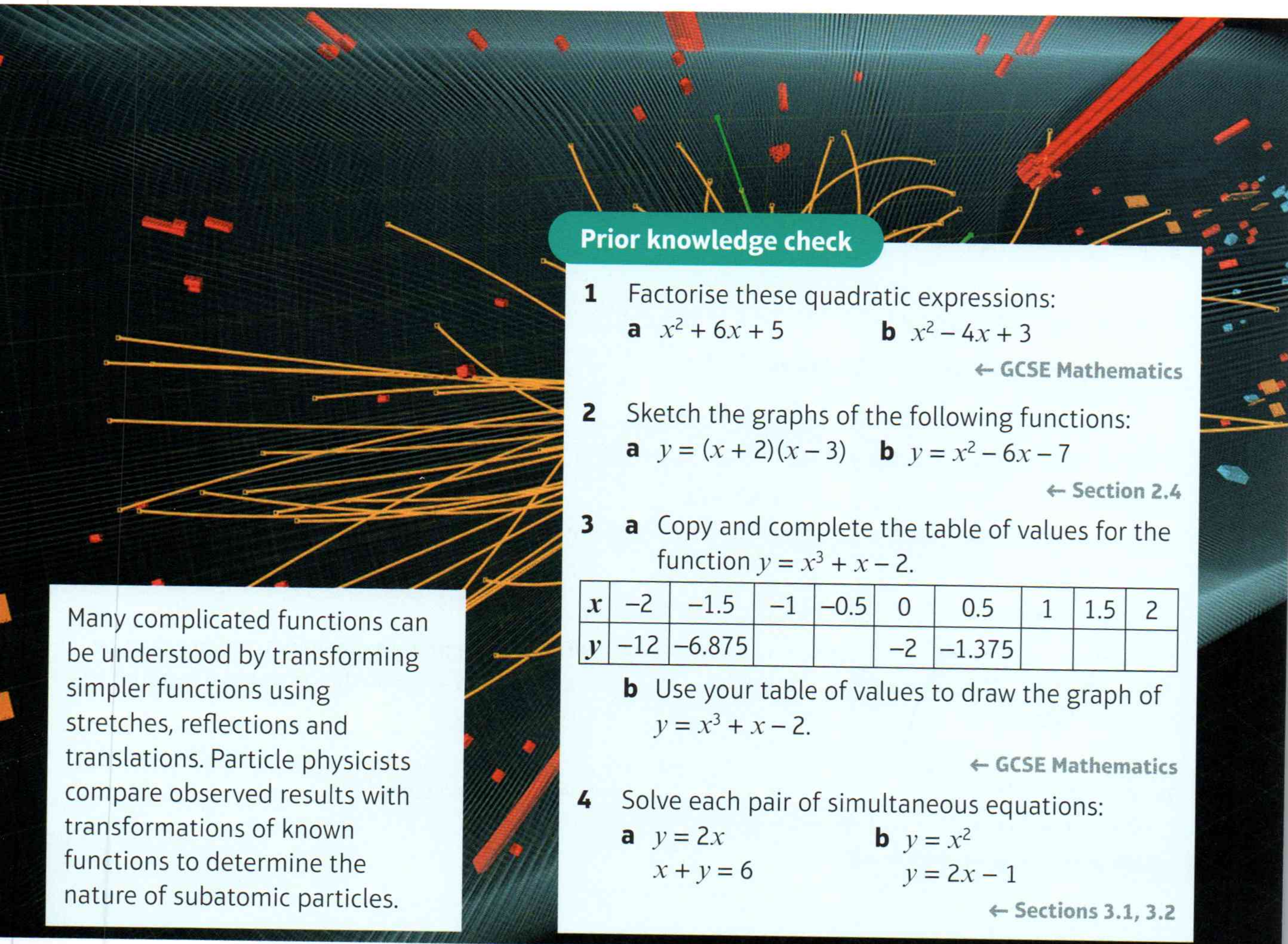
Graphs and transformations

4

Objectives

After completing this chapter you should be able to:

- Sketch cubic graphs → pages 60 – 64
- Sketch quartic graphs → pages 64 – 66
- Sketch reciprocal graphs of the form $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ → pages 66 – 67
- Use intersection points of graphs to solve equations → pages 68 – 71
- Translate graphs → pages 71 – 75
- Stretch graphs → pages 75 – 78
- Transform graphs of unfamiliar functions → pages 79 – 81



Many complicated functions can be understood by transforming simpler functions using stretches, reflections and translations. Particle physicists compare observed results with transformations of known functions to determine the nature of subatomic particles.

Prior knowledge check

1 Factorise these quadratic expressions:

a $x^2 + 6x + 5$ **b** $x^2 - 4x + 3$

← GCSE Mathematics

2 Sketch the graphs of the following functions:

a $y = (x + 2)(x - 3)$ **b** $y = x^2 - 6x - 7$

← Section 2.4

3 **a** Copy and complete the table of values for the function $y = x^3 + x - 2$.

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
y	−12	−6.875			−2	−1.375			

b Use your table of values to draw the graph of $y = x^3 + x - 2$.

← GCSE Mathematics

4 Solve each pair of simultaneous equations:

a $y = 2x$
$x + y = 6$

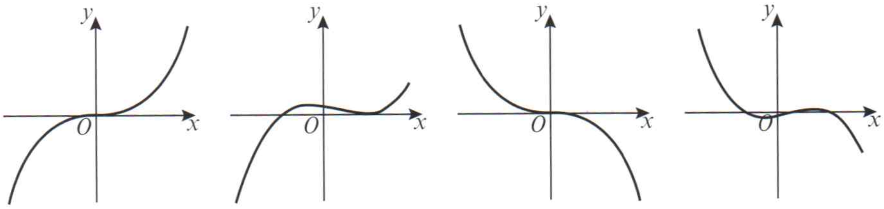
b $y = x^2$
$y = 2x - 1$

← Sections 3.1, 3.2

4.1 Cubic graphs

A **cubic function** has the form $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers and a is non-zero.

The graph of a cubic function can take several different forms, depending on the exact nature of the function.



For these two functions a is positive. For these two functions a is negative.

- **If p is a root of the function $f(x)$, then the graph of $y = f(x)$ touches or crosses the x-axis at the point $(p, 0)$.**

You can sketch the graph of a cubic function by finding the roots of the function.

Example 1

Sketch the curves with the following equations and show the points where they cross the coordinate axes.

a $y = (x - 2)(1 - x)(1 + x)$ **b** $y = x(x + 1)(x + 2)$

Online Explore the graph of $y = (x - p)(x - q)(x - r)$ where p, q and r are constants using technology.

a $y = (x - 2)(1 - x)(1 + x)$

$0 = (x - 2)(1 - x)(1 + x)$ — Put $y = 0$ and solve for x.

So $x = 2$, $x = 1$ or $x = -1$

So the curve crosses the x-axis at $(2, 0)$, $(1, 0)$ and $(-1, 0)$.

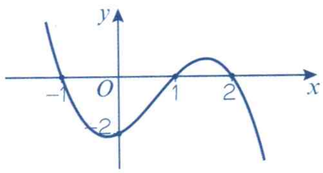
When $x = 0$, $y = -2 \times 1 \times 1 = -2$ — Find the value of y when $x = 0$.

So the curve crosses the y-axis at $(0, -2)$.

$x \to \infty$, $y \to -\infty$

$x \to -\infty$, $y \to \infty$

Check what happens to y for large positive and negative values of x.



The x^3 term in the expanded function would be $x \times (-x) \times x = -x^3$ so the curve has a negative x^3 coefficient.

b $y = x(x + 1)(x + 2)$

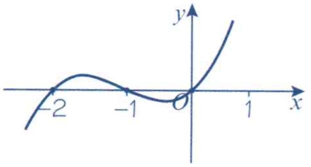
$0 = x(x + 1)(x + 2)$ — Put $y = 0$ and solve for x.

So $x = 0$, $x = -1$ or $x = -2$

So the curve crosses the x-axis at $(0, 0)$, $(-1, 0)$ and $(-2, 0)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



You know that the curve crosses the x-axis at (0, 0) so you don't need to calculate the y-intercept separately.

Check what happens to y for large positive and negative values of x.

The x^3 term in the expanded function would be $x \times x \times x = x^3$ so the curve has a positive x^3 coefficient.

Example 2

Sketch the following curves.

a $y = (x - 1)^2(x + 1)$ **b** $y = x^3 - 2x^2 - 3x$ **c** $y = (x - 2)^3$

a $y = (x - 1)^2(x + 1)$

$0 = (x - 1)^2(x + 1)$

So $x = 1$ or $x = -1$

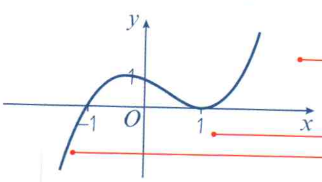
So the curve crosses the x-axis at $(-1, 0)$ and touches the x-axis at $(1, 0)$.

When $x = 0$, $y = (-1)^2 \times 1 = 1$

So the curve crosses the y-axis at $(0, 1)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



Put $y = 0$ and solve for x.

$(x - 1)$ is squared so $x = 1$ is a 'double' repeated root. This means that the curve just touches the x-axis at $(1, 0)$.

Find the value of y when $x = 0$.

Check what happens to y for large positive and negative values of x.

$x \to \infty, y \to \infty$

$x = 1$ is a 'double' repeated root.

$x \to -\infty, y \to -\infty$

b $y = x^3 - 2x^3 - 3x$

$= x(x^2 - 2x - 3)$

$= x(x - 3)(x + 1)$

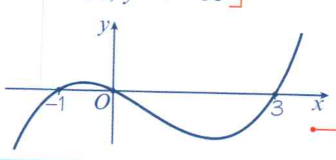
$0 = x(x - 3)(x + 1)$

So $x = 0$, $x = 3$ or $x = -1$

So the curve crosses the x-axis at $(0, 0)$, $(3, 0)$ and $(-1, 0)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



First factorise.

Check what happens to y for large positive and negative values of x.

This is a cubic curve with a positive coefficient of x^3 and three distinct roots.

c $y = (x - 2)^3$

$0 = (x - 2)^3$

So $x = 2$ and the curve crosses the x-axis at (2, 0) only.

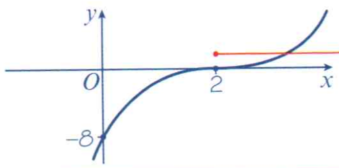
When $x = 0$, $y = (-2)^3 = -8$

So the curve crosses the y-axis at (0, −8).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

Check what happens to y for large positive and negative values of x.



$x = 2$ is a 'triple' repeated root.

Example 3

Sketch the curve with equation $y = (x - 1)(x^2 + x + 2)$.

$y = (x - 1)(x^2 + x + 2)$

$0 = (x - 1)(x^2 + x + 2)$

The quadratic factor $x^2 + x + 2$ gives no solutions since the discriminant $b^2 - 4ac = (1)^2 - 4(1)(2) = -7$.

← Section 2.5

So $x = 1$ only and the curve crosses the x-axis at (1, 0).

Watch out A cubic graph could intersect the x-axis at 1, 2 or 3 points.

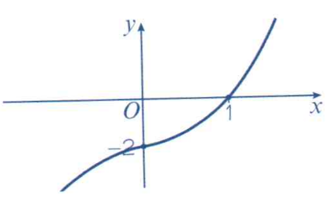
When $x = 0$, $y = (-1)(2) = -2$

So the curve crosses the y-axis at (0, −2).

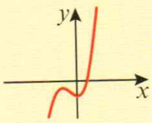
$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

Check what happens to y for large positive and negative values of x.



You haven't got enough information to know the exact shape of the graph. It could also be shaped like this:



Exercise 4A

1 Sketch the following curves and indicate clearly the points of intersection with the axes:

a $y = (x - 3)(x - 2)(x + 1)$ **b** $y = (x - 1)(x + 2)(x + 3)$

c $y = (x + 1)(x + 2)(x + 3)$ **d** $y = (x + 1)(1 - x)(x + 3)$

e $y = (x - 2)(x - 3)(4 - x)$ **f** $y = x(x - 2)(x + 1)$

g $y = x(x + 1)(x - 1)$ **h** $y = x(x + 1)(1 - x)$

i $y = (x - 2)(2x - 1)(2x + 1)$ **j** $y = x(2x - 1)(x + 3)$

2 Sketch the curves with the following equations:

a $y = (x + 1)^2(x - 1)$
b $y = (x + 2)(x - 1)^2$
c $y = (2 - x)(x + 1)^2$
d $y = (x - 2)(x + 1)^2$
e $y = x^2(x + 2)$
f $y = (x - 1)^2x$
g $y = (1 - x)^2(3 + x)$
h $y = (x - 1)^2(3 - x)$
i $y = x^2(2 - x)$
j $y = x^2(x - 2)$

3 Factorise the following equations and then sketch the curves:

a $y = x^3 + x^2 - 2x$
b $y = x^3 + 5x^2 + 4x$
c $y = x^3 + 2x^2 + x$
d $y = 3x + 2x^2 - x^3$
e $y = x^3 - x^2$
f $y = x - x^3$
g $y = 12x^3 - 3x$
h $y = x^3 - x^2 - 2x$
i $y = x^3 - 9x$
j $y = x^3 - 9x^2$

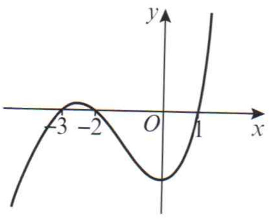
4 Sketch the following curves and indicate the coordinates of the points where the curves cross the axes:

a $y = (x - 2)^3$
b $y = (2 - x)^3$
c $y = (x - 1)^3$
d $y = (x + 2)^3$
e $y = -(x + 2)^3$
f $y = (x + 3)^3$
g $y = (x - 3)^3$
h $y = (1 - x)^3$
i $y = -(x - 2)^3$
j $y = -\left(x - \frac{1}{2}\right)^3$

P 5 The graph of $y = x^3 + bx^2 + cx + d$ is shown opposite, where b, c and d are real constants.

a Find the values of b, c and d. **(3 marks)**

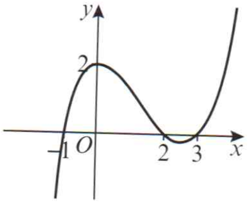
b Write down the coordinates of the point where the curve crosses the y-axis. **(1 mark)**



Problem-solving

Start by writing the equation in the form $y = (x - p)(x - q)(x - r)$.

P 6 The graph of $y = ax^3 + bx^2 + cx + d$ is shown opposite, where a, b, c and d are real constants.
Find the values of a, b, c and d. **(4 marks)**



7 Given that $f(x) = (x - 10)(x^2 - 2x) + 12x$

a Express $f(x)$ in the form $x(ax^2 + bx + c)$ where a, b and c are real constants. **(3 marks)**

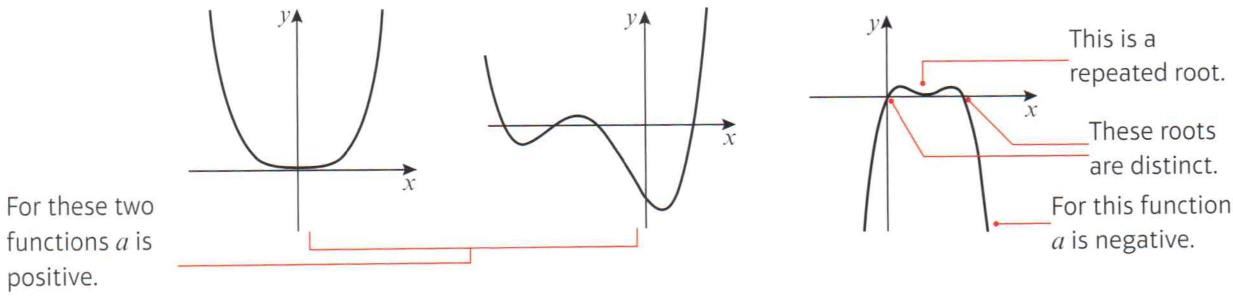
b Hence factorise $f(x)$ completely. **(2 marks)**

c Sketch the graph of $y = f(x)$ showing clearly the points where the graph intersects the axes. **(3 marks)**

4.2 Quartic graphs

A **quartic function** has the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d and e are real numbers and a is non-zero.

The graph of a quartic function can take several different forms, depending on the exact nature of the function.



You can sketch the graph of a quartic function by finding the roots of the function.

Example 4

Sketch the following curves:

a $y = (x + 1)(x + 2)(x - 1)(x - 2)$ **b** $y = x(x + 2)^2(3 - x)$ **c** $y = (x - 1)^2(x - 3)^2$

Online Explore the graph of $y = (x - p)(x - q)(x - r)(x - s)$ where p, q, r and s are constants using technology.

a $y = (x + 1)(x + 2)(x - 1)(x - 2)$

$0 = (x + 1)(x + 2)(x - 1)(x - 2)$

Set $y = 0$ and solve to find the roots of the function.

So $x = -1, -2, 1$ or 2

The curve cuts the x-axis at $(-2, 0)$, $(-1, 0)$, $(1, 0)$ and $(2, 0)$.

When $x = 0$, $y = 1 \times 2 \times (-1) \times (-2) = 4$.

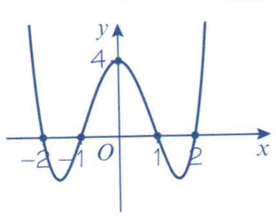
Substitute $x = 0$ into the function to find the coordinates of the y-intercept.

So the curve cuts the y-axis at $(0, 4)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to \infty$

Check what happens to y for large positive and negative values of x.



We know the general shape of the quartic graph so we can draw a smooth curve through the points.

b $y = x(x + 2)^2(3 - x)$

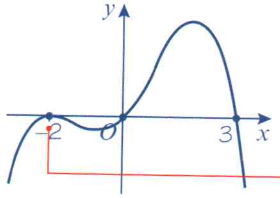
$0 = x(x + 2)^2(3 - x)$

So $x = 0, -2$ or 3

The curve cuts the x-axis at $(0, 0)$, $(-2, 0)$ and $(3, 0)$

$x \to \infty, y \to -\infty$

$x \to -\infty, y \to -\infty$



The coefficient of x^4 in the expanded function will be negative so you know the general shape of the curve.

There is a 'double' repeated root at $x = -2$ so the graph just touches the x-axis at this point.

c $y = (x - 1)^2(x - 3)^2$

$0 = (x - 1)^2(x - 3)^2$

So $x = 1$ or 3

The curve touches the x-axis at $(1, 0)$ and $(3, 0)$.

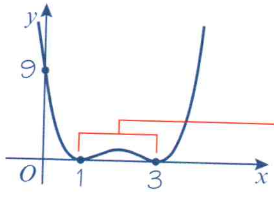
When $x = 0$, $y = 9$.

So the curve cuts the y-axis at $(0, 9)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to \infty$

These are both 'double' repeated roots, so the curve will just touch the x-axis at these points.



The coefficient of x^4 in the expanded function will be positive.

There are two 'double' repeated roots.

Exercise 4B

1 Sketch the following curves and indicate clearly the points of intersection with the axes:

a $y = (x + 1)(x + 2)(x + 3)(x + 4)$ **b** $y = x(x - 1)(x + 3)(x - 2)$

c $y = x(x + 1)^2(x + 2)$ **d** $y = (2x - 1)(x + 2)(x - 1)(x - 2)$

e $y = x^2(4x + 1)(4x - 1)$ **f** $y = -(x - 4)^2(x - 2)^2$

g $y = (x - 3)^2(x + 1)^2$ **h** $y = (x + 2)^3(x - 3)$

i $y = -(2x - 1)^3(x + 5)$ **j** $y = (x + 4)^4$

Hint In part **f** the coefficient of x^4 will be negative.

2 Sketch the following curves and indicate clearly the points of intersection with the axes:

a $y = (x + 2)(x - 1)(x^2 - 3x + 2)$ **b** $y = (x + 3)^2(x^2 - 5x + 6)$

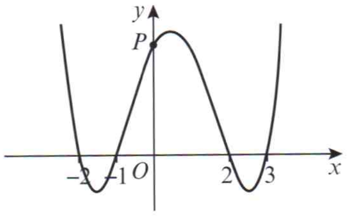
c $y = (x - 4)^2(x^2 - 11x + 30)$ **d** $y = (x^2 - 4x - 32)(x^2 + 5x - 36)$

Hint Factorise the quadratic factor first.

E/P **3** The graph of $y = x^4 + bx^3 + cx^2 + dx + e$ is shown opposite, where b, c, d and e are real constants.

a Find the coordinates of point P. **(2 marks)**

b Find the values of b, c, d and e. **(3 marks)**



E/P **4** Sketch the graph of $y = (x + 5)(x - 4)(x^2 + 5x + 14)$. **(3 marks)**

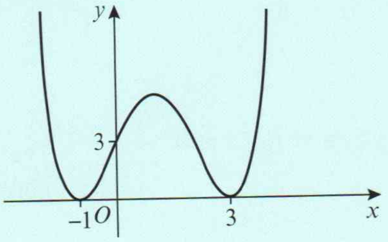
Problem-solving

Consider the discriminant of the quadratic factor.

Challenge

The graph of $y = ax^4 + bx^3 + cx^2 + dx + e$ is shown, where a, b, c, d and e are real constants.

Find the values of a, b, c, d and e.

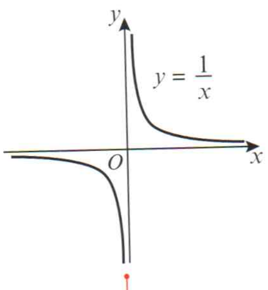


4.3 Reciprocal graphs

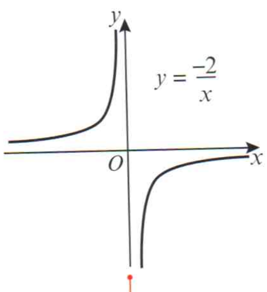
You can sketch graphs of **reciprocal functions** such as $y = \frac{1}{x}$, $y = \frac{1}{x^2}$ and $y = -\frac{2}{x}$ by considering their asymptotes.

- **The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, where k is a real constant, have asymptotes at $x = 0$ and $y = 0$.**

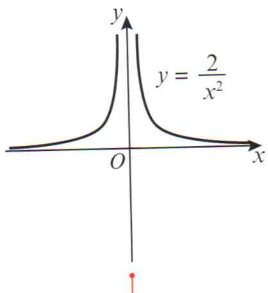
Notation An **asymptote** is a line which the graph approaches but never reaches.



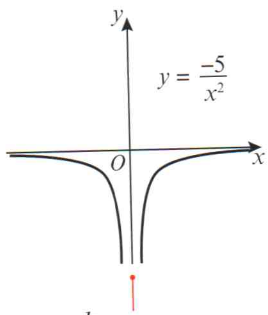
$y = \frac{k}{x}$ with $k > 0$.



$y = \frac{k}{x}$ with $k < 0$.



$y = \frac{k}{x^2}$ with $k > 0$.

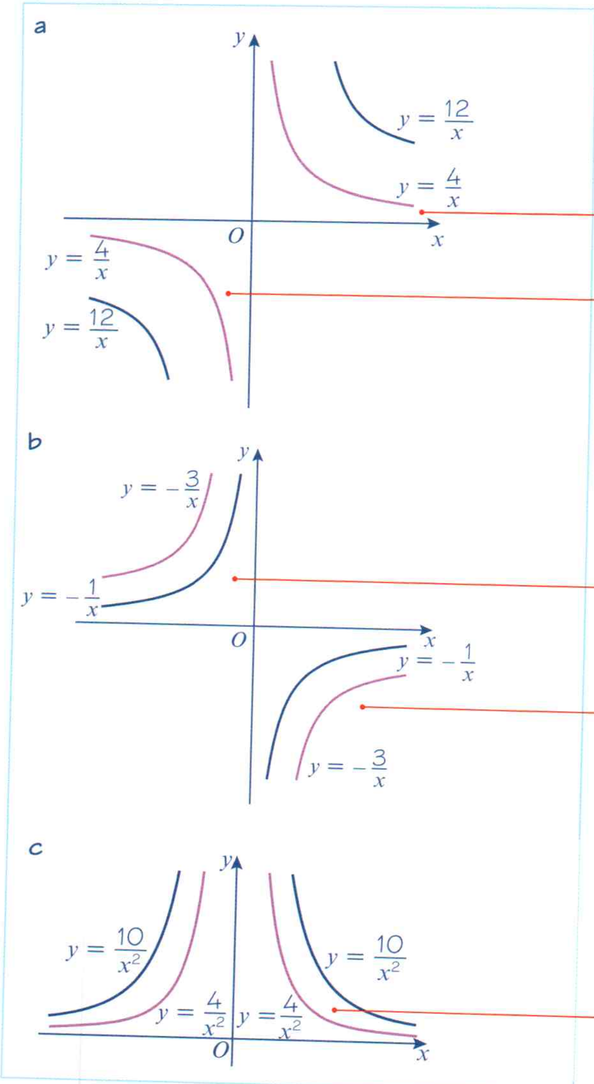


$y = \frac{k}{x^2}$ with $k < 0$.

Example 5

Sketch on the same diagram:

a $y = \frac{4}{x}$ and $y = \frac{12}{x}$ **b** $y = -\frac{1}{x}$ and $y = -\frac{3}{x}$ **c** $y = \frac{4}{x^2}$ and $y = \frac{10}{x^2}$



Online Explore the graph of $y = \frac{a}{x}$ for different values of a using technology.

This is a $y = \frac{k}{x}$ graph with $k > 0$

In this quadrant, $x > 0$

so for any values of x: $\frac{12}{x} > \frac{4}{x}$

In this quadrant, $x < 0$

so for any values of x: $\frac{12}{x} < \frac{4}{x}$

This is a $y = \frac{k}{x}$ graph with $k < 0$

In this quadrant, $x < 0$

so for any values of x: $-\frac{3}{x} > -\frac{1}{x}$

In this quadrant, $x > 0$

so for any values of x: $-\frac{3}{x} < -\frac{1}{x}$

This is a $y = \frac{k}{x^2}$ graph with $k > 0$.

x^2 is always positive and $k > 0$ so the y-values are all positive.

Exercise 4C

1 Use a separate diagram to sketch each pair of graphs.

a $y = \frac{2}{x}$ and $y = \frac{4}{x}$ **b** $y = \frac{2}{x}$ and $y = -\frac{2}{x}$ **c** $y = -\frac{4}{x}$ and $y = -\frac{2}{x}$

d $y = \frac{3}{x}$ and $y = \frac{8}{x}$ **e** $y = -\frac{3}{x}$ and $y = -\frac{8}{x}$

2 Use a separate diagram to sketch each pair of graphs.

a $y = \frac{2}{x^2}$ and $y = \frac{5}{x^2}$ **b** $y = \frac{3}{x^2}$ and $y = -\frac{3}{x^2}$ **c** $y = -\frac{2}{x^2}$ and $y = -\frac{6}{x^2}$

4.4 Points of intersection

You can sketch curves of functions to show points of intersection and solutions to equations.

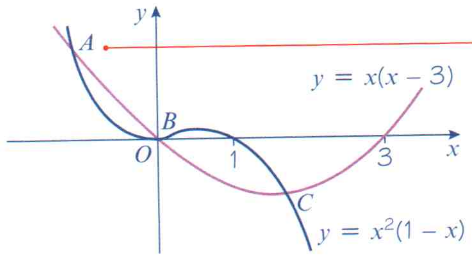
- **The x-coordinate(s) at the points of intersection of the curves with equations $y = f(x)$ and $y = g(x)$ are the solution(s) to the equation $f(x) = g(x)$.**

Example 6

a On the same diagram sketch the curves with equations $y = x(x - 3)$ and $y = x^2(1 - x)$.

b Find the coordinates of the points of intersection.

a



A cubic curve will eventually get steeper than a quadratic curve, so the graphs will intersect for some negative value of x.

b From the graph there are three points where the curves cross, labelled A, B and C. The x-coordinates are given by the solutions to the equation.

There are three points of intersection so the equation $x(x - 3) = x^2(1 - x)$ has three real roots.

$$x(x - 3) = x^2(1 - x)$$
$$x^2 - 3x = x^2 - x^3$$
$$x^3 - 3x = 0$$
$$x(x^2 - 3) = 0$$

Multiply out brackets.
Collect terms on one side.
Factorise.

So $x = 0$ or $x^2 = 3$

So $x = -\sqrt{3}, 0, \sqrt{3}$

Substitute into $y = x^2(1 - x)$

The graphs intersect for these values of x, so you can substitute into either equation to find the y-coordinates.

The points of intersection are:

$A(-\sqrt{3}, 3 + 3\sqrt{3})$

$B(0, 0)$

$C(\sqrt{3}, 3 - 3\sqrt{3})$

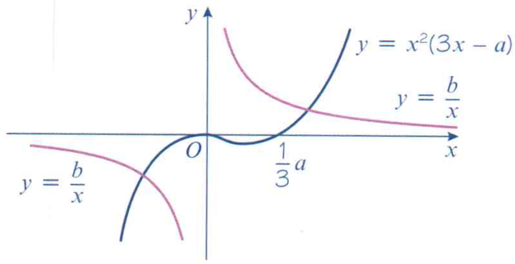
Leave your answers in surd form.

Example 7

a On the same diagram sketch the curves with equations $y = x^2(3x - a)$ and $y = \frac{b}{x}$, where a and b are positive constants.

b State, giving a reason, the number of real solutions to the equation $x^2(3x - a) - \frac{b}{x} = 0$

a



$3x - a = 0$ when $x = \frac{1}{3}a$, so the graph of $y = x^2(3x - a)$ touches the x-axis at (0, 0) and intersects it at $\left(\frac{1}{3}a, 0\right)$

b From the sketch there are only two points of intersection of the curves. This means there are only two values of x where

$$x^2(3x - a) = \frac{b}{x}$$

or $\quad x^2(3x - a) - \frac{b}{x} = 0$

So this equation has two real solutions.

Problem-solving

You can sketch curves involving unknown constants. You should give any points of intersection with the coordinate axes in terms of the constants where appropriate.

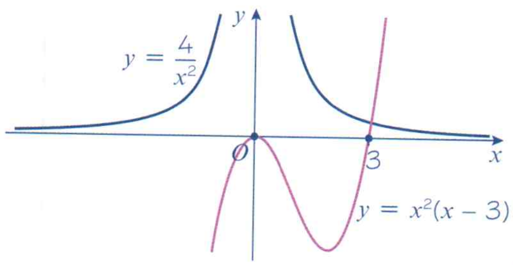
You only need to state the **number** of solutions. You don't need to find the solutions.

Example 8

a Sketch the curves $y = \frac{4}{x^2}$ and $y = x^2(x - 3)$ on the same axes.

b Using your sketch, state, with a reason, the number of real solutions to the equation $x^4(x - 3) - 4 = 0$.

a



b There is a single point of intersection so the equation $x^2(x - 3) = \frac{4}{x^2}$ has one real solution.

Rearranging:

$x^4(x - 3) = 4$

$x^4(x - 3) - 4 = 0$

So this equation has one real solution.

Problem-solving

Set the functions equal to each other to form an equation with one real solution, then rearrange the equation into the form given in the question.

You would not be expected to solve this equation in your exam.

Exercise 4D

1 In each case:

i sketch the two curves on the same axes

ii state the number of points of intersection

iii write down a suitable equation which would give the x-coordinates of these points. (You are not required to solve this equation.)

a $y = x^2, y = x(x^2 - 1)$ **b** $y = x(x + 2), y = -\frac{3}{x}$ **c** $y = x^2, y = (x + 1)(x - 1)^2$

d $y = x^2(1 - x), y = -\frac{2}{x}$ **e** $y = x(x - 4), y = \frac{1}{x}$ **f** $y = x(x - 4), y = -\frac{1}{x}$

g $y = x(x - 4), y = (x - 2)^3$ **h** $y = -x^3, y = -\frac{2}{x}$ **i** $y = -x^3, y = x^2$

j $y = -x^3, y = -x(x + 2)$ **k** $y = 4, y = x(x - 1)(x + 2)^2$ **l** $y = x^3, y = x^2(x + 1)^2$

2 **a** On the same axes sketch the curves given by $y = x^2(x - 3)$ and $y = \frac{2}{x}$

b Explain how your sketch shows that there are only two real solutions to the equation $x^3(x - 3) = 2$.

3 **a** On the same axes sketch the curves given by $y = (x + 1)^3$ and $y = 3x(x - 1)$.

b Explain how your sketch shows that there is only one real solution to the equation $x^3 + 6x + 1 = 0$.

4 **a** On the same axes sketch the curves given by $y = \frac{1}{x}$ and $y = -x(x - 1)^2$.

b Explain how your sketch shows that there are no real solutions to the equation $1 + x^2(x - 1)^2 = 0$.

E/P **5** **a** On the same axes sketch the curves given by $y = x^2(x - a)$ and $y = \frac{b}{x}$ where a and b are both positive constants. **(5 marks)**

b Using your sketch, state, giving a reason, the number of real solutions to the equation $x^4 - ax^3 - b = 0$. **(1 mark)**

Problem-solving

Even though you don't know the values of a and b, you know they are positive, so you know the shapes of the graphs. You can label the point a on the x-axis on your sketch of $y = x^2(x - a)$.

E **6** **a** On the same set of axes sketch the graphs of $y = \frac{4}{x^2}$ and $y = 3x + 7$. **(3 marks)**

b Write down the number of real solutions to the equation $\frac{4}{x^2} = 3x + 7$. **(1 mark)**

c Show that you can rearrange the equation to give $(x + 1)(x + 2)(3x - 2) = 0$. **(2 marks)**

d Hence determine the exact coordinates of the points of intersection. **(3 marks)**

7 **a** On the same axes sketch the curve $y = x^3 - 3x^2 - 4x$ and the line $y = 6x$.

b Find the coordinates of the points of intersection.

P **8** **a** On the same axes sketch the curve $y = (x^2 - 1)(x - 2)$ and the line $y = 14x + 2$.

b Find the coordinates of the points of intersection.

P **9** **a** On the same axes sketch the curves with equations $y = (x - 2)(x + 2)^2$ and $y = -x^2 - 8$.

b Find the coordinates of the points of intersection.

E/P **10** **a** Sketch the graphs of $y = x^2 + 1$ and $2y = x - 1$. **(3 marks)**

b Explain why there are no real solutions to the equation $2x^2 - x + 3 = 0$. **(2 marks)**

c Work out the range of values of a such that the graphs of $y = x^2 + a$ and $2y = x - 1$ have two points of intersection. **(5 marks)**

E/P **11** **a** Sketch the graphs of $y = x^2(x-1)(x+1)$ and $y = \frac{1}{3}x^3 + 1$. **(5 marks)**

b Find the number of real solutions to the equation $3x^2(x-1)(x+1) = x^3 + 3$. **(1 mark)**

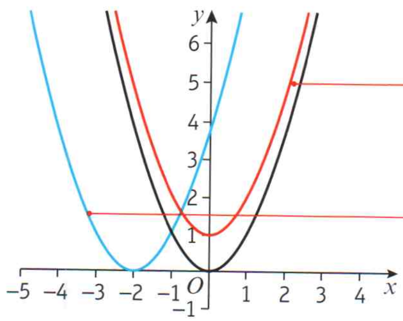
4.5 Translating graphs

You can transform the graph of a function by altering the function. Adding or subtracting a constant 'outside' the function translates a graph vertically.

- **The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.**

Adding or subtracting a constant 'inside' the function translates the graph horizontally.

- **The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.**



$y = f(x) + 1$ is a translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, or 1 unit in the direction of the positive y-axis.

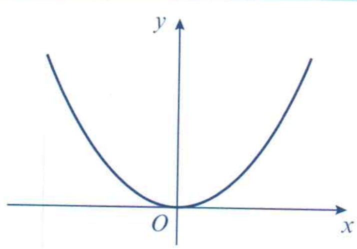
$y = f(x + 2)$ is a translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or 2 units in the direction of the negative x-axis.

Example 9

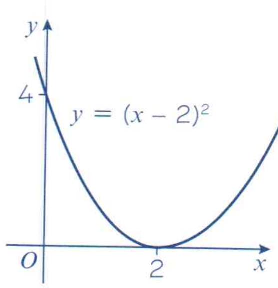
Sketch the graphs of:

a $y = x^2$ **b** $y = (x-2)^2$ **c** $y = x^2 + 2$

a



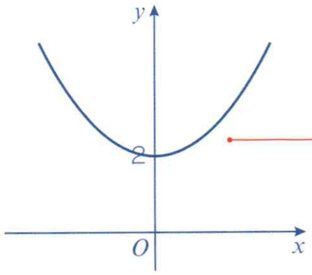
b



This is a translation by vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Remember to mark on the intersections with the axes.

c $y = x^2 + 2$



This is a translation by vector $\begin{pmatrix}0\\2\end{pmatrix}$.
Remember to mark on the y-axis intersection.

Example 10

$f(x) = x^3$

$g(x) = x(x - 2)$

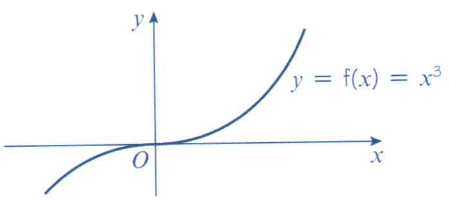
Sketch the following graphs, indicating any points where the curves cross the axes:

a $y = f(x + 1)$

b $y = g(x + 1)$

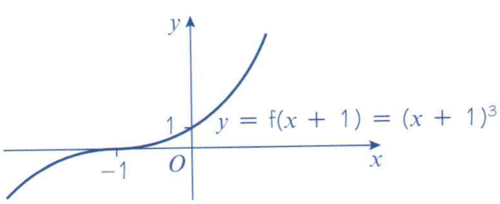
Online Explore translations of the graph of $y = x^3$ using technology.

a The graph of $f(x)$ is



First sketch $y = f(x)$.

So the graph of $y = f(x + 1)$ is



This is a translation of the graph of $y = f(x)$ by vector $\begin{pmatrix}-1\\0\end{pmatrix}$.
You could also write out the equation as $y = (x + 1)^3$ and sketch the graph directly.

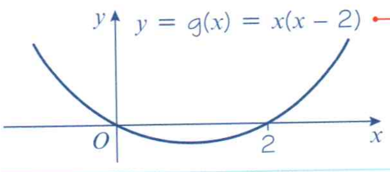
b $g(x) = x(x - 2)$

The curve is $y = x(x - 2)$

$0 = x(x - 2)$

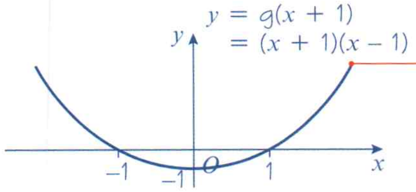
Put $y = 0$ to find where the curve crosses the x-axis.

So $x = 0$ or $x = 2$



First sketch $g(x)$.

So the graph of $y = g(x + 1)$ is



This is a translation of the graph of $y = g(x)$ by vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

You could also write out the equation and sketch the graph directly:

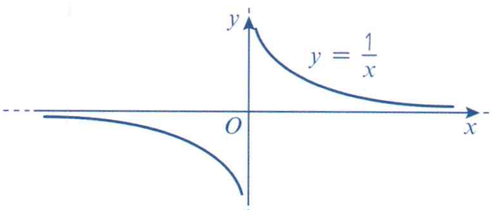
$y = g(x + 1)$
$= (x + 1)(x + 1 - 2)$
$= (x + 1)(x - 1)$

- **When you translate a function, any asymptotes are also translated.**

Example 11

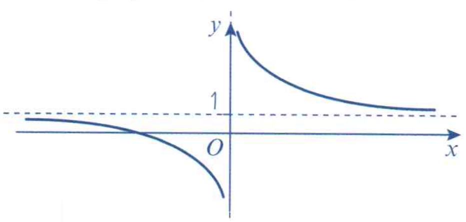
Given that $h(x) = \frac{1}{x}$, sketch the curve with equation $y = h(x) + 1$ and state the equations of any asymptotes and intersections with the axes.

The graph of $y = h(x)$ is



First sketch $y = h(x)$.

So the graph of $y = h(x) + 1$ is



The curve is translated by vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so the asymptote is translated by the same vector.

The curve crosses the x-axis once.

$y = h(x) + 1 = \frac{1}{x} + 1$

$0 = \frac{1}{x} + 1$

Put $y = 0$ to find where the curve crosses the x-axis.

$-1 = \frac{1}{x}$

$x = -1$

So the curve intersects the x-axis at $(-1, 0)$.

The horizontal asymptote is $y = 1$.

The vertical asymptote is $x = 0$.

Remember to write down the equation of the vertical asymptote as well. It is the y-axis so it has equation $x = 0$.

Exercise 4E

1 Apply the following transformations to the curves with equations $y = f(x)$ where:

i $f(x) = x^2$ **ii** $f(x) = x^3$ **iii** $f(x) = \frac{1}{x}$

In each case state the coordinates of points where the curves cross the axes and in **iii** state the equations of the asymptotes.

a $f(x + 2)$ **b** $f(x) + 2$ **c** $f(x - 1)$

d $f(x) - 1$ **e** $f(x) - 3$ **f** $f(x - 3)$

2 **a** Sketch the curve $y = f(x)$ where $f(x) = (x - 1)(x + 2)$.

b On separate diagrams sketch the graphs of **i** $y = f(x + 2)$ **ii** $y = f(x) + 2$.

c Find the equations of the curves $y = f(x + 2)$ and $y = f(x) + 2$, in terms of x, and use these equations to find the coordinates of the points where your graphs in part **b** cross the y-axis.

3 **a** Sketch the graph of $y = f(x)$ where $f(x) = x^2(1 - x)$.

b Sketch the curve with equation $y = f(x + 1)$.

c By finding the equation $f(x + 1)$ in terms of x, find the coordinates of the point in part **b** where the curve crosses the y-axis.

4 **a** Sketch the graph of $y = f(x)$ where $f(x) = x(x - 2)^2$.

b Sketch the curves with equations $y = f(x) + 2$ and $y = f(x + 2)$.

c Find the coordinates of the points where the graph of $y = f(x + 2)$ crosses the axes.

5 **a** Sketch the graph of $y = f(x)$ where $f(x) = x(x - 4)$.

b Sketch the curves with equations $y = f(x + 2)$ and $y = f(x) + 4$.

c Find the equations of the curves in part **b** in terms of x and hence find the coordinates of the points where the curves cross the axes.

6 **a** Sketch the graph of $y = f(x)$ where $f(x) = x^2(x - 1)(x - 2)$.

b Sketch the curves with equations $y = f(x + 2)$ and $y = f(x) - 1$.

(E) 7 The point $P(4, -1)$ lies on the curve with equation $y = f(x)$.

a State the coordinates that point P is transformed to on the curve with equation $y = f(x - 2)$. **(1 mark)**

b State the coordinates that point P is transformed to on the curve with equation $y = f(x) + 3$. **(1 mark)**

(E/P) 8 The graph of $y = f(x)$ where $f(x) = \frac{1}{x}$ is translated so that the asymptotes are at $x = 4$ and $y = 0$. Write down the equation for the transformed function in the form $y = \frac{1}{x + a}$ **(3 marks)**

(P) 9 **a** Sketch the graph of $y = x^3 - 5x^2 + 6x$, marking clearly the points of intersection with the axes.

b Hence sketch $y = (x - 2)^3 - 5(x - 2)^2 + 6(x - 2)$.

(P) **10** **a** Sketch the graph of $y = x^2(x - 3)(x + 2)$, marking clearly the points of intersection with the axes.

b Hence sketch $y = (x + 2)^2(x - 1)(x + 4)$.

(E/P) **11** **a** Sketch the graph of $y = x^3 + 4x^2 + 4x$. **(6 marks)**

b The point with coordinates $(-1, 0)$ lies on the curve with equation $y = (x + a)^3 + 4(x + a)^2 + 4(x + a)$ where a is a constant. Find the two possible values of a. **(3 marks)**

Problem-solving

Look at your sketch and picture the curve sliding to the left or right.

(E/P) **12** **a** Sketch the graph of $y = x(x + 1)(x + 3)^2$. **(4 marks)**

b Find the possible values of b such that the point $(2, 0)$ lies on the curve with equation $y = (x + b)(x + b + 1)(x + b + 3)^2$. **(3 marks)**

Challenge

1 Sketch the graph of $y = (x - 3)^3 + 2$ and determine the coordinates of the point of inflection. → Section 12.9

2 The point $Q(-5, -7)$ lies on the curve with equation $y = f(x)$.

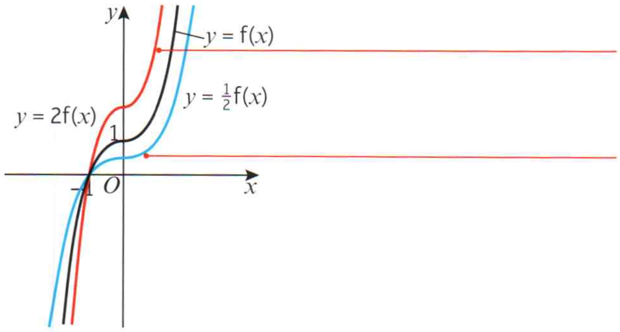
a State the coordinates that point Q is transformed to on the curve with equation $y = f(x + 2) - 5$.

b The coordinates of the point Q on a transformed curve are $(-3, -6)$. Write down the transformation in the form $y = f(x + a) - b$.

4.6 Stretching graphs

Multiplying by a constant 'outside' the function stretches the graph vertically.

- **The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.**

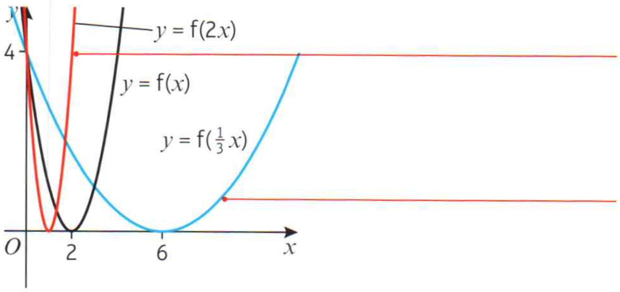


$2f(x)$ is a stretch with scale factor 2 in the y-direction. All y-coordinates are doubled.

$\frac{1}{2}f(x)$ is a stretch with scale factor $\frac{1}{2}$ in the y-direction. All y-coordinates are halved.

Multiplying by a constant 'inside' the function stretches the graph horizontally.

- **The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.**



$y = f(2x)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction. All x-coordinates are halved.

$y = f(\frac{1}{3}x)$ is a stretch with scale factor 3 in the x-direction. All x-coordinates are tripled.

Example 12

Given that $f(x) = 9 - x^2$, sketch the curves with equations:

a $y = f(2x)$ **b** $y = 2f(x)$

a $f(x) = 9 - x^2$

So $f(x) = (3 - x)(3 + x)$

You can factorise the expression.

The curve is $y = (3 - x)(3 + x)$

$0 = (3 - x)(3 + x)$

Put $y = 0$ to find where the curve crosses the x-axis.

So $x = 3$ or $x = -3$

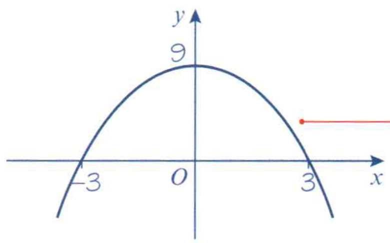
So the curve crosses the x-axis at (3, 0) and (−3, 0).

When $x = 0$, $y = 3 \times 3 = 9$

Put $x = 0$ to find where the curve crosses the y-axis.

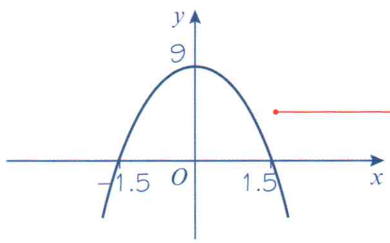
So the curve crosses the y-axis at (0, 9).

The curve $y = f(x)$ is



First sketch $y = f(x)$.

$y = f(2x)$ so the curve is



$y = f(ax)$ where $a = 2$ so it is a horizontal stretch with scale factor $\frac{1}{2}$.

Check: The curve is $y = f(2x)$.

So $y = (3 - 2x)(3 + 2x)$.

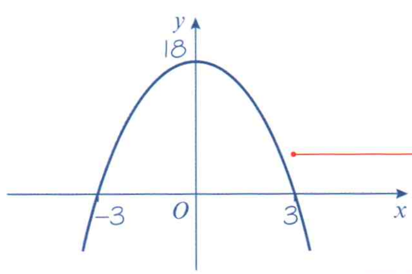
When $y = 0$, $x = -1.5$ or $x = 1.5$.

So the curve crosses the x-axis at (−1.5, 0) and (1.5, 0).

When $x = 0$, $y = 9$.

So the curve crosses the y-axis at (0, 9).

b $y = 2f(x)$ so the curve is



$y = af(x)$ where $a = 2$ so it is a vertical stretch with scale factor 2.

Check: The curve is $y = 2f(x)$.

So $y = 2(3 - x)(3 + x)$.

When $y = 0$, $x = 3$ or $x = -3$.

So the curve crosses the x-axis at (−3, 0) and (3, 0).

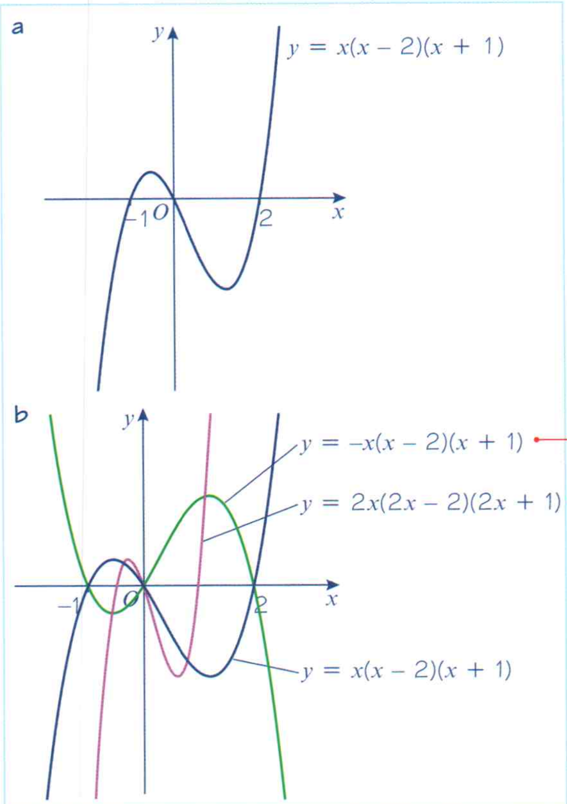
When $x = 0$, $y = 2 \times 9 = 18$.

So the curve crosses the y-axis at (0, 18).

Example 13

a Sketch the curve with equation $y = x(x - 2)(x + 1)$.

b On the same axes, sketch the curves $y = 2x(2x - 2)(2x + 1)$ and $y = -x(x - 2)(x + 1)$.



Online Explore stretches of the graph of $y = x(x - 2)(x + 1)$ using technology.

$y = -x(x - 2)(x + 1)$ is a stretch with scale factor −1 in the y-direction. Notice that this stretch has the effect of reflecting the curve in the x-axis.

$y = 2x(2x - 2)(2x + 1)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction.

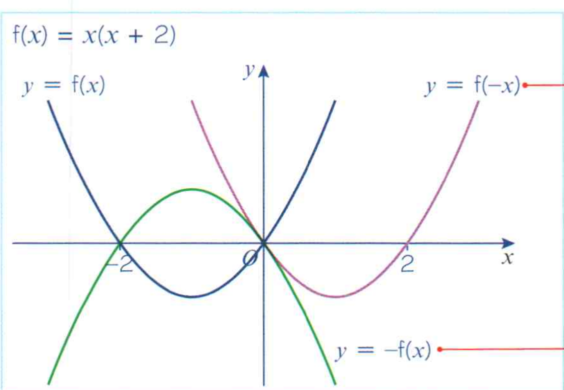
Problem-solving

You need to work out the relationship between each new function and the original function.
If $x(x - 2)(x + 1) = f(x)$ then
$2x(2x - 2)(2x + 1) = f(2x)$, and
$-x(x - 2)(x + 1) = -f(x)$.

- **The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.**
- **The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.**

Example 14

On the same axes sketch the graphs of $y = f(x)$, $y = f(-x)$ and $y = -f(x)$ where $f(x) = x(x + 2)$.



$y = f(-x)$ is $y = (-x)(-x + 2)$ which is $y = x^2 - 2x$ or $y = x(x - 2)$ and this is a reflection of the original curve in the y-axis.
Alternatively multiply each x-coordinate by −1 and leave the y-coordinates unchanged.
This is the same as a stretch parallel to the x-axis scale factor −1.

$y = -f(x)$ is $y = -x(x + 2)$ and this is a reflection of the original curve in the x-axis.
Alternatively multiply each y-coordinate by −1 and leave the x-coordinates unchanged.
This is the same as a stretch parallel to the y-axis scale factor −1.

Exercise 4F

1 Apply the following transformations to the curves with equations $y = f(x)$ where:

i $f(x) = x^2$ **ii** $f(x) = x^3$ **iii** $f(x) = \frac{1}{x}$

In each case show both $f(x)$ and the transformation on the same diagram.

a $f(2x)$ **b** $f(-x)$ **c** $f(\frac{1}{2}x)$ **d** $f(4x)$ **e** $f(\frac{1}{4}x)$

f $2f(x)$ **g** $-f(x)$ **h** $4f(x)$ **i** $\frac{1}{2}f(x)$ **j** $\frac{1}{4}f(x)$

2 **a** Sketch the curve with equation $y = f(x)$ where $f(x) = x^2 - 4$.

b Sketch the graphs of $y = f(4x)$, $\frac{1}{3}y = f(x)$, $y = f(-x)$ and $y = -f(x)$.

> **Hint** For part **b**, rearrange the second equation into the form $y = 3f(x)$.

3 **a** Sketch the curve with equation $y = f(x)$ where $f(x) = (x - 2)(x + 2)x$.

b Sketch the graphs of $y = f(\frac{1}{2}x)$, $y = f(2x)$ and $y = -f(x)$.

(P) 4 **a** Sketch the curve with equation $y = x^2(x - 3)$.

b On the same axes, sketch the curves with equations:

i $y = (2x)^2(2x - 3)$ **ii** $y = -x^2(x - 3)$

> **Problem-solving** Let $f(x) = x^2(x - 3)$ and try to write each of the equations in part **b** in terms of $f(x)$.

5 **a** Sketch the curve $y = x^2 + 3x - 4$.

b On the same axes, sketch the graph of $5y = x^2 + 3x - 4$.

6 **a** Sketch the graph of $y = x^2(x - 2)^2$.

b On the same axes, sketch the graph of $3y = -x^2(x - 2)^2$.

(E) 7 The point $P(2, -3)$ lies on the curve with equation $y = f(x)$.

a State the coordinates that point P is transformed to on the curve with equation $y = f(2x)$. **(1 mark)**

b State the coordinates that point P is transformed to on the curve with equation $y = 4f(x)$. **(1 mark)**

(E) 8 The point $Q(-2, 8)$ lies on the curve with equation $y = f(x)$.

State the coordinates that point Q is transformed to on the curve with equation $y = f(\frac{1}{2}x)$. **(1 mark)**

(E/P) 9 **a** Sketch the graph of $y = (x - 2)(x - 3)^2$. **(4 marks)**

b The graph of $y = (ax - 2)(ax - 3)^2$ passes through the point $(1, 0)$.
Find two possible values for a. **(3 marks)**

Challenge

1 The point $R(4, -6)$ lies on the curve with equation $y = f(x)$. State the coordinates that point R is transformed to on the curve with equation $y = \frac{1}{3}f(2x)$.

2 The point $S(-4, 7)$ is transformed to a point $S'(-8, 1.75)$. Write down the transformation in the form $y = af(bx)$.

4.7 Transforming functions

You can apply transformations to unfamiliar functions by considering how specific points and features are transformed.

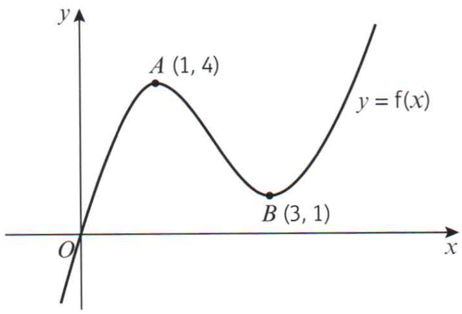
Example 15

The following diagram shows a sketch of the curve f(x) which passes through the origin.
The points $A(1, 4)$ and $B(3, 1)$ also lie on the curve.

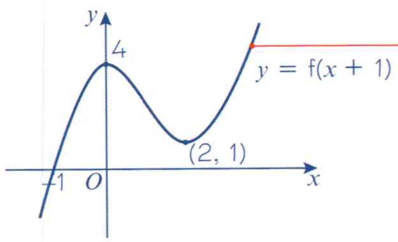
Sketch the following:

a $y = f(x + 1)$ **b** $y = f(x - 1)$ **c** $y = f(x) - 4$
d $2y = f(x)$ **e** $y - 1 = f(x)$

In each case you should show the positions of the images of the points O, A and B.

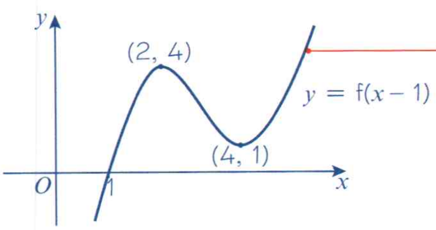


a f(x + 1)



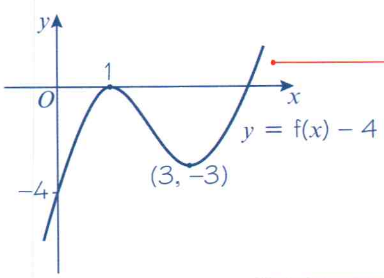
Translate f(x) 1 unit in the direction of the negative x-axis.

b f(x − 1)



Translate f(x) 1 unit in the direction of the positive x-axis.

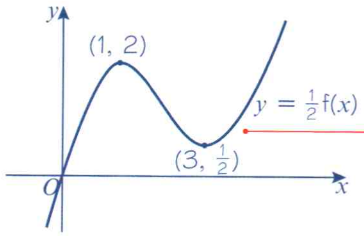
c f(x) − 4



Translate f(x) 4 units in the direction of the negative y-axis.

d $2y = \text{f}(x)$ so $y = \frac{1}{2}\text{f}(x)$

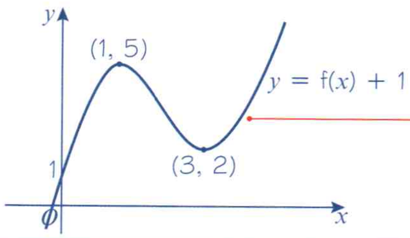
Rearrange in the form $y = \ldots$



Stretch $\text{f}(x)$ by scale factor $\frac{1}{2}$ in the y-direction.

e $y - 1 = \text{f}(x)$ so $y = \text{f}(x) + 1$

Rearrange in the form $y = \ldots$



Translate $\text{f}(x)$ 1 unit in the direction of the positive y-axis.

Exercise 4G

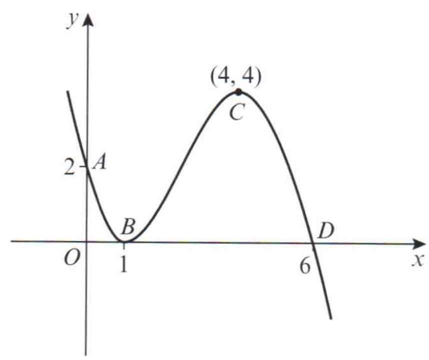
1 The following diagram shows a sketch of the curve with equation $y = \text{f}(x)$. The points $A(0, 2)$, $B(1, 0)$, $C(4, 4)$ and $D(6, 0)$ lie on the curve.

Sketch the following graphs and give the coordinates of the points, A, B, C and D after each transformation:

a $\text{f}(x + 1)$ **b** $\text{f}(x) - 4$ **c** $\text{f}(x + 4)$

d $\text{f}(2x)$ **e** $3\text{f}(x)$ **f** $\text{f}(\frac{1}{2}x)$

g $\frac{1}{2}\text{f}(x)$ **h** $\text{f}(-x)$



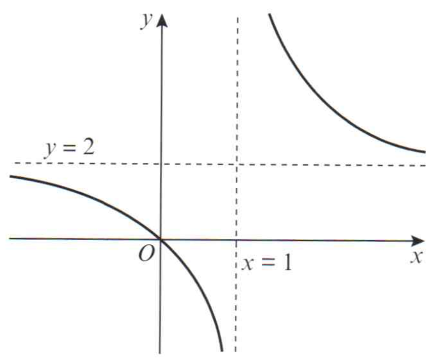
2 The curve $y = \text{f}(x)$ passes through the origin and has horizontal asymptote $y = 2$ and vertical asymptote $x = 1$, as shown in the diagram.

Sketch the following graphs. Give the equations of any asymptotes and, if possible, give the coordinates of intersections with the axes after each transformation.

a $\text{f}(x) + 2$ **b** $\text{f}(x + 1)$ **c** $2\text{f}(x)$

d $\text{f}(x) - 2$ **e** $\text{f}(2x)$ **f** $\text{f}(\frac{1}{2}x)$

g $\frac{1}{2}\text{f}(x)$ **h** $-\text{f}(x)$



3 The curve with equation $y = \mathrm{f}(x)$ passes through the points $A(-4, -6)$, $B(-2, 0)$, $C(0, -3)$ and $D(4, 0)$ as shown in the diagram.

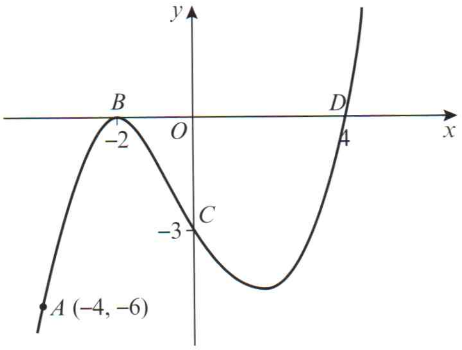
Sketch the following and give the coordinates of the points A, B, C and D after each transformation.

a $\mathrm{f}(x - 2)$ **b** $\mathrm{f}(x) + 6$ **c** $\mathrm{f}(2x)$

d $\mathrm{f}(x + 4)$ **e** $\mathrm{f}(x) + 3$ **f** $3\mathrm{f}(x)$

g $\frac{1}{3}\mathrm{f}(x)$ **h** $\mathrm{f}(\frac{1}{4}x)$ **i** $-\mathrm{f}(x)$

j $\mathrm{f}(-x)$



4 A sketch of the curve $y = \mathrm{f}(x)$ is shown in the diagram. The curve has a vertical asymptote with equation $x = -2$ and a horizontal asymptote with equation $y = 0$. The curve crosses the y-axis at $(0, 1)$.

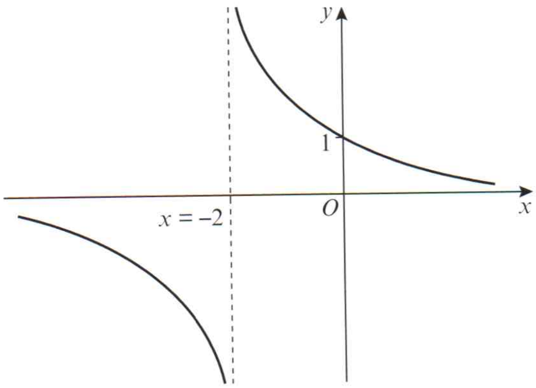
a Sketch, on separate diagrams, the graphs of:

i $2\mathrm{f}(x)$ **ii** $\mathrm{f}(2x)$ **iii** $\mathrm{f}(x - 2)$

iv $\mathrm{f}(x) - 1$ **v** $\mathrm{f}(-x)$ **vi** $-\mathrm{f}(x)$

In each case state the equations of any asymptotes and, if possible, points where the curve cuts the axes.

b Suggest a possible equation for $\mathrm{f}(x)$.



E/P **5** The point $P(2, 1)$ lies on the graph with equation $y = \mathrm{f}(x)$.

a On the graph of $y = \mathrm{f}(ax)$, the point P is mapped to the point $Q(4, 1)$. Determine the value of a. **(1 mark)**

b Write down the coordinates of the point to which P maps under each transformation

i $\mathrm{f}(x - 4)$ **ii** $3\mathrm{f}(x)$ **iii** $\frac{1}{2}\mathrm{f}(x) - 4$ **(3 marks)**

P **6** The diagram shows a sketch of a curve with equation $y = \mathrm{f}(x)$. The points $A(-1, 0)$, $B(0, 2)$, $C(1, 2)$ and $D(2, 0)$ lie on the curve.

Sketch the following graphs and give the coordinates of the points A, B, C and D after each transformation:

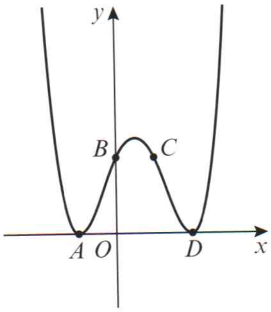
a $y + 2 = \mathrm{f}(x)$ **b** $\frac{1}{2}y = \mathrm{f}(x)$

c $y - 3 = \mathrm{f}(x)$ **d** $3y = \mathrm{f}(x)$

e $2y - 1 = \mathrm{f}(x)$

Problem-solving

Rearrange each equation into the form $y = \ldots$



Mixed exercise 4

1 a On the same axes sketch the graphs of $y = x^2(x - 2)$ and $y = 2x - x^2$.

b By solving a suitable equation find the points of intersection of the two graphs.

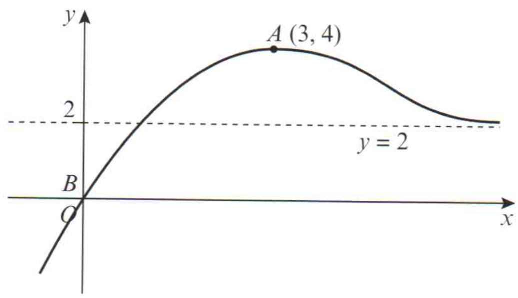
(P) 2 a On the same axes sketch the curves with equations $y = \frac{6}{x}$ and $y = 1 + x$.

b The curves intersect at the points A and B. Find the coordinates of A and B.

c The curve C with equation $y = x^2 + px + q$, where p and q are integers, passes through A and B. Find the values of p and q.

d Add C to your sketch.

3 The diagram shows a sketch of the curve $y = \mathrm{f}(x)$. The point $B(0, 0)$ lies on the curve and the point $A(3, 4)$ is a maximum point. The line $y = 2$ is an asymptote.



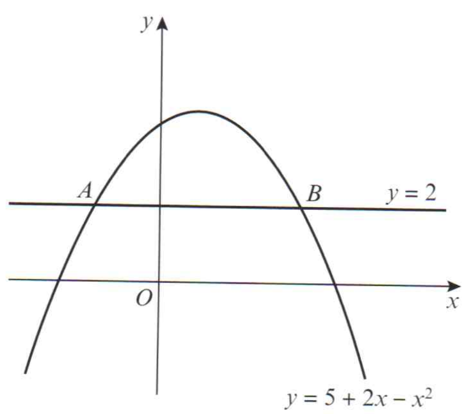
Sketch the following and in each case give the coordinates of the new positions of A and B and state the equation of the asymptote:

a $\mathrm{f}(2x)$ b $\frac{1}{2}\mathrm{f}(x)$ c $\mathrm{f}(x) - 2$

d $\mathrm{f}(x + 3)$ e $\mathrm{f}(x - 3)$ f $\mathrm{f}(x) + 1$

(E) 4 The diagram shows the curve with equation $y = 5 + 2x - x^2$ and the line with equation $y = 2$. The curve and the line intersect at the points A and B.

Find the x-coordinates of A and B. **(4 marks)**



(E/P) 5 $\mathrm{f}(x) = x^2(x - 1)(x - 3)$.

a Sketch the graph of $y = \mathrm{f}(x)$. **(2 marks)**

b On the same axes, draw the line $y = 2 - x$. **(2 marks)**

c State the number of real solutions to the equation $x^2(x - 1)(x - 3) = 2 - x$. **(1 mark)**

d Write down the coordinates of the point where the graph with equation $y = \mathrm{f}(x) + 2$ crosses the y-axis. **(1 mark)**

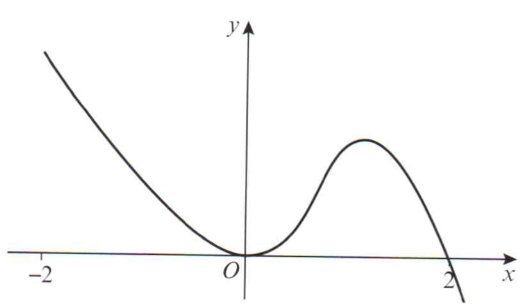
(E) 6 The figure shows a sketch of the curve with equation $y = \mathrm{f}(x)$.

On separate axes sketch the curves with equations:

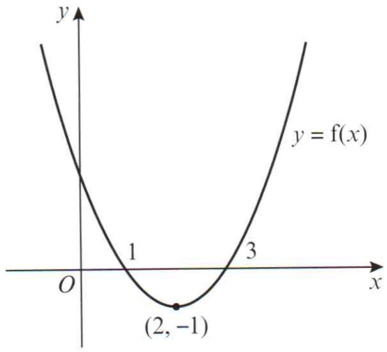
a $y = \mathrm{f}(-x)$ **(2 marks)**

b $y = -\mathrm{f}(x)$ **(2 marks)**

Mark on each sketch the x-coordinate of any point, or points, where the curve touches or crosses the x-axis.



E/P **7** The diagram shows the graph of the quadratic function f(x). The graph meets the x-axis at (1, 0) and (3, 0) and the minimum point is (2, −1).



a Find the equation of the graph in the form $y = ax^2 + bx + c$ **(2 marks)**

b On separate axes, sketch the graphs of
i $y = \text{f}(x + 2)$ **ii** $y = \text{f}(2x)$. **(4 marks)**
On each graph label the coordinates of the points at which the graph meets the x-axis and label the coordinates of the minimum point.

E/P **8** $\text{f}(x) = (x - 1)(x - 2)(x + 1)$.

a State the coordinates of the point at which the graph $y = \text{f}(x)$ intersects the y-axis. **(1 mark)**

b The graph of $y = a\text{f}(x)$ intersects the y-axis at (0, −4). Find the value of a. **(1 mark)**

c The graph of $y = \text{f}(x + b)$ passes through the origin. Find three possible values of b. **(3 marks)**

P **9** The point $P(4, 3)$ lies on a curve $y = \text{f}(x)$.

a State the coordinates of the point to which P is transformed on the curve with equation:
i $y = \text{f}(3x)$ **ii** $\frac{1}{2}y = \text{f}(x)$ **iii** $y = \text{f}(x - 5)$ **iv** $-y = \text{f}(x)$ **v** $2(y + 2) = \text{f}(x)$

b P is transformed to point (2, 3). Write down two possible transformations of f(x).

c P is transformed to point (8, 6). Write down a possible transformation of f(x) if
i f(x) is translated only **ii** f(x) is stretched only.

E/P **10** The curve C_1 has equation $y = -\frac{a}{x^2}$ where a is a positive constant. The curve C_2 has the equation $y = x^2(3x + b)$ where b is a positive constant.

a Sketch C_1 and C_2 on the same set of axes, showing clearly the coordinates of any point where the curves touch or cross the axes. **(4 marks)**

b Using your sketch state, giving reasons, the number of solutions to the equation $x^4(3x + b) + a = 0$. **(2 marks)**

E/P **11** **a** Factorise completely $x^3 - 6x^2 + 9x$. **(2 marks)**

b Sketch the curve of $y = x^3 - 6x^2 + 9x$ showing clearly the coordinates of the points where the curve touches or crosses the axes. **(4 marks)**

c The point with coordinates (−4, 0) lies on the curve with equation $y = (x - k)^3 - 6(x - k)^2 + 9(x - k)$ where k is a constant.
Find the two possible values of k. **(3 marks)**

E **12** $\text{f}(x) = x(x - 2)^2$
Sketch on separate axes the graphs of:

a $y = \text{f}(x)$ **(2 marks)**

b $y = \text{f}(x + 3)$ **(2 marks)**

Show on each sketch the coordinates of the points where each graph crosses or meets the axes.

(E) **13** Given that $f(x) = \frac{1}{x}$, $x \neq 0$,

a Sketch the graph of $y = f(x) - 2$ and state the equations of the asymptotes. **(3 marks)**

b Find the coordinates of the point where the curve $y = f(x) - 2$ cuts a coordinate axis. **(2 marks)**

c Sketch the graph of $y = f(x + 3)$. **(2 marks)**

d State the equations of the asymptotes and the coordinates of the point where the curve cuts a coordinate axis. **(2 marks)**

Challenge

The point $R(6, -4)$ lies on the curve with equation $y = f(x)$. State the coordinates that point R is transformed to on the curve with equation $y = f(x + c) - d$.

Summary of key points

1. If p is a root of the function $f(x)$, then the graph of $y = f(x)$ touches or crosses the x-axis at the point $(p, 0)$.
2. The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, where k is a real constant, have asymptotes at $x = 0$ and $y = 0$.
3. The x-coordinate(s) at the points of intersection of the curves with equations $y = f(x)$ and $y = g(x)$ are the solution(s) to the equation $f(x) = g(x)$.
4. The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.
5. The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.
6. When you translate a function, any asymptotes are also translated.
7. The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.
8. The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.
9. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.
10. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.