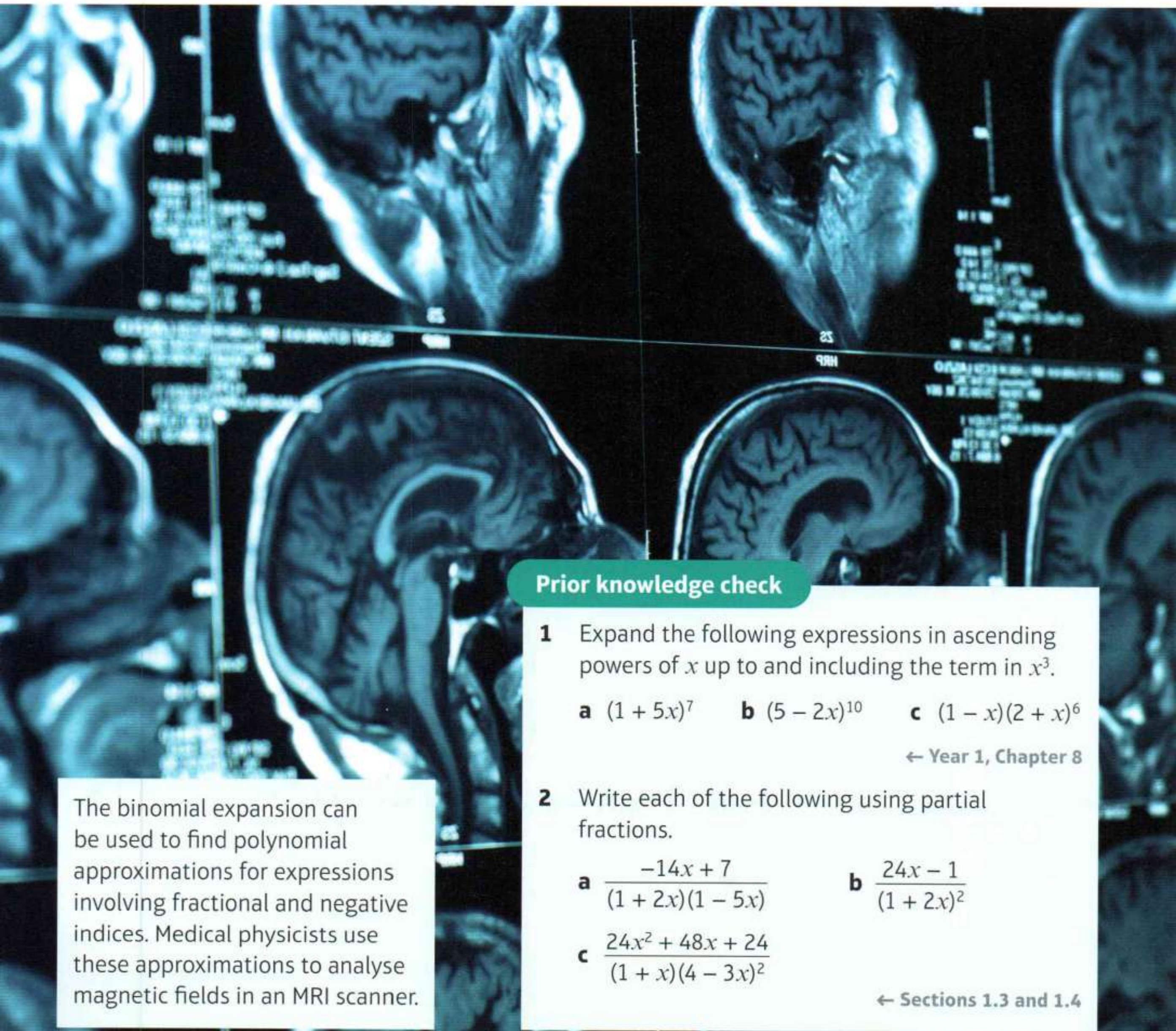
Binomial expansion

4

Objectives

After completing this chapter you should be able to:

- Expand $(1 + x)^n$ for any rational constant n and determine the range of values of x for which the expansion is valid → pages 92–97
- Expand $(a + bx)^n$ for any rational constant n and determine the range of values of x for which the expansion is valid → pages 97–100
- Use partial fractions to expand fractional expressions → pages 101–103



The binomial expansion can be used to find polynomial approximations for expressions involving fractional and negative indices. Medical physicists use these approximations to analyse magnetic fields in an MRI scanner.

Prior knowledge check

1 Expand the following expressions in ascending powers of x up to and including the term in x^3.

a $(1 + 5x)^7$ **b** $(5 - 2x)^{10}$ **c** $(1 - x)(2 + x)^6$

← Year 1, Chapter 8

2 Write each of the following using partial fractions.

a $\dfrac{-14x + 7}{(1 + 2x)(1 - 5x)}$ **b** $\dfrac{24x - 1}{(1 + 2x)^2}$

c $\dfrac{24x^2 + 48x + 24}{(1 + x)(4 - 3x)^2}$

← Sections 1.3 and 1.4

4.1 Expanding $(1 + x)^n$

If n is a natural number you can find the binomial expansion for $(a + bx)^n$ using the formula:

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n, \ (n \in \mathbb{N})$$

Hint There are $n + 1$ terms, so this formula produces a **finite** number of terms.

If n is a **fraction** or a **negative number** you need to use a different version of the binomial expansion.

- **This form of the binomial expansion can be applied to negative or fractional values of n to obtain an infinite series.**

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \left(\frac{n(n-1)\ldots(n-r+1)}{r!}\right)x^r + \ldots$$

- **The expansion is valid when $|x| < 1$, $n \in \mathbb{R}$.**

When n is not a natural number, none of the factors in the expression $n(n-1) \ldots (n-r+1)$ are equal to zero. This means that this version of the binomial expansion produces an **infinite number** of terms.

Watch out This expansion is valid for any **real value** of n, but is **only** valid for values of x that satisfy $|x| < 1$, or in other words, when $-1 < x < 1$.

Example 1

Find the first four terms in the binomial expansion of $\frac{1}{1+x}$

$$\frac{1}{1+x} = (1+x)^{-1}$$

Write in index form.

$$= 1 + (-1)x + \frac{(-1)(-2)x^2}{2!} + \frac{(-1)(-2)(-3)x^3}{3!} + \ldots$$

Replace n by -1 in the expansion.

$$= 1 - 1x + 1x^2 - 1x^3 + \ldots$$

As n is not a positive integer, no coefficient will ever be equal to zero. Therefore, the expansion is **infinite**.

$$= 1 - x + x^2 - x^3 + \ldots$$

For the series to be **convergent**, $|x| < 1$.

- **The expansion of $(1 + bx)^n$, where n is negative or a fraction, is valid for $|bx| < 1$, or $|x| < \frac{1}{|b|}$.**

Example 2

Find the binomial expansions of

a $(1 - x)^{\frac{1}{3}}$

b $\dfrac{1}{(1 + 4x)^2}$

up to and including the term in x^3. State the range of values of x for which each expansion is valid.

a $(1 - x)^{\frac{1}{3}}$

$$= 1 + (\tfrac{1}{3})(-x) + \frac{(\frac{1}{3})(\frac{1}{3} - 1)(-x)^2}{2!} + \frac{(\frac{1}{3})(\frac{1}{3} - 1)(\frac{1}{3} - 2)(-x)^3}{3!} + \ldots$$

Replace n by $\frac{1}{3}$, x by $(-x)$.

$$= 1 + (\tfrac{1}{3})(-x) + \frac{(\frac{1}{3})(-\frac{2}{3})(-x)^2}{2} + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(-x)^3}{6} + \ldots$$

Simplify brackets.

Watch out Be careful working out whether each term should be positive or negative:
- even number of negative signs means term is positive
- odd number of negative signs means term is negative

The x^3 term here has 5 negative signs in total, so it is negative.

$$= 1 - \tfrac{1}{3}x - \tfrac{1}{9}x^2 - \tfrac{5}{81}x^3 + \ldots$$

Simplify coefficients.

Expansion is valid as long as $|-x| < 1$

$\Rightarrow |x| < 1$

Terms in expansion are $(-x)$, $(-x)^2$, $(-x)^3$.

b $\dfrac{1}{(1 + 4x)^2} = (1 + 4x)^{-2}$

Write in index form.

$$= 1 + (-2)(4x) + \frac{(-2)(-2 - 1)(4x)^2}{2!} + \frac{(-2)(-2 - 1)(-2 - 2)(4x)^3}{3!} + \ldots$$

Replace n by -2, x by $4x$.

$$= 1 + (-2)(4x) + \frac{(-2)(-3)16x^2}{2} + \frac{(-2)(-3)(-4)64x^3}{6} + \ldots$$

Simplify brackets.

$$= 1 - 8x + 48x^2 - 256x^3 + \ldots$$

Simplify coefficients.

Expansion is valid as long as $|4x| < 1$

$\Rightarrow |x| < \frac{1}{4}$

Terms in expansion are $(4x)$, $(4x)^2$, $(4x)^3$.

Online Use technology to explore why the expansions are only valid for certain values of x.

Example 3

a Find the expansion of $\sqrt{1-2x}$ up to and including the term in x^3.

b By substituting in $x = 0.01$, find a decimal approximation to $\sqrt{2}$.

a $\sqrt{1-2x} = (1-2x)^{\frac{1}{2}}$

Write in index form.

$$= 1 + \left(\tfrac{1}{2}\right)(-2x) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)(-2x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(-2x)^3}{3!} + \ldots$$

Replace n by $\frac{1}{2}$, x by $(-2x)$.

$$= 1 + \left(\tfrac{1}{2}\right)(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(4x^2)}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-8x^3)}{6} + \ldots$$

Simplify brackets.

$$= 1 - x - \frac{x^2}{2} - \frac{x^3}{2} + \ldots$$

Simplify coefficients.

Expansion is valid if $|-2x| < 1$

$$\Rightarrow |x| < \tfrac{1}{2}$$

Terms in expansion are $(-2x)$, $(-2x)^2$, $(-2x)^3$.

b $\sqrt{1 - 2 \times 0.01} \approx 1 - 0.01 - \dfrac{0.01^2}{2} - \dfrac{0.01^3}{2}$

$x = 0.01$ satisfies the validity condition $|x| < \frac{1}{2}$

Substitute $x = 0.01$ into both sides of the expansion.

$$\sqrt{0.98} \approx 1 - 0.01 - 0.00005 - 0.0000005$$

Simplify both sides.
Note that the terms are getting smaller.

$$\sqrt{\frac{98}{100}} \approx 0.9899495$$

Write 0.98 as $\frac{98}{100}$

$$\sqrt{\frac{49 \times 2}{100}} \approx 0.9899495$$

$$\frac{7\sqrt{2}}{10} \approx 0.9899495$$

Use rules of surds.

$$\sqrt{2} \approx \frac{0.9899495 \times 10}{7}$$

$$\sqrt{2} \approx 1.414213571$$

This approximation is accurate to 7 decimal places.

Example 4

$$f(x) = \frac{2 + x}{\sqrt{1 + 5x}}$$

a Find the x^2 term in the series expansion of $f(x)$.

b State the range of values of x for which the expansion is valid.

a $f(x) = (2 + x)(1 + 5x)^{-\frac{1}{2}}$ — Write in index form.

$$(1 + 5x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(5x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(5x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(5x)^3 + \ldots$$

Find the binomial expansion of $(1 + 5x)^{-\frac{1}{2}}$

$$= 1 - \frac{5}{2}x + \frac{75}{8}x^2 - \frac{625}{16}x^3 + \ldots$$

Simplify coefficients.

$$f(x) = (2 + x)\left(1 - \frac{5}{2}x + \frac{75}{8}x^2 - \frac{625}{16}x^3 + \ldots\right)$$

$$2 \times \frac{75}{8} + 1 \times -\frac{5}{2} = \frac{65}{4}$$

x^2 term is $\frac{65}{4}x^2$

b The expansion is valid if $|5x| < 1$

$$\Rightarrow |x| < \frac{1}{5}$$

Online Use your calculator to calculate the coefficients of the binomial expansion.

Problem-solving

There are two ways to make an x^2 term. Either $2 \times \frac{75}{8}x^2$ or $x \times \frac{5}{2}x$. Add these together to find the term in x^2.

Example 5

In the expansion of $(1 + kx)^{-4}$ the coefficient of x is 20.

a Find the value of k.

b Find the corresponding coefficient of the x^2 term.

a $(1 + kx)^{-4} = 1 + (-4)(kx) + \frac{(-4)(-5)}{2!}(kx)^2 + \ldots$ — Find the binomial expansion of $(1 + kx)^{-4}$.

$$= 1 - 4kx + 10k^2x^2 + \ldots$$

$$-4k = 20$$
$$k = -5$$

Solve to find k.

b Coefficient of $x^2 = 10k^2 = 10(-5)^2 = 250$

Exercise 4A

1 For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $(1 + x)^{-4}$ **b** $(1 + x)^{-6}$ **c** $(1 + x)^{\frac{1}{2}}$

d $(1 + x)^{\frac{5}{3}}$ **e** $(1 + x)^{-\frac{1}{4}}$ **f** $(1 + x)^{-\frac{3}{2}}$

2 For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $(1 + 3x)^{-3}$ **b** $\left(1 + \frac{1}{2}x\right)^{-5}$ **c** $(1 + 2x)^{\frac{3}{4}}$

d $(1 - 5x)^{\frac{7}{3}}$ **e** $(1 + 6x)^{-\frac{2}{3}}$ **f** $\left(1 - \frac{3}{4}x\right)^{-\frac{5}{3}}$

3 For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $\dfrac{1}{(1 + x)^2}$ **b** $\dfrac{1}{(1 + 3x)^4}$ **c** $\sqrt{1 - x}$

d $\sqrt[3]{1 - 3x}$ **e** $\dfrac{1}{\sqrt{1 + \frac{1}{2}x}}$ **f** $\dfrac{\sqrt[3]{1 - 2x}}{1 - 2x}$

Hint In part **f**, write the fraction as a single power of $(1 - 2x)$.

E/P **4** $f(x) = \dfrac{1 + x}{1 - 2x}$

a Show that the series expansion of $f(x)$ up to and including the x^3 term is $1 + 3x + 6x^2 + 12x^3$. **(4 marks)**

b State the range of values of x for which the expansion is valid. **(1 mark)**

Hint First rewrite $f(x)$ as $(1 + x)(1 - 2x)^{-1}$.

E **5** $f(x) = \sqrt{1 + 3x}$, $-\dfrac{1}{3} < x < \dfrac{1}{3}$

a Find the series expansion of $f(x)$, in ascending powers of x, up to and including the x^3 term. Simplify each term. **(4 marks)**

b Show that, when $x = \dfrac{1}{100}$, the exact value of $f(x)$ is $\dfrac{\sqrt{103}}{10}$ **(2 marks)**

c Find the percentage error made in using the series expansion in part **a** to estimate the value of $f(0.01)$. Give your answer to 2 significant figures. **(3 marks)**

P **6** In the expansion of $(1 + ax)^{-\frac{1}{2}}$ the coefficient of x^2 is 24.

a Find the possible values of a.

b Find the possible coefficients of the x^3 term.

(P) **7** Show that if x is small, the expression $\sqrt{\frac{1+x}{1-x}}$ is approximated by $1 + x + \frac{1}{2}x^2$.

Notation 'x is small' means we can assume the expansion is valid for the x values being considered, as high powers become insignificant compared to the first few terms.

(E/P) **8** $h(x) = \frac{6}{1+5x} - \frac{4}{1-3x}$

a Find the series expansion of $h(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(6 marks)**

b Find the percentage error made in using the series expansion in part **a** to estimate the value of $h(0.01)$. Give your answer to 2 significant figures. **(3 marks)**

c Explain why it is not valid to use the expansion to find $h(0.5)$. **(1 mark)**

(E/P) **9** **a** Find the binomial expansion of $(1 - 3x)^{\frac{3}{2}}$ in ascending powers of x up to and including the x^3 term, simplifying each term. **(4 marks)**

b Show that, when $x = \frac{1}{100}$, the exact value of $(1 - 3x)^{\frac{3}{2}}$ is $\frac{97\sqrt{97}}{1000}$ **(2 marks)**

c Substitute $x = \frac{1}{100}$ into the binomial expansion in part **a** and hence obtain an approximation to $\sqrt{97}$. Give your answer to 5 decimal places. **(3 marks)**

Challenge

$h(x) = \left(1 + \frac{1}{x}\right)^{-\frac{1}{2}}, |x| > 1$

a Find the binomial expansion of $h(x)$ in ascending powers of x up to and including the x^2 term, simplifying each term.

b Show that, when $x = 9$, the exact value of $h(x)$ is $\frac{3\sqrt{10}}{10}$

c Use the expansion in part **a** to find an approximate value of $\sqrt{10}$. Write your answer to 2 decimal places.

Hint Replace x with $\frac{1}{x}$

4.2 Expanding $(a + bx)^n$

The binomial expansion of $(1 + x)^n$ can be used to expand $(a + bx)^n$ for any constants a and b.

You need to take a factor of a^n out of the expression:

$$(a + bx)^n = \left(a\left(1 + \frac{b}{a}x\right)\right)^n = a^n\left(1 + \frac{b}{a}x\right)^n$$

Watch out Make sure you multiply a^n by **every term** in the expansion of $\left(1 + \frac{b}{a}x\right)^n$.

- **The expansion of $(a + bx)^n$, where n is negative or a fraction, is valid for $\left|\frac{b}{a}x\right| < 1$ or $|x| < \left|\frac{a}{b}\right|$.**

Example 6

Find the first four terms in the binomial expansion of **a** $\sqrt{4 + x}$ **b** $\frac{1}{(2 + 3x)^2}$

State the range of values of x for which each of these expansions is valid.

a $\sqrt{4 + x} = (4 + x)^{\frac{1}{2}}$ — Write in index form.

$= \left(4\left(1 + \frac{x}{4}\right)\right)^{\frac{1}{2}}$

$= 4^{\frac{1}{2}}\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$ — Take out a factor of $4^{\frac{1}{2}}$.

$= 2\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$ — Write $4^{\frac{1}{2}}$ as 2.

$$= 2\left(1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{x}{4}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left(\frac{x}{4}\right)^3}{3!} + \ldots\right)$$

Expand $\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$ using the binomial expansion with $n = \frac{1}{2}$ and $x = \frac{x}{4}$

$$= 2\left(1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x^2}{16}\right)}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^3}{64}\right)}{6} + \ldots\right)$$

$$= 2\left(1 + \frac{x}{8} - \frac{x^2}{128} + \frac{x^3}{1024} + \ldots\right)$$

Simplify coefficients.

$$= 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \ldots$$

Multiply every term in the expansion by 2.

Expansion is valid if $\left|\frac{x}{4}\right| < 1$

$\Rightarrow |x| < 4$

The expansion is infinite, and converges when $\left|\frac{x}{4}\right| < 1$, or $|x| < 4$.

b $\frac{1}{(2+3x)^2} = (2+3x)^{-2}$

Write in index form.

$= \left(2\left(1+\frac{3x}{2}\right)\right)^{-2}$

$= 2^{-2}\left(1+\frac{3x}{2}\right)^{-2}$

Take out a factor of 2^{-2}.

$= \frac{1}{4}\left(1+\frac{3x}{2}\right)^{-2}$

Write $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

$= \frac{1}{4}\left(1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-2-1)\left(\frac{3x}{2}\right)^2}{2!} + \frac{(-2)(-2-1)(-2-2)\left(\frac{3x}{2}\right)^3}{3!} + \ldots\right)$

Expand $\left(1+\frac{3x}{2}\right)^{-2}$ using the binomial expansion with $n = -2$ and $x = \frac{3x}{2}$

$= \frac{1}{4}\left((1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-3)\left(\frac{9x^2}{4}\right)}{2} + \frac{(-2)(-3)(-4)\left(\frac{27x^3}{8}\right)}{6} + \ldots\right)$

Simplify coefficients.

$= \frac{1}{4}\left(1 - 3x + \frac{27x^2}{4} - \frac{27x^3}{2} + \ldots\right)$

Multiply every term by $\frac{1}{4}$

$= \frac{1}{4} - \frac{3}{4}x + \frac{27x^2}{16} - \frac{27x^3}{8} + \ldots$

Expansion is valid if $\left|\frac{3x}{2}\right| < 1$

$\Rightarrow |x| < \frac{2}{3}$

The expansion is infinite, and converges when $\left|\frac{3x}{2}\right| < 1$, $|x| < \frac{2}{3}$

Exercise 4B

(P) **1** For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $\sqrt{4+2x}$ **b** $\frac{1}{2+x}$ **c** $\frac{1}{(4-x)^2}$ **d** $\sqrt{9+x}$

e $\frac{1}{\sqrt{2+x}}$ **f** $\frac{5}{3+2x}$ **g** $\frac{1+x}{2+x}$ **h** $\sqrt{\frac{2+x}{1-x}}$

Hint Write part **g** as $1 - \frac{1}{x+2}$

(E) **2** $f(x) = (5 + 4x)^{-2}$, $|x| < \frac{5}{4}$

Find the binomial expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction. **(5 marks)**

(E) **3** $m(x) = \sqrt{4 - x}$, $|x| < 4$

a Find the series expansion of $m(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(4 marks)**

b Show that, when $x = \frac{1}{9}$, the exact value of $m(x)$ is $\frac{\sqrt{35}}{3}$ **(2 marks)**

c Use your answer to part **a** to find an approximate value for $\sqrt{35}$, and calculate the percentage error in your approximation. **(4 marks)**

(P) **4** The first three terms in the binomial expansion of $\frac{1}{\sqrt{a + bx}}$ are $3 + \frac{1}{3}x + \frac{1}{18}x^2 + \ldots$

a Find the values of the constants a and b.

b Find the coefficient of the x^3 term in the expansion.

(P) **5** $f(x) = \frac{3 + 2x - x^2}{4 - x}$

Prove that if x is sufficiently small, $f(x)$ may be approximated by $\frac{3}{4} + \frac{11}{16}x - \frac{5}{64}x^2$.

(E/P) **6 a** Expand $\frac{1}{\sqrt{5 + 2x}}$, where $|x| < \frac{5}{2}$, in ascending powers of x up to and including the term in x^2, giving each coefficient in simplified surd form. **(5 marks)**

b Hence or otherwise, find the first 3 terms in the expansion of $\frac{2x - 1}{\sqrt{5 + 2x}}$ as a series in ascending powers of x. **(4 marks**

(E/P) **7 a** Use the binomial theorem to expand $(16 - 3x)^{\frac{1}{4}}$, $|x| < \frac{16}{3}$ in ascending powers of x, up to and including the term in x^2, giving each term as a simplified fraction. **(4 marks)**

b Use your expansion, with a suitable value of x, to obtain an approximation to $\sqrt[4]{15.7}$. Give your answer to 3 decimal places. **(2 marks)**

8 $g(x) = \frac{3}{4 - 2x} - \frac{2}{3 + 5x}$, $|x| < \frac{1}{2}$

a Show that the first three terms in the series expansion of $g(x)$ can be written as $\frac{1}{12} + \frac{107}{72}x - \frac{719}{432}x^2$. **(5 marks)**

b Find the exact value of $g(0.01)$. Round your answer to 7 decimal places. **(2 marks)**

c Find the percentage error made in using the series expansion in part **a** to estimate the value of $g(0.01)$. Give your answer to 2 significant figures. **(3 marks)**

4.3 Using partial fractions

Partial fractions can be used to simplify the expansions of more difficult expressions.

Links You need to be confident expressing algebraic fractions as sums of partial fractions.
← Chapter 1

Example 7

a Express $\frac{4-5x}{(1+x)(2-x)}$ as partial fractions.

b Hence show that the cubic approximation of $\frac{4-5x}{(1+x)(2-x)}$ is $2 - \frac{7x}{2} + \frac{11}{4}x^2 - \frac{25}{8}x^3$.

c State the range of values of x for which the expansion is valid.

a $\frac{4-5x}{(1+x)(2-x)} \equiv \frac{A}{1+x} + \frac{B}{2-x}$ — The denominators must be $(1 + x)$ and $(2 - x)$.

$\equiv \frac{A(2-x)+B(1+x)}{(1+x)(2-x)}$ — Add the fractions.

$4 - 5x \equiv A(2-x) + B(1+x)$ — Set the numerators equal.

Substitute $x = 2$:

$4 - 10 = A \times 0 + B \times 3$ — Set $x = 2$ to find B.

$-6 = 3B$

$B = -2$

Substitute $x = -1$:

$4 + 5 = A \times 3 + B \times 0$ — Set $x = -1$ to find A.

$9 = 3A$

$A = 3$

so $\frac{4-5x}{(1+x)(2-x)} = \frac{3}{1+x} - \frac{2}{2-x}$

b $\frac{4-5x}{(1+x)(2-x)} = \frac{3}{1+x} - \frac{2}{2-x}$

$= 3(1+x)^{-1} - 2(2-x)^{-1}$ — Write in index form.

Problem-solving
Use headings to keep track of your working. This will help you stay organised and check your answers.

The expansion of $3(1+x)^{-1}$

$= 3\left(1 + (-1)x + (-1)(-2)\frac{x^2}{2!} + (-1)(-2)(-3)\frac{x^3}{3!} + \ldots\right)$ — Expand $3(1 + x)^{-1}$ using the binomial expansion with $n = -1$.

$= 3(1 - x + x^2 - x^3 + \ldots)$

$= 3 - 3x + 3x^2 - 3x^3 + \ldots$

The expansion of $2(2 - x)^{-1}$

$= 2\left(2\left(1 - \frac{x}{2}\right)\right)^{-1}$

$= 2 \times 2^{-1}\left(1 - \frac{x}{2}\right)^{-1}$

Take out a factor of 2^{-1}.

$= 1 \times \left(1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(-1)(-2)(-3)\left(-\frac{x}{2}\right)^3}{3!} + \ldots\right)$

Expand $\left(1 - \frac{x}{2}\right)^{-1}$ using the binomial expansion with $n = -1$ and $x = \frac{x}{2}$

$= 1 \times \left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots\right)$

$= 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8}$

Hence $\frac{4 - 5x}{(1 + x)(2 - x)}$

$= 3(1 + x)^{-1} - 2(2 - x)^{-1}$

'Add' both expressions.

$= (3 - 3x + 3x^2 - 3x^3)$

$-\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8}\right)$

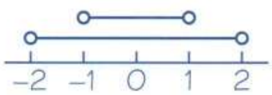
$= 2 - \frac{7}{2}x + \frac{11}{4}x^2 - \frac{25}{8}x^3$

c $\frac{3}{1 + x}$ is valid if $|x| < 1$

The expansion is infinite, and converges when $|x| < 1$.

$\frac{2}{2 - x}$ is valid if $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

The expansion is infinite, and converges when $\left|\frac{x}{2}\right| < 1$ or $|x| < 2$.



The expansion is valid when $|x| < 1$.

Watch out You need to find the range of values of x that satisfy **both** inequalities.

Exercise 4C

(P) **1** **a** Express $\frac{8x + 4}{(1 - x)(2 + x)}$ as partial fractions.

b Hence or otherwise expand $\frac{8x + 4}{(1 - x)(2 + x)}$ in ascending powers of x as far as the term in x^2.

c State the set of values of x for which the expansion is valid.

(P) **2** **a** Express $-\frac{2x}{(2+x)^2}$ as partial fractions.

b Hence prove that $-\frac{2x}{(2+x)^2}$ can be expressed in the form $-\frac{1}{2}x + Bx^2 + Cx^3$ where constants B and C are to be determined.

c State the set of values of x for which the expansion is valid.

(P) **3** **a** Express $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ as partial fractions.

b Hence or otherwise expand $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ in ascending powers of x as far as the term in x^3.

c State the set of values of x for which the expansion is valid.

(E/P) **4** $g(x) = \frac{12x-1}{(1+2x)(1-3x)}, |x| < \frac{1}{3}$

Given that g(x) can be expressed in the form $g(x) = \frac{A}{1+2x} + \frac{B}{1-3x}$

a Find the values of A and B. **(3 marks)**

b Hence, or otherwise, find the series expansion of g(x), in ascending powers of x, up to and including the x^2 term. Simplify each term. **(6 marks)**

(P) **5** **a** Express $\frac{2x^2+7x-6}{(x+5)(x-4)}$ in partial fractions.

Hint First divide the numerator by the denominator.

b Hence, or otherwise, expand $\frac{2x^2+7x-6}{(x+5)(x-4)}$ in ascending powers of x as far as the term in x^2.

c State the set of values of x for which the expansion is valid.

(E/P) **6** $\frac{3x^2+4x-5}{(x+3)(x-2)} = A + \frac{B}{x+3} + \frac{C}{x-2}$

a Find the values of the constants A, B and C. **(4 marks)**

b Hence, or otherwise, expand $\frac{3x^2+4x-5}{(x+3)(x-2)}$ in ascending powers of x, as far as the term in x^2. Give each coefficient as a simplified fraction. **(7 marks)**

(E/P) **7** $f(x) = \frac{2x^2+5x+11}{(2x-1)^2(x+1)}, |x| < \frac{1}{2}$

f(x) can be expressed in the form $f(x) = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x+1}$

a Find the values of A, B and C. **(4 marks)**

b Hence or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the term in x^2. Simplify each term. **(6 marks)**

c Find the percentage error made in using the series expansion in part **b** to estimate the value of f(0.05). Give your answer to 2 significant figures. **(4 marks)**

Mixed exercise 4

P 1 For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $(1 - 4x)^3$ b $\sqrt{16 + x}$ c $\frac{1}{1 - 2x}$ d $\frac{4}{2 + 3x}$

e $\frac{4}{\sqrt{4 - x}}$ f $\frac{1 + x}{1 + 3x}$ g $\left(\frac{1 + x}{1 - x}\right)^2$ h $\frac{x - 3}{(1 - x)(1 - 2x)}$

E 2 Use the binomial expansion to expand $\left(1 - \frac{1}{2}x\right)^{\frac{1}{2}}$, $|x| < 2$ in ascending powers of x, up to and including the term in x^3, simplifying each term. **(5 marks)**

3 a Give the binomial expansion of $(1 + x)^{\frac{1}{2}}$ up to and including the term in x^3.

b By substituting $x = \frac{1}{4}$, find an approximation to $\sqrt{5}$ as a fraction.

E/P 4 The binomial expansion of $(1 + 9x)^{\frac{2}{3}}$ in ascending powers of x up to and including the term in x^3 is $1 + 6x + cx^2 + dx^3$, $|x| < \frac{1}{9}$

a Find the value of c and the value of d. **(4 marks)**

b Use this expansion with your values of c and d together with an appropriate value of x to obtain an estimate of $(1.45)^{\frac{2}{3}}$. **(2 marks)**

c Obtain $(1.45)^{\frac{2}{3}}$ from your calculator and hence make a comment on the accuracy of the estimate you obtained in part **b**. **(1 mark)**

P 5 In the expansion of $(1 + ax)^{\frac{1}{2}}$ the coefficient of x^2 is -2.

a Find the possible values of a.

b Find the corresponding coefficients of the x^3 term.

E 6 $f(x) = (1 + 3x)^{-1}$, $|x| < \frac{1}{3}$

a Expand $f(x)$ in ascending powers of x up to and including the term in x^3. **(5 marks)**

b Hence show that, for small x:

$$\frac{1 + x}{1 + 3x} \approx 1 - 2x + 6x^2 - 18x^3.$$ **(4 marks)**

c Taking a suitable value for x, which should be stated, use the series expansion in part **b** to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places. **(3 marks)**

E/P 7 When $(1 + ax)^n$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 27 respectively.

a Find the values of a and n. **(4 marks)**

b Find the coefficient of x^3. **(3 marks)**

c State the values of x for which the expansion is valid. **(1 mark)**

8 Show that if x is sufficiently small then $\frac{3}{\sqrt{4+x}}$ can be approximated by $\frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$.

E 9 a Expand $\frac{1}{\sqrt{4-x}}$, where $|x| < 4$, in ascending powers of x up to and including the term in x^2. Simplify each term. **(5 marks)**

b Hence, or otherwise, find the first 3 terms in the expansion of $\frac{1+2x}{\sqrt{4-x}}$ as a series in ascending powers of x. **(4 marks)**

E 10 a Find the first four terms of the expansion, in ascending powers of x, of $(2 + 3x)^{-1}$, $|x| < \frac{2}{3}$ **(4 marks)**

b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of:

$\frac{1+x}{2+3x}$, $|x| < \frac{2}{3}$ **(3 marks)**

E/P 11 a Use the binomial theorem to expand $(4 + x)^{-\frac{1}{2}}$, $|x| < 4$, in ascending powers of x, up to and including the x^3 term, giving each answer as a simplified fraction. **(5 marks)**

b Use your expansion, together with a suitable value of x, to obtain an approximation to $\frac{\sqrt{2}}{2}$. Give your answer to 4 decimal places. **(3 marks)**

E 12 $q(x) = (3 + 4x)^{-3}$, $|x| < \frac{3}{4}$

Find the binomial expansion of $q(x)$ in ascending powers of x, up to and including the term in the x^2. Give each coefficient as a simplified fraction. **(5 marks)**

E/P 13 $g(x) = \frac{39x + 12}{(x+1)(x+4)(x-8)}$, $|x| < 1$

$g(x)$ can be expressed in the form $g(x) = \frac{A}{x+1} + \frac{B}{x+4} + \frac{C}{x-8}$

a Find the values of A, B and C. **(4 marks)**

b Hence, or otherwise, find the series expansion of $g(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(7 marks)**

E/P 14 $f(x) = \frac{12x + 5}{(1 + 4x)^2}$, $|x| < \frac{1}{4}$

For $x \neq -\frac{1}{4}$, $\frac{12x+5}{(1+4x)^2} = \frac{A}{1+4x} + \frac{B}{(1+4x)^2}$, where A and B are constants.

a Find the values of A and B. **(3 marks)**

b Hence, or otherwise, find the series expansion of $f(x)$, in ascending powers of x, up to and including the term x^2, simplifying each term. **(6 marks)**

E/P **15** $q(x) = \dfrac{9x^2 + 26x + 20}{(1 + x)(2 + x)}, |x| < 1$

a Show that the expansion of $q(x)$ in ascending powers of x can be approximated to $10 - 2x + Bx^2 + Cx^3$ where B and C are constants to be found. **(7 marks)**

b Find the percentage error made in using the series expansion in part **a** to estimate the value of $q(0.1)$. Give your answer to 2 significant figures. **(4 marks)**

Challenge

Obtain the first four non-zero terms in the expansion, in ascending powers of x, of the function $f(x)$ where $f(x) = \dfrac{1}{\sqrt{1 + 3x^2}}, 3x^2 < 1$.

Summary of key points

1 This form of the binomial expansion can be applied to negative or fractional values of n to obtain an infinite series:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots + \frac{n(n-1)\ldots(n-r+1)x^r}{r!} + \ldots$$

The expansion is valid when $|x| < 1, n \in \mathbb{R}$.

2 The expansion of $(1 + bx)^n$, where n is negative or a fraction, is valid for $|bx| < 1$, or $|x| < \dfrac{1}{|b|}$.

3 The expansion of $(a + bx)^n$, where n is negative or a fraction, is valid for $\left|\dfrac{b}{a}x\right| < 1$ or $|x| < \left|\dfrac{a}{b}\right|$.