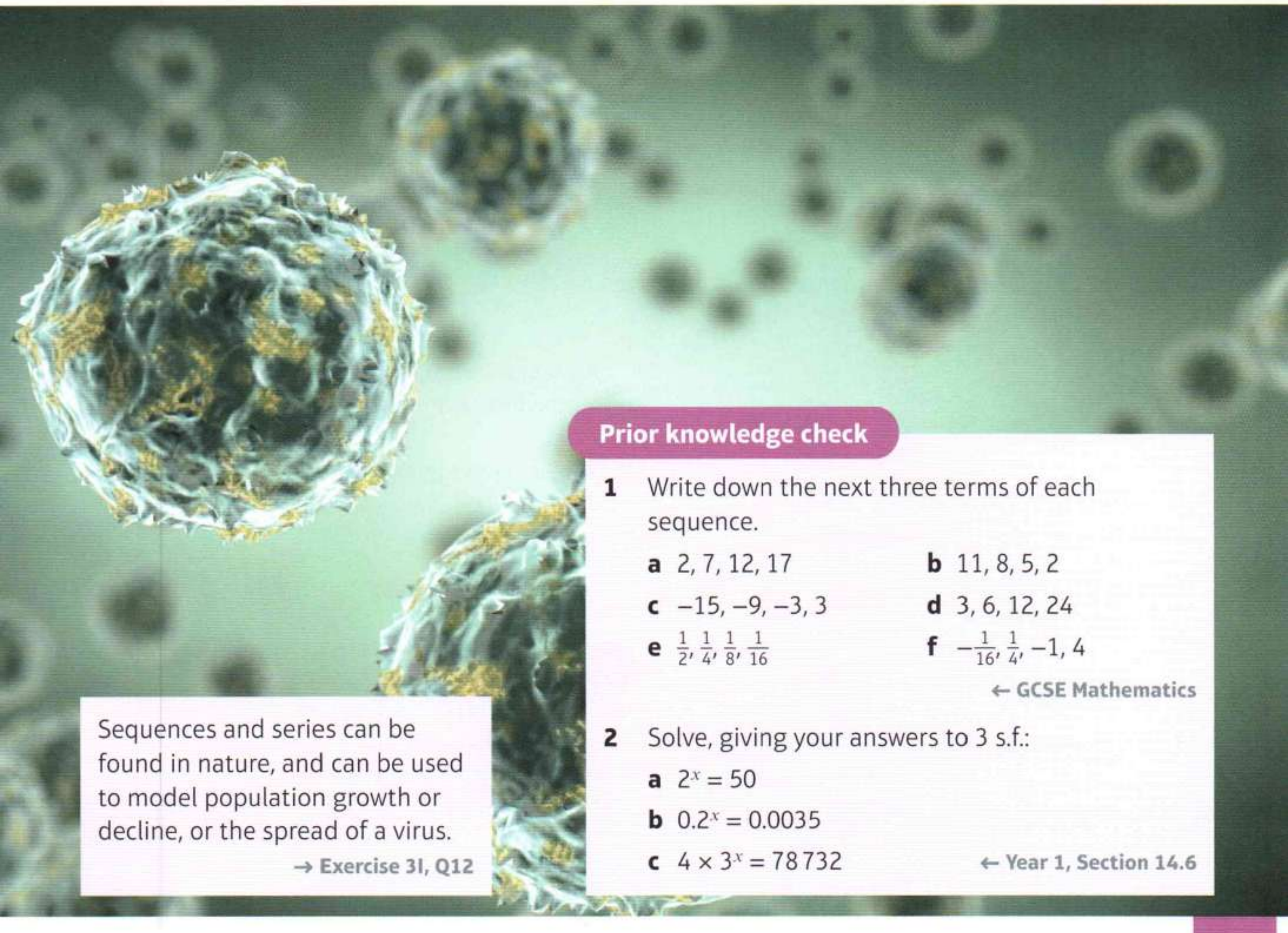
Sequences and series

3

Objectives

After completing this chapter you should be able to:

- Find the nth term of an arithmetic sequence → pages 60–62
- Prove and use the formula for the sum of the first n terms of an arithmetic series → pages 63–66
- Find the nth term of a geometric sequence → pages 66–70
- Prove and use the formula for the sum of a finite geometric series → pages 70–73
- Prove and use the formula for the sum to infinity of a convergent geometric series → pages 73–76
- Use sigma notation to describe series → pages 76–78
- Generate sequences from recurrence relations → pages 79–83
- Model real-life situations with sequences and series → pages 83–86



Sequences and series can be found in nature, and can be used to model population growth or decline, or the spread of a virus.

→ Exercise 3I, Q12

Prior knowledge check

1 Write down the next three terms of each sequence.

a 2, 7, 12, 17 **b** 11, 8, 5, 2

c −15, −9, −3, 3 **d** 3, 6, 12, 24

e $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ **f** $-\frac{1}{16}, \frac{1}{4}, -1, 4$

← GCSE Mathematics

2 Solve, giving your answers to 3 s.f.:

a $2^x = 50$

b $0.2^x = 0.0035$

c $4 \times 3^x = 78\,732$

← Year 1, Section 14.6

3.1 Arithmetic sequences

- **In an arithmetic sequence, the difference between consecutive terms is constant.**

Notation An arithmetic sequence is sometimes called an **arithmetic progression**.

5, 7, 9, 11,
+2 +2 +2

This sequence is arithmetic. The difference between consecutive terms is +2. The sequence is **increasing**.

12.5, 10, 7.5, 5,
−2.5 −2.5 −2.5

This sequence is arithmetic. The difference between consecutive terms is −2.5. The sequence is **decreasing**.

4, 7, 12, 19,
+3 +5 +7

The difference is not constant so the sequence is not arithmetic.

- **The formula for the *n*th term of an arithmetic sequence is:**
  $u_n = a + (n - 1)d$
  where a is the first term and d is the common difference.

Notation In questions on sequences and series:
- u_n is the *n*th term
- a is the first term
- d is the common difference

Example 1

The *n*th term of an arithmetic sequence is $u_n = 55 - 2n$.

a Write down the first 5 terms of the sequence.

b Find the 99th term in the sequence.

c Find the first term in the sequence that is negative.

Online Use the table function on your calculator to generate terms in the sequence for this function, or to check an *n*th term.

a $u_n = 55 - 2n$

$n = 1 \rightarrow u_1 = 55 - 2(1) = 53$

$n = 2 \rightarrow u_2 = 55 - 2(2) = 51$

$n = 3 \rightarrow u_3 = 55 - 2(3) = 49$

$n = 4 \rightarrow u_4 = 55 - 2(4) = 47$

$n = 5 \rightarrow u_5 = 55 - 2(5) = 45$

Remember, n is the position in the sequence, so for the first term substitute $n = 1$.

For the second term substitute $n = 2$.

b $u_{99} = 55 - 2(99) = -143$

For the 99th term substitute $n = 99$.

c $55 - 2n < 0$

$-2n < -55$

$n > 27.5$

$n = 28$

$u_{28} = 55 - 2(28) = -1$

Problem-solving

To find the first negative term, set $u_n < 0$ and solve the inequality. n is the term number so it must be a positive integer.

Example 2

Find the nth term of each arithmetic sequence.

a 6, 20, 34, 48, 62

b 101, 94, 87, 80, 73

a $a = 6, d = 14$

$u_n = 6 + 14(n - 1)$

$u_n = 6 + 14n - 14$

$u_n = 14n - 8$

b $a = 101, d = -7$

$u_n = 101 - 7(n - 1)$

$u_n = 101 - 7n + 7$

$u_n = 108 - 7n$

Write down the values of a and d.

Substitute the values of a and d into the formula $a + (n - 1)d$ and simplify.

Watch out If the sequence is decreasing then d is negative.

Example 3

A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.
Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b.

Problem-solving

You know two terms and there are two unknowns in the expression for the nth term. You can use this information to form two simultaneous equations. ← Year 1 Section 3.1

$u_3 = 5$, so $3a + b = 5$. (1)

$u_8 = 20$, so $8a + b = 20$. (2)

(2) − (1) gives:

$5a = 15$

$a = 3$

Substitute $a = 3$ in (1):

$9 + b = 5$

$b = -4$

Constants are $a = 3$ and $b = -4$.

Substitute $n = 3$ and $u_3 = 5$ in $u_n = an + b$.

Substitute $n = 8$ and $u_8 = 20$ in $u_n = an + b$.

Solve simultaneously.

Exercise 3A

1 For each sequence:

i write down the first 4 terms of the sequence

ii write down a and d.

a $u_n = 5n + 2$ **b** $u_n = 9 - 2n$ **c** $u_n = 7 + 0.5n$ **d** $u_n = n - 10$

2 Find the nth terms and the 10th terms in the following arithmetic progressions:

a 5, 7, 9, 11, …
b 5, 8, 11, 14, …
c 24, 21, 18, 15, …
d −1, 3, 7, 11, …
e $x, 2x, 3x, 4x, \ldots$
f $a, a + d, a + 2d, a + 3d, \ldots$

(P) **3** Calculate the number of terms in each of the following arithmetic sequences.

a 3, 7, 11, …, 83, 87
b 5, 8, 11, …, 119, 122
c 90, 88, 86, …, 16, 14
d 4, 9, 14, …, 224, 229
e $x, 3x, 5x, \ldots, 35x$
f $a, a + d, a + 2d, \ldots, a + (n - 1)d$

Problem-solving

Find an expression for u_n and set it equal to the final term in the sequence. Solve the equation to find the value of n.

(P) **4** The first term of an arithmetic sequence is 14. The fourth term is 32. Find the common difference.

(P) **5** A sequence is generated by the formula $u_n = pn + q$ where p and q are constants to be found. Given that $u_6 = 9$ and $u_9 = 11$, find the constants p and q.

(P) **6** For an arithmetic sequence $u_3 = 30$ and $u_9 = 9$. Find the first negative term in the sequence.

(P) **7** The 20th term of an arithmetic sequence is 14. The 40th term is −6. Find the value of the 10th term.

(P) **8** The first three terms of an arithmetic sequence are $5p$, 20 and $3p$, where p is a constant. Find the 20th term in the sequence.

(E/P) **9** The first three terms in an arithmetic sequence are $-8, k^2, 17k \ldots$
Find two possible values of k. **(3 marks)**

(E/P) **10** An arithmetic sequence has first term k^2 and common difference k, where $k > 0$. The fifth term of the sequence is 41. Find the value of k, giving your answer in the form $p + q\sqrt{5}$, where p and q are integers to be found. **(4 marks)**

Problem-solving

You will need to make use of the condition $k > 0$ in your answer.

Challenge

The nth term of an arithmetic sequence is $u_n = \ln a + (n - 1) \ln b$ where a and b are integers. $u_3 = \ln 16$ and $u_7 = \ln 256$. Find the values of a and b.

3.2 Arithmetic series

- **An arithmetic series is the sum of the terms of an arithmetic sequence.**

5, 7, 9, 11 is an arithmetic **sequence**.
5 + 7 + 9 + 11 is an arithmetic **series**.

Notation S_n is used for the sum of the first n terms of a series.

Example 4

Prove that the sum of the first 100 natural numbers is 5050.

$S_{100} = 1 + 2 + 3 + \ldots + 98 + 99 + 100$ (1)
$S_{100} = 100 + 99 + 98 + \ldots + 3 + 2 + 1$ (2)
Adding (1) and (2):
$2 \times S_{100} = 100 \times 101$
$S_{100} = \frac{100 \times 101}{2}$
$= 5050$

The **natural numbers** are the positive integers: 1, 2, 3, 4, …

Problem-solving

Write out the sum longhand, then write it out in reverse. You can pair up the numbers so that each pair has a sum of 101. There are 100 pairs in total.

- **The sum of the first n terms of an arithmetic series is given by the formula**

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

where a is the first term and d is the common difference.
You can also write this formula as

$$S_n = \frac{n}{2}(a + l)$$

where l is the last term.

Example 5

Prove that the sum of the first n terms of an arithmetic series is $\frac{n}{2}(2a + (n-1)d)$.

$S_n = a + (a + d) + (a + 2d) + \ldots$
$+ (a + (n-2)d) + (a + (n-1)d)$ (1) — Write out the terms of the sum.
$S_n = (a + (n-1)d) + (a + (n-2)d) + \ldots.$
$+ (a + 2d) + (a + d) + a$ (2) — This is the sum reversed.
Adding (1) and (2): — Adding together the two sums.
$2 \times S_n = n(2a + (n-1)d)$
$S_n = \frac{n}{2}(2a + (n-1)d)$

Problem-solving

You need to learn this proof for your exam.

Example 6

Find the sum of the first 50 terms of the arithmetic series 32 + 27 + 22 + 17 + 12 + …

$a = 32, d = -5$ — Write down a and d.

$S_{50} = \frac{50}{2}(2(32) + (50 - 1)(-5))$ — Substitute into the formula.

$S_{50} = -4525$ — Simplify.

Example 7

Find the least number of terms required for the sum of 4 + 9 + 14 + 19 + … to exceed 2000.

$4 + 9 + 14 + 19 + \ldots > 2000$ — Always establish what you are given in a question. As you are adding on positive terms, it is easier to solve the equality $S_n = 2000$.

Using $S_n = \frac{n}{2}(2a + (n-1)d)$ — Knowing $a = 4$, $d = 5$ and $S_n = 2000$, you need to find n. Substitute into $S_n = \frac{n}{2}(2a + (n-1)d)$.

$2000 = \frac{n}{2}(2 \times 4 + (n-1)5)$

$4000 = n(8 + 5n - 5)$

$4000 = n(5n + 3)$

$4000 = 5n^2 + 3n$

$0 = 5n^2 + 3n - 4000$

$n = \frac{-3 \pm \sqrt{9 + 80\,000}}{10}$ — Solve using the quadratic formula.

$n = 27.99$ or -28.59

28 terms are needed. — n is the number of terms, so must be a positive integer.

Exercise 3B

1 Find the sums of the following series.

a 3 + 7 + 11 + 15 + … (20 terms)

b 2 + 6 + 10 + 14 + … (15 terms)

c 30 + 27 + 24 + 21 + … (40 terms)

d 5 + 1 + −3 + −7 + … (14 terms)

e 5 + 7 + 9 + … + 75

f 4 + 7 + 10 + … + 91

g 34 + 29 + 24 + 19 + … + −111

h $(x + 1) + (2x + 1) + (3x + 1) + \ldots + (21x + 1)$

Hint For parts **e** to **h**, start by using the last term to work out the number of terms in the series.

2 Find how many terms of the following series are needed to make the given sums.

a 5 + 8 + 11 + 14 + … = 670

b 3 + 8 + 13 + 18 + … = 1575

c 64 + 62 + 60 + … = 0

d 34 + 30 + 26 + 22 + … = 112

Hint Set the expression for S_n equal to the total and solve the resulting equation to find n.

(P) 3 Find the sum of the first 50 even numbers.

(P) 4 Find the least number of terms for the sum of $7 + 12 + 17 + 22 + 27 + \dots$ to exceed 1000.

(P) 5 The first term of an arithmetic series is 4. The sum to 20 terms is -15. Find, in any order, the common difference and the 20th term.

(P) 6 The sum of the first three terms of an arithmetic series is 12. If the 20th term is -32, find the first term and the common difference.

(P) 7 Prove that the sum of the first 50 natural numbers is 1275.

Problem-solving

Use the same method as Example 4.

(P) 8 Show that the sum of the first $2n$ natural numbers is $n(2n + 1)$.

(P) 9 Prove that the sum of the first n odd numbers is n^2.

(E/P) 10 The fifth term of an arithmetic series is 33. The tenth term is 68. The sum of the first n terms is 2225.

a Show that $7n^2 + 3n - 4450 = 0$. **(4 marks)**

b Hence find the value of n. **(1 mark)**

(E/P) 11 An arithmetic series is given by $(k + 1) + (2k + 3) + (3k + 5) + \dots + 303$

a Find the number of terms in the series in terms of k. **(1 mark)**

b Show that the sum of the series is given by $\dfrac{152k + 46208}{k + 2}$ **(3 marks)**

c Given that $S_n = 2568$, find the value of k. **(1 mark)**

(E/P) 12 **a** Calculate the sum of all the multiples of 3 from 3 to 99 inclusive,

$$3 + 6 + 9 + \dots + 99$$ **(3 marks)**

b In the arithmetic series

$$4p + 8p + 12p + \dots + 400$$

where p is a positive integer and a factor of 100,

i find, in terms of p, an expression for the number of terms in this series.

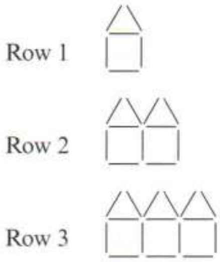
ii Show that the sum of this series is $200 + \dfrac{20\,000}{p}$ **(4 marks)**

c Find, in terms of p, the 80th term of the arithmetic sequence

$$(3p + 2), (5p + 3), (7p + 4), \dots,$$

giving your answer in its simplest form. **(2 marks)**

E/P **13** Joanna has some sticks that are all of the same length.
She arranges them in shapes as shown opposite and has made the following 3 rows of patterns.



She notices that 6 sticks are required to make the single pentagon in the first row, 11 sticks in the second row and for the third row she needs 16 sticks.

a Find an expression, in terms of n, for the number of sticks required to make a similar arrangement of n pentagons in the nth row. **(3 marks)**

Joanna continues to make pentagons following the same pattern. She continues until she has completed 10 rows.

b Find the total number of sticks Joanna uses in making these 10 rows. **(3 marks)**

Joanna started with 1029 sticks. Given that Joanna continues the pattern to complete k rows but does not have enough sticks to complete the $(k + 1)$th row:

c show that k satisfies $(5k - 98)(k + 21) \leqslant 0$ **(4 marks)**

d find the value of k. **(2 marks)**

Challenge

An arithmetic sequence has nth term $u_n = \ln 9 + (n - 1) \ln 3$. Show that the sum of the first n terms $= a \ln 3^{n^2+3n}$ where a is a rational number to be found.

3.3 Geometric sequences

- **A geometric sequence has a common ratio between consecutive terms.**

Notation A geometric sequence is sometimes called a **geometric progression**.

To get from one term to the next you **multiply** by the common ratio.

$2, 4, 8, 16$ ($\times 2$, $\times 2$, $\times 2$)

This is a geometric sequence with common ratio 2. This sequence is increasing.

$\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}$ ($\times \frac{1}{3}$, $\times \frac{1}{3}$, $\times \frac{1}{3}$)

This is a geometric sequence with common ratio $\frac{1}{3}$. This sequence is decreasing but will never get to zero.

Notation A geometric sequence with a common ratio $|r| < 1$ converges. This means it tends to a certain value. You call the value the **limit** of the sequence.

$5, -10, 20, -40, 80$ ($\times(-2)$, $\times(-2)$, $\times(-2)$, $\times(-2)$)

Here the common ratio is -2. The sequence alternates between positive and negative terms.

Notation An **alternating sequence** is a sequence in which terms are alternately positive and negative.

- **The formula for the nth term of a geometric sequence is:**

$$u_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

Example 8

Find the **i** 10th and **ii** nth terms in the following geometric sequences:

a 3, 6, 12, 24, …

b 40, −20, 10, −5, …

a 3, 6, 12, 24, …

For this sequence $a = 3$ and $r = \frac{6}{3} = 2$.

i 10th term $= 3 \times 2^9$
$= 3 \times 512$
$= 1536$

For the 10th term use ar^{n-1} with $a = 3$, $r = 2$ and $n = 10$.

ii nth term $= 3 \times 2^{n-1}$

For the nth term use ar^{n-1} with $a = 3$ and $r = 2$.

b 40, −20, 10, −5, …

For this sequence $a = 40$ and $r = -\frac{20}{40} = -\frac{1}{2}$

i 10th term $= 40 \times \left(-\frac{1}{2}\right)^9$
$= 40 \times -\frac{1}{512}$
$= -\frac{5}{64}$

Use ar^{n-1} with $a = 40$, $r = -\frac{1}{2}$ and $n = 10$.

ii nth term $= 40 \times \left(-\frac{1}{2}\right)^{n-1}$

Use ar^{n-1} with $a = 40$, $r = -\frac{1}{2}$ and $n = n$.

$= 5 \times 8 \times \left(-\frac{1}{2}\right)^{n-1}$
$= 5 \times 2^3 \times \left(-\frac{1}{2}\right)^{n-1}$
$= (-1)^{n-1} \times \frac{5}{2^{n-4}}$

Use laws of indices $\frac{x^m}{x^n} = x^{(m-n)} = \frac{1}{x^{n-m}}$

So $2^3 \times \frac{1}{2^{n-1}} = \frac{1}{2^{n-1-3}}$

Example 9

The 2nd term of a geometric sequence is 4 and the 4th term is 8. Given that the common ratio is positive, find the exact value of the 11th term in the sequence.

Problem-solving

You can use the general term of a geometric sequence to write two equations. Solve these simultaneously to find a and r, then find the 11th term in the sequence.

nth term $= ar^{n-1}$, so the 2nd term is ar, and the 4th term is ar^3

$ar = 4$ (1)
$ar^3 = 8$ (2)

Dividing equation (2) by equation (1):

$$\frac{ar^3}{ar} = \frac{8}{4}$$

$r^2 = 2$

$r = \sqrt{2}$

You are told in the question that $r > 0$ so use the positive square root.

Substituting back into equation (1):

$a\sqrt{2} = 4$

$a = \frac{4}{\sqrt{2}}$

$a = 2\sqrt{2}$

nth term $= ar^{n-1}$, so

11th term $= (2\sqrt{2})(\sqrt{2})^{10}$

$= 64\sqrt{2}$

Rationalise the denominator.

Simplify your answer as much as possible.

Example 10

The numbers 3, x and $(x + 6)$ form the first three terms of a geometric sequence with all positive terms. Find:

a the possible values of x, **b** the 10th term of the sequence.

a $\frac{u_2}{u_1} = \frac{u_3}{u_2}$

$\frac{x}{3} = \frac{x+6}{x}$

$x^2 = 3(x + 6)$

$x^2 = 3x + 18$

$x^2 - 3x - 18 = 0$

$(x - 6)(x + 3) = 0$

$x = 6$ or -3

So x is either 6 or -3, but there are no negative terms so $x = 6$.

b 10th term $= ar^9$

$= 3 \times 2^9$

$= 3 \times 512$

$= 1536$

The 10th term is 1536.

Problem-solving

In a geometric sequence the ratio between consecutive terms is the same, so $\frac{u_2}{u_1} = \frac{u_3}{u_2}$

Simplify the algebraic fraction to form a quadratic equation. ← Year 1, Section 3.2

Factorise.

If there are no negative terms then -3 cannot be an answer.

Use the formula nth term $= ar^{n-1}$ with $n = 10$, $a = 3$ and $r = \frac{x}{3} = \frac{6}{3} = 2$.

Example 11

What is the first term in the geometric progression 3, 6, 12, 24, … to exceed 1 million?

nth term $= ar^{n-1}$

$= 3 \times 2^{n-1}$

We want nth term $> 1\,000\,000$

Problem-solving

Determine a and r, then write an inequality using the formula for the general term of a geometric sequence.

Sequence has $a = 3$ and $r = 2$.

So $3 \times 2^{n-1} > 1000000$

$$2^{n-1} > \frac{1000000}{3}$$

Divide by 3.

$$\log 2^{n-1} > \log\left(\frac{1000000}{3}\right)$$

To solve this inequality take logs of both sides.

$$(n-1)\log 2 > \log\left(\frac{1000000}{3}\right)$$

$\log a^n = n \log a$ ← Year 1, Chapter 14

$$n - 1 > \frac{\log\left(\frac{1000000}{3}\right)}{\log(2)}$$

Divide by log 2.

$$n - 1 > 18.35 \text{ (2 d.p.)}$$

$$n > 19.35$$

$$n \geqslant 20$$

n has to be an integer.

The 20th term is the first to exceed 1000000.

Online Use your calculator to check your answer.

Exercise 3C

1 Which of the following are geometric sequences? For the ones that are, give the value of the common ratio, r.

a 1, 2, 4, 8, 16, 32, …
b 2, 5, 8, 11, 14, …
c 40, 36, 32, 28, …
d 2, 6, 18, 54, 162, …
e 10, 5, 2.5, 1.25, …
f 5, −5, 5, −5, 5, …
g 3, 3, 3, 3, 3, 3, 3, …
h 4, −1, 0.25, −0.0625, …

2 Continue the following geometric sequences for three more terms.

a 5, 15, 45, …
b 4, −8, 16, …
c 60, 30, 15, …
d $1, \frac{1}{4}, \frac{1}{16}, \ldots$
e $1, p, p^2, \ldots$
f $x, -2x^2, 4x^3, \ldots$

(P) **3** If 3, x and 9 are the first three terms of a geometric sequence, find:

a the exact value of x,
b the exact value of the 4th term.

Problem-solving

In a geometric sequence the common ratio can be calculated by $\frac{u_2}{u_1}$ or $\frac{u_3}{u_2}$

4 Find the sixth and nth terms of the following geometric sequences.

a 2, 6, 18, 54, …
b 100, 50, 25, 12.5, …
c 1, −2, 4, −8, …
d 1, 1.1, 1.21, 1.331, …

5 The nth term of a geometric sequence is 2×5^n. Find the first and 5th terms.

6 The sixth term of a geometric sequence is 32 and the 3rd term is 4. Find the first term and the common ratio.

7 A geometric sequence has first term 4 and third term 1. Find the two possible values of the 6th term.

E/P 8 The first three terms of a geometric sequence are given by $8 - x$, $2x$, and x^2 respectively where $x > 0$.

a Show that $x^3 - 4x^2 = 0$. **(2 marks)**

b Find the value of the 20th term. **(3 marks)**

c State, with a reason, whether 4096 is a term in the sequence. **(1 mark)**

E/P 9 A geometric sequence has first term 200 and a common ratio p where $p > 0$. The 6th term of the sequence is 40.

a Show that p satisfies the equation $5 \log p + \log 5 = 0$. **(3 marks)**

b Hence or otherwise, find the value of p correct to 3 significant figures. **(1 mark)**

P 10 A geometric sequence has first term 4 and fourth term 108. Find the smallest value of k for which the kth term in this sequence exceeds 500 000.

P 11 The first three terms of a geometric sequence are 9, 36, 144. State, with a reason, whether 383 616 is a term in the sequence.

Problem-solving

Determine the values of a and r and find the general term of the sequence. Set the number given equal to the general term and solve to find n. If n is an integer, then the number is in the sequence.

P 12 The first three terms of a geometric sequence are 3, −12, 48. State, with a reason, whether 49 152 is a term in the sequence.

P 13 Find which term in the geometric progression 3, 12, 48, …is the first to exceed 1 000 000.

3.4 Geometric series

A geometric **series** is the sum of the terms of a geometric **sequence**. 3, 6, 12, 24, … is a geometric sequence. 3 + 6 + 12 + 24 + … is a geometric series.

- **The sum of the first n terms of a geometric series is given by the formula**

$$S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1$$

or $$S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

where a is the first term and r is the common ratio.

Hint These two formulae are equivalent. It is often easier to use the first one if $r < 1$ and the second one if $r > 1$.

Example 12

A geometric series has first term a and common difference r. Prove that the sum of the first n terms of this series is given by $S_n = \frac{a(1 - r^n)}{1 - r}$

Let $S_n = a + ar + ar^2 + ar^3 + \dots ar^{n-2} + ar^{n-1}$ (1)

$rS_n = ar + ar^2 + ar^3 + \dots ar^{n-1} + ar^n$ (2)

Multiply by r.

(1) − (2) gives $S_n - rS_n = a - ar^n$

Subtract rS_n from S_n.

$S_n(1 - r) = a(1 - r^n)$

Take out the common factor.

$S_n = \frac{a(1 - r^n)}{1 - r}$

Divide by $(1 - r)$.

Problem-solving

You need to learn this proof for your exam.

Example 13

Find the sums of the following geometric series.

a $2 + 6 + 18 + 54 + \dots$ (for 10 terms)

b $1024 - 512 + 256 - 128 + \dots + 1$

a Series is

$2 + 6 + 18 + 54 + \dots$ (for 10 terms)

As in all questions, write down what is given.

So $a = 2$, $r = \frac{6}{2} = 3$ and $n = 10$

So $S_{10} = \frac{2(3^{10} - 1)}{3 - 1} = 59\,048$

When $r > 1$ it is easier to use the formula $S_n = \frac{a(r^n - 1)}{r - 1}$

b Series is

$1024 - 512 + 256 - 128 + \dots + 1$

So $a = 1024$, $r = -\frac{512}{1024} = -\frac{1}{2}$

and the nth term $= 1$

$1024\left(-\frac{1}{2}\right)^{n-1} = 1$

First solve $ar^{n-1} = 1$ to find n.

$(-2)^{n-1} = 1024$

$2^{n-1} = 1024$

$(-2)^{n-1} = (-1)^{n-1}(2^{n-1}) = 1024$, so $(-1)^{n-1}$ must be positive and $2^{n-1} = 1024$.

$n - 1 = \frac{\log 1024}{\log 2}$

$n - 1 = 10$

$1024 = 2^{10}$

$n = 11$

So $S_n = \frac{1024\left(1 - \left(-\frac{1}{2}\right)^{11}\right)}{1 - \left(-\frac{1}{2}\right)}$

When $r < 1$, it is easier to use the formula $S_n = \frac{a(1 - r^n)}{1 - r}$

$= \frac{1024\left(1 + \frac{1}{2048}\right)}{1 + \frac{1}{2}}$

$= \frac{1024.5}{\frac{3}{2}} = 683$

Example 14

Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \ldots$ to n terms exceeds 2 000 000.

Sum to n terms is $S_n = \frac{1(2^n - 1)}{2 - 1}$
$= 2^n - 1$

If this is to exceed 2 000 000 then

$S_n > 2\,000\,000$
$2^n - 1 > 2\,000\,000$
$2^n > 2\,000\,001$ — Add 1.
$n \log 2 > \log(2\,000\,001)$ — Use laws of logs: $\log a^n = n \log a$.
$n > \frac{\log(2\,000\,001)}{\log(2)}$
$n > 20.9$

It needs 21 terms to exceed 2 000 000. — Round n up to the nearest integer.

Problem-solving

Determine the values of a and r, then use the formula for the sum of a geometric series to form an inequality.

Exercise 3D

1 Find the sum of the following geometric series (to 3 d.p. if necessary).

a $1 + 2 + 4 + 8 + \ldots$ (8 terms)

b $32 + 16 + 8 + \ldots$ (10 terms)

c $\frac{2}{3} + \frac{4}{15} + \frac{8}{75} + \ldots + \frac{256}{234\,375}$

d $4 - 12 + 36 - 108 + \ldots$ (6 terms)

e $729 - 243 + 81 - \ldots - \frac{1}{3}$

f $-\frac{5}{2} + \frac{5}{4} - \frac{5}{8} \ldots - \frac{5}{32\,768}$

2 A geometric series has first three terms $3 + 1.2 + 0.48\ldots$ Evaluate S_{10}, giving your answer to 4 d.p.

3 A geometric series has first term 5 and common ratio $\frac{2}{3}$. Find the value of S_8.

(P) **4** The sum of the first three terms of a geometric series is 30.5. If the first term is 8, find possible values of r.

(P) **5** Find the least value of n such that the sum $3 + 6 + 12 + 24 + \ldots$ to n terms exceeds 1.5 million.

(P) **6** Find the least value of n such that the sum $5 + 4.5 + 4.05 + \ldots$ to n terms exceeds 45.

(E) **7** A geometric series has first term 25 and common ratio $\frac{3}{5}$

Given that the sum to k terms of the series is greater than 61,

a show that $k > \frac{\log(0.024)}{\log(0.6)}$ **(4 marks)**

b find the smallest possible value of k. **(1 mark)**

E/P **8** A geometric series has first term a and common ratio r.
The sum of the first two terms of the series is 4.48.
The sum of the first four terms is 5.1968. Find the two possible values of r. **(4 marks)**

Problem-solving One value will be positive and one value will be negative.

E/P **9** The first term of a geometric series is a and the common ratio is $\sqrt{3}$.
Show that $S_{10} = 121a(\sqrt{3} + 1)$. **(4 marks)**

E/P **10** A geometric series has first term a and common ratio 2. A different geometric series has first term b and common ratio 3. Given that the sum of the first 4 terms of both series is the same, show that $a = \frac{8}{3}b$. **(4 marks)**

E/P **11** The first three terms of a geometric series are $(k - 6)$, k, $(2k + 5)$, where k is a positive constant.

a Show that $k^2 - 7k - 30 = 0$. **(4 marks)**

b Hence find the value of k. **(2 marks)**

c Find the common ratio of this series. **(1 mark)**

d Find the sum of the first 10 terms of this series, giving your answer to the nearest whole number. **(2 marks)**

3.5 Sum to infinity

You can work out the sum of the first n terms of a geometric series. As n tends to infinity, the sum of the series is called the **sum to infinity**.

Notation You can write the sum to infinity of a geometric series as S_∞

Consider the sum of the first n terms of the geometric series $2 + 4 + 8 + 16 + \dots$

The terms of this series are getting larger, so as n tends to infinity, S_n also tends to infinity. This is called a **divergent** series.

Now consider the sum of the first n terms of the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

The terms of this series are getting smaller. As n tends to infinity, S_n gets closer and closer to a finite value, S_∞. This is called a **convergent** series.

- **A geometric series is convergent if and only if $|r| < 1$, where r is the common ratio.**

Hint You can also write this condition as $-1 < r < 1$.

The sum of the first n terms of a geometric series is given by $S_n = \frac{a(1 - r^n)}{1 - r}$

When $|r| < 1$, $\lim_{n\to\infty}\left(\frac{a(1 - r^n)}{1 - r}\right) = \frac{a}{1 - r}$

This is because $r^n \to 0$ as $n \to \infty$.

Notation $\lim_{n\to\infty}$ means 'the limit as n tends to ∞'. You can't evaluate the expression when n is ∞, but as n gets larger the expression gets closer to a fixed (or **limiting**) value.

- **The sum to infinity of a convergent geometric series is given by $S_\infty = \frac{a}{1 - r}$**

Watch out You can only use this formula for a convergent series, i.e. when $|r| < 1$.

Example 15

The fourth term of a geometric series is 1.08 and the seventh term is 0.233 28.

a Show that this series is convergent.

b Find the sum to infinity of the series.

a $ar^3 = 1.08$ (1)

$ar^6 = 0.233\,28$ (2)

Dividing (2) by (1):

$\frac{ar^6}{ar^3} = \frac{0.233\,28}{1.08}$

$r^3 = 0.216$

$r = 0.6$

The series is convergent as $|r| = 0.6 < 1$.

b Substituting the value of r^3 into equation (1) to find a

$0.216a = 1.08$

$a = \frac{1.08}{0.216}$

$a = 5$

Substituting into S_∞ formula:

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{5}{1-0.6}$

$S_\infty = 12.5$

Use the nth term of a geometric sequence ar^{n-1} to write down 2 simultaneous equations.

Divide equation (2) by equation (1) to eliminate a.

Problem-solving

To show that a series is convergent you need to find r, then state that the series is convergent if $|r| < 1$.

Example 16

For a geometric series with first term a and common ratio r, $S_4 = 15$ and $S_\infty = 16$.

a Find the possible values of r.

b Given that all the terms in the series are positive, find the value of a.

a $\frac{a(1-r^4)}{1-r} = 15$ (1)

$\frac{a}{1-r} = 16$ (2)

$16(1-r^4) = 15$

$1 - r^4 = \frac{15}{16}$

$r^4 = \frac{1}{16}$

$r = \pm\frac{1}{2}$

$S_4 = 15$ so use the formula $S_n = \frac{a(1-r^n)}{1-r}$ with $n = 4$.

$S_\infty = 16$ so use the formula $S_\infty = \frac{a}{1-r}$ with $S_\infty = 16$.

Solve equations simultaneously.

Replace $\frac{a}{1-r}$ by 16 in equation (1).

Take the 4th root of $\frac{1}{16}$

b As all terms are positive, $r = +\frac{1}{2}$

$$\frac{a}{1 - \frac{1}{2}} = 16$$

Substitute $r = \frac{1}{2}$ into equation (2) to fnd a.

$16(1 - \frac{1}{2}) = a$

$a = 8$

The first term in the series is 8.

Exercise 3E

1 For each of the following series:

i state, with a reason, whether the series is convergent.

ii If the series is convergent, find the sum to infinity.

a $1 + 0.1 + 0.01 + 0.001 + \dots$

b $1 + 2 + 4 + 8 + 16 + \dots$

c $10 - 5 + 2.5 - 1.25 + \dots$

d $2 + 6 + 10 + 14 + \dots$

e $1 + 1 + 1 + 1 + 1 + \dots$

f $3 + 1 + \frac{1}{3} + \frac{1}{9} + \dots$

g $0.4 + 0.8 + 1.2 + 1.6 + \dots$

h $9 + 8.1 + 7.29 + 6.561 + \dots$

2 A geometric series has first term 10 and sum to infinity 30. Find the common ratio.

3 A geometric series has first term −5 and sum to infinity −3. Find the common ratio.

4 A geometric series has sum to infinity 60 and common ratio $\frac{2}{3}$. Find the first term.

5 A geometric series has common ratio $-\frac{1}{3}$ and $S_\infty = 10$. Find the first term.

(P) 6 Find the fraction equal to the recurring decimal $0.\dot{2}\dot{3}$.

Hint $0.\dot{2}\dot{3} = \frac{23}{100} + \frac{23}{10\,000} + \frac{23}{1\,000\,000} + \dots$

7 For a geometric series $a + ar + ar^2 + \dots$, $S_3 = 9$ and $S_\infty = 8$, find the values of a and r.

(E/P) 8 Given that the geometric series $1 - 2x + 4x^2 - 8x^3 + \dots$ is convergent,

a find the range of possible values of x **(3 marks)**

b find an expression for S_∞ in terms of x. **(1 mark)**

(E/P) 9 In a convergent geometric series the common ratio is r and the first term is 2.
Given that $S_\infty = 16 \times S_3$,

a find the value of the common ratio, giving your answer to 4 significant figures **(3 marks)**

b find the value of the fourth term. **(2 marks)**

(E/P) 10 The first term of a geometric series is 30. The sum to infinity of the series is 240.

a Show that the common ratio, r, is $\frac{7}{8}$ **(2 marks)**

b Find to 3 significant figures, the difference between the 4th and 5th terms. **(2 marks)**

c Calculate the sum of the first 4 terms, giving your answer to 3 significant figures. **(2 marks)**

The sum of the first n terms of the series is greater than 180.

d Calculate the smallest possible value of n. **(4 marks)**

E/P **11** A geometric series has first term a and common ratio r. The second term of the series is $\frac{15}{8}$ and the sum to infinity of the series is 8.

a Show that $64r^2 - 64r + 15 = 0$. **(4 marks)**

b Find the two possible values of r. **(2 marks)**

c Find the corresponding two possible values of a. **(2 marks)**

Given that r takes the smaller of its two possible values,

d find the smallest value of n for which S_n exceeds 7.99. **(2 marks)**

Challenge

The sum to infinity of a geometric series is 7. A second series is formed by squaring every term in the first geometric series.

a Show that the second series is also geometric.

b Given that the sum to infinity of the second series is 35, show that the common ratio of the original series is $\frac{1}{6}$

3.6 Sigma notation

- **The Greek capital letter 'sigma' is used to signify a sum. You write it as $\sum$. You write limits on the top and bottom to show which terms you are summing.**

This tells you that are summing the expression in brackets with $r = 1, r = 2, \ldots$ up to $r = 5$.

$$\sum_{r=1}^{5}(2r - 3) = -1 + 1 + 3 + 5 + 7$$

Substitute $r = 1, r = 2, r = 3, r = 4, r = 5$ to find the five terms in this arithmetic series.

Look at the limits carefully: they don't have to start at 1.

$$\sum_{r=3}^{7}(5 \times 2^r) = 40 + 80 + 160 + 320 + 640$$

To find the terms in this geometric series, you substitute $r = 3, r = 4, r = 5, r = 6, r = 7$.

You can write some results that you already know using sigma notation:

- $\sum_{r=1}^{n} 1 = n$
- $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

Hint $\sum_{r=1}^{n} 1 = \underbrace{1 + 1 + 1 + \ldots + 1}_{n \text{ times}}$

Example 17

Calculate $\sum_{r=1}^{20}(4r + 1)$

$\sum_{r=1}^{20}(4r + 1) = 5 + 9 + 13 + \ldots + 81$

$a = 5, d = 4$ and $n = 20$

Problem-solving

Substitute $r = 1, 2$, etc. to find the terms in the series.

$S = \frac{n}{2}(2a + (n - 1)d)$

$= \frac{20}{2}(2 \times 5 + (20 - 1)4)$

$= 10(10 + 19 \times 4)$

$= 10 \times 86$

$= 860$

Use the formula for the sum to n terms of an arithmetic series.
Substitute $a = 5$, $d = 4$ and $n = 20$ into $S = \frac{n}{2}(2a + (n - 1)d)$.

Online Check your answer by using your calculator to calculate the sum of the series.

Example 18

Find the values of:

a $\sum_{k=1}^{12} 5 \times 3^{k-1}$ **b** $\sum_{k=5}^{12} 5 \times 3^{k-1}$

a $\sum_{k=1}^{12} 5 \times 3^{k-1}$

$= 5 + 15 + 45 + \ldots$

$a = 5$, $r = 3$ and $n = 12$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{12} = \frac{5(3^{12} - 1)}{3 - 1}$

$S_{12} = 1328600$

Substitute $k = 1$, $k = 2$ and so on to write out the first few terms of the series. This will help you determine the correct values for a, r and n.

Since $r > 1$ use the formula $S_n = \frac{a(r^n - 1)}{r - 1}$ and substitute in $a = 5$, $r = 3$ and $n = 12$.

b $\sum_{k=5}^{12} 5 \times 3^{k-1} = \sum_{k=1}^{12} 5 \times 3^{k-1} - \sum_{k=1}^{4} 5 \times 3^{k-1}$

$S_{12} = 1328600$

$S_4 = \frac{5(3^4 - 1)}{3 - 1} = 200$

$\sum_{k=5}^{12} 5 \times 3^{k-1} = 1328600 - 200 = 1328400$

Problem-solving

When we are summing series from k to n, we can consider the sum of the terms from 1 to n and subtract the terms from 1 to $k - 1$.

Exercise 3F

1 For each series:

i write out every term in the series

ii hence find the value of the sum.

a $\sum_{r=1}^{5}(3r + 1)$ **b** $\sum_{r=1}^{6} 3r^2$ **c** $\sum_{r=1}^{5} \sin(90r°)$ **d** $\sum_{r=5}^{8} 2\left(-\frac{1}{3}\right)^r$

2 For each series:

i write the series using sigma notation

ii evaluate the sum.

a $2 + 4 + 6 + 8$ **b** $2 + 6 + 18 + 54 + 162$ **c** $6 + 4.5 + 3 + 1.5 + 0 - 1.5$

3 For each series:

i find the number of terms in the series

ii write the series using sigma notation.

a $7 + 13 + 19 + \ldots + 157$ **b** $\frac{1}{3} + \frac{2}{15} + \frac{4}{75} + \ldots + \frac{64}{46875}$ **c** $8 - 1 - 10 - 19 \ldots - 127$

4 Evaluate:

a $\sum_{r=1}^{20}(7 - 2r)$ **b** $\sum_{r=1}^{10} 3 \times 4^r$ **c** $\sum_{r=1}^{100}(2r - 8)$ **d** $\sum_{r=1}^{\infty} 7\left(-\frac{1}{3}\right)^r$

(P) **5** Evaluate:

a $\sum_{r=9}^{30}\left(5r - \frac{1}{2}\right)$ **b** $\sum_{r=100}^{200}(3r + 4)$ **c** $\sum_{r=5}^{100} 3 \times 0.5^r$ **d** $\sum_{i=5}^{100} 1$

Problem-solving

$\sum_{r=k}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{k-1} u_r$

(P) **6** Show that $\sum_{r=1}^{n} 2r = n + n^2$.

(P) **7** Show that $\sum_{r=1}^{n} 2r - \sum_{r=1}^{n}(2r - 1) = n$.

8 Find in terms of k:

a $\sum_{r=1}^{k} 4(-2)^r$ **b** $\sum_{r=1}^{k}(100 - 2r)$ **c** $\sum_{r=10}^{k}(7 - 2r)$

(P) **9** Find the value of $\sum_{r=10}^{\infty} 200 \times \left(\frac{1}{4}\right)^r$

(E/P) **10** Given that $\sum_{r=1}^{k}(8 + 3r) = 377$,

a show that $(3k + 58)(k - 13) = 0$ **(3 marks)**

b hence find the value of k. **(1 mark)**

(E/P) **11** Given that $\sum_{r=1}^{k} 2 \times 3^r = 59\,046$,

a show that $k = \frac{\log 19\,683}{\log 3}$ **(4 marks)**

b For this value of k, calculate $\sum_{r=k+1}^{13} 2 \times 3^r$. **(3 marks)**

(E/P) **12** A geometric series is given by $1 + 3x + 9x^2 + \ldots$

The series is convergent.

a Write down the range of possible values of x. **(3 marks)**

Given that $\sum_{r=1}^{\infty}(3x)^{r-1} = 2$

b calculate the value of x. **(3 marks)**

Challenge

Given that $\sum_{r=1}^{10}(a + (r - 1)d) = \sum_{r=11}^{14}(a + (r - 1)d)$, show that $d = 6a$.

3.7 Recurrence relations

If you know the rule to get from one term to the next in a sequence you can write a recurrence relation.

- **A recurrence relation of the form $u_{n+1} = f(u_n)$ defines each term of a sequence as a function of the previous term.**

For example, the recurrence relation $u_{n+1} = 2u_n + 3$, $u_1 = 6$ produces the following sequence:

6, 15, 33, 69, … $u_2 = 2u_1 + 3 = 2(6) + 3 = 15$

Watch out In order to generate a sequence from a recurrence relation like this, you need to know the **first term** of the sequence.

Example 19

Find the first four terms of the following sequences.

a $u_{n+1} = u_n + 4$, $u_1 = 7$ **b** $u_{n+1} = u_n + 4$, $u_1 = 5$

a $u_{n+1} = u_n + 4$, $u_1 = 7$
Substituting $n = 1$, $u_2 = u_1 + 4 = 7 + 4 = 11$.
Substituting $n = 2$, $u_3 = u_2 + 4 = 11 + 4 = 15$.
Substituting $n = 3$, $u_4 = u_3 + 4 = 15 + 4 = 19$.
Sequence is 7, 11, 15, 19, …

Substitute $n = 1$, 2 and 3. Use u_1 to find u_2, and then u_2 to find u_3.

b $u_{n+1} = u_n + 4$, $u_1 = 5$
Substituting $n = 1$, $u_2 = u_1 + 4 = 5 + 4 = 9$.
Substituting $n = 2$, $u_3 = u_2 + 4 = 9 + 4 = 13$.
Substituting $n = 3$, $u_4 = u_3 + 4 = 13 + 4 = 17$.
Sequence is 5, 9, 13, 17, …

This is the same recurrence formula. It produces a different sequence because u_1 is different.

Example 20

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = p$$
$$a_{n+1} = (a_n)^2 - 1, n \geqslant 1$$

where $p < 0$.

a Show that $a_3 = p^4 - 2p^2$.

b Given that $a_2 = 0$, find the value of p.

c Find $\sum_{r=1}^{200} a_r$

d Write down the value of a_{199}

a $a_1 = p$
$a_2 = (a_1)^2 - 1 = p^2 - 1$
$a_3 = (a_2)^2 - 1$
$= (p^2 - 1)^2 - 1$
$= p^4 - 2p^2 + 1 - 1$
$= p^4 - 2p^2$

Use $a_2 = (a_1)^2 - 1$ and substitute $a_1 = p$.

Now substitute the expression for a_2 to find a_3.

b $p^2 - 1 = 0$
$p^2 = 1$
$p = \pm 1$ but since $p < 0$ is given, $p = -1$

Set the expression for a_2 equal to zero and solve.

c $a_1 = -1$, $a_2 = 0$, $a_3 = -1$ series alternates between -1 and 0

In 200 terms, there will be one hundred -1s and one hundred 0s.

$$\sum_{r=1}^{200} a_r = -100$$

Since this is a recurrence relation, we can see that the sequence is going to alternate between -1 and 0. The first 200 terms will have one hundred -1s and one hundred 0s.

d $a_{199} = -1$ as 199 is odd

Problem-solving

For an alternating series, consider the sums of the odd and even terms separately. Write the first few terms of the series. The odd terms are -1 and the even terms are 0. Only the odd terms contribute to the sum.

Exercise 3G

1 Find the first four terms of the following recurrence relationships.

a $u_{n+1} = u_n + 3$, $u_1 = 1$
b $u_{n+1} = u_n - 5$, $u_1 = 9$
c $u_{n+1} = 2u_n$, $u_1 = 3$
d $u_{n+1} = 2u_n + 1$, $u_1 = 2$
e $u_{n+1} = \frac{u_n}{2}$, $u_1 = 10$
f $u_{n+1} = (u_n)^2 - 1$, $u_1 = 2$

2 Suggest possible recurrence relationships for the following sequences. (Remember to state the first term.)

a 3, 5, 7, 9, …
b 20, 17, 14, 11, …
c 1, 2, 4, 8, …
d 100, 25, 6.25, 1.5625, …
e 1, −1, 1, −1, 1, …
f 3, 7, 15, 31, …
g 0, 1, 2, 5, 26, …
h 26, 14, 8, 5, 3.5, …

3 By writing down the first four terms or otherwise, find the recurrence formula that defines the following sequences:

a $u_n = 2n - 1$
b $u_n = 3n + 2$
c $u_n = n + 2$
d $u_n = \frac{n+1}{2}$
e $u_n = n^2$
f $u_n = 3^n - 1$

(P) **4** A sequence of terms is defined for $n \geqslant 1$ by the recurrence relation $u_{n+1} = ku_n + 2$, where k is a constant. Given that $u_1 = 3$,

a find an expression in terms of k for u_2

b hence find an expression for u_3

Given that $u_3 = 42$:

c find the possible values of k.

(E/P) **5** A sequence is defined for $n \geqslant 1$ by the recurrence relation

$$u_{n+1} = pu_n + q, \ u_1 = 2$$

Given that $u_2 = -1$ and $u_3 = 11$, find the values of p and q. **(4 marks)**

E/P **6** A sequence is given by

$$x_1 = 2$$
$$x_{n+1} = x_n(p - 3x_n)$$

where p is an integer.

a Show that $x_3 = -10p^2 + 132p - 432$. **(2 marks)**

b Given that $x_3 = -288$ find the value of p. **(1 mark)**

c Hence find the value of x_4. **(1 mark)**

E/P **7** A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k$$
$$a_{n+1} = 4a_n + 5$$

a Find a_3 in terms of k. **(2 marks)**

b Show that $\sum_{r=1}^{4} a_r$ is a multiple of 5. **(3 marks)**

- **A sequence is increasing if $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.**
- **A sequence is decreasing if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.**
- **A sequence is periodic if the terms repeat in a cycle. For a periodic sequence there is an integer k such that $u_{n+k} = u_n$ for all $n \in \mathbb{N}$. The value k is called the order of the sequence.**

Notation The order of a periodic sequence is sometimes called its **period**.

- 2, 3, 4, 5… is an increasing sequence.
- −3, −6, −12, −24… is a decreasing sequence.
- −2, 1, −2, 1, −2, 1 is a periodic sequence with a period of 2.
- 1, −2, 3, −4, 5, −6… is not increasing, decreasing or periodic.

Example 21

For each sequence:

i state whether the sequence is increasing, decreasing, or periodic.

ii If the sequence is periodic, write down its order.

a $u_{n+1} = u_n + 3,\ u_1 = 7$ **b** $u_{n+1} = (u_n)^2,\ u_1 = \frac{1}{2}$ **c** $u_n = \sin(90n°)$

a 7, 10, 13, 16, …

$u_{n+1} > u_n$ for all n, so the sequence is increasing.

Write out the first few terms of the sequence.

State the condition for an increasing sequence. You could also write that $k + 3 > k$ for all numbers k.

b $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots$

$u_{n+1} < u_n$ for all n, so the sequence is decreasing.

The starting value in the sequence makes a big difference. Because $u_1 < 1$ the numbers get smaller every time you square them.

c $u_1 = \sin(90°) = 1$

$u_2 = \sin(180°) = 0$

$u_3 = \sin(270°) = -1$

$u_4 = \sin(360°) = 0$

$u_5 = \sin(450°) = 1$

$u_6 = \sin(540°) = 0$

$u_7 = \sin(630°) = -1$

The sequence is periodic, with order 4.

To find u_1 substitute $n = 1$ into $\sin(90n°)$.

Watch out Although every even term of the sequence is 0, the period is not 2 because the odd terms alternate between 1 and –1.

The graph of $y = \sin x$ repeats with period 360°. So $\sin(x + 360°) = \sin x$. ← Year 1, Chapter 9

Exercise 3H

1 For each sequence:

i state whether the sequence is increasing, decreasing, or periodic.

ii If the sequence is periodic, write down its order.

a 2, 5, 8, 11, 14 **b** $3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ **c** 5, 9, 15, 23, 33 **d** 3, –3, 3, –3, 3

2 For each sequence:

i write down the first 5 terms of the sequence

ii state whether the sequence is increasing, decreasing, or periodic.

iii If the sequence is periodic, write down its order.

a $u_n = 20 - 3n$ **b** $u_n = 2^{n-1}$ **c** $u_n = \cos(180n°)$

d $u_n = (-1)^n$ **e** $u_{n+1} = u_n - 5, u_1 = 20$ **f** $u_{n+1} = 5 - u_n, u_1 = 20$

g $u_{n+1} = \frac{2}{3}u_n, u_1 = k$

3 The sequence of numbers $u_1, u_2, u_3, \ldots$ is given by $u_{n+1} = ku_n, u_1 = 5$.
Find the range of values of k for which the sequence is strictly decreasing.

E/P **4** The sequence with recurrence relation $u_{k+1} = pu_k + q, u_1 = 5$, where p is a constant and $q = 13$, is periodic with order 2.
Find the value of p. **(5 marks)**

E/P **5** A sequence has nth term $a_n = \cos(90n°), n \geqslant 1$.

a Find the order of the sequence. **(1 mark)**

b Find $\sum_{r=1}^{444} a_r$ **(2 marks)**

Challenge

The sequence of numbers $u_1, u_2, u_3, \ldots$ is given by $u_{n+2} = \frac{1 + u_{n+1}}{u_n}$, $u_1 = a$, $u_2 = b$, where a and b are positive integers.

a Show that the sequence is periodic for all positive a and b.

b State the order of the sequence.

Hint Each term in this sequence is defined in terms of the **previous two** terms.

3.8 Modelling with series

You can model real-life situations with series. For example if a person's salary increases by the same percentage every year, their salaries each year would form a **geometric sequence** and the amount they had been paid in total over n years would be modelled by the corresponding **geometric series**.

Example 22

Bruce starts a new company. In year 1 his profits will be £20 000. He predicts his profits to increase by £5000 each year, so that his profits in year 2 are modelled to be £25 000, in year 3 £30 000, and so on. He predicts this will continue until he reaches annual profits of £100 000. He then models his annual profits to remain at £100 000.

a Calculate the profits for Bruce's business in the first 20 years.

b State one reason why this may not be a suitable model.

c Bruce's financial advisor says the yearly profits are likely to increase by 5% per annum. Using this model, calculate the profits for Bruce's business in the first 20 years.

a Year 1 $P = 20\,000$, Year 2 $P = 25\,000$, Year 3 $P = 30\,000$

This is an arithmetic sequence as the difference is constant.

$a = 20\,000$, $d = 5000$

Write down the values of a and d.

$u_n = a + (n-1)d$

$100\,000 = 20\,000 + (n-1)(5000)$

Use the nth term of an arithmetic sequence to work out n when profits will reach £100 000.

$100\,000 = 20\,000 + 5000n - 5000$

$85\,000 = 5000n$

$n = \frac{85\,000}{5000} = 17$

Solve to find n.

$S_{17} = \frac{17}{2}(2(20\,000) + (17-1)(5000))$

$= 1\,020\,000$

You want to know how much he made overall in the 17 years, so find the sum of the arithmetic series.

$S_{20} = 1\,020\,000 + 3(100\,000)$

$= 1\,320\,000$

In the 18th, 19th and 20th year he makes £100 000 each year, so add on 3 × £100 000 to the sum of the first 17 years.

So Bruce's total profit after 20 years is £1 320 000.

b It is unlikely that Bruce's profits will increase by exactly the same amount each year.

c $a = £20\,000$, $r = 1.05$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{20} = \frac{20\,000(1.05^{20} - 1)}{1.05 - 1}$$

$S_{20} = 661\,319.08$

So Bruce's total profit after 20 years is £661 319.08.

This is a geometric series, as to get the next term you multiply the current term by 1.05.

Use the formula for the sum of the first n terms of a geometric series $S_n = \frac{a(r^n - 1)}{r - 1}$

Example 23

A piece of A4 paper is folded in half repeatedly. The thickness of the A4 paper is 0.5 mm.

a Work out the thickness of the paper after four folds.

b Work out the thickness of the paper after 20 folds.

c State one reason why this might be an unrealistic model.

a $a = 0.5$ mm, $r = 2$

After 4 folds:

$u_5 = 0.5 \times 2^4 = 8$ mm

b After 20 folds

$u_{21} = 0.5 \times 2^{20} = 524\,288$ mm

c It is impossible to fold the paper that many times so the model is unrealistic.

This is a geometric sequence, as each time we fold the paper the thickness doubles.

Since u_1 is the first term (after 0 folds), u_2 is after 1 fold, so u_5 is after 4 folds.

Problem-solving

If you have to comment on the validity of a model, always refer to the context given in the question.

Exercise 3I

1 An investor puts £4000 in an account. Every month thereafter she deposits another £200. How much money in total will she have invested at the start of **a** the 10th month and **b** the mth month?

Hint At the start of the 6th month she will have only made 5 deposits of £200.

(P) 2 Carol starts a new job on a salary of £20 000. She is given an annual wage rise of £500 at the end of every year until she reaches her maximum salary of £25 000. Find the total amount she earns (assuming no other rises), **a** in the first 10 years, **b** over 15 years and **c** state one reason why this may be an unsuitable model.

Problem-solving

This is an arithmetic series with $a = 20\,000$ and $d = 500$. First find how many years it will take her to reach her maximum salary.

(P) **3** James decides to save some money during the six-week holiday. He saves 1p on the first day, 2p on the second, 3p on the third and so on.

a How much will he have at the end of the holiday (42 days)?

b If he carried on, how long would it be before he has saved £100?

(P) **4** A population of ants is growing at a rate of 10% a year. If there were 200 ants in the initial population, write down the number of ants after:

a 1 year **b** 2 years **c** 3 years **d** 10 years.

Problem-solving

This is a geometric sequence. $a = 200$ and $r = 1.1$

(P) **5** A motorcycle has four gears. The maximum speed in bottom gear is $40\,\text{km h}^{-1}$ and the maximum speed in top gear is $120\,\text{km h}^{-1}$. Given that the maximum speeds in each successive gear form a geometric progression, calculate, in km h^{-1} to one decimal place, the maximum speeds in the two intermediate gears.

(P) **6** A car depreciates in value by 15% a year. After 3 years it is worth £11 054.25.

a What was the car's initial price?

b When will the car's value first be less than £5000?

Problem-solving

Use your answer to part **a** to write an inequality, then solve it using logarithms.

(E) **7** A salesman is paid commission of £10 per week for each life insurance policy that he has sold. Each week he sells one new policy so that he is paid £10 commission in the first week, £20 commission in the second week, £30 commission in the third week and so on.

a Find his total commission in the first year of 52 weeks. **(2 marks)**

b In the second year the commission increases to £11 per week on new policies sold, although it remains at £10 per week for policies sold in the first year. He continues to sell one policy per week. Show that he is paid £542 in the second week of his second year. **(3 marks)**

c Find the total commission paid to him in the second year. **(2 marks)**

(E) **8** Prospectors are drilling for oil. The cost of drilling to a depth of 50 m is £500. To drill a further 50 m costs £640 and, hence, the total cost of drilling to a depth of 100 m is £1140. Each subsequent extra depth of 50 m costs £140 more to drill than the previous 50 m.

a Show that the cost of drilling to a depth of 500 m is £11 300. **(3 marks)**

b The total sum of money available for drilling is £76 000. Find, to the nearest 50 m, the greatest depth that can be drilled. **(3 marks)**

(E) **9** Each year, for 40 years, Anne will pay money into a savings scheme. In the first year she pays in £500. Her payments then increase by £50 each year, so that she pays in £550 in the second year, £600 in the third year, and so on.

a Find the amount that Anne will pay in the 40th year. **(2 marks)**

b Find the total amount that Anne will pay in over the 40 years. **(3 marks)**

c Over the same 40 years, Brian will also pay money into the savings scheme. In the first year he pays in £890 and his payments then increase by £d each year. Given that Brian and Anne will pay in exactly the same amount over the 40 years, find the value of d. **(4 marks)**

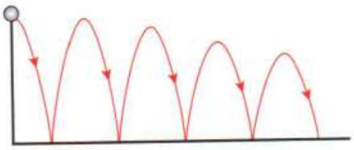
(P) **10** A virus is spreading such that the number of people infected increases by 4% a day. Initially 100 people were diagnosed with the virus. How many days will it be before 1000 are infected?

(P) **11** I invest £A in the bank at a rate of interest of 3.5% per annum. How long will it be before I double my money?

(P) **12** The fish in a particular area of the North Sea are being reduced by 6% each year due to overfishing. How long will it be before the fish stocks are halved?

(P) **13** The man who invented the game of chess was asked to name his reward. He asked for 1 grain of corn to be placed on the first square of his chessboard, 2 on the second, 4 on the third and so on until all 64 squares were covered. He then said he would like as many grains of corn as the chessboard carried. How many grains of corn did he claim as his prize?

(P) **14** A ball is dropped from a height of 10 m. It bounces to a height of 7 m and continues to bounce. Subsequent heights to which it bounces follow a geometric sequence. Find out:



a how high it will bounce after the fourth bounce

b the total vertical distance travelled up to the point when the ball hits the ground for the sixth time.

(P) **15** Richard is doing a sponsored cycle. He plans to cycle 1000 miles over a number of days. He plans to cycle 10 miles on day 1 and increase the distance by 10% a day.

a How long will it take Richard to complete the challenge?

b What will be his greatest number of miles completed in a day?

(P) **16** A savings scheme is offering a rate of interest of 3.5% per annum for the lifetime of the plan. Alan wants to save up £20 000. He works out that he can afford to save £500 every year, which he will deposit on 1 January. If interest is paid on 31 December, how many years will it be before he has saved up his £20 000?

Mixed exercise 3

(E/P) **1** A geometric series has third term 27 and sixth term 8.

a Show that the common ratio of the series is $\frac{2}{3}$ **(2 marks)**

b Find the first term of the series. **(2 marks)**

c Find the sum to infinity of the series. **(2 marks)**

d Find the difference between the sum of the first 10 terms of the series and the sum to infinity. Give your answer to 3 significant figures. **(2 marks)**

(E/P) **2** The second term of a geometric series is 80 and the fifth term of the series is 5.12.

a Show that the common ratio of the series is 0.4. **(2 marks)**

Calculate:

b the first term of the series **(2 marks)**

c the sum to infinity of the series, giving your answer as an exact fraction **(1 mark)**

d the difference between the sum to infinity of the series and the sum of the first 14 terms of the series, giving your answer in the form $a \times 10^n$, where $1 \leqslant a < 10$ and n is an integer. **(2 marks)**

E/P **3** The nth term of a sequence is u_n, where $u_n = 95\left(\frac{4}{5}\right)^n$, $n = 1, 2, 3, \ldots$

a Find the values of u_1 and u_2. **(2 marks)**

Giving your answers to 3 significant figures, calculate:

b the value of u_{21} **(1 mark)**

c $\sum_{n=1}^{15} u_n$ **(2 marks)**

d the sum to infinity of the series whose first term is u_1 and whose nth term is u_n. **(1 mark)**

E/P **4** A sequence of numbers $u_1, u_2, \ldots, u_n, \ldots$ is given by the formula $u_n = 3\left(\frac{2}{3}\right)^n - 1$ where n is a positive integer.

a Find the values of u_1, u_2 and u_3. **(2 marks)**

b Show that $\sum_{n=1}^{15} u_n = -9.014$ to 4 significant figures. **(2 marks)**

c Prove that $u_{n+1} = \dfrac{2u_n - 1}{3}$ **(2 marks)**

E/P **5** The third and fourth terms of a geometric series are 6.4 and 5.12 respectively. Find:

a the common ratio of the series, **(2 marks)**

b the first term of the series, **(2 marks)**

c the sum to infinity of the series. **(2 marks)**

d Calculate the difference between the sum to infinity of the series and the sum of the first 25 terms of the series. **(2 marks)**

E/P **6** The price of a car depreciates by 15% per annum. Its price when new is £20 000.

a Find the value of the car after 5 years. **(2 marks)**

b Find when the value will be less than £4000. **(3 marks)**

E/P **7** The first three terms of a geometric series are $p(3q + 1)$, $p(2q + 2)$ and $p(2q - 1)$, where p and q are non-zero constants.

a Show that one possible value of q is 5 and find the other possible value. **(2 marks)**

b Given that $q = 5$, and the sum to infinity of the series is 896, find the sum of the first 12 terms of the series. Give your answer to 2 decimal places. **(4 marks)**

E/P **8** **a** Prove that the sum of the first n terms in an arithmetic series is

$$S = \frac{n}{2}(2a + (n - 1)d)$$

where a = first term and d = common difference. **(3 marks)**

b Use this to find the sum of the first 100 natural numbers. **(2 marks)**

E/P **9** Find the least value of n for which $\sum_{r=1}^{n}(4r - 3) > 2000$. **(2 marks)**

E/P **10** The sum of the first two terms of an arithmetic series is 47.
The thirtieth term of this series is –62. Find:

a the first term of the series and the common difference **(3 marks)**

b the sum of the first 60 terms of the series. **(2 marks)**

E/P **11** **a** Find the sum of the integers which are divisible by 3 and lie between 1 and 400. **(3 marks)**

b Hence, or otherwise, find the sum of the integers, from 1 to 400 inclusive, which are **not** divisible by 3. **(2 marks)**

E/P **12** A polygon has 10 sides. The lengths of the sides, starting with the shortest, form an arithmetic series. The perimeter of the polygon is 675 cm and the length of the longest side is twice that of the shortest side. Find the length of the shortest side of the polygon. **(4 marks)**

E/P **13** Prove that the sum of the first $2n$ multiples of 4 is $4n(2n + 1)$. **(4 marks)**

E/P **14** A sequence of numbers is defined, for $n \geqslant 1$, by the recurrence relation $u_{n+1} = ku_n - 4$, where k is a constant. Given that $u_1 = 2$:

a find expressions, in terms of k, for u_2 and u_3. **(2 marks)**

b Given also that $u_3 = 26$, use algebra to find the possible values of k. **(2 marks)**

E/P **15** The fifth term of an arithmetic series is 14 and the sum of the first three terms of the series is –3.

a Use algebra to show that the first term of the series is –6 and calculate the common difference of the series. **(3 marks)**

b Given that the nth term of the series is greater than 282, find the least possible value of n. **(3 marks)**

E/P **16** The fourth term of an arithmetic series is $3k$, where k is a constant, and the sum of the first six terms of the series is $7k + 9$.

a Show that the first term of the series is $9 - 8k$. **(3 marks)**

b Find an expression for the common difference of the series in terms of k. **(2 marks)**

Given that the seventh term of the series is 12, calculate:

c the value of k **(2 marks)**

d the sum of the first 20 terms of the series. **(2 marks)**

E/P **17** A sequence is defined by the recurrence relation

$$a_{n+1} = \frac{1}{a_n},\ a_1 = p$$

a Show that the sequence is periodic and state its order. **(2 marks)**

b Find $\sum_{r=1}^{1000} a_n$ in terms of p. **(2 marks)**

E/P **18** A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k$$

$$a_{n+1} = 2a_n + 6,\ n \geqslant 1$$

where k is an integer.

a Given that the sequence is increasing for the first 3 terms, show that $k > p$, where p is an integer to be found. **(2 marks)**

b Find a_4 in terms of k. **(2 marks)**

c Show that $\sum_{r=1}^{4} a_r$ is divisible by 3. **(3 marks)**

E/P **19** The first term of a geometric series is 130. The sum to infinity of the series is 650.

a Show that the common ratio, r, is $\frac{4}{5}$ **(3 marks)**

b Find, to 2 decimal places, the difference between the 7th and 8th terms. **(2 marks)**

c Calculate the sum of the first 7 terms. **(2 marks)**

The sum of the first n terms of the series is greater than 600.

d Show that $n > \dfrac{-\log 13}{\log 0.8}$ **(4 marks)**

E/P **20** The adult population of a town is 25 000 at the beginning of 2012.
A model predicts that the adult population of the town will increase by 2% each year, forming a geometric sequence.

a Show that the predicted population at the beginning of 2014 is 26 010. **(1 mark)**

The model predicts that after n years, the population will first exceed 50 000.

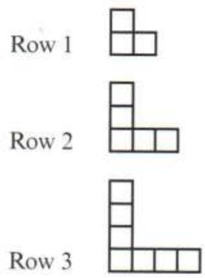
b Show that $n > \dfrac{\log 2}{\log 1.02}$ **(3 marks)**

c Find the year in which the population first exceeds 50 000. **(2 marks)**

d Every member of the adult population is modelled to visit the doctor once per year. Calculate the number of appointments the doctor has from the beginning of 2012 to the end of 2019. **(4 marks)**

e Give a reason why this model for doctors' appointments may not be appropriate. **(1 mark)**

E/P **21** Kyle is making some patterns out of squares. He has made 3 rows so far.



a Find an expression, in terms of n, for the number of squares required to make a similar arrangement in the nth row. **(3 marks)**

b Kyle counts the number of squares used to make the pattern in the kth row. He counts 301 squares. Write down the value of k. **(1 mark)**

c In the first q rows, Kyle uses a total of p squares.

i Show that $q^2 + 2q - p = 0$. **(3 marks)**

ii Given that $p > 1520$, find the minimum number of rows that Kyle makes. **(3 marks)**

E/P **22** A convergent geometric series has first term a and common ratio r. The second term of the series is -3 and the sum to infinity of the series is 6.75.

a Show that $27r^2 - 27r - 12 = 0$. **(4 marks)**

b Given that the series is convergent, find the value of r. **(2 marks)**

c Find the sum of the first 5 terms of the series, giving your answer to 2 decimal places. **(3 marks)**

Challenge

A sequence is defined by the recurrence relation $u_{n+2} = 5u_{n+1} - 6u_n$.

a Prove that any sequence of the form $u_n = p \times 3^n + q \times 2^n$, where p and q are constants, satisfies this recurrence relation.

Given that $u_1 = 5$ and $u_2 = 12$,

b find an expression for u_n in terms of n only.

c Hence determine the number of digits in u_{100}.

Summary of key points

1 In an **arithmetic sequence**, the difference between consecutive terms is constant.

2 The formula for the nth term of an arithmetic sequence is $\boldsymbol{u_n = a + (n-1)d}$, where a is the first term and d is the common difference.

3 An arithmetic series is the sum of the terms of an arithmetic sequence.

The sum of the first n terms of an arithmetic series is given by $\boldsymbol{S_n = \frac{n}{2}(2a + (n-1)d)}$, where where a is the first term and d is the common difference.

You can also write this formula as $\boldsymbol{S_n = \frac{n}{2}(a + l)}$, where l is the last term.

4 A **geometric sequence** has a **common ratio** between consecutive terms.

5 The formula for the nth term of a geometric sequence is $\boldsymbol{u_n = ar^{n-1}}$, where a is the first term and r is the common ratio.

6 The sum of the first n terms of a geometric series is given by

$$S_n = \frac{a(1-r^n)}{1-r}, r \neq 1 \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1}, r \neq 1$$

where a is the first term and r is the common ratio.

7 A geometric series is convergent if and only if $|r| < 1$, where r is the common ratio.

The **sum to infinity** of a convergent geometric series is given by $\boldsymbol{S_\infty = \frac{a}{1-r}}$

8 The Greek capital letter 'sigma' is used to signify a sum. You write it as $\sum$. You write limits on the top and bottom to show which terms you are summing.

9 A recurrence relation of the form $u_{n+1} = \mathrm{f}(u_n)$ defines each term of a sequence as a function of the previous term.

10 A sequence is **increasing** if $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.

A sequence is **decreasing** if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.

A sequence is **periodic** if the terms repeat in a cycle. For a periodic sequence there is an integer k such that $u_{n+k} = u_n$ for all $n \in \mathbb{N}$. The value k is called the **order** of the sequence.