3 Equations and inequalities

Objectives

After completing this chapter you should be able to:

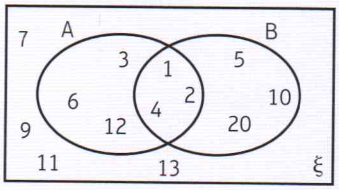
- Solve linear simultaneous equations using elimination or substitution → pages 39 – 40
- Solve simultaneous equations: one linear and one quadratic → pages 41 – 42
- Interpret algebraic solutions of equations graphically → pages 42 – 45
- Solve linear inequalities → pages 46 – 48
- Solve quadratic inequalities → pages 48 – 51
- Interpret inequalities graphically → pages 51 – 53
- Represent linear and quadratic inequalities graphically → pages 53 – 55

Prior knowledge check

1 A = {factors of 12}

B = {factors of 20}

Write down the numbers in each of these sets:



a $A \cap B$ **b** $(A \cup B)'$

← GCSE Mathematics

2 Simplify these expressions.

a $\sqrt{75}$ **b** $\dfrac{2\sqrt{45} + 3\sqrt{32}}{6}$

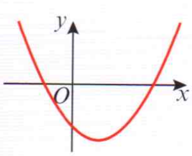
← Section 1.5

3 Match the equations to the correct graph. Label the points of intersection with the axes and the coordinates of the turning point.

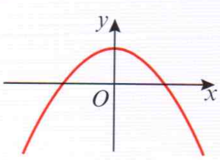
a $y = 9 - x^2$ **b** $y = (x - 2)^2 + 4$

c $y = (x - 7)(2x + 5)$

i



ii



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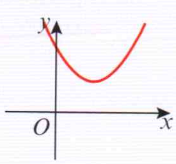
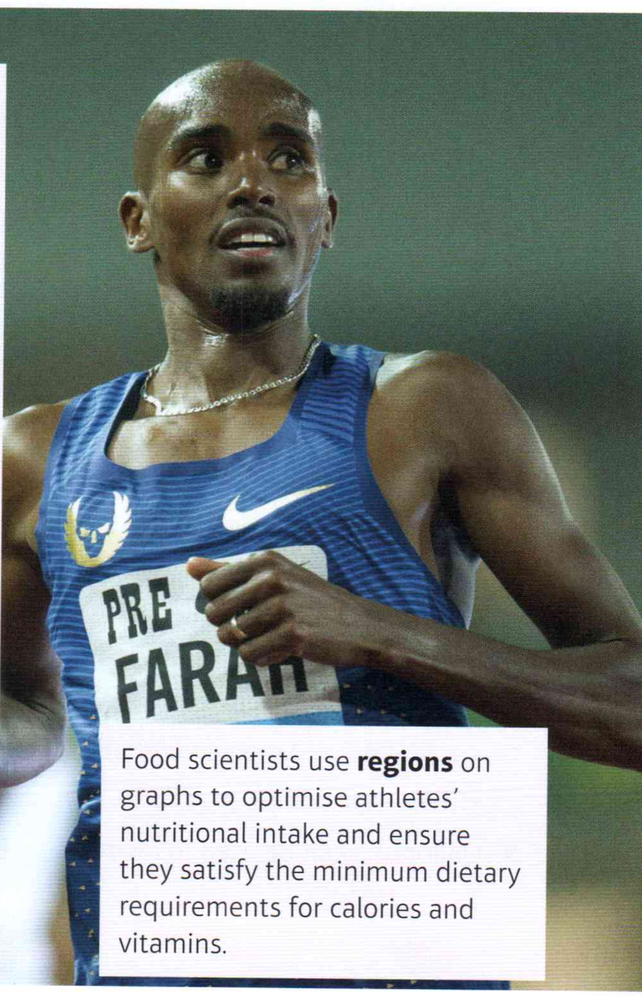


← Section 2.4



Food scientists use **regions** on graphs to optimise athletes' nutritional intake and ensure they satisfy the minimum dietary requirements for calories and vitamins.

3.1 Linear simultaneous equations

Linear simultaneous equations in two unknowns have **one set of values** that will make a pair of equations true at the same time.

The solution to this pair of simultaneous equations is $x = 5$, $y = 2$

$x + 3y = 11$ (1) — $5 + 3(2) = 5 + 6 = 11$ ✓

$4x - 5y = 10$ (2) — $4(5) - 5(2) = 20 - 10 = 10$ ✓

- **Linear simultaneous equations can be solved using elimination or substitution.**

Example 1

Solve the simultaneous equations:

a $2x + 3y = 8$
$3x - y = 23$

b $4x - 5y = 4$
$6x + 2y = 25$

a

$2x + 3y = 8$ (1)

$3x - y = 23$ (2) — First look for a way to eliminate x or y. Multiply equation (2) by 3 to get $3y$ in each equation.

$9x - 3y = 69$ (3) — Number this new equation (3).

$11x = 77$ — Then add equations (1) and (3), since the $3y$ terms have different signs and y will be eliminated.

$x = 7$

$14 + 3y = 8$ — Substitute $x = 7$ into equation (1) to find y.

$3y = 8 - 14$

$y = -2$

The solution is $x = 7$, $y = -2$. — Remember to check your solution by substituting into equation (2). $3(7) - (-2) = 21 + 2 = 23$ ✓

Note that you could also multiply equation (1) by 3 and equation (2) by 2 to get $6x$ in both equations. You could then subtract to eliminate x.

b

$4x - 5y = 4$ (1) — Multiply equation (1) by 3 and multiply equation (2) by 2 to get $12x$ in each equation.

$6x + 2y = 25$ (2)

$12x - 15y = 12$ (3)

$12x + 4y = 50$ (4)

$-19y = -38$ — Subtract, since the $12x$ terms have the same sign (both positive).

$y = 2$

$4x - 10 = 4$ — Substitute $y = 2$ into equation (1) to find x.

$4x = 14$

$x = 3\frac{1}{2}$

The solution is $x = 3\frac{1}{2}$, $y = 2$.

Example 2

Solve the simultaneous equations:

$2x - y = 1$
$4x + 2y = -30$

$2x - y = 1 \quad (1)$
$4x + 2y = -30 \quad (2)$

$y = 2x - 1$ — Rearrange an equation, in this case equation (1), to get either $x = \ldots$ or $y = \ldots$ (here $y = \ldots$).

$4x + 2(2x - 1) = -30$ — Substitute this into the other equation (here into equation (2) in place of y).

$4x + 4x - 2 = -30$
$8x = -28$
$x = -3\frac{1}{2}$ — Solve for x.

$y = 2(-3\frac{1}{2}) - 1 = -8$ — Substitute $x = -3\frac{1}{2}$ into equation (1) to find the value of y.

The solution is $x = -3\frac{1}{2}$, $y = -8$ — Remember to check your solution in equation (2). $4(-3.5) + 2(-8) = -14 - 16 = -30$ ✓

Exercise 3A

1 Solve these simultaneous equations by elimination:

a $2x - y = 6$, $4x + 3y = 22$
b $7x + 3y = 16$, $2x + 9y = 29$
c $5x + 2y = 6$, $3x - 10y = 26$
d $2x - y = 12$, $6x + 2y = 21$
e $3x - 2y = -6$, $6x + 3y = 2$
f $3x + 8y = 33$, $6x = 3 + 5y$

2 Solve these simultaneous equations by substitution:

a $x + 3y = 11$, $4x - 7y = 6$
b $4x - 3y = 40$, $2x + y = 5$
c $3x - y = 7$, $10x + 3y = -2$
d $2y = 2x - 3$, $3y = x - 1$

3 Solve these simultaneous equations:

a $3x - 2y + 5 = 0$, $5(x + y) = 6(x + 1)$
b $\frac{x - 2y}{3} = 4$, $2x + 3y + 4 = 0$
c $3y = 5(x - 2)$, $3(x - 1) + y + 4 = 0$

Hint First rearrange both equations into the same form e.g. $ax + by = c$.

E/P **4** $3x + ky = 8$
$x - 2ky = 5$
are simultaneous equations where k is a constant.

a Show that $x = 3$. **(3 marks)**
b Given that $y = \frac{1}{2}$ determine the value of k. **(1 mark)**

Problem-solving k is a constant, so it has the same value in both equations.

E/P **5** $2x - py = 5$
$4x + 5y + q = 0$
are simultaneous equations where p and q are constants.
The solution to this pair of simultaneous equations is $x = q$, $y = -1$.
Find the value of p and the value of q. **(5 marks)**

3.2 Quadratic simultaneous equations

You need to be able to solve simultaneous equations where one equation is linear and one is quadratic.

To solve simultaneous equations involving one linear equation and one quadratic equation, you need to use a substitution method from the linear equation into the quadratic equation.

- **Simultaneous equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the solutions are paired correctly.**

The solutions to this pair of simultaneous equations are $x = 4$, $y = -3$ and $x = 5.5$, $y = -1.5$.

$x - y = 7$ (1) — $4 - (-3) = 7$ ✓ and $5.5 - (-1.5) = 7$ ✓

$y^2 + xy + 2x = 5$ (2) — $(-3)^2 + (4)(-3) + 2(4) = 9 - 12 + 8 = 5$ ✓ and $(-1.5)^2 + (5.5)(-1.5) + 2(5.5) = 2.25 - 8.25 + 11 = 5$ ✓

Example 3

Solve the simultaneous equations:

$x + 2y = 3$

$x^2 + 3xy = 10$

$x + 2y = 3$ (1)

$x^2 + 3xy = 10$ (2) — The quadratic equation can contain terms involving y^2 and xy.

$x = 3 - 2y$ — Rearrange linear equation (1) to get $x = \ldots$ or $y = \ldots$ (here $x = \ldots$).

$(3 - 2y)^2 + 3y(3 - 2y) = 10$ — Substitute this into quadratic equation (2) (here in place of x).

$(3 - 2y)^2$ means $(3 - 2y)(3 - 2y)$ ← Section 1.2

$9 - 12y + 4y^2 + 9y - 6y^2 = 10$

$-2y^2 - 3y - 1 = 0$

$2y^2 + 3y + 1 = 0$

$(2y + 1)(y + 1) = 0$ — Solve for y using factorisation.

$y = -\frac{1}{2}$ or $y = -1$

So $x = 4$ or $x = 5$ — Find the corresponding x-values by substituting the y-values into linear equation (1), $x = 3 - 2y$.

Solutions are $x = 4$, $y = -\frac{1}{2}$ and $x = 5$, $y = -1$. — There are two solution pairs for x and y.

Exercise 3B

1 Solve the simultaneous equations:

a $x + y = 11$
$xy = 30$

b $2x + y = 1$
$x^2 + y^2 = 1$

c $y = 3x$
$2y^2 - xy = 15$

d $3a + b = 8$
$3a^2 + b^2 = 28$

e $2u + v = 7$
$uv = 6$

f $3x + 2y = 7$
$x^2 + y = 8$

2 Solve the simultaneous equations:

a $2x + 2y = 7$
$x^2 - 4y^2 = 8$

b $x + y = 9$
$x^2 - 3xy + 2y^2 = 0$

c $5y - 4x = 1$
$x^2 - y^2 + 5x = 41$

3 Solve the simultaneous equations, giving your answers in their simplest surd form:

a $x - y = 6$
$xy = 4$

b $2x + 3y = 13$
$x^2 + y^2 = 78$

Watch out Use brackets when you are substituting an expression into an equation.

E/P 4 Solve the simultaneous equations:

$x + y = 3$
$x^2 - 3y = 1$ **(6 marks)**

E/P 5 a By eliminating y from the equations

$y = 2 - 4x$
$3x^2 + xy + 11 = 0$

show that $x^2 - 2x - 11 = 0$. **(2 marks)**

b Hence, or otherwise, solve the simultaneous equations

$y = 2 - 4x$
$3x^2 + xy + 11 = 0$

giving your answers in the form $a \pm b\sqrt{3}$, where a and b are integers. **(5 marks)**

P 6 One pair of solutions for the simultaneous equations

$y = kx - 5$
$4x^2 - xy = 6$

is $(1, p)$ where k and p are constants.

a Find the values of k and p.

b Find the second pair of solutions for the simultaneous equations.

Problem-solving
If $(1, p)$ is a solution, then $x = 1$, $y = p$ satisfies both equations.

Challenge

$y - x = k$
$x^2 + y^2 = 4$

Given that the simultaneous equations have exactly one pair of solutions, show that

$k = \pm 2\sqrt{2}$

3.3 Simultaneous equations on graphs

You can represent the solutions of simultaneous equations graphically. As every point on a line or curve satisfies the equation of that line or curve, the points of intersection of two lines or curves satisfy both equations simultaneously.

- **The solutions to a pair of simultaneous equations represent the points of intersection of their graphs.**

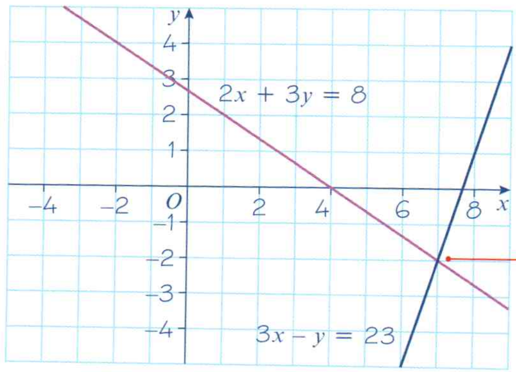
Example 4

a On the same axes, draw the graphs of:

$2x + 3y = 8$
$3x - y = 23$

b Use your graph to write down the solutions to the simultaneous equations.

a



b The solution is (7, −2) or $x = 7$, $y = -2$.

Online Find the point of intersection graphically using technology.

The point of intersection is the solution to the simultaneous equations

$2x + 3y = 8$

$3x - y = 23$

This solution matches the algebraic solution to the simultaneous equations.

Example 5

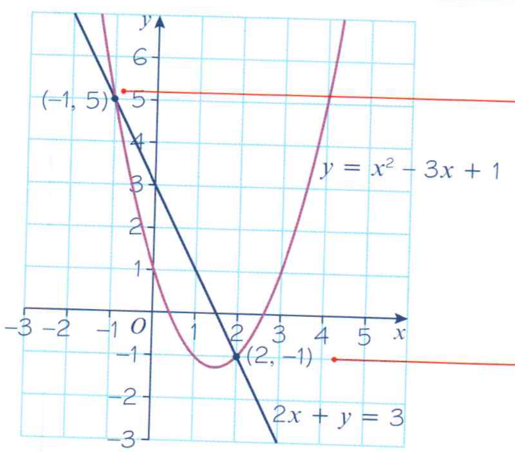
a On the same axes, draw the graphs of:

$2x + y = 3$

$y = x^2 - 3x + 1$

b Use your graph to write down the solutions to the simultaneous equations.

a



b The solutions are (−1, 5) or $x = -1$, $y = 5$ and (2, −1) or $x = 2$, $y = -1$.

There are **two** solutions. Each solution will have an x-value and a y-value.

Check your solutions by substituting into both equations.

$2(-1) + (5) = -2 + 5 = 3$ ✓ and

$5 = (-1)^2 - 3(-1) + 1 = 1 + 3 + 1 = 5$ ✓

$2(2) + (-1) = 4 - 1 = 3$ ✓ and

$-1 = (2)^2 - 3(2) + 1 = 4 - 6 + 1 = -1$ ✓

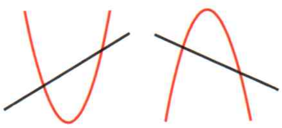
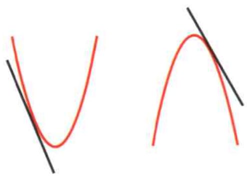
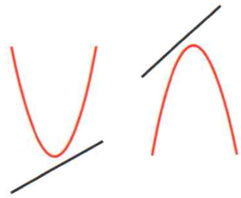
Online Plot the curve and the line using technology to find the two points of intersection.

The graph of a linear equation and the graph of a quadratic equation can either:

- intersect twice
- intersect once
- not intersect

After substituting, you can use the **discriminant** of the resulting quadratic equation to determine the number of points of intersection.

- **For a pair of simultaneous equations that produce a quadratic equation of the form $ax^2 + bx + c = 0$:**
  - $b^2 - 4ac > 0$ **two real solutions**
  - $b^2 - 4ac = 0$ **one real solution**
  - $b^2 - 4ac < 0$ **no real solutions**

Example 6

The line with equation $y = 2x + 1$ meets the curve with equation $kx^2 + 2y + (k - 2) = 0$ at exactly one point. Given that k is a positive constant

a find the value of k

b for this value of k, find the coordinates of the point of intersection.

Online Explore how the value of k affects the line and the curve using technology.

a

$$y = 2x + 1 \quad (1)$$
$$kx^2 + 2y + (k - 2) = 0 \quad (2)$$
$$kx^2 + 2(2x + 1) + (k - 2) = 0$$
$$kx^2 + 4x + 2 + k - 2 = 0$$
$$kx^2 + 4x + k = 0$$

Substitute $y = 2x + 1$ into equation (2) and simplify the quadratic equation. The resulting quadratic equation is in the form $ax^2 + bx + c = 0$ with $a = k$, $b = 4$ and $c = k$.

$$4^2 - 4 \times k \times k = 0$$
$$16 - 4k^2 = 0$$
$$k^2 - 4 = 0$$
$$(k - 2)(k + 2) = 0$$

Problem-solving

You are told that the line meets the curve at exactly one point, so use the discriminant of the resulting quadratic. There will be exactly one solution, so $b^2 - 4ac = 0$.

Factorise the quadratic to find the values of k.

$k = 2$ or $k = -2$

So $k = 2$

The solution is $k = +2$ as k is a positive constant.

b

$$2x^2 + 4x + 2 = 0$$
$$x^2 + 2x + 1 = 0$$
$$(x + 1)(x + 1) = 0$$
$$x = -1$$

Substitute $k = +2$ into the quadratic equation $kx^2 + 4x + k = 0$. Simplify and factorise to find the x-coordinate.

$$y = 2(-1) + 1 = -1$$

Substitute $x = -1$ into linear equation (1) to find the y-coordinate.

Point of intersection is $(-1, -1)$.

Check your answer by substituting into equation (2):

$$2x^2 + 2y = 0$$
$$2(-1)^2 + 2(-1) = 2 - 2 = 0 \checkmark$$

Exercise 3C

1 In each case:

i draw the graphs for each pair of equations on the same axes

ii find the coordinates of the point of intersection.

a $y = 3x - 5$
$y = 3 - x$

b $y = 2x - 7$
$y = 8 - 3x$

c $y = 3x + 2$
$3x + y + 1 = 0$

2 **a** Use graph paper to draw accurately the graphs of $2y = 2x + 11$ and $y = 2x^2 - 3x - 5$ on the same axes.

b Use your graph to find the coordinates of the points of intersection.

c Verify your solutions by substitution.

3 **a** On the same axes sketch the curve with equation $x^2 + y = 9$ and the line with equation $2x + y = 6$.

b Find the coordinates of the points of intersection.

c Verify your solutions by substitution.

4 **a** On the same axes sketch the curve with equation $y = (x - 2)^2$ and the line with equation $y = 3x - 2$.

b Find the coordinates of the point of intersection.

Hint You need to use algebra in part **b** to find the coordinates.

5 Find the coordinates of the points at which the line with equation $y = x - 4$ intersects the curve with equation $y^2 = 2x^2 - 17$.

6 Find the coordinates of the points at which the line with equation $y = 3x - 1$ intersects the curve with equation $y^2 = xy + 15$.

7 Determine the number of points of intersection for these pairs of simultaneous equations.

a $y = 6x^2 + 3x - 7$
$y = 2x + 8$

b $y = 4x^2 - 18x + 40$
$y = 10x - 9$

c $y = 3x^2 - 2x + 4$
$7x + y + 3 = 0$

8 Given the simultaneous equations

$$2x - y = 1$$
$$x^2 + 4ky + 5k = 0$$

where k is a non-zero constant

a show that $x^2 + 8kx + k = 0$. **(2 marks)**

Given that $x^2 + 8kx + k = 0$ has equal roots,

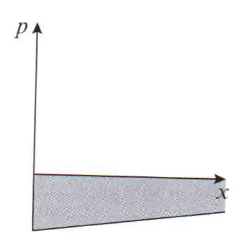
b find the value of k **(3 marks)**

c for this value of k, find the solution of the simultaneous equations. **(3 marks)**

9 A swimmer dives into a pool. Her position, p m, underwater can be modelled in relation to her horizontal distance, x m, from the point she entered the water as a quadratic equation $p = \frac{1}{2}x^2 - 3x$.

The position of the bottom of the pool can be modelled by the linear equation $p = 0.3x - 6$.

Determine whether this model predicts that the swimmer will touch the bottom of the pool.



(5 marks)

3.4 Linear inequalities

You can solve linear inequalities using similar methods to those for solving linear equations.

- **The solution of an inequality is the set of all real numbers x that make the inequality true.**

Example 7

Find the set of values of x for which:

a $5x + 9 \geqslant x + 20$ **b** $12 - 3x < 27$
c $3(x - 5) > 5 - 2(x - 8)$

a $5x + 9 \geqslant x + 20$
$4x + 9 \geqslant 20$
$4x \geqslant 11$
$x \geqslant 2.75$

Rearrange to get $x \geqslant \dots$

Notation You can write the solution to this inequality using set notation as $\{x : x \geqslant 2.75\}$. This means the set of all values x for which x is greater than or equal to 2.75.

b $12 - 3x < 27$
$-3x < 15$
$x > -5$

Subtract 12 from both sides.
Divide both sides by −3. (You therefore need to turn round the inequality sign.)
In set notation $\{x : x > -5\}$.

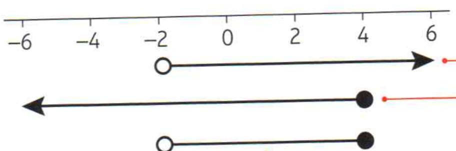
c $3(x - 5) > 5 - 2(x - 8)$
$3x - 15 > 5 - 2x + 16$
$5x > 5 + 16 + 15$
$5x > 36$
$x > 7.2$

Multiply out (note: $-2 \times -8 = +16$).
Rearrange to get $x > \dots$
In set notation $\{x : x > 7.2\}$.

You may sometimes need to find the set of values for which **two** inequalities are true together. Number lines can be useful to find your solution.

Notation In set notation
$x > -2$ **and** $x \leqslant 4$ is written $\{x : -2 < x \leqslant 4\}$
or alternatively $\{x : x > -2\} \cap \{x : x \leqslant 4\}$
$x \leqslant -1$ **or** $x > 3$ is written $\{x : x \leqslant -1\} \cup \{x : x > 3\}$

For example, in the number line below the solution set is $x > -2$ **and** $x \leqslant 4$.

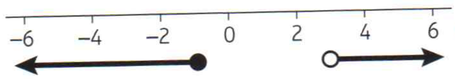


○ is used for < and > and means the end value is not included.

● is used for ⩽ and ⩾ and means the end value is included.

These are the only real values that satisfy both equalities simultaneously so the solution is $-2 < x \leqslant 4$.

Here the solution sets are $x \leqslant -1$ **or** $x > 3$.



Here there is no overlap and the two inequalities have to be written separately as $x \leqslant -1$ or $x > 3$.

Example 8

Find the set of values of x for which:

a $3x - 5 < x + 8$ and $5x > x - 8$

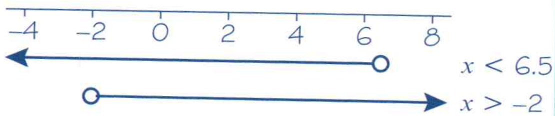
b $x - 5 > 1 - x$ or $15 - 3x > 5 + 2x$.

a $3x - 5 < x + 8$ $\quad$ $5x > x - 8$

$2x - 5 < 8$ $\quad$ $4x > -8$

$2x < 13$ $\quad$ $x > -2$

$x < 6.5$



So the required set of values is $-2 < x < 6.5$.

Draw a number line to illustrate the two inequalities.

The two sets of values overlap (intersect) where $-2 < x < 6.5$.

Notice here how this is written when x lies between two values.

In set notation this can be written as $\{x : -2 < x < 6.5\}$.

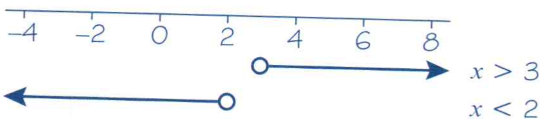
b $x - 5 > 1 - x$ $\quad$ $15 - 3x > 5 + 2x$

$2x - 5 > 1$ $\quad$ $10 - 3x > 2x$

$2x > 6$ $\quad$ $10 > 5x$

$x > 3$ $\quad$ $2 > x$

$x < 2$



The solution is $x > 3$ or $x < 2$.

Draw a number line. Note that there is no overlap between the two sets of values.

In set notation this can be written as $\{x : x < 2\} \cup \{x : x > 3\}$.

Exercise 3D

1 Find the set of values of x for which:

a $2x - 3 < 5$

b $5x + 4 \geqslant 39$

c $6x - 3 > 2x + 7$

d $5x + 6 \leqslant -12 - x$

e $15 - x > 4$

f $21 - 2x > 8 + 3x$

g $1 + x < 25 + 3x$

h $7x - 7 < 7 - 7x$

i $5 - 0.5x \geqslant 1$

j $5x + 4 > 12 - 2x$

2 Find the set of values of x for which:

a $2(x-3) \geqslant 0$ **b** $8(1-x) > x-1$ **c** $3(x+7) \leqslant 8-x$

d $2(x-3)-(x+12) < 0$ **e** $1+11(2-x) < 10(x-4)$ **f** $2(x-5) \geqslant 3(4-x)$

g $12x-3(x-3) < 45$ **h** $x-2(5+2x) < 11$ **i** $x(x-4) \geqslant x^2+2$

j $x(5-x) \geqslant 3+x-x^2$ **k** $3x+2x(x-3) \leqslant 2(5+x^2)$ **l** $x(2x-5) \leqslant \frac{4x(x+3)}{2} - 9$

3 Use set notation to describe the set of values of x for which:

a $3(x-2) > x-4$ and $4x+12 > 2x+17$

b $2x-5 < x-1$ and $7(x+1) > 23-x$

c $2x-3 > 2$ and $3(x+2) < 12+x$

d $15-x < 2(11-x)$ and $5(3x-1) > 12x+19$

e $3x+8 \leqslant 20$ and $2(3x-7) \geqslant x+6$

f $5x+3 < 9$ or $5(2x+1) > 27$

g $4(3x+7) \leqslant 20$ or $2(3x-5) \geqslant \frac{7-6x}{2}$

Challenge

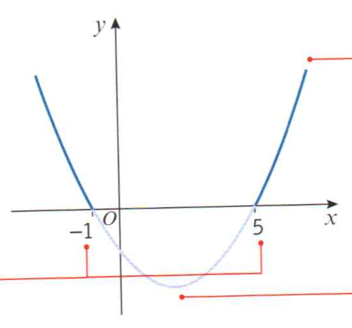
$A = \{x : 3x+5 > 2\}$ $B = \left\{x : \frac{x}{2}+1 \leqslant 3\right\}$ $C = \{x : 11 < 2x-1\}$

Given that $A \cap (B \cup C) = \{x : p < x \leqslant q\} \cup \{x : x > r\}$, find the values of p, q and r.

3.5 Quadratic inequalities

- **To solve a quadratic inequality:**
  - **Rearrange so that the right-hand side of the inequality is 0**
  - **Solve the corresponding quadratic equation to find the critical values**
  - **Sketch the graph of the quadratic function**
  - **Use your sketch to find the required set of values.**

The sketch shows the graph of $f(x) = x^2 - 4x - 5$
$= (x+1)(x-5)$



The solutions to f(x) = 0 are $x = -1$ and $x = 5$. These are called the critical values.

The solutions to the quadratic inequality $x^2 - 4x - 5 > 0$ are the x-values when the curve is **above** the x-axis (the darker part of the curve). This is when $x < -1$ or $x > 5$. In set notation the solution is $\{x : x < -1\} \cup \{x : x > 5\}$.

The solutions to the quadratic inequality $x^2 - 4x - 5 < 0$ are the x-values when the curve is **below** the x-axis (the lighter part of the curve). This is when $x > -1$ and $x < 5$ or $-1 < x < 5$. In set notation the solution is $\{x : -1 < x < 5\}$.

Example 9

Find the set of values of x for which:

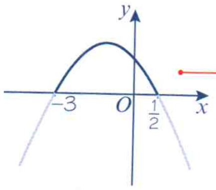
$3 - 5x - 2x^2 < 0$.

$3 - 5x - 2x^2 = 0$

$2x^2 + 5x - 3 = 0$

$(2x - 1)(x + 3) = 0$

$x = \frac{1}{2}$ or $x = -3$



So the required set of values is

$x < -3$ or $x > \frac{1}{2}$.

Quadratic equation.

Multiply by −1 (so it's easier to factorise).

$\frac{1}{2}$ and −3 are the critical values.

Draw a sketch to show the shape of the graph and the critical values.

Since the coefficient of x^2 is negative, the graph is 'upside-down ∪-shaped'. It crosses the x-axis at −3 and $\frac{1}{2}$. ← Section 2.4

$3 - 5x - 2x^2 < 0$ ($y < 0$) for the outer parts of the graph, below the x-axis, as shown by the paler parts of the curve.

In set notation this can be written as

$\{x : x < -3\} \cup \{x : x > \frac{1}{2}\}$.

Example 10

a Find the set of values of x for which $12 + 4x > x^2$.

b Hence find the set of values for which $12 + 4x > x^2$ and $5x - 3 > 2$.

a $12 + 4x > x^2$

$0 > x^2 - 4x - 12$

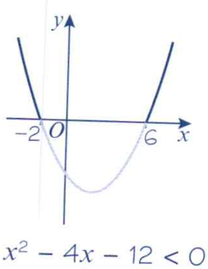
$x^2 - 4x - 12 < 0$

$x^2 - 4x - 12 = 0$

$(x + 2)(x - 6) = 0$

$x = -2$ or $x = 6$

Sketch of $y = x^2 - 4x - 12$



$x^2 - 4x - 12 < 0$

Solution: $-2 < x < 6$

You can use a table to check your solution.

$-2 < x < 6$

Use the critical values to split the real number line into sets.

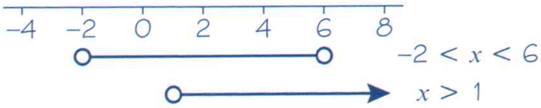
	$x < -2$	$-2 < x < 6$	$x > 6$
$x + 2$	−	+	+
$x - 6$	−	−	+
$(x + 2)(x - 6)$	+	−	+

For each set, check whether the set of values makes the value of the bracket positive or negative. For example, if $x < -2$, $(x + 2)$ is negative, $(x - 6)$ is negative, and $(x + 2)(x - 6)$ is (neg) × (neg) = positive.

In set notation the solution is $\{x : -2 < x < 6\}$.

b Solving $12 + 4x > x^2$ gives $-2 < x < 6$.

Solving $5x - 3 > 2$ gives $x > 1$.



The two sets of values overlap where $1 < x < 6$.

So the solution is $1 < x < 6$.

Problem-solving

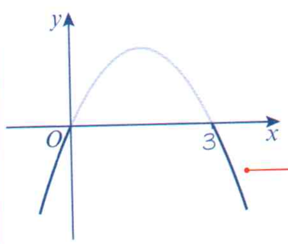
This question is easier if you represent the information in more than one way. Use a sketch graph to solve the quadratic inequality, and use a number line to combine it with the linear inequality.

In set notation this can be written as $\{x : 1 < x < 6\}$.

Example 11

Find the set of values for which $\frac{6}{x} > 2$, $x \neq 0$

$$\frac{6}{x} > 2$$
$$6x > 2x^2$$
$$6x - 2x^2 > 0$$
$$6x - 2x^2 = 0$$
$$x(6 - 2x) = 0$$
$$x = 0 \text{ or } x = 3$$



The solution is $0 < x < 3$.

Watch out x could be either positive or negative, so you can't multiply both sides of this inequality by x. Instead, multiply both sides by x^2. Because x^2 is never negative, and $x \neq 0$ so $x^2 \neq 0$, the inequality sign stays the same.

Solve the corresponding quadratic equation to find the critical values.

$x = 0$ can still be a critical value even though $x \neq 0$. But it would not be part of the solution set, even if the inequality was $\geqslant$ rather than $>$.

Sketch $y = x(6 - 2x)$. You are interested in the values of x where the graph is above the x-axis.

In set notation this can be written as $\{x : 0 < x < 3\}$.

Exercise 3E

1 Find the set of values of x for which:

a $x^2 - 11x + 24 < 0$ **b** $12 - x - x^2 > 0$ **c** $x^2 - 3x - 10 > 0$

d $x^2 + 7x + 12 \geqslant 0$ **e** $7 + 13x - 2x^2 > 0$ **f** $10 + x - 2x^2 < 0$

g $4x^2 - 8x + 3 \leqslant 0$ **h** $-2 + 7x - 3x^2 < 0$ **i** $x^2 - 9 < 0$

j $6x^2 + 11x - 10 > 0$ **k** $x^2 - 5x > 0$ **l** $2x^2 + 3x \leqslant 0$

2 Find the set of values of x for which:

a $x^2 < 10 - 3x$ **b** $11 < x^2 + 10$

c $x(3 - 2x) > 1$ **d** $x(x + 11) < 3(1 - x^2)$

3 Use set notation to describe the set of values of x for which:

a $x^2 - 7x + 10 < 0$ and $3x + 5 < 17$

b $x^2 - x - 6 > 0$ and $10 - 2x < 5$

c $4x^2 - 3x - 1 < 0$ and $4(x + 2) < 15 - (x + 7)$

d $2x^2 - x - 1 < 0$ and $14 < 3x - 2$

e $x^2 - x - 12 > 0$ and $3x + 17 > 2$

f $x^2 - 2x - 3 < 0$ and $x^2 - 3x + 2 > 0$

P **4** Given that $x \neq 0$, find the set of values of x for which:

a $\frac{2}{x} < 1$

b $5 > \frac{4}{x}$

c $\frac{1}{x} + 3 > 2$

d $6 + \frac{5}{x} > \frac{8}{x}$

e $25 > \frac{1}{x^2}$

f $\frac{6}{x^2} + \frac{7}{x} \leqslant 3$

5 **a** Find the range of values of k for which the equation $x^2 - kx + (k + 3) = 0$ has no real roots.

b Find the range of values of p for which the roots of the equation $px^2 + px - 2 = 0$ are real.

Hint The quadratic equation $ax^2 + bx + c = 0$ has real roots if $b^2 - 4ac \geqslant 0$. ← Section 2.5

E **6** Find the set of values of x for which $x^2 - 5x - 14 > 0$. **(4 marks)**

E **7** Find the set of values of x for which

a $2(3x - 1) < 4 - 3x$ **(2 marks)**

b $2x^2 - 5x - 3 < 0$ **(4 marks)**

c **both** $2(3x - 1) < 4 - 3x$ **and** $2x^2 - 5x - 3 < 0$. **(2 marks)**

P **8** Given that $x \neq 3$, find the set of values for which $\frac{5}{x - 3} < 2$. **(6 marks)**

Problem-solving

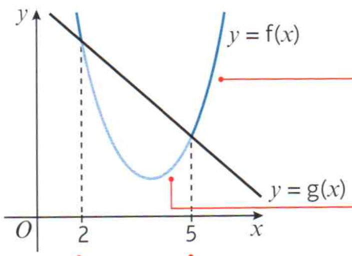
Multiply both sides of the inequality by $(x - 3)^2$.

P **9** The equation $kx^2 - 2kx + 3 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leqslant k < 3$. **(4 marks)**

3.6 Inequalities on graphs

You may be asked to interpret graphically the solutions to inequalities by considering the graphs of functions that are related to them.

- **The values of x for which the curve $y = \mathrm{f}(x)$ is below the curve $y = \mathrm{g}(x)$ satisfy the inequality $\mathrm{f}(x) < \mathrm{g}(x)$.**
- **The values of x for which the curve $y = \mathrm{f}(x)$ is above the curve $y = \mathrm{g}(x)$ satisfy the inequality $\mathrm{f}(x) > \mathrm{g}(x)$.**



f(x) is above g(x) when $x < 2$ and when $x > 5$. These values of x satisfy f(x) > g(x).

f(x) is below g(x) when $2 < x < 5$. These values of x satisfy f(x) < g(x).

The solutions to f(x) = g(x) are $x = 2$ and $x = 5$.

Example 12

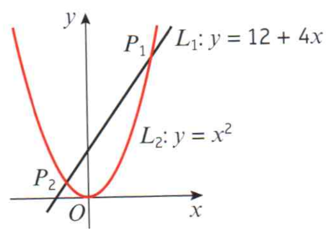
L_1 has equation $y = 12 + 4x$.

L_2 has equation $y = x^2$.

The diagram shows a sketch of L_1 and L_2 on the same axes.

a Find the coordinates of P_1 and P_2, the points of intersection.

b Hence write down the solution to the inequality $12 + 4x > x^2$.



a $x^2 = 12 + 4x$

$x^2 - 4x - 12 = 0$

$(x - 6)(x + 2) = 0$

$x = 6$ and $x = -2$

substitute into $y = x^2$

when $x = 6$, $y = 36$ $\quad P_1\ (6, 36)$

when $x = -2$, $y = 4$ $\quad P_2\ (-2, 4)$

b $12 + 4x > x^2$ when the graph of L_1 is above the graph of L_2

$-2 < x < 6$

Equate to find the points of intersection, then rearrange to solve the quadratic equation.

Factorise to find the x-coordinates at the points of intersection.

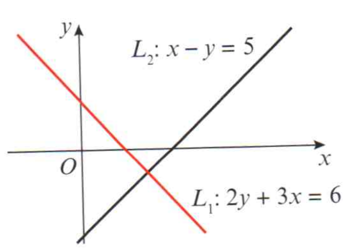
This is the range of values of x for which the graph of $y = 12 + 4x$ is above the graph of $y = x^2$ i.e. between the two points of intersection.
In set notation this is $\{x : -2 < x < 6\}$.

Exercise 3F

1 L_1 has equation $2y + 3x = 6$.
L_2 has the equation $x - y = 5$.
The diagram shows a sketch of L_1 and L_2.

a Find the coordinates of P, the point of intersection.

b Hence write down the solution to the inequality
$-\frac{3}{2}x + 3 > x - 5$.



2 For each pair of functions:

i Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes.

ii Find the coordinates of any points of intersection.

iii Write down the solutions to the inequality $f(x) \leqslant g(x)$.

a $f(x) = 3x - 7$
$g(x) = 13 - 2x$

b $f(x) = 8 - 5x$
$g(x) = 14 - 3x$

c $f(x) = x^2 + 5$
$g(x) = 5 - 2x$

d $f(x) = 3 - x^2$
$g(x) = 2x - 12$

e $f(x) = x^2 - 5$
$g(x) = 7x + 13$

f $f(x) = 7 - x^2$
$g(x) = 2x - 8$

(P) **3** Find the set of values of x for which the curve with equation $y = f(x)$ is below the line with equation $y = g(x)$.

a $f(x) = 3x^2 - 2x - 1$
$g(x) = x + 5$

b $f(x) = 2x^2 - 4x + 1$
$g(x) = 3x - 2$

c $f(x) = 5x - 2x^2 - 4$
$g(x) = -2x - 1$

d $f(x) = \frac{2}{x}, x \neq 0$
$g(x) = 1$

e $f(x) = \frac{3}{x^2} - \frac{4}{x}, x \neq 0$
$g(x) = -1$

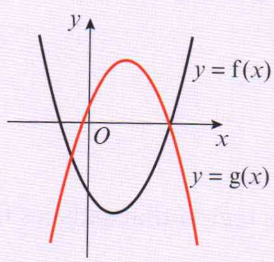
f $f(x) = \frac{2}{x + 1}, x \neq -1$
$g(x) = 8$

Challenge

The sketch shows the graphs of
$f(x) = x^2 - 4x - 12$
$g(x) = 6 + 5x - x^2$

a Find the coordinates of the points of intersection.

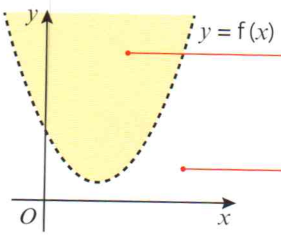
b Find the set of values of x for which $f(x) < g(x)$. Give your answer in set notation.



3.7 Regions

You can use shading on graphs to identify regions that satisfy linear and quadratic inequalities.

- **$y < f(x)$ represents the points on the coordinate grid below the curve $y = f(x)$.**
- **$y > f(x)$ represents the points on the coordinate grid above the curve $y = f(x)$.**



All the shaded points in this region satisfy the inequality $y > f(x)$.

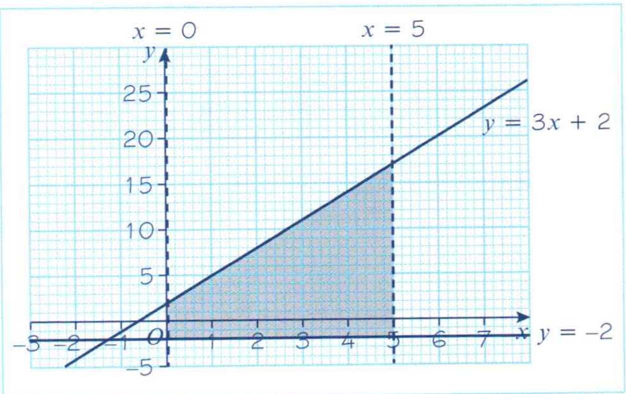
All the unshaded points in this region satisfy the inequality $y < f(x)$.

- **If $y > f(x)$ or $y < f(x)$ then the curve $y = f(x)$ is not included in the region and is represented by a dotted line.**
- **If $y \geqslant f(x)$ or $y \leqslant f(x)$ then the curve $y = f(x)$ is included in the region and is represented by a solid line.**

Example 13

On graph paper, shade the region that satisfies the inequalities:

$y \geqslant -2$, $x < 5$, $y \leqslant 3x + 2$ and $x > 0$.



Draw dotted lines for $x = 0$, $x = 5$.

Shade the required region.
Test a point in the region. Try (1, 2).
For $x = 1$: $1 < 5$ and $1 > 0$ ✓
For $y = 2$: $2 \geqslant -2$ and $2 \leqslant 3 + 2$ ✓

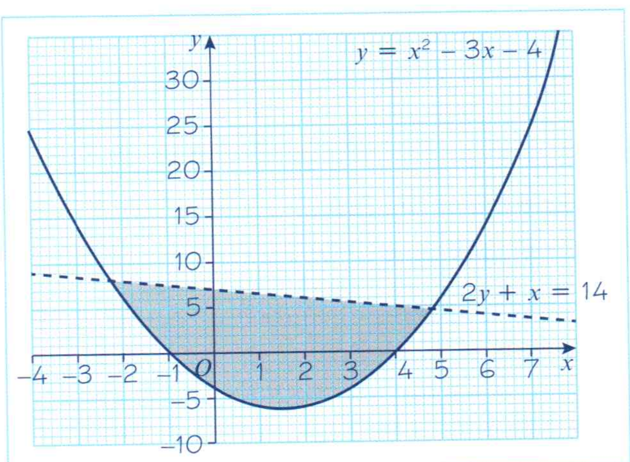
Draw solid lines for $y = -2$, $y = 3x + 2$.

Example 14

On graph paper, shade the region that satisfies the inequalities:

$2y + x < 14$

$y \geqslant x^2 - 3x - 4$



Online Explore which regions on the graph satisfy which inequalities using technology.

Draw a dotted line for $2y + x = 14$ and a solid line for $y = x^2 - 3x - 4$.

Shade the required region.
Test a point in the region. Try (0, 0).
$0 + 0 < 14$ and $0 > 0 - 0 - 4$ ✓

Exercise 3G

1 On a coordinate grid, shade the region that satisfies the inequalities:
$y > x - 2$, $y < 4x$ and $y \leqslant 5 - x$.

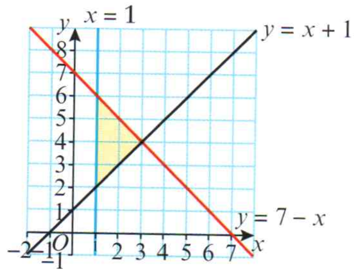
2 On a coordinate grid, shade the region that satisfies the inequalities:
$x \geqslant -1$, $y + x < 4$, $2x + y \leqslant 5$ and $y > -2$.

3 On a coordinate grid, shade the region that satisfies the inequalities:
$y < (3 - x)(2 + x)$ and $y + x \geqslant 3$.

4 On a coordinate grid, shade the region that satisfies the inequalities:
$y > x^2 - 2$ and $y \leqslant 9 - x^2$.

5 On a coordinate grid, shade the region that satisfies the inequalities:
$y > (x - 3)^2$, $y + x \geqslant 5$ and $y < x - 1$.

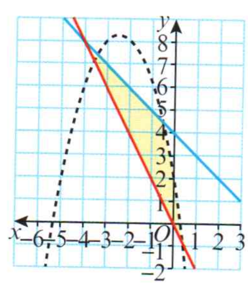
6 The diagram shows the graphs of the straight lines with equations:
$y = x + 1$, $y = 7 - x$ and $x = 1$.



a Write down the coordinates of the points of intersection of the functions.

b Write down the set of inequalities that represent the shaded region shown in the diagram.

7 The diagram shows the graphs of the curves with equations:
$y = 2 - 5x - x^2$, $2x + y = 0$ and $x + y = 4$.
Write down the set of inequalities that represent the shaded region shown in the diagram.



P **8** **a** On a coordinate grid, shade the region that satisfies the inequalities
$y < x + 4$, $y + 5x + 3 \geqslant 0$, $y \geqslant -1$ and $x < 2$.

b Work out the coordinates of the vertices of the shaded region.

c Which of the vertices lie within the region identified by the inequalities?

d Work out the area of the shaded region.

Problem-solving

A vertex is only included if both intersecting lines are included.

Mixed exercise 3

E **1** $2kx - y = 4$
$4kx + 3y = -2$

are two simultaneous equations, where k is a constant.

a Show that $y = -2$. **(3 marks)**

b Find an expression for x in terms of the constant k. **(1 mark)**

E **2** Solve the simultaneous equations:

$x + 2y = 3$
$x^2 - 4y^2 = -33$ **(7 marks)**

E **3** Given the simultaneous equations

$x - 2y = 1$
$3xy - y^2 = 8$

a Show that $5y^2 + 3y - 8 = 0$. **(2 marks)**

b Hence find the pairs (x, y) for which the simultaneous equations are satisfied. **(5 marks)**

E **4 a** By eliminating y from the equations

$x + y = 2$
$x^2 + xy - y^2 = -1$

show that $x^2 - 6x + 3 = 0$. **(2 marks)**

b Hence, or otherwise solve the simultaneous equations

$x + y = 2$
$x^2 + xy - y^2 = -1$

giving x and y in the form $a \pm b\sqrt{6}$, where a and b are integers. **(5 marks)**

E **5 a** Given that $3^x = 9^{y-1}$, show that $x = 2y - 2$. **(1 mark)**

b Solve the simultaneous equations:

$x = 2y - 2$
$x^2 = y^2 + 7$ **(6 marks)**

E **6** Solve the simultaneous equations:

$x + 2y = 3$
$x^2 - 2y + 4y^2 = 18$ **(7 marks)**

E/P **7** The curve and the line given by the equations

$kx^2 - xy + (k + 1)x = 1$
$-\frac{k}{2}x + y = 1$

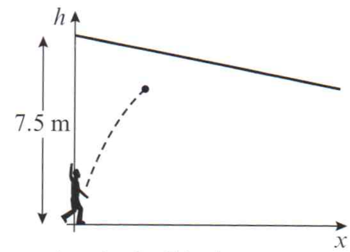
where k is a non-zero constant, intersect at a single point.

a Find the value of k. **(5 marks)**

b Give the coordinates of the point of intersection of the line and the curve. **(3 marks)**

E/P **8** A person throws a ball in a sports hall. The height of the ball, h m, can be modelled in relation to the horizontal distance from the point it was thrown from by the quadratic equation:

$$h = -\tfrac{3}{10}x^2 + \tfrac{5}{2}x + \tfrac{3}{2}$$



The hall has a ceiling which slopes downwards in a straight line from an initial height of 7.5 m at the point the ball is thrown. The height of the ceiling reduces by 20 cm for every metre of horizontal distance from the point the ball is thrown.

Determine whether the model predicts that the ball will hit the ceiling. **(5 marks)**

E **9** Give your answers in set notation.

a Solve the inequality $3x - 8 > x + 13$. **(2 marks)**

b Solve the inequality $x^2 - 5x - 14 > 0$. **(4 marks)**

E **10** Find the set of values of x for which $(x - 1)(x - 4) < 2(x - 4)$. **(6 marks)**

E **11** **a** Use algebra to solve $(x - 1)(x + 2) = 18$. **(2 marks)**

b Hence, or otherwise, find the set of values of x for which $(x - 1)(x + 2) > 18$. Give your answer in set notation. **(2 marks)**

12 Find the set of values of x for which:

a $6x - 7 < 2x + 3$ **(2 marks)**

b $2x^2 - 11x + 5 < 0$ **(4 marks)**

c $5 < \dfrac{20}{x}$ **(4 marks)**

d both $6x - 7 < 2x + 3$ and $2x^2 - 11x + 5 < 0$. **(2 marks)**

E **13** Find the set of values of x that satisfy $\dfrac{8}{x^2} + 1 \leqslant \dfrac{9}{x}$, $x \neq 0$ **(5 marks)**

E **14** Find the values of k for which $kx^2 + 8x + 5 = 0$ has real roots. **(3 marks)**

/P **15** The equation $2x^2 + 4kx - 5k = 0$, where k is a constant, has no real roots.

Prove that k satisfies the inequality $-\frac{5}{2} < k < 0$. **(3 marks)**

E **16** **a** Sketch the graphs of $y = f(x) = x^2 + 2x - 15$ and $g(x) = 6 - 2x$ on the same axes. **(4 marks)**

b Find the coordinates of any points of intersection. **(3 marks)**

c Write down the set of values of x for which $f(x) > g(x)$. **(1 mark)**

E **17** Find the set of values of x for which the curve with equation $y = 2x^2 + 3x - 15$ is below the line with equation $y = 8 + 2x$. **(5 marks)**

E **18** On a coordinate grid, shade the region that satisfies the inequalities:

$y > x^2 + 4x - 12$ and $y < 4 - x^2$. **(5 marks)**

/P **19** **a** On a coordinate grid, shade the region that satisfies the inequalities

$y + x < 6$, $y < 2x + 9$, $y > 3$ and $x > 0$. **(6 marks)**

b Work out the area of the shaded region. **(2 marks)**

Challenge

1 Find the possible values of k for the quadratic equation $2kx^2 + 5kx + 5k - 3 = 0$ to have real roots.

2 A straight line has equation $y = 2x - k$ and a parabola has equation $y = 3x^2 + 2kx + 5$ where k is a constant. Find the range of values of k for which the line and the parabola do not intersect.

Summary of key points

1 Linear simultaneous equations can be solved using elimination or substitution.

2 Simultaneous equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the solutions are paired correctly.

3 The solutions of a pair of simultaneous equations represent the points of intersection of their graphs.

4 For a pair of simultaneous equations that produce a quadratic equation of the form $ax^2 + bx + c = 0$:

- $b^2 - 4ac > 0$ two real solutions
- $b^2 - 4ac = 0$ one real solution
- $b^2 - 4ac < 0$ no real solutions

5 The solution of an inequality is the set of all real numbers x that make the inequality true.

6 To solve a quadratic inequality:

- Rearrange so that the right-hand side of the inequality is 0
- Solve the corresponding quadratic equation to find the critical values
- Sketch the graph of the quadratic function
- Use your sketch to find the required set of values.

7 The values of x for which the curve $y = f(x)$ is **below** the curve $y = g(x)$ satisfy the inequality $f(x) < g(x)$.

The values of x for which the curve $y = f(x)$ is **above** the curve $y = g(x)$ satisfy the inequality $f(x) > g(x)$.

8 $y < f(x)$ represents the points on the coordinate grid below the curve $y = f(x)$.

$y > f(x)$ represents the points on the coordinate grid above the curve $y = f(x)$.

9 If $y > f(x)$ or $y < f(x)$ then the curve $y = f(x)$ is not included in the region and is represented by a dotted line.

If $y \geqslant f(x)$ or $y \leqslant f(x)$ then the curve $y = f(x)$ is included in the region and is represented by a solid line.