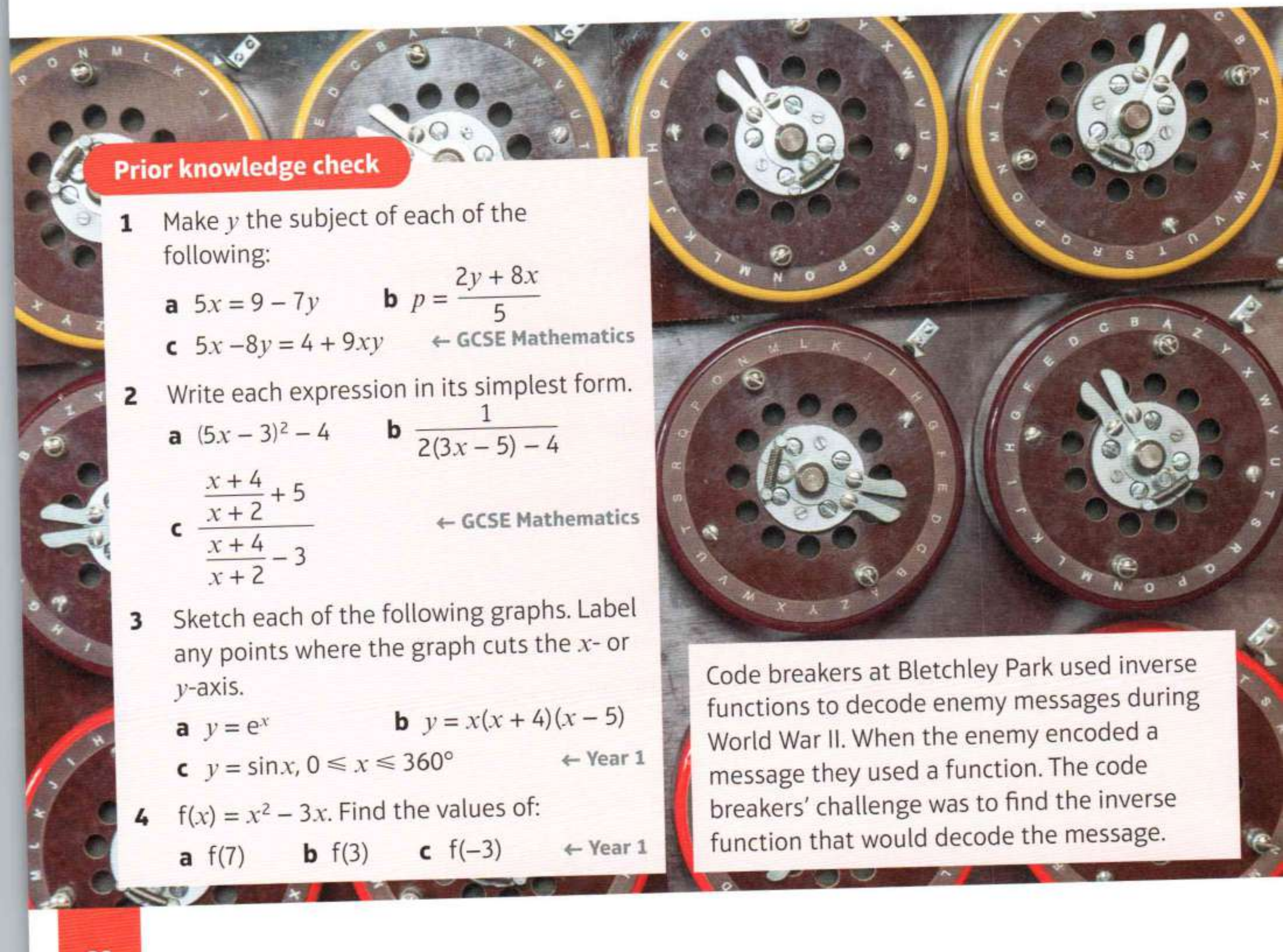
2 Functions and graphs

Objectives

After completing this chapter you should be able to:

- Understand and use the modulus function → pages 23–27
- Understand mappings and functions, and use domain and range → pages 27–32
- Combine two or more functions to make a composite function → pages 32–35
- Know how to find the inverse of a function graphically and algebraically → pages 36–39
- Sketch the graphs of the modulus functions $y = |f(x)|$ and $y = f(|x|)$ → pages 40–44
- Apply a combination of two (or more) transformations to the same curve → pages 44–48
- Transform the modulus function → pages 48–52



Prior knowledge check

1 Make y the subject of each of the following:

a $5x = 9 - 7y$ **b** $p = \dfrac{2y + 8x}{5}$

c $5x - 8y = 4 + 9xy$ ← GCSE Mathematics

2 Write each expression in its simplest form.

a $(5x - 3)^2 - 4$ **b** $\dfrac{1}{2(3x - 5) - 4}$

c $\dfrac{\frac{x+4}{x+2} + 5}{\frac{x+4}{x+2} - 3}$ ← GCSE Mathematics

3 Sketch each of the following graphs. Label any points where the graph cuts the x- or y-axis.

a $y = e^x$ **b** $y = x(x + 4)(x - 5)$

c $y = \sin x, 0 \leqslant x \leqslant 360°$ ← Year 1

4 $f(x) = x^2 - 3x$. Find the values of:

a f(7) **b** f(3) **c** f(−3) ← Year 1

Code breakers at Bletchley Park used inverse functions to decode enemy messages during World War II. When the enemy encoded a message they used a function. The code breakers' challenge was to find the inverse function that would decode the message.

2.1 The modulus function

The modulus of a number a, written as $|a|$, is its **non-negative** numerical value.
So, for example, $|5| = 5$ and also $|-5| = 5$.

- **A modulus function is, in general, a function of the type $y = |f(x)|$.**
  - **When $f(x) \geqslant 0$, $|f(x)| = f(x)$**
  - **When $f(x) < 0$, $|f(x)| = -f(x)$**

Notation The modulus function is also known as the **absolute value** function. On a calculator, the button is often labelled 'Abs'.

Example 1

Write down the values of

a $|-2|$ **b** $|6.5|$ **c** $\left|\frac{1}{3} - \frac{4}{5}\right|$

a $|-2| = 2$ — The positive numerical value of −2 is 2.

b $|6.5| = 6.5$ — 6.5 is a positive number.

c $\left|\frac{1}{3} - \frac{4}{5}\right| = \left|\frac{5}{15} - \frac{12}{15}\right| = \left|-\frac{7}{15}\right| = \frac{7}{15}$ — Work out the value inside the modulus.

Example 2

$f(x) = |2x - 3| + 1$

Write down the values of

a $f(5)$ **b** $f(-2)$ **c** $f(1)$

a $f(5) = |2 \times 5 - 3| + 1$
$= |7| + 1 = 7 + 1 = 8$

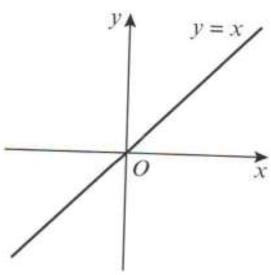
b $f(-2) = |2(-2) - 3| + 1$
$= |-7| + 1 = 7 + 1 = 8$

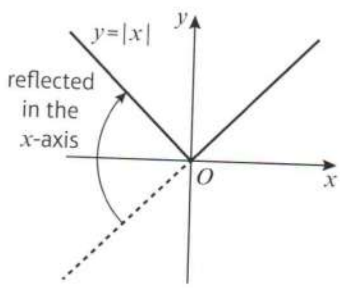
c $f(1) = |2 \times 1 - 3| + 1$
$= |-1| + 1 = 1 + 1 = 2$

Watch out The modulus function acts like a pair of brackets. Work out the value inside the modulus function first.

Online Use your calculator to work out values of modulus functions.

- **To sketch the graph of $y = |ax + b|$, sketch $y = ax + b$ then reflect the section of the graph below the x-axis in the x-axis.**

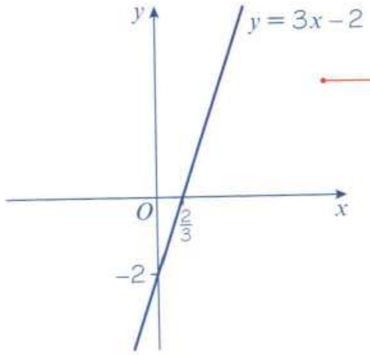




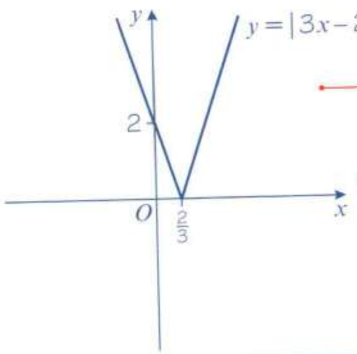
Example 3

Sketch the graph of $y = |3x - 2|$.

Online Explore graphs of $f(x)$ and $|f(x)|$ using technology.



Step 1
Sketch the graph of $y = 3x - 2$.
(Ignore the modulus.)



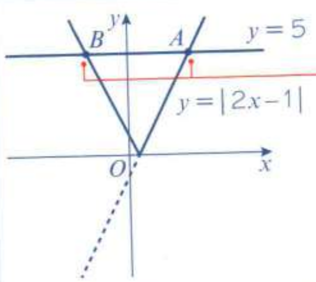
Step 2
For the part of the line below the x-axis (the negative values of y), reflect in the x-axis.
For example, this will change the y-value −2 into the y-value 2.

You could check your answer using a table of values:

x	−1	0	1	2
$y = \|3x - 2\|$	5	2	1	4

Example 4

Solve the equation $|2x - 1| = 5$.



Start by sketching the graphs of $y = |2x - 1|$ and $y = 5$.

The graphs intersect at two points, A and B, so there will be two solutions to the equation.

At A, $2x - 1 = 5$
$2x = 6$
$x = 3$

A is the point of intersection on the original part of the graph.

At B, $-(2x - 1) = 5$
$-2x + 1 = 5$
$2x = -4$
$x = -2$

B is the point of intersection on the reflected part of the graph.

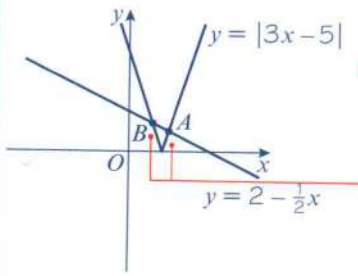
The solutions are $x = 3$ and $x = -2$.

Notation The function inside the modulus is called the **argument** of the modulus. You can solve modulus equations algebraically by considering the positive argument and the negative argument separately.

Example 5

Solve the equation $|3x - 5| = 2 - \frac{1}{2}x$.

Online Explore intersections of straight lines and modulus graphs using technology.



First draw a sketch of the line $y = |3x - 5|$ and the line $y = 2 - \frac{1}{2}x$.

The sketch shows there are two solutions, at A and B, the points of intersection.

At A: $3x - 5 = 2 - \frac{1}{2}x$

$\frac{7}{2}x = 7$

$x = 2$

This is the solution on the original part of the graph.

At B: $-(3x - 5) = 2 - \frac{1}{2}x$

$-3x + 5 = 2 - \frac{1}{2}x$

$-\frac{5}{2}x = -3$

$x = \frac{6}{5}$

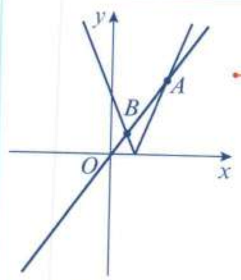
When $f(x) < 0$, $|f(x)| = -f(x)$, so $-(3x - 5) = 2 - \frac{1}{2}x$ gives you the second solution.

This is the solution on the reflected part of the graph.

The solutions are $x = 2$ and $x = \frac{6}{5}$

Example 6

Solve the inequality $|5x - 1| > 3x$.



First draw a sketch of the line $y = |5x - 1|$ and the line $y = 3x$.

Solve the equation $|5x - 1| = 3x$ to find the x-coordinates of the points of intersection, A and B.

At A, $5x - 1 = 3x$

$2x = 1$

$x = \frac{1}{2}$

This is the intersection on the original part of the graph.

At B, $-(5x - 1) = 3x$

$-5x + 1 = 3x$

$8x = 1$

$x = \frac{1}{8}$

Consider the negative argument to find the point of intersection on the reflected part of the graph.

The points of intersection are $x = \frac{1}{2}$ and $x = \frac{1}{8}$

So the solution to $|5x - 1| > 3x$ is $x < \frac{1}{8}$ or $x > \frac{1}{2}$

Problem-solving

Look at the sketch to work out which values of x satisfy the inequality. $y = |5x - 1|$ is above $y = 3x$ when $x > \frac{1}{2}$ or $x < \frac{1}{8}$. You could write the solution in set notation as $\left\{x : x > \frac{1}{2}\right\} \cup \left\{x : x < \frac{1}{8}\right\}$.

Exercise 2A

1 Write down the values of

a $\left|\frac{3}{4}\right|$ **b** $|-0.28|$ **c** $|3 - 11|$ **d** $\left|\frac{5}{7} - \frac{3}{8}\right|$ **e** $|20 - 6 \times 4|$ **f** $|4^2 \times 2 - 3 \times 7|$

2 $f(x) = |7 - 5x| + 3$. Write down the values of:

a $f(1)$ **b** $f(10)$ **c** $f(-6)$

3 $g(x) = |x^2 - 8x|$. Write down the values of:

a $g(4)$ **b** $g(-5)$ **c** $g(8)$

4 Sketch the graph of each of the following. In each case, write down the coordinates of any points at which the graph meets the coordinate axes.

a $y = |x - 1|$ **b** $y = |2x + 3|$ **c** $y = |4x - 7|$ **d** $y = \left|\frac{1}{2}x - 5\right|$

e $y = |7 - x|$ **f** $y = |6 - 4x|$

g $y = -|x|$ **h** $y = -|3x - 1|$

Hint $y = -|x|$ is a reflection of $y = |x|$ in the x-axis. ← Year 1, Chapter 4

5 $g(x) = \left|4 - \frac{3}{2}x\right|$ and $h(x) = 5$

a On the same axes, sketch the graphs of $y = g(x)$ and $y = h(x)$.

b Hence solve the equation $\left|4 - \frac{3}{2}x\right| = 5$.

6 Solve:

a $|3x - 1| = 5$ **b** $\left|\frac{x - 5}{2}\right| = 1$ **c** $|4x + 3| = -2$

d $|7x - 3| = 4$ **e** $\left|\frac{4 - 5x}{3}\right| = 2$ **f** $\left|\frac{x}{6} - 1\right| = 3$

7 **a** On the same diagram, sketch the graphs $y = -2x$ and $y = \left|\frac{1}{2}x - 2\right|$.

b Solve the equation $-2x = \left|\frac{1}{2}x - 2\right|$.

(E) **8** Solve $|3x - 5| = 11 - x$. **(4 marks)**

9 **a** On the same set of axes, sketch $y = |6 - x|$ and $y = \frac{1}{2}x - 5$.

b State with a reason whether there are any solutions to the equation $|6 - x| = \frac{1}{2}x - 5$.

(P) **10** A student attempts to solve the equation $|3x + 4| = x$. The student writes the following working:

$$3x + 4 = x \qquad \text{or} \qquad -(3x+4) = x$$
$$4 = -2x \qquad\qquad -3x - 4 = x$$
$$x = -2 \qquad\qquad -4 = 4x$$
$$x = -1$$

Solutions are $x = -2$ and $x = -1$.

Explain the error made by the student.

11 **a** On the same diagram, sketch the graphs of $y = -|3x + 4|$ and $y = 2x - 9$.

b Solve the inequality $-|3x + 4| < 2x - 9$.

(E) **12** Solve the inequality $|2x + 9| < 14 - x$. **(4 marks)**

(E/P) **13** The equation $|6 - x| = \frac{1}{2}x + k$ has exactly one solution.

a Find the value of k. **(2 marks)**

b State the solution to the equation. **(2 marks)**

Problem-solving

The solution must be at the vertex of the graph of the modulus function.

Challenge

$f(x) = |x^2 + 9x + 8|$ and $g(x) = 1 - x$

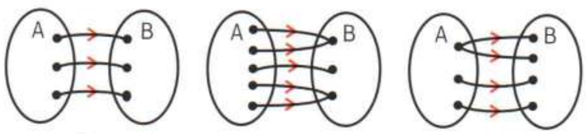
a On the same axes, sketch graphs of $y = f(x)$ and $y = g(x)$.

b Use your sketch to find all the solutions to $|x^2 + 9x + 8| = 1 - x$.

2.2 Functions and mappings

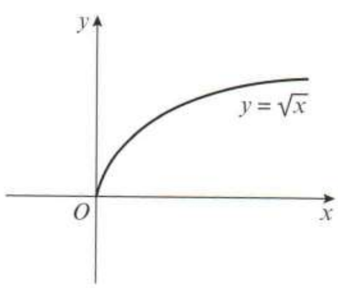
A **mapping** transforms one set of numbers into a different set of numbers. The mapping can be described in words or through an algebraic equation. It can also be represented by a graph.

- **A mapping is a function if every input has a distinct output. Functions can either be one-to-one or many-to-one.**



one-to-one function   many-to-one function   not a function

Many mappings can be made into functions by changing the domain. Consider $y = \sqrt{x}$:



Notation The **domain** is the set of all possible inputs for a mapping.

The **range** is the set of all possible outputs for the mapping.

If the domain were all of the real numbers, $\mathbb{R}$, then $y = \sqrt{x}$ would not be a function because values of x less than 0 would not be mapped anywhere.

If you restrict the domain to $x \geqslant 0$, every element in the domain is mapped to exactly one element in the range.

We can write this function together with its domain as $f(x) = \sqrt{x}, x \in \mathbb{R}, x \geqslant 0$.

Notation You can also write this function as: $f: x \mapsto \sqrt{x}, x \in \mathbb{R}, x \geqslant 0$

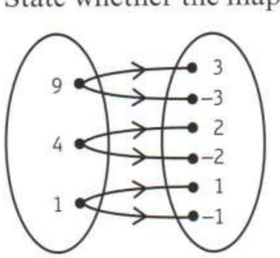
Example 7

For each of the following mappings:

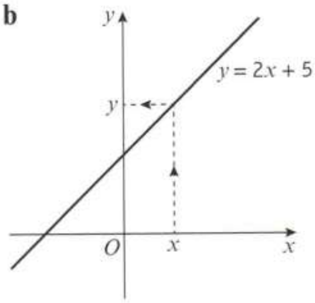
i State whether the mapping is one-to-one, many-to-one or one-to-many.

ii State whether the mapping is a function.

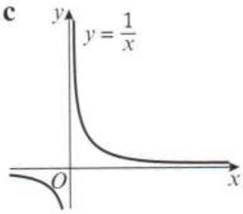
a



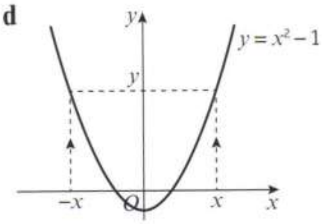
b



c



d



a i Every element in set A gets mapped to two elements in set B, so the mapping is **one-to-many**.

ii The mapping is not a function.

b i Every value of x gets mapped to one value of y, so the mapping is **one-to-one**.

ii The mapping is a function.

c i The mapping is **one-to-one**.

ii $x = 0$ does not get mapped to a value of y so the mapping is not a function.

d i On the graph, you can see that x and $-x$ both get mapped to the same value of y. Therefore, this is a **many-to-one** mapping.

ii The mapping is a function.

You couldn't write down a single value for f(9).

For a mapping to be a function, every input in the domain must map onto exactly one output.

The mapping in part **c** could be a function if $x = 0$ were omitted from the domain. You could write this as a function as $f(x) = \frac{1}{x}, x \in \mathbb{R}, x \neq 0$.

Example 8

Find the range of each of the following functions:

a $f(x) = 3x - 2$, domain $\{x = 1, 2, 3, 4\}$

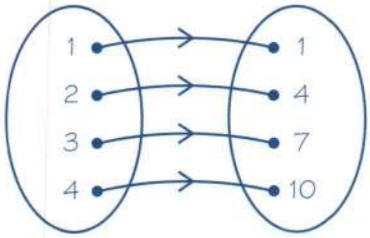
b $g(x) = x^2$, domain $\{x \in \mathbb{R}, -5 \leqslant x \leqslant 5\}$

c $h(x) = \frac{1}{x}$, domain $\{x \in \mathbb{R}, 0 < x \leqslant 3\}$

State if the functions are one-to-one or many-to-one.

a $f(x) = 3x - 2, \{x = 1, 2, 3, 4\}$

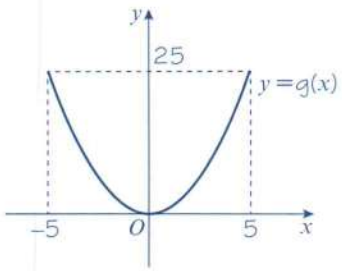
The domain contains a finite number of elements, so you can draw a mapping diagram showing the whole function.



Range of $f(x)$ is $\{1, 4, 7, 10\}$.
$f(x)$ is one-to-one.

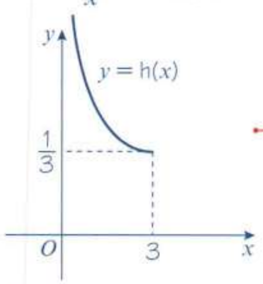
b $g(x) = x^2, \{-5 \leqslant x \leqslant 5\}$

The domain is the set of all the x-values that correspond to points on the graph. The range is the set of y-values that correspond to points on the graph.



Range of $g(x)$ is $0 \leqslant g(x) \leqslant 25$.
$g(x)$ is many-to-one.

c $h(x) = \frac{1}{x}, \{x \in \mathbb{R}, 0 < x \leqslant 3\}$



Calculate $h(3) = \frac{1}{3}$ to find the minimum value in the range of h. As x approaches 0, $\frac{1}{x}$ approaches ∞, so there is no maximum value in the range of h.

Range of $h(x)$ is $h(x) \geqslant \frac{1}{3}$
$h(x)$ is one-to-one.

Example 9

The function $f(x)$ is defined by

$$f: x \mapsto \begin{cases} 5 - 2x, & x < 1 \\ x^2 + 3, & x \geqslant 1 \end{cases}$$

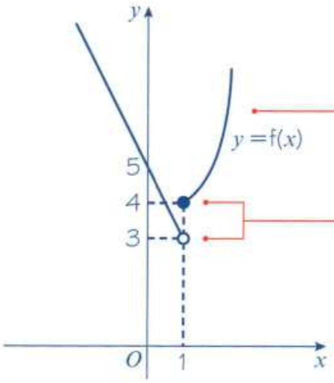
a Sketch $y = f(x)$, and state the range of $f(x)$.

b Solve $f(x) = 19$.

Notation This is an example of a **piecewise-defined function**. It consists of two parts: one linear (for $x < 1$) and one quadratic (for $x \geqslant 1$).

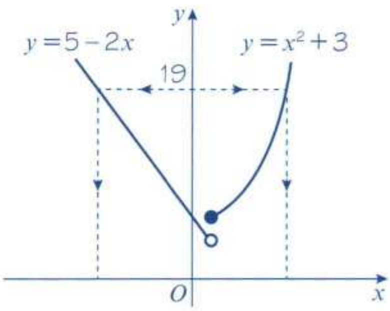
Online Explore graphs of functions on a given domain using technology.

a



The range is the set of values that y takes and therefore $f(x) > 3$.

b



The positive solution is where

$x^2 + 3 = 19$

$x^2 = 16$

$x = \pm 4$

$x = 4$

The negative solution is where

$5 - 2x = 19$

$-2x = 14$

$x = -7$

The solutions are $x = 4$ and $x = -7$.

Watch out Although the graph jumps at $x = 1$, the function is still defined for all real values of x:
$f(0.9) = 5 - 2(0.9) = 3.2$
$f(1) = (1)^2 + 3 = 4$

Sketch the graph of $y = 5 - 2x$ for $x < 1$, and the graph of $y = x^2 + 3$ for $x \geqslant 1$.

f(1) lies on the quadratic curve, so use a solid dot on the quadratic curve, and an open dot on the line.

Note that $f(x) \neq 3$ at $x = 1$
so $f(x) > 3$
not $f(x) \geqslant 3$

There are 2 values of x such that $f(x) = 19$.

Problem-solving

Use $x^2 + 3 = 19$ to find the solution when $x \geqslant 1$ and use $5 - 2x = 19$ to find the solution when $x < 1$.

Ignore $x = -4$ because the function is only equal to $x^2 + 3$ for $x \geqslant 1$.

Exercise 2B

1 For each of the following functions:

i draw the mapping diagram

ii state if the function is one-to-one or many-to-one

iii find the range of the function.

a $f(x) = 5x - 3$, domain $\{x = 3, 4, 5, 6\}$

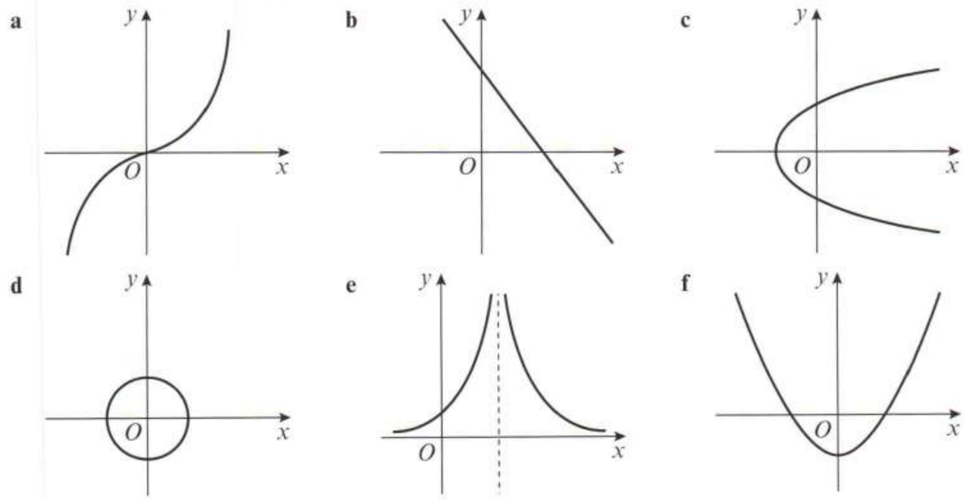
b $g(x) = x^2 - 3$, domain $\{x = -3, -2, -1, 0, 1, 2, 3\}$

c $h(x) = \dfrac{7}{4 - 3x}$, domain $\{x = -1, 0, 1\}$

2 For each of the following mappings:

i State whether the mapping is one-to-one, many-to-one or one-to-many.

ii State whether the mapping could represent a function.



3 Calculate the value(s) of a, b, c and d given that:

a $p(a) = 16$ where $p: x \mapsto 3x - 2, x \in \mathbb{R}$

b $q(b) = 17$ where $q: x \mapsto x^2 - 3, x \in \mathbb{R}$

c $r(c) = 34$ where $r: x \mapsto 2(2^x) + 2, x \in \mathbb{R}$

d $s(d) = 0$ where $s: x \mapsto x^2 + x - 6, x \in \mathbb{R}$

4 For each function:

i represent the function on a mapping diagram, writing down the elements in the range

ii state whether the function is one-to-one or many-to-one.

a $f(x) = 2x + 1$ for the domain $\{x = 1, 2, 3, 4, 5\}$

b $g: x \mapsto \sqrt{x}$ for the domain $\{x = 1, 4, 9, 16, 25, 36\}$

c $h(x) = x^2$ for the domain $\{x = -2, -1, 0, 1, 2\}$

d $j: x \mapsto \frac{2}{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$

e $k(x) = e^x + 3$ for the domain $\{x = -2, -1, 0, 1, 2\}$

Notation Remember, $\sqrt{x}$ means the positive square root of x.

5 For each function:

i sketch the graph of $y = f(x)$

ii state the range of $f(x)$

iii state whether $f(x)$ is one-to-one or many-to-one.

a $f: x \mapsto 3x + 2$ for the domain $\{x \geqslant 0\}$

b $f(x) = x^2 + 5$ for the domain $\{x \geqslant 2\}$

c $f: x \mapsto 2\sin x$ for the domain $\{0 \leqslant x \leqslant 180\}$

d $f: x \mapsto \sqrt{x + 2}$ for the domain $\{x \geqslant -2\}$

e $f(x) = e^x$ for the domain $\{x \geqslant 0\}$

f $f(x) = 7\log x$, for the domain, $\{x \in \mathbb{R}, x > 0\}$

6 The following mappings f and g are defined on all the real numbers by

$$f(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x \geqslant 4 \end{cases} \qquad g(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x > 4 \end{cases}$$

a Explain why $f(x)$ is a function and $g(x)$ is not.

b Sketch $y = f(x)$.

c Find the values of: **i** $f(3)$ **ii** $f(10)$

d Solve $f(a) = 90$.

(P) **7** The function s is defined by

$$s(x) = \begin{cases} x^2 - 6, & x < 0 \\ 10 - x, & x \geqslant 0 \end{cases}$$

a Sketch $y = s(x)$.

b Find the value(s) of a such that $s(a) = 43$.

c Solve $s(x) = x$.

Problem-solving

The solutions of $s(x) = x$ are the values in the domain that get mapped to themselves in the range.

(E/P) **8** The function p is defined by

$$p(x) = \begin{cases} e^{-x}, & -5 \leqslant x < 0 \\ x^3 + 4, & 0 \leqslant x \leqslant 4 \end{cases}$$

a Sketch $y = p(x)$. **(3 marks)**

b Find the values of a, to 2 decimal places, such that $p(a) = 50$. **(4 marks)**

(E/P) **9** The function h has domain $-10 \leqslant x \leqslant 6$, and is linear from $(-10, 14)$ to $(-4, 2)$ and from $(-4, 2)$ to $(6, 27)$.

a Sketch $y = h(x)$. **(2 marks)**

b Write down the range of $h(x)$. **(1 mark)**

c Find the values of a, such that $h(a) = 12$. **(4 marks)**

Problem-solving

The graph of $y = h(x)$ will consist of two line segments which meet at $(-4, 2)$.

(P) **10** The function g is defined by $g(x) = cx + d$ where c and d are constants to be found. Given $g(3) = 10$ and $g(8) = 12$ find the values of c and d.

(P) **11** The function f is defined by $f(x) = ax^3 + bx - 5$ where a and b are constants to be found. Given that $f(1) = -4$ and $f(2) = 9$, find the values of the constants a and b.

(E/P) **12** The function h is defined by $h(x) = x^2 - 6x + 20$ and has domain $x \geqslant a$. Given that $h(x)$ is a one-to-one function find the smallest possible value of the constant a. **(6 marks)**

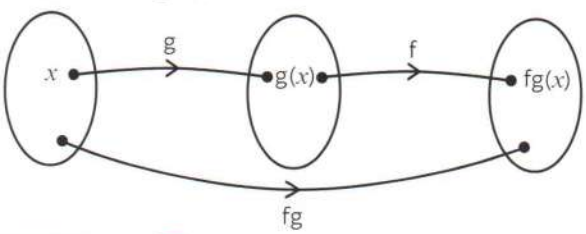
Problem-solving

First complete the square for $h(x)$.

2.3 Composite functions

Two or more functions can be combined to make a new function. The new function is called a **composite function**.

- **fg(x) means apply g first, then apply f.**
- **fg(x) = f(g(x))**



Watch out The order in which the functions are combined is important: $fg(x)$ is not necessarily the same as $gf(x)$.

Example 10

Given $f(x) = x^2$ and $g(x) = x + 1$, find:

a fg(1) **b** gf(3) **c** ff(−2)

a $fg(1) = f(1 + 1)$
$= 2^2$
$= 4$

b $gf(3) = g(3^2)$
$= g(9)$
$= 9 + 1$
$= 10$

c $ff(-2) = f((-2)^2)$
$= f(4)$
$= 4^2$
$= 16$

$g(1) = 1 + 1$

$f(2) = 2^2$

$f(3) = 3^2$

$g(9) = 9 + 1$

$f(-2) = (-2)^2$

$f(4) = 4^2$

Example 11

The functions f and g are defined by $f(x) = 3x + 2$ and $g(x) = x^2 + 4$. Find:

a the function $fg(x)$

b the function $gf(x)$

c the function $f^2(x)$

d the values of b such that $fg(b) = 62$.

Notation $f^2(x)$ is $ff(x)$.

a $fg(x) = f(x^2 + 4)$
$= 3(x^2 + 4) + 2$
$= 3x^2 + 14$

b $gf(x) = g(3x + 2)$
$= (3x + 2)^2 + 4$
$= 9x^2 + 12x + 8$

c $f^2(x) = f(3x + 2)$
$= 3(3x + 2) + 2$
$= 9x + 8$

d $fg(x) = 3x^2 + 14$
If $fg(b) = 62$
then $3b^2 + 14 = 62$
$b^2 = 16$
$b = \pm 4$

g acts on x first, mapping it to $x^2 + 4$.

f acts on the result.

Simplify answer.

f acts on x first, mapping it to $3x + 2$.

g acts on the result.

f maps x to $3x + 2$.

f acts on the result.

Set up and solve an equation in b.

Example 12

The functions f and g are defined by

$f: x \mapsto |2x - 8|$

$g: x \mapsto \dfrac{x + 1}{2}$

a Find $fg(3)$. **b** Solve $fg(x) = x$.

a $\text{fg}(3) = \text{f}\left(\frac{3+1}{2}\right)$
$= \text{f}(2)$
$= |2 \times 2 - 8|$
$= |-4|$
$= 4$

$g(3) = \left(\frac{3+1}{2}\right)$

$f(2) = |2 \times 2 - 8|$

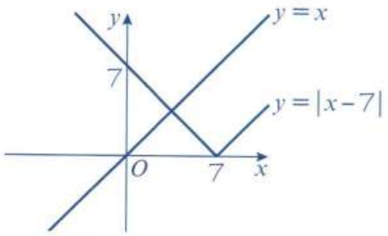
b First find $\text{fg}(x)$:

$\text{fg}(x) = \text{f}\left(\frac{x+1}{2}\right)$
$= \left|2\left(\frac{x+1}{2}\right) - 8\right|$
$= |x - 7|$
$\text{fg}(x) = x$
$|x - 7| = x$

g acts on x first, mapping it to $\frac{x+1}{2}$

f acts on the result.

Simplify the answer.



Draw a sketch of $y = |x - 7|$ and $y = x$.
The sketch shows there is only one solution to the equation $|x - 7| = x$ and that it occurs on the reflected part of the graph.

$-(x - 7) = x$
$-x + 7 = x$
$2x = 7$
$x = 3.5$

When $f(x) < 0$, $|f(x)| = -f(x)$. The solution is on the reflected part of the graph so use $-(x - 7)$.

This is the x-coordinate at the point of intersection marked on the graph.

Exercise 2C

1 Given the functions $p(x) = 1 - 3x$, $q(x) = \frac{x}{4}$ and $r(x) = (x - 2)^2$, find:

a $pq(-8)$ **b** $qr(5)$ **c** $rq(6)$ **d** $p^2(-5)$ **e** $pqr(8)$

2 Given the functions $f(x) = 4x + 1$, $g(x) = x^2 - 4$ and $h(x) = \frac{1}{x}$, find expressions for the functions:

a $fg(x)$ **b** $gf(x)$ **c** $gh(x)$ **d** $fh(x)$ **e** $f^2(x)$

(E) **3** The functions f and g are defined by
$f(x) = 3x - 2, x \in \mathbb{R}$
$g(x) = x^2, x \in \mathbb{R}$

a Find an expression for $fg(x)$. **(2 marks)**

b Solve $fg(x) = gf(x)$. **(4 marks)**

(E) **4** The functions p and q are defined by
$p(x) = \frac{1}{x - 2}, x \in \mathbb{R}, x \neq 2$
$q(x) = 3x + 4, x \in \mathbb{R}$

a Find an expression for $qp(x)$ in the form $\frac{ax + b}{cx + d}$ **(3 marks)**

b Solve $qp(x) = 16$. **(3 marks)**

(E) **5** The functions f and g are defined by:

f: $x \mapsto |9 - 4x|$

g: $x \mapsto \dfrac{3x - 2}{2}$

a Find fg(6). **(2 marks)**

b Solve $\text{fg}(x) = x$. **(5 marks)**

(P) **6** Given $f(x) = \dfrac{1}{x + 1}, x \neq -1$

a Prove that $f^2(x) = \dfrac{x + 1}{x + 2}$

b Find an expression for $f^3(x)$.

7 The functions s and t are defined by

$s(x) = 2^x, x \in \mathbb{R}$

$t(x) = x + 3, x \in \mathbb{R}$

a Find an expression for $\text{st}(x)$.

b Find an expression for $\text{ts}(x)$.

c Solve $\text{st}(x) = \text{ts}(x)$, leaving your answer in the form $\dfrac{\ln a}{\ln b}$

Hint Rearrange the equation in part **c** into the form $2^x = k$ where k is a real number, then take natural logs of both sides. ← Year 1, Section 14.5

(E) **8** Given $f(x) = e^{5x}$ and $g(x) = 4 \ln x$, find in its simplest form:

a $\text{gf}(x)$ **(2 marks)**

b $\text{fg}(x)$ **(2 marks)**

(E/P) **9** The functions p and q are defined by

p: $x \mapsto \ln(x + 3), x \in \mathbb{R}, x > -3$

q: $x \mapsto e^{3x} - 1, x \in \mathbb{R}$

Hint The range of p will be the set of possible inputs for q in the function qp.

a Find $\text{qp}(x)$ and state its range. **(3 marks)**

b Find the value of qp(7). **(1 mark)**

c Solve $\text{qp}(x) = 124$. **(3 marks)**

(E/P) **10** The function t is defined by

t: $x \mapsto 5 - 2x$

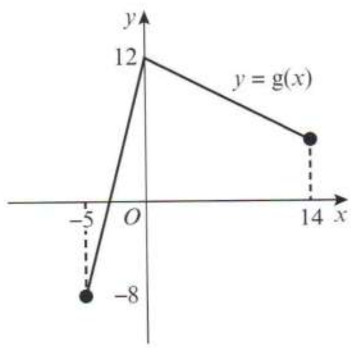
Solve the equation $t^2(x) - (t(x))^2 = 0$. **(5 marks)**

Problem-solving

You need to work out the intermediate steps for this problem yourself, so plan your answer before you start. You could start by finding an expression for $\text{tt}(x)$.

(E) **11** The function g has domain $-5 \leqslant x \leqslant 14$ and is linear from $(-5, -8)$ to $(0, 12)$ and from $(0, 12)$ to $(14, 5)$.

A sketch of the graph of $y = g(x)$ is shown in the diagram.



a Write down the range of g. **(1 mark)**

b Find gg(0). **(2 marks)**

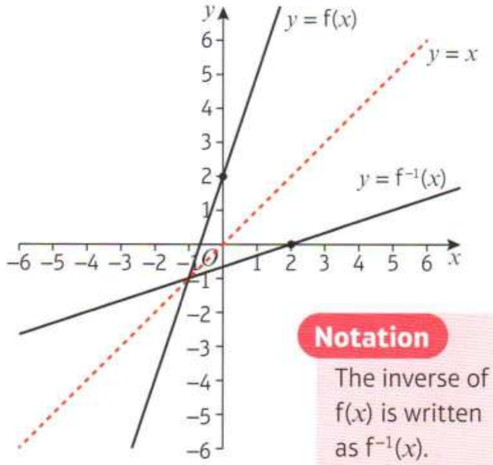
The function h is defined by h: $x \mapsto \dfrac{2x - 5}{10 - x}$

c Find gh(7). **(2 marks)**

2.4 Inverse functions

The **inverse** of a function performs the opposite operation to the original function. It takes the elements in the range of the original function and maps them back into elements of the domain of the original function. For this reason, inverse functions only exist for one-to-one functions.

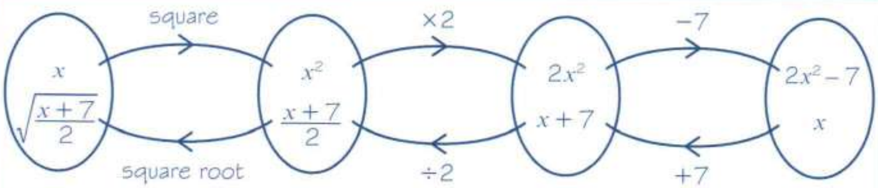
- **Functions $f(x)$ and $f^{-1}(x)$ are inverses of each other. $ff^{-1}(x) = f^{-1}f(x) = x$.**
- **The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each another in the line $y = x$.**
- **The domain of $f(x)$ is the range of $f^{-1}(x)$.**
- **The range of $f(x)$ is the domain of $f^{-1}(x)$.**



Notation The inverse of $f(x)$ is written as $f^{-1}(x)$.

Example 13

Find the inverse of the function $h(x) = 2x^2 - 7, x \geqslant 0$.



Range of $h(x)$ is $h(x) \geqslant -7$, so domain of $h^{-1}(x)$ is $x \geqslant -7$.

Therefore, $h^{-1}(x) = \sqrt{\frac{x + 7}{2}}, x \geqslant -7$

An inverse function can often be found using a flow diagram.

The range of $h(x)$ is the domain of $h^{-1}(x)$.

Example 14

Find the inverse of the function $f(x) = \frac{3}{x - 1}, \{x \in \mathbb{R}, x \neq 1\}$ by changing the subject of the formula.

Let $y = f(x)$

$$y = \frac{3}{x - 1}$$

$$y(x - 1) = 3$$

$$yx - y = 3$$

$$yx = 3 + y$$

$$x = \frac{3 + y}{y}$$

Range of $f(x)$ is $f(x) \neq 0$, so domain of $f^{-1}(x)$ is $x \neq 0$.

Therefore $f^{-1}(x) = \frac{3 + x}{x}, x \neq 0$

$f(4) = \frac{3}{4 - 1} = \frac{3}{3} = 1$

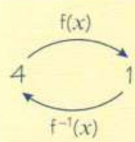
$f^{-1}(1) = \frac{3 + 1}{1} = \frac{4}{1} = 4$

You can rearrange to find an inverse function. Start by letting $y = f(x)$.

Rearrange to make x the subject of the formula.

Define $f^{-1}(x)$ in terms of x.

Check to see that at least one element works. Try 4. Note that $f^{-1}f(4) = 4$.



Example 15

The function, $f(x) = \sqrt{x-2}$, $x \in \mathbb{R}$, $x \geqslant 2$.

a State the range of $f(x)$.

b Find the function $f^{-1}(x)$ and state its domain and range.

c Sketch $y = f(x)$ and $y = f^{-1}(x)$ and the line $y = x$.

a The range of $f(x)$ is $y \in \mathbb{R}$, $y \geqslant 0$.

> $f(2) = 0$. As x increases from 2, $f(x)$ also increases without limit, so the range is $f(x) \geqslant 0$, or $y \geqslant 0$.

b $y = \sqrt{x-2}$

$y^2 = x - 2$

$x^2 = y - 2$

$y = x^2 + 2$

> Interchange x and y.

The inverse function is $f^{-1}(x) = x^2 + 2$.

> Always write your function in terms of x.

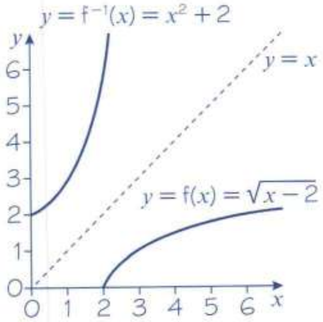
The domain of $f^{-1}(x)$ is $x \in \mathbb{R}$, $x \geqslant 0$.

> The range of $f(x)$ is the same as the domain of $f^{-1}(x)$.

The range of $f^{-1}(x)$ is $y \in \mathbb{R}$, $y \geqslant 2$.

> The range of $f^{-1}(x)$ is the same as the domain of $f(x)$.

c



> The graph of $f^{-1}(x)$ is a reflection of $f(x)$ in the line $y = x$. This is because the reflection transforms y to x and x to y.

Example 16

The function $f(x)$ is defined by $f(x) = x^2 - 3$, $x \in \mathbb{R}$, $x \geqslant 0$.

a Find $f^{-1}(x)$.

b Sketch $y = f^{-1}(x)$ and state its domain.

c Solve the equation $f(x) = f^{-1}(x)$.

a Let $y = f(x)$

$y = x^2 - 3$

$y + 3 = x^2$

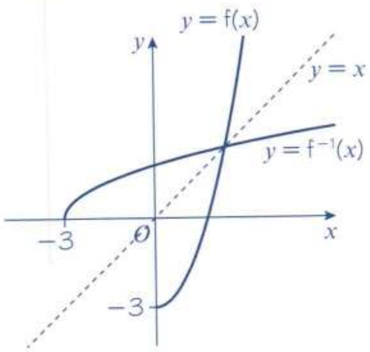
$x = \sqrt{y+3}$

> Change the subject of the formula.

$f^{-1}(x) = \sqrt{x+3}$

> **Online** Explore functions and their inverses using technology.

b



> First sketch $f(x)$. Then reflect $f(x)$ in the line $y = x$.

The domain of $f^{-1}(x)$ is $x \in \mathbb{R}$, $x \geqslant -3$.

> The range of the original function is $f(x) \geqslant -3$.

c When $f(x) = f^{-1}(x)$

$f(x) = x$

$x^2 - 3 = x$

$x^2 - x - 3 = 0$

So $x = \frac{1 + \sqrt{13}}{2}$

Problem-solving

$y = f(x)$ and $y = f^{-1}(x)$ intersect on the line $y = x$. This means that the solution to $f(x) = f^{-1}(x)$ is the same as the solution to $f(x) = x$.

From the graph you can see that the solution must be positive, so ignore the negative solution to the equation.

Exercise 2D

1 For each of the following functions $f(x)$:

i state the range of $f(x)$

ii determine the equation of the inverse function $f^{-1}(x)$

iii state the domain and range of $f^{-1}(x)$

iv sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

a $f: x \mapsto 2x + 3, x \in \mathbb{R}$

b $f: x \mapsto \frac{x + 5}{2}, x \in \mathbb{R}$

c $f: x \mapsto 4 - 3x, x \in \mathbb{R}$

d $f: x \mapsto x^3 - 7, x \in \mathbb{R}$

2 Find the inverse of each function:

a $f(x) = 10 - x, x \in \mathbb{R}$

b $g(x) = \frac{x}{5}, x \in \mathbb{R}$

c $h(x) = \frac{3}{x}, x \neq 0, x \in \mathbb{R}$

d $k(x) = x - 8, x \in \mathbb{R}$

Notation Two of these functions are **self-inverse**. A function is self-inverse if $f^{-1}(x) = f(x)$. In this case $ff(x) = x$.

(P) **3** Explain why the function $g: x \mapsto 4 - x, \{x \in \mathbb{R}, x > 0\}$ is not identical to its inverse.

4 For each of the following functions $g(x)$ with a restricted domain:

i state the range of $g(x)$

ii determine the equation of the inverse function $g^{-1}(x)$

iii state the domain and range of $g^{-1}(x)$

iv sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes.

a $g(x) = \frac{1}{x}, \{x \in \mathbb{R}, x \geqslant 3\}$

b $g(x) = 2x - 1, \{x \in \mathbb{R}, x \geqslant 0\}$

c $g(x) = \frac{3}{x - 2}, \{x \in \mathbb{R}, x > 2\}$

d $g(x) = \sqrt{x - 3}, \{x \in \mathbb{R}, x \geqslant 7\}$

e $g(x) = x^2 + 2, \{x \in \mathbb{R}, x > 2\}$

f $g(x) = x^3 - 8, \{x \in \mathbb{R}, x \geqslant 2\}$

(E) **5** The function $t(x)$ is defined by
$t(x) = x^2 - 6x + 5, x \in \mathbb{R}, x \geqslant 5$
Find $t^{-1}(x)$. **(5 marks)**

Hint First complete the square for the function $t(x)$.

(E/P) **6** The function $m(x)$ is defined by $m(x) = x^2 + 4x + 9, x \in \mathbb{R}, x > a$, for some constant a.

a State the least value of a for which $m^{-1}(x)$ exists. **(4 marks)**

b Determine the equation of $m^{-1}(x)$. **(3 marks)**

c State the domain of $m^{-1}(x)$. **(1 mark)**

7 The function h(x) is defined by $h(x) = \frac{2x+1}{x-2}$, $\{x \in \mathbb{R}, x \neq 2\}$.

a What happens to the function as x approaches 2?

b Find $h^{-1}(3)$.

c Find $h^{-1}(x)$, stating clearly its domain.

d Find the elements of the domain that get mapped to themselves by the function.

8 The functions m and n are defined by

m: $x \mapsto 2x + 3$, $x \in \mathbb{R}$

n: $x \mapsto \frac{x-3}{2}$, $x \in \mathbb{R}$

a Find nm(x)

b What can you say about the functions m and n?

(P) **9** The functions s and t are defined by

$s(x) = \frac{3}{x+1}$, $x \neq -1$

$t(x) = \frac{3-x}{x}$, $x \neq 0$

Show that the functions are inverses of each other.

(E/P) **10** The function f(x) is defined by $f(x) = 2x^2 - 3$, $\{x \in \mathbb{R}, x < 0\}$.
Determine:

a $f^{-1}(x)$ clearly stating its domain **(4 marks)**

b the value(s) of a for which $f(a) = f^{-1}(a)$. **(4 marks)**

(E) **11** The functions f and g are defined by

f: $x \mapsto e^x - 5$, $x \in \mathbb{R}$

g: $x \mapsto \ln(x - 4)$, $x > 4$

a State the range of f. **(1 mark)**

b Find f^{-1}, the inverse function of f, stating its domain. **(3 marks)**

c On the same axes, sketch the curves with equation $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes. **(4 marks)**

d Find g^{-1}, the inverse function of g, stating its domain. **(3 marks)**

e Solve the equation $g^{-1}(x) = 11$, giving your answer to 2 decimal places. **(3 marks)**

(E/P) **12** The function f is defined by

f: $x \mapsto \frac{3(x+2)}{x^2 + x - 20} - \frac{2}{x-4}$, $x > 4$

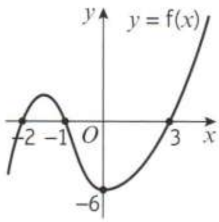
a Show that f: $x \mapsto \frac{1}{x+5}$, $x > 4$. **(4 marks)**

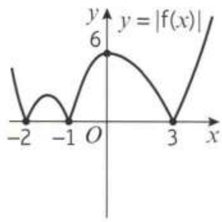
b Find the range of f. **(2 marks)**

c Find $f^{-1}(x)$. State the domain of this inverse function. **(4 marks)**

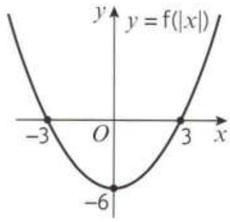
2.5 $y = |f(x)|$ and $y = f(|x|)$

- **To sketch the graph of $y = |f(x)|$:**
  - **Sketch the graph of $y = f(x)$.**
  - **Reflect any parts where $f(x) < 0$ (parts below the x-axis) in the x-axis.**
  - **Delete the parts below the x-axis.**





- **To sketch the graph of $y = f(|x|)$:**
  - **Sketch the graph of $y = f(x)$ for $x \geqslant 0$.**
  - **Reflect this in the y-axis.**



Example 17

$f(x) = x^2 - 3x - 10$

a Sketch the graph of $y = f(x)$.

b Sketch the graph of $y = |f(x)|$.

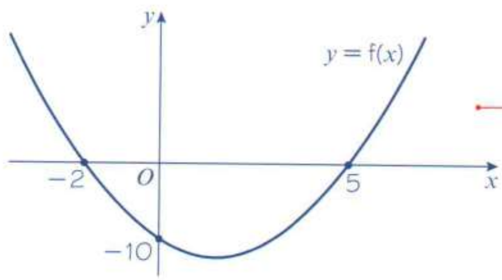
c Sketch the graph of $y = f(|x|)$.

a $f(x) = x^2 - 3x - 10 = (x - 5)(x + 2)$

$f(x) = 0$ implies $(x - 5)(x + 2) = 0$

So $x = 5$ or $x = -2$

$f(0) = -10$



The graph of $y = x^2 - 3x - 10$ cuts the x-axis at −2 and 5.

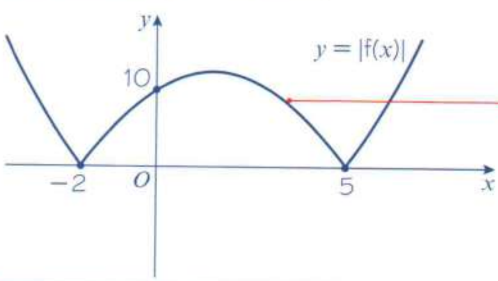
The graph cuts the y-axis at -10.

This is the sketch of $y = x^2 - 3x - 10$.

The sketch includes the points where the graph intercepts the coordinate axes.

A sketch does not have to be to scale.

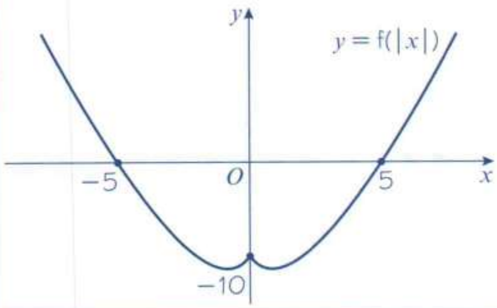
b $y = |f(x)| = |x^2 - 3x - 10|$



Reflect the part of the curve where $y = f(x) < 0$ (the negative values of y) in the x-axis.

Online Explore graphs of modulus functions using technology.

c $y = \mathrm{f}(|x|) = |x|^2 - 3|x| - 10$



Reflect the part of the curve where $x \geqslant 0$ (the positive values of x) in the y-axis.

Example 18

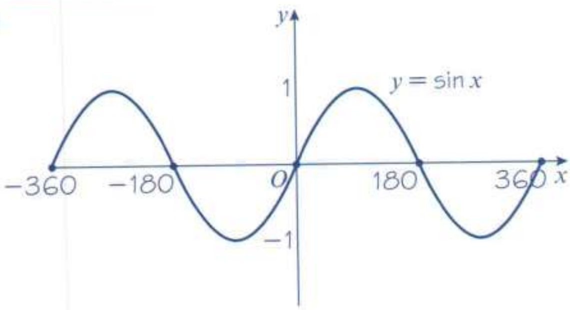
$\mathrm{g}(x) = \sin x, -360° \leqslant x \leqslant 360°$

a Sketch the graph of $y = \mathrm{g}(x)$.

b Sketch the graph of $y = |\mathrm{g}(x)|$.

c Sketch the graph of $y = \mathrm{g}(|x|)$.

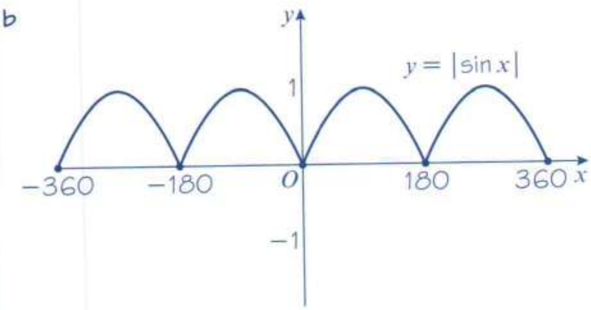
a



The graph is periodic and passes through the origin, (±180, 0) and (±360, 0).

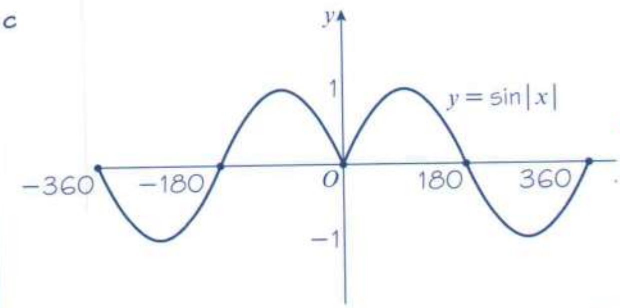
← Year 1, Section 9.5

b



Reflect the part of the curve below the x-axis in the x-axis.

c



Reflect the part of the curve where $x \geqslant 0$ in the y-axis.

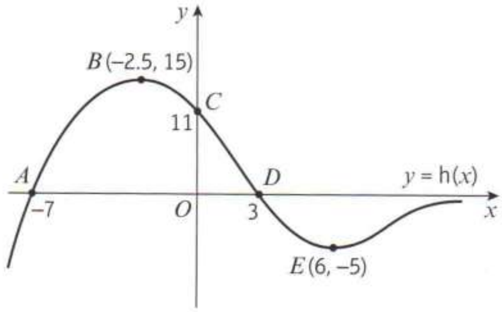
Example 19

The diagram shows the graph of $y = \mathrm{h}(x)$, with five points labelled.

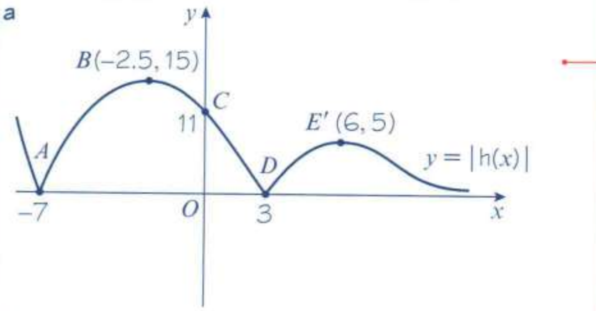
Sketch each of the following graphs, labelling the points corresponding to A, B, C, D and E, and any points of intersection with the coordinate axes.

a $y = |\mathrm{h}(x)|$

b $y = \mathrm{h}(|x|)$



a

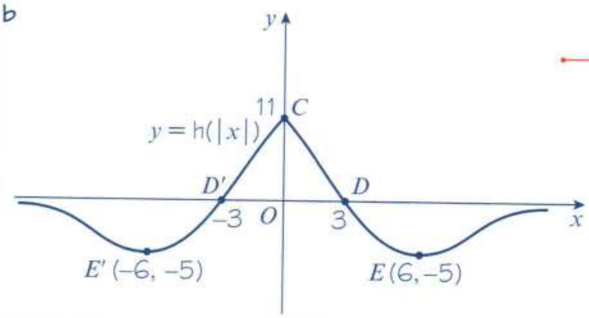


The parts of the curve below the x-axis are reflected in the x-axis.

The points A, B, C and D are unchanged.

The point E was reflected, so the new coordinates are $E'(6, 5)$.

b



The part of the curve to the right of the y-axis is reflected in the y-axis.

The old points A and B had negative x-values so they are no longer part of the graph.

The points C, D and E are unchanged.

There is a new point of intersection with the x-axis at $(-3, 0)$.

The point E was reflected, so the new coordinates are $E'(-6, -5)$.

Exercise 2E

1 $\mathrm{f}(x) = x^2 - 7x - 8$

a Sketch the graph of $y = \mathrm{f}(x)$.

b Sketch the graph of $y = |\mathrm{f}(x)|$.

c Sketch the graph of $y = \mathrm{f}(|x|)$.

2 $\mathrm{g}: x \mapsto \cos x, -360° \leqslant x \leqslant 360°$

a Sketch the graph of $y = \mathrm{g}(x)$.

b Sketch the graph of $y = |\mathrm{g}(x)|$.

c Sketch the graph of $y = \mathrm{g}(|x|)$.

3 $\mathrm{h}: x \mapsto (x - 1)(x - 2)(x + 3)$

a Sketch the graph of $y = \mathrm{h}(x)$.

b Sketch the graph of $y = |\mathrm{h}(x)|$.

c Sketch the graph of $y = \mathrm{h}(|x|)$.

(P) **4** The function k is defined by $\mathrm{k}(x) = \frac{a}{x^2}, a > 0, x \in \mathbb{R}, x \neq 0$.

a Sketch the graph of $y = \mathrm{k}(x)$.

b Explain why it is not necessary to sketch $y = |\mathrm{k}(x)|$ and $y = \mathrm{k}(|x|)$.

The function m is defined by $\mathrm{m}(x) = \frac{a}{x^2}, a < 0, x \in \mathbb{R}, x \neq 0$.

c Sketch the graph of $y = \mathrm{m}(x)$.

d State with a reason whether the following statements are true or false.

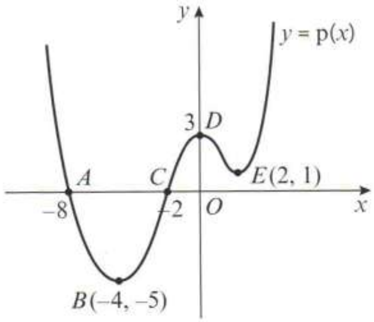
i $|\mathrm{k}(x)| = |\mathrm{m}(x)|$ **ii** $\mathrm{k}(|x|) = \mathrm{m}(|x|)$ **iii** $\mathrm{m}(x) = \mathrm{m}(|x|)$

(E) **5** The diagram shows the graph of $y = \mathrm{p}(x)$ with 5 points labelled.

Sketch each of the following graphs, labelling the points corresponding to A, B, C, D and E, and any points of intersection with the coordinate axes.

a $y = |\mathrm{p}(x)|$ **(3 marks)**

b $y = \mathrm{p}(|x|)$ **(3 marks)**

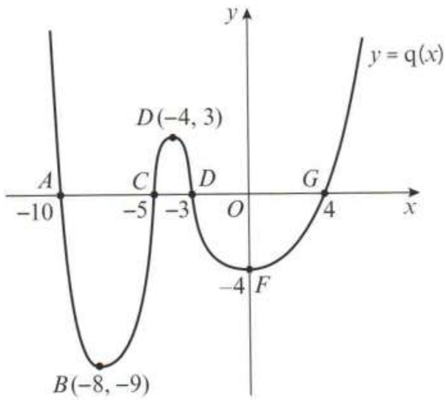


(E) **6** The diagram shows the graph of $y = \mathrm{q}(x)$ with 7 points labelled.

Sketch each of the following graphs, labelling the points corresponding to A, B, C, D and E, and any points of intersection with the coordinate axes.

a $y = |\mathrm{q}(x)|$ **(4 marks)**

b $y = \mathrm{q}(|x|)$ **(3 marks)**



7 $\mathrm{k}(x) = \frac{a}{x}, a > 0, x \neq 0$

a Sketch the graph of $y = \mathrm{k}(x)$.

b Sketch the graph of $y = |\mathrm{k}(x)|$.

c Sketch the graph of $y = \mathrm{k}(|x|)$.

8 $\mathrm{m}(x) = \frac{a}{x}, a < 0, x \neq 0$

a Sketch the graph of $y = \mathrm{m}(x)$.

b Describe the relationship between $y = |\mathrm{m}(x)|$ and $y = \mathrm{m}(|x|)$.

9 $\mathrm{f}(x) = \mathrm{e}^x$ and $\mathrm{g}(x) = \mathrm{e}^{-x}$

a Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ on the same axes.

b Explain why it is not necessary to sketch $y = |\mathrm{f}(x)|$ and $y = |\mathrm{g}(x)|$.

c Sketch the graphs of $y = \mathrm{f}(|x|)$ and $y = \mathrm{g}(|x|)$ on the same axes.

(E/P) **10** The function f(x) is defined by

$$f(x) = \begin{cases} -2x - 6, & -5 \leqslant x < -1 \\ (x+1)^2, & -1 \leqslant x \leqslant 2 \end{cases}$$

a Sketch f(x) stating its range. **(5 marks)**

b Sketch the graph of $y = |f(x)|$. **(3 marks)**

c Sketch the graph of $y = f(|x|)$. **(3 marks)**

Problem-solving

A piecewise function like this does not have to be continuous. Work out the value of both expressions when $x = -1$ to help you with your sketch.

2.6 Combining transformations

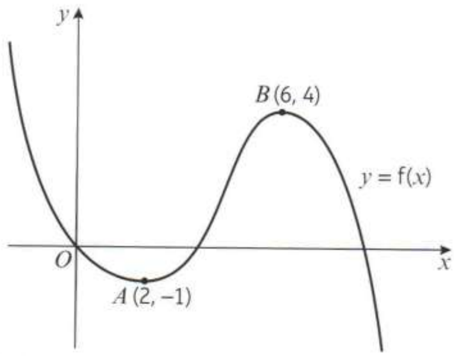
You can use combinations of the following transformations of a function to sketch graphs of more complicated transformations.

- **f($x + a$) is a translation by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$**
- **f(x) + a is a translation by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$**
- **f($-x$) reflects f(x) in the y-axis.**
- **$-$f(x) reflects f(x) in the x-axis.**
- **f(ax) is a horizontal stretch of scale factor $\frac{1}{a}$**
- **af(x) is a vertical stretch of scale factor a**

Links You can think of f($-x$) and $-$f(x) as stretches with scale factor -1. ← **Year 1, Sections 4.6, 4.7**

Example 20

The diagram shows a sketch of the graph of $y = f(x)$. The curve passes through the origin O, the point $A(2, -1)$ and the point $B(6, 4)$.



Sketch the graphs of:

a $y = 2f(x) - 1$

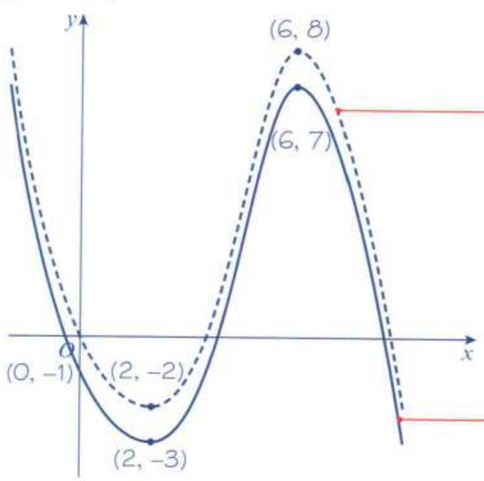
b $y = f(x + 2) + 2$

c $y = \frac{1}{4}f(2x)$

d $y = -f(x - 1)$

In each case, find the coordinates of the images of the points O, A and B.

a $y = 2f(x) - 1$



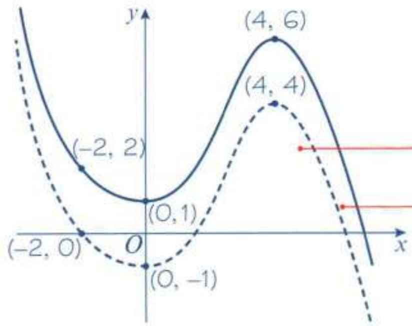
The images of O, A and B are (0, −1), (2, −3) and (6, 7) respectively.

Apply the stretch first. The dotted curve is the graph of $y = 2f(x)$, which is a vertical stretch with scale factor 2.

Next apply the translation. The solid curve is the graph of $y = 2f(x) - 1$, as required. This is a translation of $y = 2f(x)$ by vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Watch out The order is important. If you applied the transformations in the opposite order you would have the graph of $y = 2(f(x) - 1)$ or $y = 2f(x) - 2$.

b $y = f(x + 2) + 2$

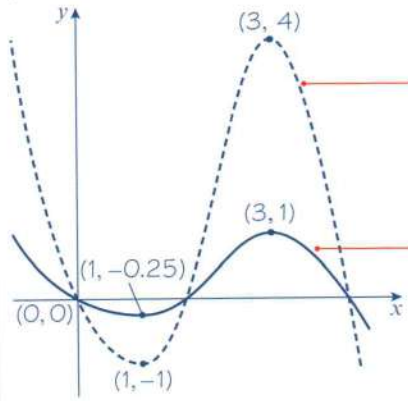


The images of O, A and B are (−2, 2), (0, 1) and (4, 6) respectively.

Apply the translation **inside** the brackets first. The dotted curve is the graph of $y = f(x + 2)$, which is a translation of $y = f(x)$ by vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$.

Next apply the translation **outside** the brackets. The solid curve is the graph of $y = f(x + 2) + 2$, as required. This is a translation of $y = f(x + 2)$ by vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

c $y = \frac{1}{4}f(2x)$

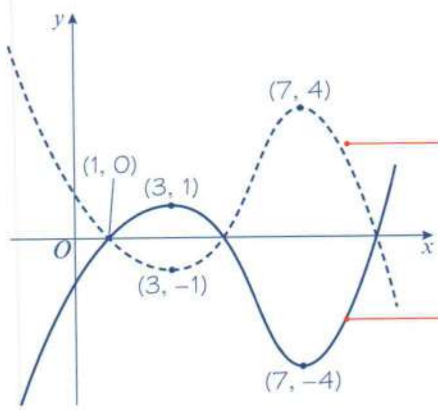


The images of O, A and B are (0, 0), (1, −0.25) and (3, 1) respectively.

Apply the stretch **inside** the brackets first. The dotted curve is the graph of $y = f(2x)$, which is a horizontal stretch with scale factor $\frac{1}{2}$

Then apply the stretch **outside** the brackets. The solid curve is the graph of $y = \frac{1}{4}f(2x)$, as required. This is a vertical stretch of $y = f(2x)$ with scale factor $\frac{1}{4}$

d $y = -f(x - 1)$



The images of O, A and B are (1, 0), (3, 1) and (7, −4) respectively.

Apply the translation **inside** the brackets first. The dotted curve is the graph of $y = f(x - 1)$, which is a translation of $y = f(x)$ by vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then apply the reflection **outside** the brackets. The solid curve is the graph of $y = -f(x - 1)$, as required. This is a reflection of $y = f(x - 1)$ in the x-axis.

Example 21

$f(x) = \ln x, x > 0$

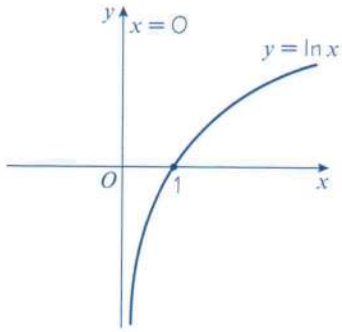
Sketch the graphs of

a $y = 2f(x) - 3$ **b** $y = |f(-x)|$

Show, on each diagram, the point where the graph meets or crosses the x-axis.

In each case, state the equation of the asymptote.

a



$2\ln x - 3 = 0$

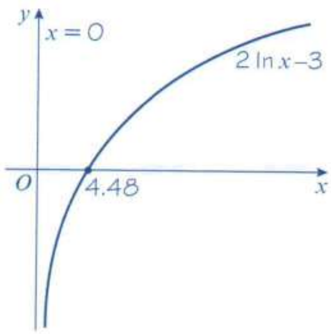
$\ln x = \frac{3}{2}$

$x = e^{\frac{3}{2}}$

$= 4.48$ (3 s.f.)

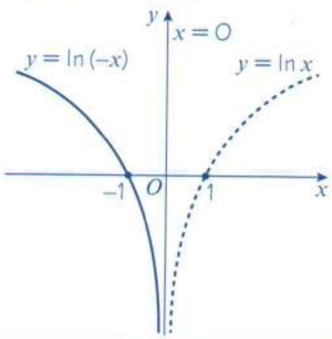
Solve this equation to find the x-intercept of $y = 2f(x) - 3$.

The graph $y = 2\ln x - 3$ will cross the x-axis at (4.48, 0).



The original graph underwent a vertical stretch by a scale factor of 2 and then a vertical translation by vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

b The graph of $y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis.



The original graph is first reflected in the y-axis.

The x-intercept becomes (–1, 0).

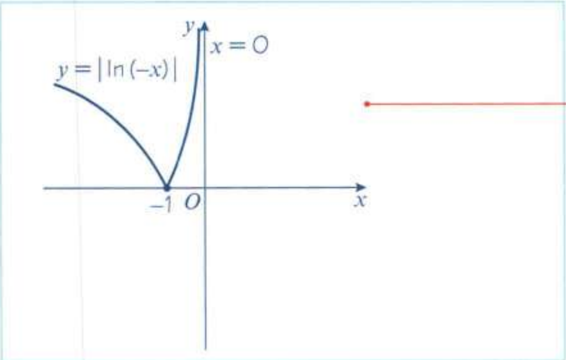
The asymptote is unchanged.

Online Explore combinations of transformations using technology.

Problem-solving

You have not been asked to sketch $y = f(x)$ in this question, but it is a good idea to do this before sketching transformations of this graph.

Sketch $y = f(x)$, labelling its asymptote and the coordinates of the point where it crosses the x-axis. ← Year 1, Section 14.3



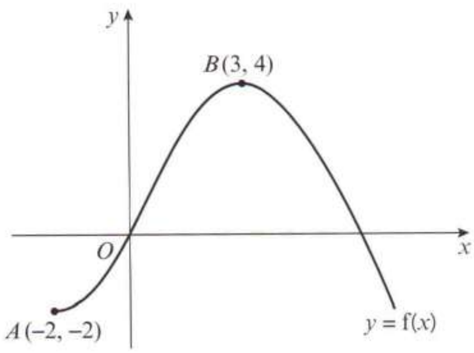
To sketch the graph of $y = |f(-x)|$ reflect any negative y-values of $y = f(-x)$ in the x-axis.

Exercise 2F

1 The diagram shows a sketch of the graph $y = f(x)$. The curve passes through the origin O, the point $A(-2, -2)$ and the point $B(3, 4)$.
On separate axes, sketch the graphs of:

a $y = 3f(x) + 2$ **b** $y = f(x - 2) - 5$
c $y = \frac{1}{2}f(x + 1)$ **d** $y = -f(2x)$
e $y = |f(x)|$ **f** $y = |f(-x)|$

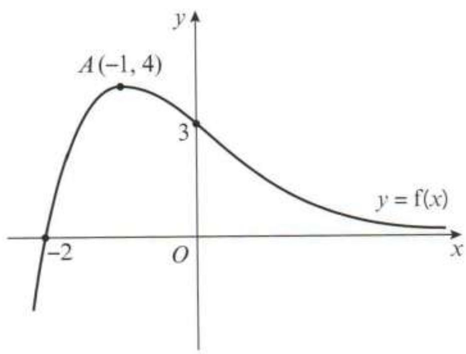
In each case find the coordinates of the images of the points O, A and B.



2 The diagram shows a sketch of the graph $y = f(x)$. The curve has a maximum at the point $A(-1, 4)$ and crosses the axes at the points $(0, 3)$ and $(-2, 0)$.
On separate axes, sketch the graphs of:

a $y = 3f(x - 2)$ **b** $y = \frac{1}{2}f\left(\frac{1}{2}x\right)$
c $y = -f(x) + 4$ **d** $y = -2f(x + 1)$
e $y = 2f(|x|)$

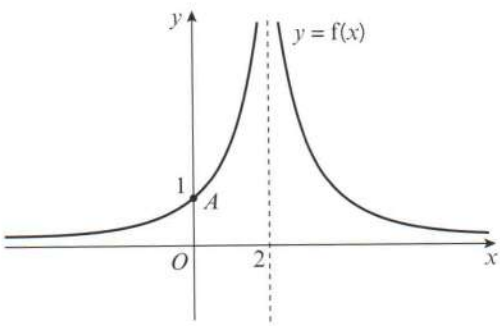
For each graph, find, where possible, the coordinates of the maximum or minimum and the coordinates of the intersection points with the axes.



3 The diagram shows a sketch of the graph $y = f(x)$. The lines $x = 2$ and $y = 0$ (the x-axis) are asymptotes to the curve.
On separate axes, sketch the graphs of:

a $y = 3f(x) - 1$ **b** $y = f(x + 2) + 4$
c $y = -f(2x)$ **d** $y = f(|x|)$

For each part, state the equations of the asymptotes and the new coordinates of the point A.



(E) **4** The function g is defined by

g: $x \mapsto (x-2)^2 - 9, x \in \mathbb{R}$.

a Draw a sketch of the graph of $y = \text{g}(x)$, labelling the turning point and the x- and y-intercepts. **(3 marks)**

b Write down the coordinates of the turning point when the curve is transformed as follows:

i $2\text{g}(x-4)$ **(2 marks)**

ii $\text{g}(2x)$ **(2 marks)**

iii $|\text{g}(x)|$ **(2 marks)**

c Sketch the curve with equation $y = \text{g}(|x|)$. On your sketch show the coordinates of all turning points and all x- and y-intercepts. **(4 marks)**

5 $\text{h}(x) = 2\sin x, -180° \leqslant x \leqslant 180°$.

a Sketch the graph of $y = \text{h}(x)$.

b Write down the coordinates of the minimum, A, and the maximum, B.

c Sketch the graphs of:

i $\text{h}(x - 90°) + 1$ **ii** $\frac{1}{4}\text{h}\left(\frac{1}{2}x\right)$ **iii** $\frac{1}{2}|\text{h}(-x)|$

In each case find the coordinates of the images of the points O, A and B.

2.7 Solving modulus problems

You can use combinations of transformations together with $|\text{f}(x)|$ and $\text{f}(|x|)$ and an understanding of domain and range to solve problems.

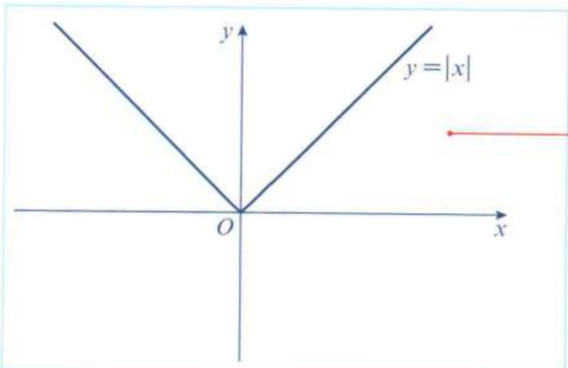
Example 22

Given the function $\text{t}(x) = 3|x-1| - 2, x \in \mathbb{R}$,

a sketch the graph of the function

b state the range of the function

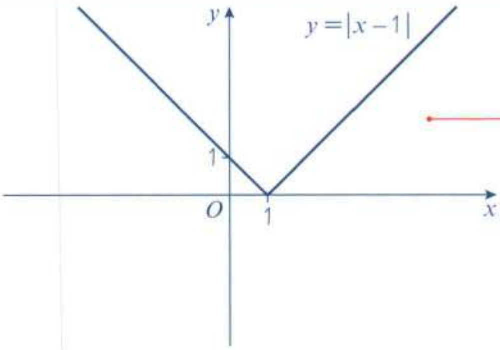
c solve the equation $\text{t}(x) = \frac{1}{2}x + 3$.



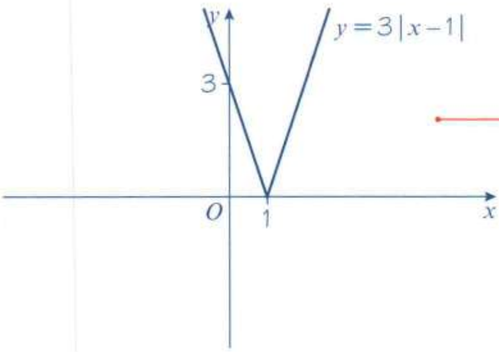
Problem-solving

Use transformations to sketch the graph of $y = 3|x-1| - 2$.

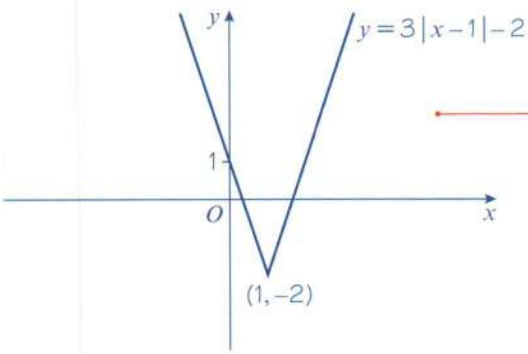
Sketch the graph of $y = |x|$.



Step 1
Horizontal translation by vector $\begin{pmatrix}1\\0\end{pmatrix}$.



Step 2
Vertical stretch, scale factor 3.

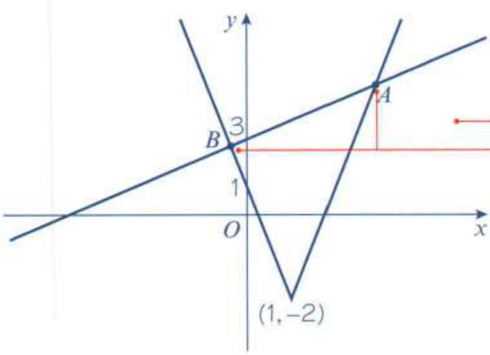


Step 3
Vertical translation by vector $\begin{pmatrix}0\\-2\end{pmatrix}$.

b The range of the function $t(x)$ is $y \in \mathbb{R}$, $y \geqslant -2$.

The graph has a minimum at (1, −2).

c



First draw a sketch of $y = 3|x - 1| - 2$ and the line $y = \frac{1}{2}x + 3$.

The sketch shows there are two solutions, at A and B, the points of intersection.

At A, $3(x - 1) - 2 = \frac{1}{2}x + 3$

$$3x - 5 = \frac{1}{2}x + 3$$

$$\frac{5}{2}x = 8$$

$$x = \frac{16}{5}$$

This is the solution on the original part of the graph.

At B, $-3(x - 1) - 2 = \frac{1}{2}x + 3$

When $f(x) < 0$, $|f(x)| = -f(x)$, so use $-(3x - 1) - 2$ to find the solution on the reflected part of the graph.

$$-3x + 3 - 2 = \frac{1}{2}x + 3$$

$$-\frac{7}{2}x = 2$$

$$x = -\frac{4}{7}$$

This is the solution corresponding to point B on the sketch.

The solutions are $x = \frac{16}{5}$ and $x = -\frac{4}{7}$

Example 23

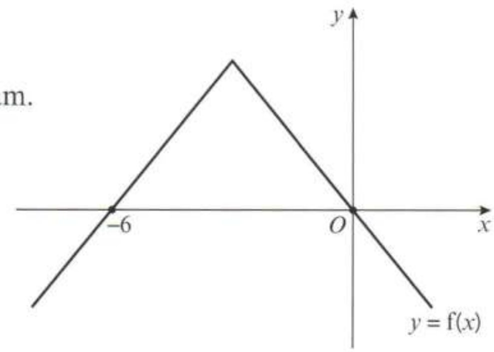
The function f is defined by f: $x \mapsto 6 - 2|x + 3|$.

A sketch of the graph of the function is shown in the diagram.

a State the range of f.

b Give a reason why f^{-1} does not exist.

c Solve the inequality $f(x) > 5$.



a The range of $f(x)$ is $f(x) \leqslant 6$.

The greatest value $f(x)$ can take is 6.

b $f(x)$ is a many-to-one function. Therefore, f^{-1} does not exist.

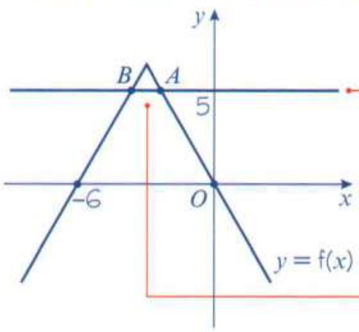
For example, $f(0) = f(-6) = 0$.

Problem-solving

Only one-to-one functions have inverses.

c $f(x) = 5$ at the points A and B.

$f(x) > 5$ between the points A and B.



Add the line $y = 5$ to the graph of $y = f(x)$.

Between the points A and B, the graph of $y = f(x)$ is above the line $y = 5$.

At A, $6 - 2(x + 3) = 5$

$$-2(x + 3) = -1$$

$$x + 3 = \frac{1}{2}$$

$$x = -\frac{5}{2}$$

This is the solution on the original part of the graph.

At B, $6 - (-2(x + 3)) = 5$

$2(x + 3) = -1$

$x + 3 = -\frac{1}{2}$

$x = -\frac{7}{2}$

The solution to the inequality $f(x) > 5$ is

$-\frac{7}{2} < x < -\frac{5}{2}$

When $f(x) < 0$, $|f(x)| = -f(x)$, so use the negative argument, $-2(x + 3)$.

This is the solution on the reflected part of the graph.

Online Explore the solution using technology.

Exercise 2G

(P) **1** For each function

i sketch the graph of $y = f(x)$

ii state the range of the function.

a $f: x \mapsto 4|x| - 3, x \in \mathbb{R}$

b $f(x) = \frac{1}{3}|x + 2| - 1, x \in \mathbb{R}$

c $f(x) = -2|x - 1| + 6, x \in \mathbb{R}$

d $f: x \mapsto -\frac{5}{2}|x| + 4, x \in \mathbb{R}$

Hint For part **b** transform the graph of $y = |x|$ by:

- A translation by vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
- A vertical sketch with scale factor $\frac{1}{3}$
- A translation by vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

2 Given that $p(x) = 2|x + 4| - 5, x \in \mathbb{R}$,

a sketch the graph of $y = p(x)$

b shade the region of the graph that satisfies $y \geqslant p(x)$.

3 Given that $q(x) = -3|x| + 6, x \in \mathbb{R}$,

a sketch the graph of $y = q(x)$

b shade the region of the graph that satisfies $y < q(x)$.

4 The function f is defined as

$f: x \mapsto 4|x + 6| + 1, x \in \mathbb{R}$.

a Sketch the graph of $y = f(x)$.

b State the range of the function.

c Solve the equation $f(x) = -\frac{1}{2}x + 1$.

5 Given that $g(x) = -\frac{5}{2}|x - 2| + 7, x \in \mathbb{R}$,

a sketch the graph of $y = g(x)$

b state the range of the function

c solve the equation $g(x) = x + 1$.

E/P **6** The functions m and n are defined as

$m(x) = -2x + k, x \in \mathbb{R}$

$n(x) = 3|x - 4| + 6, x \in \mathbb{R}$

where k is a constant.

The equation $m(x) = n(x)$ has no real roots.
Find the range of possible values for the constant k. **(4 marks)**

Problem-solving

$m(x) = n(x)$ has no real roots means that $y = m(x)$ and $y = n(x)$ do not intersect.

E/P **7** The functions s and t are defined as

$s(x) = -10 - x, x \in \mathbb{R}$

$t(x) = 2|x + b| - 8, x \in \mathbb{R}$

where b is a constant.

The equation $s(x) = t(x)$ has exactly one real root. Find the value of b. **(4 marks)**

E/P **8** The function h is defined by

$h(x) = \frac{2}{3}|x - 1| - 7, x \in \mathbb{R}$

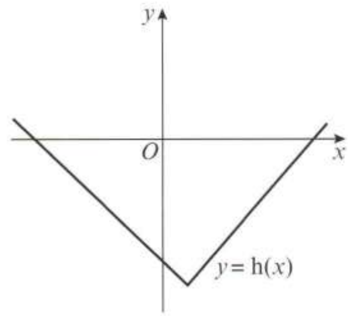
The diagram shows a sketch of the graph $y = h(x)$.

a State the range of h. **(1 mark)**

b Give a reason why h^{-1} does not exist. **(1 mark)**

c Solve the inequality $h(x) < -6$. **(4 marks)**

d State the range of values of k for which the equation $h(x) = \frac{2}{3}x + k$ has no solutions. **(4 marks)**

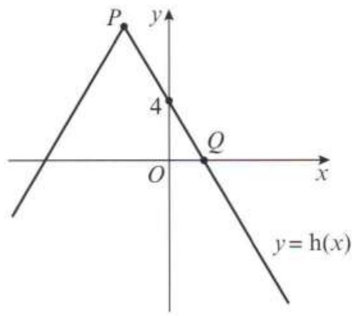


E/P **9** The diagram shows a sketch of part of the graph $y = h(x)$, where $h(x) = a - 2|x + 3|, x \in \mathbb{R}$.
The graph intercepts the y-axis at $(0, 4)$.

a Find the value of a. **(2 marks)**

b Find the coordinates of P and Q. **(3 marks)**

c Solve $h(x) = \frac{1}{3}x + 6$. **(5 marks)**



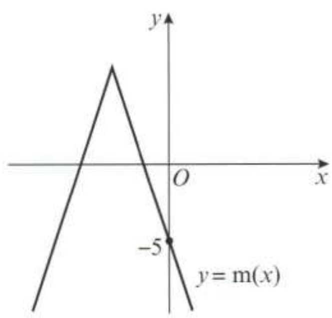
E/P **10** The diagram shows a sketch of part of the graph $y = m(x)$, where $m(x) = -4|x + 3| + 7, x \in \mathbb{R}$.

a State the range of m. **(1 mark)**

b Solve the equation $m(x) = \frac{3}{5}x + 2$. **(4 marks)**

Given that $m(x) = k$, where k is a constant, has two distinct roots

c state the set of possible values for k. **(4 marks)**



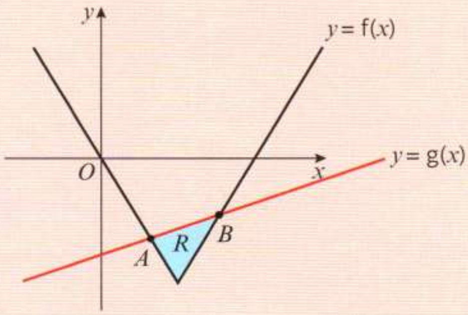
Challenge

1 The functions f and g are defined by

$f(x) = 2|x - 4| - 8, x \in \mathbb{R}$

$g(x) = x - 9, x \in \mathbb{R}$

The diagram shows a sketch of the graphs of $y = f(x)$ and $y = g(x)$.



a Find the coordinates of the points A and B.

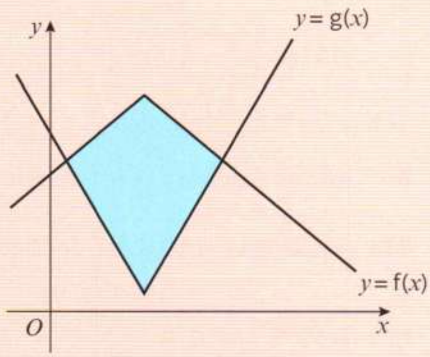
b Find the area of the region R.

2 The functions f and g are defined as:

$f(x) = -|x - 3| + 10, x \in \mathbb{R}$

$g(x) = 2|x - 3| + 2, x \in \mathbb{R}$

Show that the area of the shaded region is $\frac{64}{3}$



Mixed exercise 2

1 **a** On the same axes, sketch the graphs of $y = 2 - x$ and $y = 2|x + 1|$.

b Hence, or otherwise, find the values of x for which $2 - x = 2|x + 1|$.

E/P **2** The equation $|2x - 11| = \frac{1}{2}x + k$ has exactly two distinct solutions.

Find the range of possible values of k. **(4 marks)**

E/P **3** Solve $|5x - 2| = -\frac{1}{4}x + 8$. **(4 marks)**

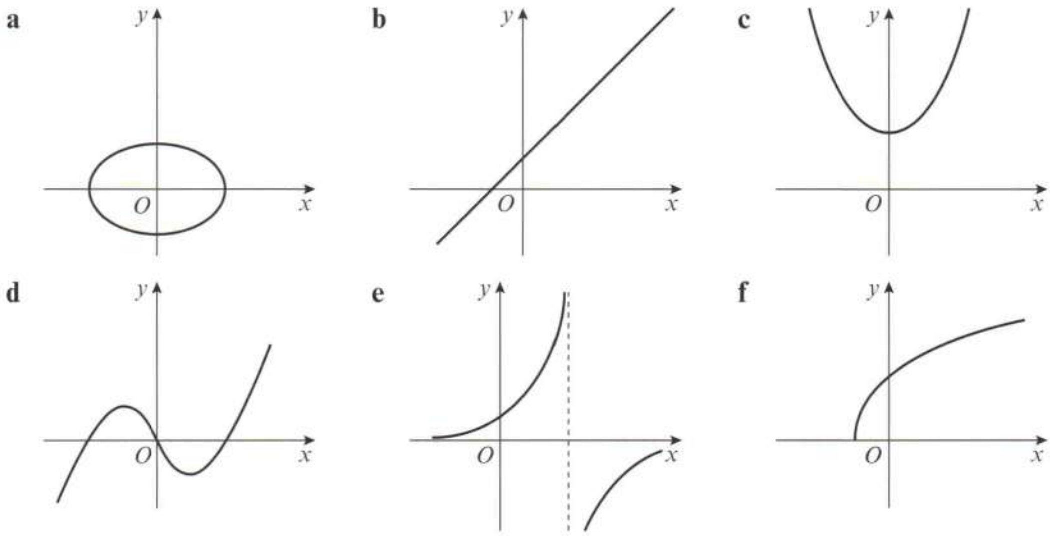
E/P **4** **a** On the same set of axes, sketch $y = |12 - 5x|$ and $y = -2x + 3$. **(3 marks)**

b State with a reason whether there are any solutions to the equation $|12 - 5x| = -2x + 3$ **(2 marks)**

5 For each of the following mappings:

i state whether the mapping is one-to-one, many-to-one or one-to-many

ii state whether the mapping could represent a function.



E **6** The function f(x) is defined by

$$f(x) = \begin{cases} -x, & x \leqslant 1 \\ x - 2, & x > 1 \end{cases}$$

a Sketch the graph of f(x) for $-2 \leqslant x \leqslant 6$. **(4 marks)**

b Find the values of x for which $f(x) = -\frac{1}{2}$ **(3 marks)**

E **7** The functions p and q are defined by

p: $x \mapsto x^2 + 3x - 4, x \in \mathbb{R}$

q: $x \mapsto 2x + 1, x \in \mathbb{R}$

a Find an expression for pq(x). **(2 marks)**

b Solve pq(x) = qq(x). **(3 marks)**

E **8** The function g(x) is defined as $g(x) = 2x + 7$, $\{x \in \mathbb{R}, x \geqslant 0\}$.

a Sketch $y = g(x)$ and find the range. **(3 marks)**

b Determine $y = g^{-1}(x)$, stating its domain. **(3 marks)**

c Sketch $y = g^{-1}(x)$ on the same axes as $y = g(x)$, stating the relationship between the two graphs. **(2 marks)**

E **9** The function f is defined by

$$f: x \mapsto \frac{2x + 3}{x - 1}, \quad \{x \in \mathbb{R}, x > 1\}$$

a Find $f^{-1}(x)$. **(4 marks)**

b Find: **i** the range of $f^{-1}(x)$

ii the domain of $f^{-1}(x)$ **(2 marks)**

E/P **10** The functions f and g are given by

$$\text{f}: x \mapsto \frac{x}{x^2 - 1} - \frac{1}{x + 1}, \quad \{x \in \mathbb{R}, x > 1\}$$

$$\text{g}: x \mapsto \frac{2}{x}, \quad \{x \in \mathbb{R}, x > 0\}$$

a Show that $\text{f}(x) = \frac{1}{(x - 1)(x + 1)}$ **(3 marks)**

b Find the range of $\text{f}(x)$. **(1 mark)**

c Solve $\text{gf}(x) = 70$. **(4 marks)**

P **11** The following functions $\text{f}(x)$, $\text{g}(x)$ and $\text{h}(x)$ are defined by

$$\text{f}(x) = 4(x - 2), \quad \{x \in \mathbb{R}, x \geqslant 0\}$$

$$\text{g}(x) = x^3 + 1, \quad \{x \in \mathbb{R}\}$$

$$\text{h}(x) = 3^x, \quad \{x \in \mathbb{R}\}$$

a Find $\text{f}(7)$, $\text{g}(3)$ and $\text{h}(-2)$.

b Find the range of $\text{f}(x)$ and the range of $\text{g}(x)$.

c Find $\text{g}^{-1}(x)$.

d Find the composite function $\text{fg}(x)$.

e Solve $\text{gh}(a) = 244$.

E/P **12** The function $\text{f}(x)$ is defined by $\text{f}: x \mapsto x^2 + 6x - 4$, $x \in \mathbb{R}$, $x > a$, for some constant a.

a State the least value of a for which f^{-1} exists. **(4 marks)**

b Given that $a = 0$, find f^{-1}, stating its domain. **(4 marks)**

E/P **13** The functions f and g are given by

$$\text{f}: x \mapsto 4x - 1, \{x \in \mathbb{R}\}$$

$$\text{g}: x \mapsto \frac{3}{2x - 1}, \left\{x \in \mathbb{R}, x \neq \frac{1}{2}\right\}$$

Find in its simplest form:

a the inverse function f^{-1} **(2 marks)**

b the composite function gf, stating its domain **(3 marks)**

c the values of x for which $2\text{f}(x) = \text{g}(x)$, giving your answers to 3 decimal places. **(4 marks)**

E **14** The functions f and g are given by

$$\text{f}: x \mapsto \frac{x}{x - 2}, \quad \{x \in \mathbb{R}, x \neq 2\}$$

$$\text{g}: x \mapsto \frac{3}{x}, \quad \{x \in \mathbb{R}, x \neq 0\}$$

a Find an expression for $\text{f}^{-1}(x)$. **(2 marks)**

b Write down the range of $\text{f}^{-1}(x)$. **(1 mark)**

c Calculate $\text{gf}(1.5)$. **(2 marks)**

d Use algebra to find the values of x for which $\text{g}(x) = \text{f}(x) + 4$. **(4 marks)**

15 The function $\text{n}(x)$ is defined by

$$\text{n}(x) = \begin{cases} 5 - x, & x \leqslant 0 \\ x^2, & x > 0 \end{cases}$$

a Find $\text{n}(-3)$ and $\text{n}(3)$.

b Solve the equation $\text{n}(x) = 50$.

16 $g(x) = \tan x$, $-180° \leqslant x \leqslant 180°$

a Sketch the graph of $y = g(x)$.

b Sketch the graph of $y = |g(x)|$.

c Sketch the graph of $y = g(|x|)$.

(E) **17** The diagram shows the graph of $f(x)$.

The points $A(4, -3)$ and $B(9, 3)$ are turning points on the graph.

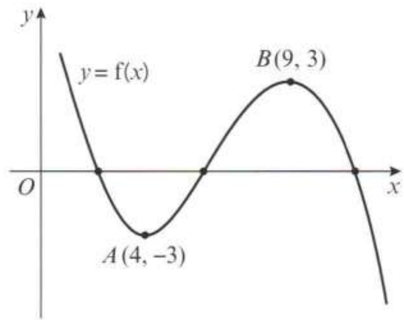
Sketch on separate diagrams, the graphs of

a $y = f(2x) + 1$ **(3 marks)**

b $y = |f(x)|$ **(3 marks)**

c $y = -f(x - 2)$ **(3 marks)**

Indicate on each diagram the coordinates of any turning points on your sketch.



(E) **18** Functions f and g are defined by

$f: x \mapsto 4 - x, \quad \{x \in \mathbb{R}\}$

$g: x \mapsto 3x^2, \quad \{x \in \mathbb{R}\}$

a Write down the range of g. **(1 mark)**

b Solve $gf(x) = 48$. **(4 marks)**

c Sketch the graph of $y = |f(x)|$ and hence find the values of x for which $|f(x)| = 2$. **(4 marks)**

(E/P) **19** The function f is defined by $f: x \mapsto |2x - a|$, $\{x \in \mathbb{R}\}$, where a is a positive constant.

a Sketch the graph of $y = f(x)$, showing the coordinates of the points where the graph cuts the axes. **(3 marks)**

b On a separate diagram, sketch the graph of $y = f(2x)$, showing the coordinates of the points where the graph cuts the axes. **(2 marks)**

c Given that a solution of the equation $f(x) = \frac{1}{2}x$ is $x = 4$, find the two possible values of a. **(4 marks)**

(E/P) **20** **a** Sketch the graph of $y = |x - 2a|$, where a is a positive constant. Show the coordinates of the points where the graph meets the axes. **(3 marks)**

b Using algebra solve, for x in terms of a, $|x - 2a| = \frac{1}{3}x$. **(4 marks)**

c On a separate diagram, sketch the graph of $y = a - |x - 2a|$, where a is a positive constant. Show the coordinates of the points where the graph cuts the axes. **(4 marks)**

(E/P) **21** **a** Sketch the graph of $y = |2x + a|$, $a > 0$, showing the coordinates of the points where the graph meets the coordinate axes. **(3 marks)**

b On the same axes, sketch the graph of $y = \frac{1}{x}$ **(2 marks)**

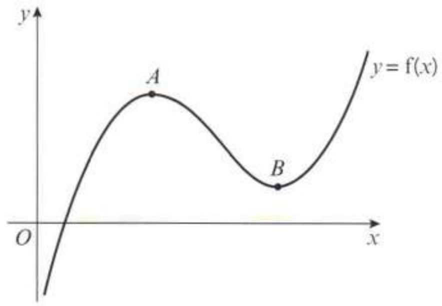
c Explain how your graphs show that there is only one solution of the equation $x|2x + a| - 1 = 0$ **(2 marks)**

d Find, using algebra, the value of x for which $x|2x + a| - 1 = 0$. **(3 marks)**

(E/P) **22** The diagram shows part of the curve with equation $y = f(x)$, where

$$f(x) = x^2 - 7x + 5\ln x + 8, \quad x > 0$$

The points A and B are the stationary points of the curve.

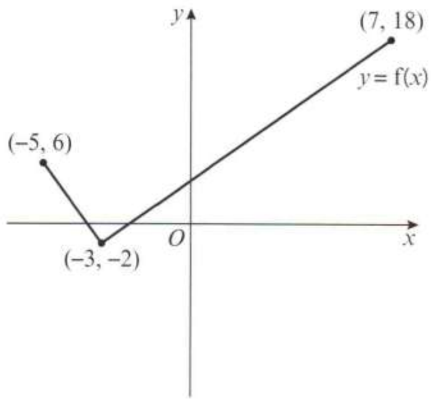


a Using calculus and showing your working, find the coordinates of the points A and B. **(4 marks)**

b Sketch the curve with equation $y = -3f(x - 2)$. **(3 marks)**

c Find the coordinates of the stationary points of the curve with equation $y = -3f(x - 2)$. State, without proof, which point is a maximum and which point is a minimum. **(3 marks)**

(E/P) **23** The function f has domain $-5 \leqslant x \leqslant 7$ and is linear from $(-5, 6)$ to $(-3, -2)$ and from $(-3, -2)$ to $(7, 18)$. The diagram shows a sketch of the function.



a Write down the range of f. **(1 mark)**

b Find ff(−3). **(2 marks)**

c Sketch the graph of $y = |f(x)|$, marking the points at which the graph meets or cuts the axes. **(3 marks)**

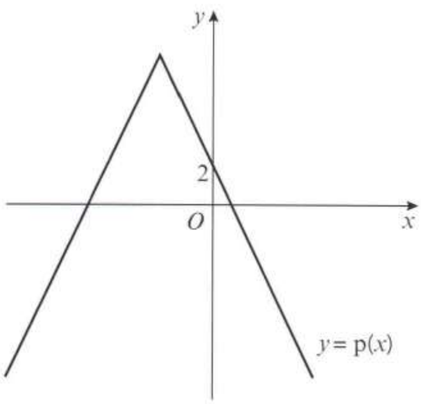
The function g is defined by $g: x \mapsto x^2 - 7x + 10$.

d Solve the equation $fg(x) = 2$. **(3 marks)**

(P) **24** The function p is defined by

$p: x \mapsto -2|x + 4| + 10$

The diagram shows a sketch of the graph.



a State the range of p. **(1 mark)**

b Give a reason why p^{-1} does not exist. **(1 mark)**

c Solve the inequality $p(x) > -4$. **(4 marks)**

d State the range of values of k for which the equation $p(x) = -\frac{1}{2}x + k$ has no solutions. **(4 marks)**

Challenge

a Sketch, on a single diagram, the graphs of $y = a^2 - x^2$ and $y = |x + a|$, where a is a constant and $a > 1$.

b Write down the coordinates of the points where the graph of $y = a^2 - x^2$ cuts the coordinate axes.

c Given that the two graphs intersect at $x = 4$, calculate the value of a.

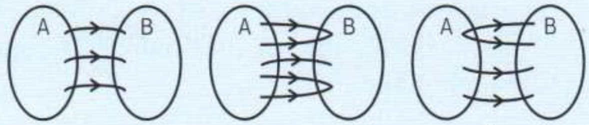
Summary of key points

1 A modulus function is, in general, a function of the type $y = |f(x)|$.

- When $f(x) \geqslant 0$, $|f(x)| = f(x)$
- When $f(x) < 0$, $|f(x)| = -f(x)$

2 To sketch the graph of $y = |ax + b|$, sketch $y = ax + b$ then reflect the section of the graph below the x-axis in the x-axis.

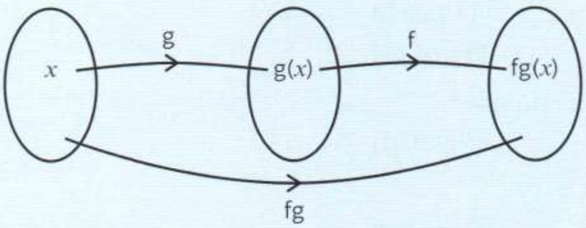
3 A mapping is a **function** if every input has a distinct output. Functions can either be **one-to-one** or **many-to-one**.



one-to-one function | many-to-one function | not a function

4 $fg(x)$ means apply g first, then apply f.

$fg(x) = f(g(x))$



5 Functions $f(x)$ and $f^{-1}(x)$ are inverses of each other. $ff^{-1}(x) = x$ and $f^{-1}f(x) = x$.

6 The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each another in the line $y = x$.

7 The domain of $f(x)$ is the range of $f^{-1}(x)$.

8 The range of $f(x)$ is the domain of $f^{-1}(x)$.

9 To sketch the graph of $y = |f(x)|$

- Sketch the graph of $y = f(x)$
- Reflect any parts where $f(x) < 0$ (parts below the x-axis) in the x-axis
- Delete the parts below the x-axis

10 To sketch the graph of $y = f(|x|)$

- Sketch the graph of $y = f(x)$ for $x \geqslant 0$
- Reflect this in the y-axis

11 $f(x + a)$ is a horizontal translation of $-a$.

12 $f(x) + a$ is a vertical translation of $+a$.

13 $f(ax)$ is a horizontal stretch of scale factor $\frac{1}{a}$

14 $af(x)$ is a vertical stretch of scale factor a.

15 $f(-x)$ reflects $f(x)$ in the y-axis.

16 $-f(x)$ reflects $f(x)$ in the x-axis.