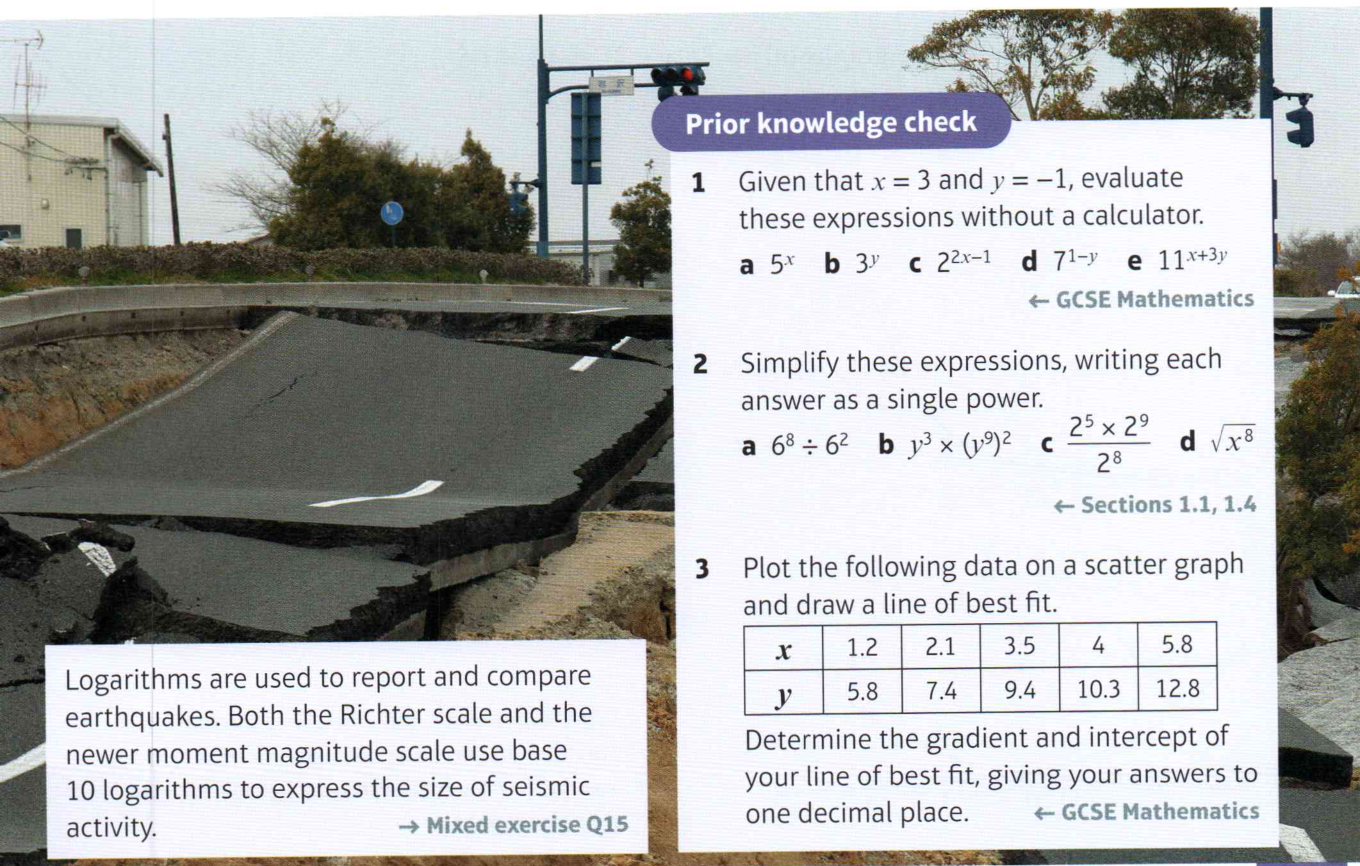
Exponentials and logarithms

14

Objectives

After completing this unit you should be able to:

- Sketch graphs of the form $y = a^x$, $y = e^x$, and transformations of these graphs → pages 312–317
- Differentiate e^{kx} and understand why this result is important → pages 314–317
- Use and interpret models that use exponential functions → pages 317–319
- Recognise the relationship between exponents and logarithms → pages 319–321
- Recall and apply the laws of logarithms → pages 321–324
- Solve equations of the form $a^x = b$ → pages 324–325
- Describe and use the natural logarithm function → pages 326–328
- Use logarithms to estimate the values of constants in non-linear models → pages 328–333



Logarithms are used to report and compare earthquakes. Both the Richter scale and the newer moment magnitude scale use base 10 logarithms to express the size of seismic activity. → Mixed exercise Q15

Prior knowledge check

1 Given that $x = 3$ and $y = -1$, evaluate these expressions without a calculator.

a 5^x **b** 3^y **c** 2^{2x-1} **d** 7^{1-y} **e** 11^{x+3y}

← GCSE Mathematics

2 Simplify these expressions, writing each answer as a single power.

a $6^8 \div 6^2$ **b** $y^3 \times (y^9)^2$ **c** $\dfrac{2^5 \times 2^9}{2^8}$ **d** $\sqrt{x^8}$

← Sections 1.1, 1.4

3 Plot the following data on a scatter graph and draw a line of best fit.

x	1.2	2.1	3.5	4	5.8
y	5.8	7.4	9.4	10.3	12.8

Determine the gradient and intercept of your line of best fit, giving your answers to one decimal place. ← GCSE Mathematics

14.1 Exponential functions

Functions of the form $f(x) = a^x$, where a is a constant, are called **exponential functions**. You should become familiar with these functions and the shapes of their graphs.

For an example, look at a table of values of $y = 2^x$.

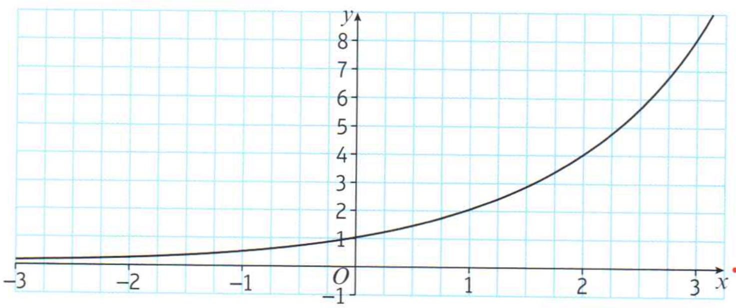
x	−3	−2	−1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notation In the expression 2^x, x can be called an **index**, a **power** or an **exponent.**

Links Recall that $2^0 = 1$ and that $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ ← Section 1.4

The value of 2^x tends towards 0 as x decreases, and grows without limit as x increases.

The graph of $y = 2^x$ is a smooth curve that looks like this:



The x-axis is an asymptote to the curve.

Example 1

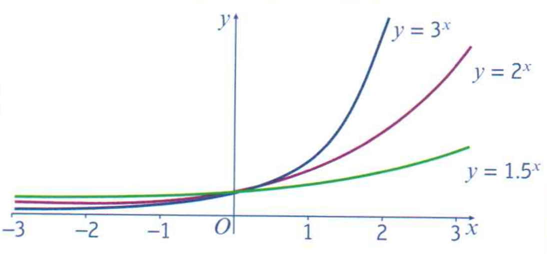
a On the same axes sketch the graphs of $y = 3^x$, $y = 2^x$ and $y = 1.5^x$.

b On another set of axes sketch the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^x$.

a For all three graphs, $y = 1$ when $x = 0$.
When $x > 0$, $3^x > 2^x > 1.5^x$.
When $x < 0$, $3^x < 2^x < 1.5^x$.

$a^0 = 1$

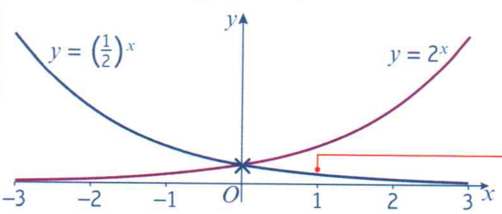
Work out the relative positions of the three graphs.



Whenever $a > 1$, $f(x) = a^x$ is an increasing function. In this case, the value of a^x grows without limit as x **increases**, and tends towards 0 as x **decreases**.

b The graph of $y = \left(\frac{1}{2}\right)^x$ is a reflection in the y-axis of the graph of $y = 2^x$.

Since $\frac{1}{2} = 2^{-1}$, $y = \left(\frac{1}{2}\right)^x$ is the same as $y = (2^{-1})^x = 2^{-x}$.



Whenever $0 < a < 1$, $f(x) = a^x$ is a decreasing function. In this case, the value of a^x tends towards 0 as x **increases**, and grows without limit as x **decreases**.

Example 2

Sketch the graph of $y = \left(\frac{1}{2}\right)^{x-3}$. Give the coordinates of the point where the graph crosses the y-axis.

If $f(x) = \left(\frac{1}{2}\right)^x$ then $y = f(x - 3)$.

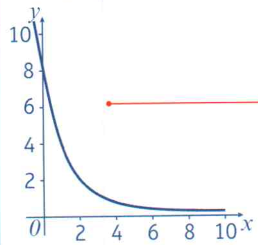
The graph is a translation of the graph $y = \left(\frac{1}{2}\right)^x$ by the vector $\begin{pmatrix}3\\0\end{pmatrix}$.

The graph crosses the y-axis when $x = 0$.

$y = \left(\frac{1}{2}\right)^{0-3}$

$y = 8$

The graph crosses the y-axis at (0, 8).



Problem-solving

If you have to sketch the graph of an unfamiliar function, try writing it as a transformation of a familiar function. ← Section 4.5

You can also consider this graph as a stretch of the graph $y = \left(\frac{1}{2}\right)^x$

$$\begin{aligned} y &= \left(\tfrac{1}{2}\right)^{x-3} \\ &= \left(\tfrac{1}{2}\right)^x \times \left(\tfrac{1}{2}\right)^{-3} \\ &= \left(\tfrac{1}{2}\right)^x \times 8 \\ &= 8\left(\tfrac{1}{2}\right)^x = 8f(x) \end{aligned}$$

So the graph of $y = \left(\frac{1}{2}\right)^{x-3}$ is a vertical stretch of the graph of $y = \left(\frac{1}{2}\right)^x$ with scale factor 8.

Exercise 14A

1 **a** Draw an accurate graph of $y = (1.7)^x$, for $-4 \leqslant x \leqslant 4$.

b Use your graph to solve the equation $(1.7)^x = 4$.

2 **a** Draw an accurate graph of $y = (0.6)^x$, for $-4 \leqslant x \leqslant 4$.

b Use your graph to solve the equation $(0.6)^x = 2$.

3 Sketch the graph of $y = 1^x$.

(P) **4** For each of these statements, decide whether it is true or false, justifying your answer or offering a counter-example.

a The graph of $y = a^x$ passes through (0, 1) for all positive real numbers a.

b The function $f(x) = a^x$ is always an increasing function for $a > 0$.

c The graph of $y = a^x$, where a is a positive real number, never crosses the x-axis.

5 The function f(x) is defined as $f(x) = 3^x$, $x \in \mathbb{R}$. On the same axes, sketch the graphs of:

a $y = f(x)$ **b** $y = 2f(x)$ **c** $y = f(x) - 4$ **d** $y = f\left(\frac{1}{2}x\right)$

Write down the coordinates of the point where each graph crosses the y-axis, and give the equations of any asymptotes.

(P) **6** The graph of $y = ka^x$ passes through the points (1, 6) and (4, 48). Find the values of the constants k and a.

Problem-solving

Substitute the coordinates into $y = ka^x$ to create two simultaneous equations. Use division to eliminate one of the two unknowns.

(P) **7** The graph of $y = pq^x$ passes through the points $(-3, 150)$ and $(2, 0.048)$.

a By drawing a sketch or otherwise, explain why $0 < q < 1$.

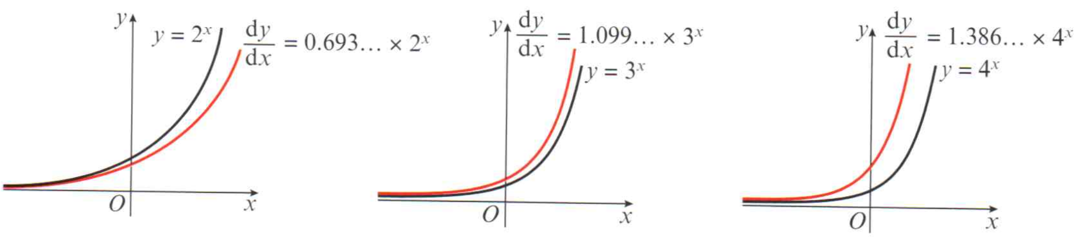
b Find the values of the constants p and q.

Challenge

Sketch the graph of $y = 2^{x-2} + 5$. Give the coordinates of the point where the graph crosses the y-axis.

14.2 $y = e^x$

Exponential functions of the form $f(x) = a^x$ have a special mathematical property. The graphs of their gradient functions are a similar shape to the graphs of the functions themselves.



In each case $f'(x) = kf(x)$, where k is a constant. As the value of a increases, so does the value of k.

Something unique happens between $a = 2$ and $a = 3$. There is going to be a value of a where the gradient function is exactly the same as the original function. This occurs when a is approximately equal to 2.71828. The exact value is represented by the letter e. Like π, e is both an important mathematical constant and an irrational number.

Function	Gradient function
$f(x) = 1^x$	$f'(x) = 0 \times 1^x$
$f(x) = 2^x$	$f'(x) = 0.693\ldots \times 2^x$
$f(x) = 3^x$	$f'(x) = 1.099\ldots \times 3^x$
$f(x) = 4^x$	$f'(x) = 1.386\ldots \times 4^x$

Online Explore the relationship between exponential functions and their derivatives using technology.

- **For all real values of x:**
  - **If $f(x) = e^x$ then $f'(x) = e^x$**
  - **If $y = e^x$ then $\frac{dy}{dx} = e^x$**

A similar result holds for functions such as e^{5x}, e^{-x} and $e^{\frac{1}{2}x}$.

- **For all real values of x and for any constant k:**
  - **If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$**
  - **If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$**

Example 3

Differentiate with respect to x.

a e^{4x} **b** $e^{-\frac{1}{2}x}$ **c** $3e^{2x}$

a $y = e^{4x}$

$\frac{dy}{dx} = 4e^{4x}$

Use the rule for differentiating e^{kx} with $k = 4$.

b $y = e^{-\frac{1}{2}x}$

$\frac{dy}{dx} = -\frac{1}{2}e^{-\frac{1}{2}x}$

c $y = 3e^{2x}$

$\frac{dy}{dx} = 2 \times 3e^{2x} = 6e^{2x}$

To differentiate ae^{kx} multiply the whole function by k. The derivative is kae^{kx}.

Example 4

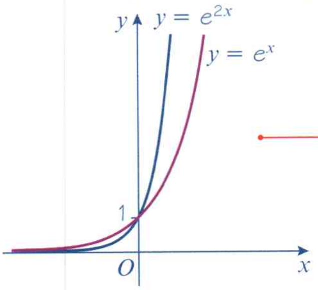
Sketch the graphs of the following equations. Give the coordinates of any points where the graphs cross the axes, and state the equations of any asymptotes.

a $y = e^{2x}$ **b** $y = 10e^{-x}$ **c** $y = 3 + 4e^{\frac{1}{2}x}$

a $y = e^{2x}$

When $x = 0$, $y = e^{2 \times 0} = 1$ so the graph crosses the y-axis at $(0, 1)$.

The x-axis ($y = 0$) is an asymptote.



The graph of $y = e^x$ has been shown in purple on this sketch.

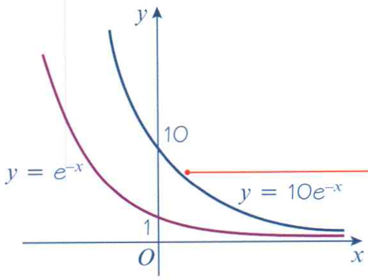
This is a stretch of the graph of $y = e^x$, parallel to the x-axis and with scale factor $\frac{1}{2}$

← Section 4.6

b $y = 10e^{-x}$

When $x = 0$, $y = 10e^{-0}$. So the graph crosses the y-axis at $(0, 10)$.

The x-axis ($y = 0$) is an asymptote.



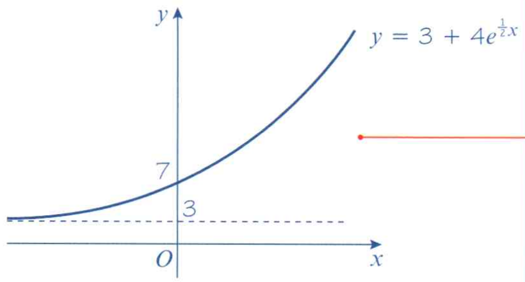
Negative powers of e^x, such as e^{-x} or e^{-4x}, give rise to decreasing functions.

The graph of $y = e^x$ has been reflected in the y-axis and stretched parallel to the y-axis with scale factor 10.

c $y = 3 + 4e^{\frac{1}{2}x}$

When $x = 0$, $y = 3 + 4e^{\frac{1}{2}\times 0} = 7$ so the graph crosses the y-axis at (0, 7).

The line $y = 3$ is an asymptote.



The graph of $y = e^{\frac{1}{2}x}$ has been stretched parallel to the y-axis with scale factor 4 and then translated by $\begin{pmatrix}0\\3\end{pmatrix}$.

Problem-solving

If you have to sketch a transformed graph with an asymptote, it is often easier to sketch the asymptote first.

Online Use technology to draw transformations of $y = e^x$.

Exercise 14B

1 Use a calculator to find the value of e^x to 5 decimal places when

a $x = 1$ **b** $x = 4$ **c** $x = -10$ **d** $x = 0.2$

2 a Draw an accurate graph of $y = e^x$ for $-4 \leqslant x \leqslant 4$.

b By drawing appropriate tangent lines, estimate the gradient at $x = 1$ and $x = 3$.

c Compare your answers to the actual values of e and e^3.

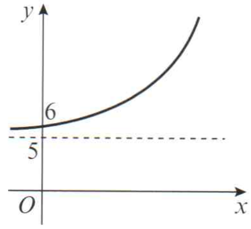
3 Sketch the graphs of:

a $y = e^x + 1$ **b** $y = 4e^{-2x}$ **c** $y = 2e^x - 3$

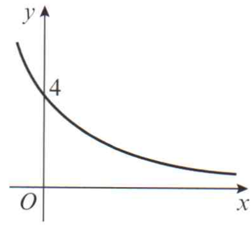
d $y = 4 - e^x$ **e** $y = 6 + 10e^{\frac{1}{2}x}$ **f** $y = 100e^{-x} + 10$

4 Each of the sketch graphs below is of the form $y = Ae^{bx} + C$, where A, b and C are constants. Find the values of A and C for each graph, and state whether b is positive or negative.

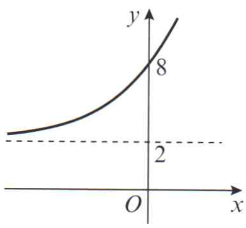
a



b



c



Hint You do not have enough information to work out the value of b, so simply state whether it is positive or negative.

5 Rearrange $f(x) = e^{3x+2}$ into the form $f(x) = Ae^{bx}$, where A and b are constants whose values are to be found. Hence, or otherwise, sketch the graph of $y = f(x)$.

Hint $e^{m+n} = e^m \times e^n$

6 Differentiate the following with respect to x.

a e^{6x} **b** $e^{-\frac{1}{3}x}$ **c** $7e^{2x}$

d $5e^{0.4x}$ **e** $e^{3x} + 2e^x$ **f** $e^x(e^x + 1)$

Hint For part **f**, start by expanding the bracket.

7 Find the gradient of the curve with equation $y = e^{3x}$ at the point where

a $x = 2$ **b** $x = 0$ **c** $x = -0.5$

(P) **8** The function f is defined as $f(x) = e^{0.2x}$, $x \in \mathbb{R}$. Show that the tangent to the curve at the point (5, e) goes through the origin.

14.3 Exponential modelling

You can use e^x to model situations such as population growth, where the rate of **increase** is proportional to the size of the population at any given moment. Similarly, e^{-x} can be used to model situations such as radioactive decay, where the rate of **decrease** is proportional to the number of atoms remaining.

Example 5

The density of a pesticide in a given section of field, P mg/m², can be modelled by the equation

$$P = 160e^{-0.006t}$$

where t is the time in days since the pesticide was first applied.

a Use this model to estimate the density of pesticide after 15 days.

b Interpret the meaning of the value 160 in this model.

c Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k.

d Interpret the significance of the sign of your answer to part **c**.

e Sketch the graph of P against t.

a After 15 days, $t = 15$.

$P = 160e^{-0.006 \times 15}$

$P = 146.2$ mg/m²

Substitute $t = 15$ into the model.

Online Work this out in one go using the e□ button on your calculator.

b When $t = 0$, $P = 160e^0 = 160$, so 160 mg/m² is the initial density of pesticide in the field.

Notation The value given by a model when $t = 0$ is called the **initial value**.

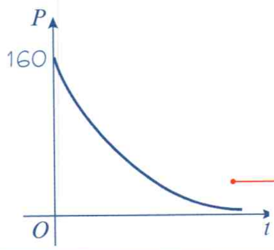
c $P = 160e^{-0.006t}$

$\frac{dP}{dt} = -0.96e^{-0.006t}$, so $k = -0.006$

If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

d As k is negative, the density of pesticide is decreasing (there is exponential decay).

e



Use your answers to parts **b** and **d** to help you draw the graph. To check what happens to P in the long term, substitute in a very large value of t.

Exercise 14C

1 The value of a car is modelled by the formula

$$V = 20\,000e^{-\frac{t}{12}}$$

where V is the value in £s and t is its age in years from new.

a State its value when new.

b Find its value (to the nearest £) after 4 years.

c Sketch the graph of V against t.

(P) 2 The population of a country is modelled using the formula

$$P = 20 + 10e^{\frac{t}{50}}$$

where P is the population in thousands and t is the time in years after the year 2000.

a State the population in the year 2000.

b Use the model to predict the population in the year 2030.

c Sketch the graph of P against t for the years 2000 to 2100.

d Do you think that it would be valid to use this model to predict the population in the year 2500? Explain your answer.

(P) 3 The number of people infected with a disease is modelled by the formula

$$N = 300 - 100e^{-0.5t}$$

where N is the number of people infected with the disease and t is the time in years after detection.

a How many people were first diagnosed with the disease?

b What is the long term prediction of how this disease will spread?

c Sketch the graph of N against t for $t > 0$.

(P) 4 The number of rabbits, R, in a population after m months is modelled by the formula

$$R = 12e^{0.2m}$$

a Use this model to estimate the number of rabbits after

i 1 month   ii 1 year

b Interpret the meaning of the constant 12 in this model.

c Show that after 6 months, the rabbit population is increasing by almost 8 rabbits per month.

d Suggest one reason why this model will stop giving valid results for large enough values of t.

Problem-solving

Your answer to part **b** must refer to the context of the model.

(P) **5** On Earth, the atmospheric pressure, p, in bars can be modelled approximately by the formula $p = e^{-0.13h}$ where h is the height above sea level in kilometres.

a Use this model to estimate the pressure at the top of Mount Rainier, which has an altitude of 4.394 km. **(1 mark)**

b Demonstrate that $\frac{dp}{dh} = kp$ where k is a constant to be found. **(2 marks)**

c Interpret the significance of the sign of k in part **b**. **(1 mark)**

d This model predicts that the atmospheric pressure will change by s% for every kilometre gained in height. Calculate the value of s. **(3 marks)**

(P) **6** Nigel has bought a tractor for £20 000. He wants to model the depreciation of the value of his tractor, £T, in t years. His friend suggests two models:

Model 1: $T = 20\,000e^{-0.24t}$

Model 2: $T = 19\,000e^{-0.255t} + 1000$

a Use both models to predict the value of the tractor after one year. Compare your results. **(2 marks)**

b Use both models to predict the value of the tractor after ten years. Compare your results. **(2 marks)**

c Sketch a graph of T against t for both models. **(2 marks)**

d Interpret the meaning of the 1000 in model 2, and suggest why this might make model 2 more realistic. **(1 mark)**

14.4 Logarithms

The inverses of exponential functions are called **logarithms**. A relationship which is expressed using an exponent can also be written in terms of logarithms.

- **$\log_a n = x$ is equivalent to $a^x = n$ $\quad (a \neq 1)$**

Notation a is called the base of the logarithm.

Example 6

Write each statement as a logarithm.

a $3^2 = 9$ **b** $2^7 = 128$ **c** $64^{\frac{1}{2}} = 8$

a $3^2 = 9$, so $\log_3 9 = 2$

b $2^7 = 128$, so $\log_2 128 = 7$

c $64^{\frac{1}{2}} = 8$, so $\log_{64} 8 = \frac{1}{2}$

In words, you would say 'the logarithm of 9 to the base 3 is 2'.

Logarithms can take fractional or negative values.

Example 7

Rewrite each statement using a power.

a $\log_3 81 = 4$ **b** $\log_2 \left(\frac{1}{8}\right) = -3$

a $\log_3 81 = 4$, so $3^4 = 81$

b $\log_2 \left(\frac{1}{8}\right) = -3$, so $2^{-3} = \frac{1}{8}$

Example 8

Without using a calculator, find the value of:

a $\log_3 81$ **b** $\log_4 0.25$ **c** $\log_{0.5} 4$ **d** $\log_a (a^5)$

a $\log_3 81 = 4$ — Because $3^4 = 81$.

b $\log_4 0.25 = -1$ — Because $4^{-1} = \frac{1}{4} = 0.25$.

c $\log_{0.5} 4 = -2$ — Because $0.5^{-2} = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$.

d $\log_a (a^5) = 5$ — Because $a^5 = a^5$.

You can use your calculator to find logarithms of any base. Some calculators have a specific [log□□] key for this function. Most calculators also have separate buttons for logarithms to the base 10 (usually written as [log]) and logarithms to the base e (usually written as [ln]).

Notation Logarithms to the base e are typically called **natural logarithms**. This is why the calculator key is labelled [ln].

Online Use the logarithm buttons on your calculator.

Example 9

Use your calculator to find the following logarithms to 3 decimal places.

a $\log_3 40$ **b** $\log_e 8$ **c** $\log_{10} 75$

a 3.358 — For part **a** use [log□□].

b 2.079 — For part **b** you can use either [ln] or [log□□].

c 1.875 — For part **c** you can use either [log] or [log□□].

Exercise 14D

1 Rewrite using a logarithm.

a $4^4 = 256$ **b** $3^{-2} = \frac{1}{9}$ **c** $10^6 = 1\,000\,000$

d $11^1 = 11$ **e** $(0.2)^3 = 0.008$

2 Rewrite using a power.

a $\log_2 16 = 4$ **b** $\log_5 25 = 2$ **c** $\log_9 3 = \frac{1}{2}$

d $\log_5 0.2 = -1$ **e** $\log_{10} 100\,000 = 5$

3 Without using a calculator, find the value of

a $\log_2 8$ **b** $\log_5 25$ **c** $\log_{10} 10\,000\,000$ **d** $\log_{12} 12$

e $\log_3 729$ **f** $\log_{10}\sqrt{10}$ **g** $\log_4(0.25)$ **h** $\log_{0.25} 16$

i $\log_a(a^{10})$ **j** $\log_{\frac{2}{3}}\left(\frac{9}{4}\right)$

4 Without using a calculator, find the value of x for which

a $\log_5 x = 4$ **b** $\log_x 81 = 2$ **c** $\log_7 x = 1$

d $\log_2(x-1) = 3$ **e** $\log_3(4x+1) = 4$ **f** $\log_x(2x) = 2$

5 Use your calculator to evaluate these logarithms to three decimal places.

a $\log_9 230$ **b** $\log_5 33$ **c** $\log_{10} 1020$ **d** $\log_e 3$

P **6 a** Without using a calculator, justify why the value of $\log_2 50$ must be between 5 and 6.

b Use a calculator to find the exact value of $\log_2 50$ to 4 significant figures.

Hint Use corresponding statements involving powers of 2.

7 a Find the values of:

i $\log_2 2$ **ii** $\log_3 3$ **iii** $\log_{17} 17$

b Explain why $\log_a a$ has the same value for all positive values of a ($a \neq 1$).

8 a Find the values of:

i $\log_2 1$ **ii** $\log_3 1$ **iii** $\log_{17} 1$

b Explain why $\log_a 1$ has the same value for all positive values of a ($a \neq 1$).

14.5 Laws of logarithms

Expressions involving more than one logarithm can often be rearranged or simplified. For instance:

$\log_a x = m$ and $\log_a y = n$ — Take two logarithms with the same base

$x = a^m$ and $y = a^n$ — Rewrite these expressions using powers

$xy = a^m \times a^n = a^{m+n}$ — Multiply these powers

$\log_a xy = m + n = \log_a x + \log_a y$ — Rewrite your result using logarithms

This result is one of the **laws of logarithms**.

You can use similar methods to prove two further laws.

- **The laws of logarithms:**
  - $\log_a x + \log_a y = \log_a xy$ **(the multiplication law)**
  - $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ **(the division law)**
  - $\log_a(x^k) = k\log_a x$ **(the power law)**

Watch out You need to learn these three laws of logarithms, and the special cases below.

- **You should also learn to recognise the following special cases:**
  - $\log_a\left(\frac{1}{x}\right) = \log_a(x^{-1}) = -\log_a x$ **(the power law when $k = -1$)**
  - $\log_a a = 1$ **($a > 0$, $a \neq 1$)**
  - $\log_a 1 = 0$ **($a > 0$, $a \neq 1$)**

Example 10

Write as a single logarithm.

a $\log_3 6 + \log_3 7$ **b** $\log_2 15 - \log_2 3$ **c** $2\log_5 3 + 3\log_5 2$ **d** $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$

a $\log_3 (6 \times 7)$
$= \log_3 42$

Use the multiplication law.

b $\log_2 (15 \div 3)$
$= \log_2 5$

Use the division law.

c $2\log_5 3 = \log_5 (3^2) = \log_5 9$
$3\log_5 2 = \log_5 (2^3) = \log_5 8$
$\log_5 9 + \log_5 8 = \log_5 72$

First apply the power law to both parts of the expression.
Then use the multiplication law.

d $4\log_{10}\left(\frac{1}{2}\right) = \log_{10}\left(\frac{1}{2}\right)^4 = \log_{10}\left(\frac{1}{16}\right)$

$\log_{10} 3 - \log_{10}\left(\frac{1}{16}\right) = \log_{10}\left(3 \div \frac{1}{16}\right)$
$= \log_{10} 48$

Use the power law first.
Then use the division law.

Example 11

Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a (x^2yz^3)$ **b** $\log_a\left(\frac{x}{y^3}\right)$ **c** $\log_a\left(\frac{x\sqrt{y}}{z}\right)$ **d** $\log_a\left(\frac{x}{a^4}\right)$

a $\log_a (x^2yz^3)$
$= \log_a (x^2) + \log_a y + \log_a (z^3)$
$= 2\log_a x + \log_a y + 3\log_a z$

b $\log_a\left(\frac{x}{y^3}\right)$
$= \log_a x - \log_a (y^3)$
$= \log_a x - 3\log_a y$

c $\log_a\left(\frac{x\sqrt{y}}{z}\right)$
$= \log_a (x\sqrt{y}) - \log_a z$
$= \log_a x + \log_a \sqrt{y} - \log_a z$
$= \log_a x + \frac{1}{2}\log_a y - \log_a z$

Use the power law ($\sqrt{y} = y^{\frac{1}{2}}$).

d $\log_a\left(\frac{x}{a^4}\right)$
$= \log_a x - \log_a (a^4)$
$= \log_a x - 4\log_a a$
$= \log_a x - 4$

$\log_a a = 1$.

Example 12

Solve the equation $\log_{10} 4 + 2\log_{10} x = 2$.

$\log_{10} 4 + 2\log_{10} x = 2$

$\log_{10} 4 + \log_{10} x^2 = 2$ — Use the power law.

$\log_{10} 4x^2 = 2$ — Use the multiplication law.

$4x^2 = 10^2$ — Rewrite the logarithm using powers.

$4x^2 = 100$

$x^2 = 25$

$x = 5$

Watch out $\log_{10} x$ is only defined for **positive** values of x, so $x = -5$ cannot be a solution of the equation.

Example 13

Solve the equation $\log_3 (x + 11) - \log_3 (x - 5) = 2$

$\log_3 (x + 11) - \log_3 (x - 5) = 2$

$\log_3 \left(\frac{x + 11}{x - 5}\right) = 2$ — Use the division law.

$\frac{x + 11}{x - 5} = 3^2$ — Rewrite the logarithm using powers.

$x + 11 = 9(x - 5)$

$x + 11 = 9x - 45$

$56 = 8x$

$x = 7$

Exercise 14E

1 Write as a single logarithm.

a $\log_2 7 + \log_2 3$ **b** $\log_2 36 - \log_2 4$ **c** $3\log_5 2 + \log_5 10$

d $2\log_6 8 - 4\log_6 3$ **e** $\log_{10} 5 + \log_{10} 6 - \log_{10} \left(\frac{1}{4}\right)$

2 Write as a single logarithm, then simplify your answer.

a $\log_2 40 - \log_2 5$ **b** $\log_6 4 + \log_6 9$ **c** $2\log_{12} 3 + 4\log_{12} 2$

d $\log_8 25 + \log_8 10 - 3\log_8 5$ **e** $2\log_{10} 2 - (\log_{10} 5 + \log_{10} 8)$

3 Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a (x^3y^4z)$ **b** $\log_a \left(\frac{x^5}{y^2}\right)$ **c** $\log_a (a^2x^2)$

d $\log_a \left(\frac{x}{z\sqrt{y}}\right)$ **e** $\log_a \sqrt{ax}$

4 Solve the following equations:

a $\log_2 3 + \log_2 x = 2$ **b** $\log_6 12 - \log_6 x = 3$

c $2\log_5 x = 1 + \log_5 6$ **d** $2\log_9 (x + 1) = 2\log_9 (2x - 3) + 1$

Hint Move the logarithms onto the same side if necessary and use the division law.

(P) **5 a** Given that $\log_3 (x + 1) = 1 + 2\log_3 (x - 1)$, show that $3x^2 - 7x + 2 = 0$. **(5 marks)**

b Hence, or otherwise, solve $\log_3 (x + 1) = 1 + 2\log_3 (x - 1)$. **(2 marks)**

(P) **6** Given that a and b are positive constants, and that $a > b$, solve the simultaneous equations

$a + b = 13$

$\log_6 a + \log_6 b = 2$

Problem-solving

Pay careful attention to the conditions on a and b given in the question.

Challenge

By writing $\log_a x = m$ and $\log_a y = n$, prove that $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$.

14.6 Solving equations using logarithms

You can use logarithms and your calculator to solve equations of the form $a^x = b$.

Example 14

Solve the following equations, giving your answers to 3 decimal places.

a $3^x = 20$ **b** $5^{4x-1} = 61$

a $3^x = 20$,
so $x = \log_3 20 = 2.727$

Use the $\log_{\square}\square$ button on your calculator.

b $5^{4x-1} = 61$, so $4x - 1 = \log_5 61$

$4x = \log_5 61 + 1$

$x = \frac{\log_5 61 + 1}{4}$

$= 0.889$

You can evaluate the final answer in one step on your calculator.

Example 15

Solve the equation $5^{2x} - 12(5^x) + 20 = 0$, giving your answer to 3 significant figures.

$5^{2x} - 12(5^x) + 20$ is a quadratic function of 5^x

An alternative method is to rewrite the equation using the substitution $y = 5^x$: $y^2 - 12y + 20 = 0$.

$(5^x - 10)(5^x - 2) = 0$

$5^x = 10$ or $5^x = 2$

$5^x = 10 \Rightarrow x = \log_5 10 \Rightarrow x = 1.43$

$5^x = 2 \Rightarrow x = \log_5 2 \Rightarrow x = 0.431$

Watch out Solving the quadratic equation gives you two possible values for 5^x. Make sure you calculate both corresponding values of x for your final answer.

You can solve more complicated equations by 'taking logs' of both sides.

- **Whenever $f(x) = g(x)$, $\log_a f(x) = \log_a g(x)$**

Example 16

Find the solution to the equation $3^x = 2^{x+1}$, giving your answer to four decimal places.

$3^x = 2^{x+1}$

$\log 3^x = \log 2^{x+1}$

This step is called 'taking logs of both sides'. The logs on both sides must be to the **same base**. Here 'log' is used to represent $\log_{10}$.

$x \log 3 = (x + 1) \log 2$

Use the power law.

$x \log 3 = x \log 2 + \log 2$

$x \log 3 - x \log 2 = \log 2$

$x(\log 3 - \log 2) = \log 2$

Move all the terms in x to one side then factorise.

$x = \dfrac{\log 2}{\log 3 - \log 2} = 1.7095$

Exercise 14F

1 Solve, giving your answers to 3 significant figures.

a $2^x = 75$ **b** $3^x = 10$ **c** $5^x = 2$ **d** $4^{2x} = 100$

e $9^{x+5} = 50$ **f** $7^{2x-1} = 23$ **g** $11^{3x-2} = 65$ **h** $2^{3-2x} = 88$

2 Solve, giving your answers to 3 significant figures.

a $2^{2x} - 6(2^x) + 5 = 0$ **b** $3^{2x} - 15(3^x) + 44 = 0$

c $5^{2x} - 6(5^x) - 7 = 0$ **d** $3^{2x} + 3^{x+1} - 10 = 0$

e $7^{2x} + 12 = 7^{x+1}$ **f** $2^{2x} + 3(2^x) - 4 = 0$

g $3^{2x+1} - 26(3^x) - 9 = 0$ **h** $4(3^{2x+1}) + 17(3^x) - 7 = 0$

Hint $3^{x+1} = 3^x \times 3^1 = 3(3^x)$

Problem-solving

Consider these equations as functions of functions. Part **a** is equivalent to $u^2 - 6u + 5 = 0$, with $u = 2^x$.

E **3** Solve the following equations, giving your answers to 3 significant figures where appropriate.

a $3^{x+1} = 2000$ **(2 marks)**

b $\log_5(x - 3) = -1$ **(2 marks)**

E/P **4 a** Sketch the graph of $y = 4^x$, stating the coordinates of any points where the graph crosses the axes. **(2 marks)**

b Solve the equation $4^{2x} - 10(4^x) + 16 = 0$. **(4 marks)**

Hint Attempt this question without a calculator.

5 Solve the following equations, giving your answers to four decimal places.

a $5^x = 2^{x+1}$ **b** $3^{x+5} = 6^x$ **c** $7^{x+1} = 3^{x+2}$

Hint Take logs of both sides.

14.7 Working with natural logarithms

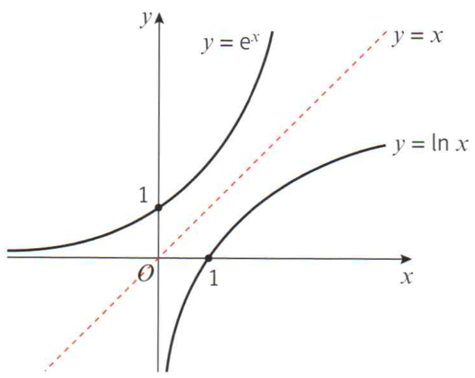
- **The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$.**

The graph of $y = \ln x$ passes through (1, 0) and does not cross the y-axis.

The y-axis is an asymptote of the graph $y = \ln x$. This means that $\ln x$ is only defined for positive values of x.

As x increases, $\ln x$ grows without limit, but relatively slowly.

You can also use the fact that logarithms are the inverses of exponential functions to solve equations involving powers and logarithms.



- **$e^{\ln x} = \ln(e^x) = x$**

Notation $\ln x = \log_e x$

Example 17

Solve these equations, giving your answers in exact form.

a $e^x = 5$ **b** $\ln x = 3$

a When $e^x = 5$

$\ln(e^x) = \ln 5$

$x = \ln 5$

b When $\ln x = 3$

$e^{\ln x} = e^3$

$x = e^3$

The inverse operation of raising e to the power x is taking natural logarithms (logarithms to the base e) and vice versa.

You can write the natural logarithm on both sides. $\ln(e^x) = x$

Leave your answer as a logarithm or a power of e so that it is exact.

Example 18

Solve these equations, giving your answers in exact form.

a $e^{2x+3} = 7$ **b** $2\ln x + 1 = 5$ **c** $e^{2x} + 5e^x = 14$

a $e^{2x+3} = 7$

$2x + 3 = \ln 7$

$2x = \ln 7 - 3$

$x = \frac{1}{2}\ln 7 - \frac{3}{2}$

Take natural logarithms of both sides and use the fact that the inverse of e^x is $\ln x$.

b $2\ln x + 1 = 5$

$2\ln x = 4$

$\ln x = 2$

$x = e^2$

Rearrange to make $\ln x$ the subject.

The inverse of $\ln x$ is e^x.

c $e^{2x} + 5e^x = 14$

$e^{2x} + 5e^x - 14 = 0$

$(e^x + 7)(e^x - 2) = 0$

$e^x = -7$ or $e^x = 2$

$e^x = 2$

$x = \ln 2$

$e^{2x} = (e^x)^2$, so this is a quadratic function of e^x. Start by setting the equation equal to 0 and factorise. You could also use the substitution $u = e^x$ and write the equation as $u^2 + 5u - 14 = 0$.

Watch out e^x is always positive, so you can't have $e^x = -7$. You need to discard this solution.

Exercise 14G

1 Solve these equations, giving your answers in exact form.

a $e^x = 6$ **b** $e^{2x} = 11$ **c** $e^{-x+3} = 20$

d $3e^{4x} = 1$ **e** $e^{2x+6} = 3$ **f** $e^{5-x} = 19$

2 Solve these equations, giving your answers in exact form.

a $\ln x = 2$ **b** $\ln(4x) = 1$ **c** $\ln(2x + 3) = 4$

d $2\ln(6x - 2) = 5$ **e** $\ln(18 - x) = \frac{1}{2}$ **f** $\ln(x^2 - 7x + 11) = 0$

3 Solve these equations, giving your answers in exact form.

a $e^{2x} - 8e^x + 12 = 0$ **b** $e^{4x} - 3e^{2x} = -2$

c $(\ln x)^2 + 2\ln x - 15 = 0$ **d** $e^x - 5 + 4e^{-x} = 0$

e $3e^{2x} + 5 = 16e^x$ **f** $(\ln x)^2 = 4(\ln x + 3)$

Hint All of the equations in question 3 are quadratic equations in a function of x.

Hint First in part **d** multiply each term by e^x.

P 4 Find the exact solutions to the equation $e^x + 12e^{-x} = 7$. **(4 marks)**

5 Solve these equations, giving your answers in exact form.

a $\ln(8x - 3) = 2$ **b** $e^{5(x-8)} = 3$ **c** $e^{10x} - 8e^{5x} + 7 = 0$

d $(\ln x - 1)^2 = 4$

P 6 Solve $3^x e^{4x-1} = 5$, giving your answer in the form $\frac{a + \ln b}{c + \ln d}$ **(5 marks)**

Hint Take natural logarithms of both sides and then apply the laws of logarithms.

P 7 Officials are testing athletes for doping at a sporting event. They model the concentration of a particular drug in an athlete's bloodstream using the equation $D = 6e^{\frac{-t}{10}}$ where D is the concentration of the drug in mg/l and t is the time in hours since the athlete took the drug.

a Interpret the meaning of the constant 6 in this model.

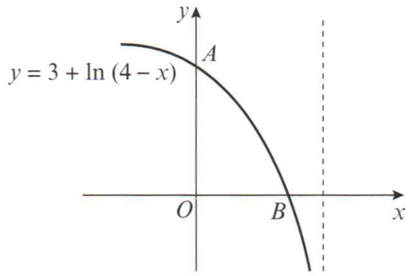
b Find the concentration of the drug in the bloodstream after 2 hours.

c It is impossible to detect this drug in the bloodstream if the concentration is lower than 3 mg/l. Show that this happens after $t = -10\ln\left(\frac{1}{2}\right)$ and convert this result into hours and minutes.

E/P **8** The graph of $y = 3 + \ln(4 - x)$ is shown to the right.

a State the exact coordinates of point A. **(1 mark)**

b Calculate the exact coordinates of point B. **(3 marks)**



E **9** The value of a boat £V, depends on its age, t years. The value of the boat when new is £27 000 and its value after 5 years is £18 000.

a Use an exponential model of the form $V = Ae^{kt}$, where A and k are constants to be found correct to 3 significant figures, to find an equation linking V and t. **(3 marks)**

Hint Substitute $t = 0$ and $V = 27000$ into the equation to find A. Then substitute the other given value to find k.

When the boat is 8 years old it is sold for £14 000.

b Evaluate your model in light of this information. **(2 marks)**

E/P **10** In 1960 the population of Mozambique was 7.6 million.
In 1980 the population of Mozambuque was 12.1 million.

A researcher wants to model the population of Mozambique, P million people, as a function of t, where t is the time in years since 1960.

a Use a linear model to form an equation linking P and t. **(3 marks)**

b Use an exponential model to form a different equation linking P and t. **(4 marks)**

Problem-solving For part **a** find a model in the form $P = a + bt$ where a and b are constants. For part **b** find a model in the form $P = Ae^{kt}$ where A and k are constants.

In 2010 the population of Mozambique was 23.4 million.

c Explain which of your two models is best supported by this fact. **(3 marks)**

Challenge

The graph of the function $g(x) = Ae^{Bx} + C$ passes through (0, 5) and (6, 10). Given that the line $y = 2$ is an asymptote to the graph, show that $B = \frac{1}{6}\ln\left(\frac{8}{3}\right)$.

14.8 Logarithms and non-linear data

Logarithms can also be used to manage and explore non-linear trends in data.

Case 1: $y = ax^n$

Start with a non-linear relationship — $y = ax^n$
Take logs of both sides ($\log = \log_{10}$) — $\log y = \log ax^n$
Use the multiplication law — $\log y = \log a + \log x^n$
Use the power law — $\log y = \log a + n \log x$

Compare this equation to the straight line equation, $Y = MX + C$.

$\log y$ variable	=	n constant (gradient)	$\log x$ variable	+	$\log a$ constant (intercept)
Y variable	=	M constant (gradient)	X variable	+	C constant (intercept)

- **If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$.**

Example 19

The table below gives the rank (by size) and population of the UK's largest cities and districts (London is ranked number 1 but has been excluded as an outlier).

City	**Birmingham**	**Leeds**	**Glasgow**	**Sheffield**	**Bradford**
Rank, R	2	3	4	5	6
Population, P (2 s.f.)	1 000 000	730 000	620 000	530 000	480 000

The relationship between the rank and population can be modelled by the formula

$P = aR^n$ where a and n are constants.

a Draw a table giving values of $\log R$ and $\log P$ to 2 decimal places.

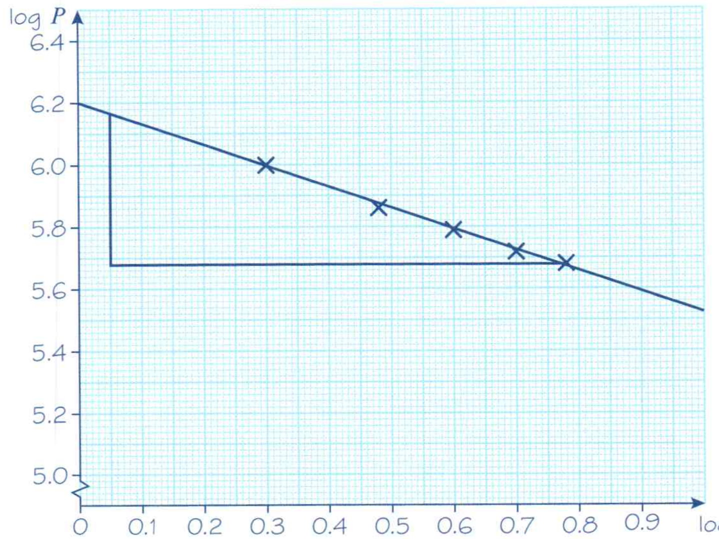
b Plot a graph of $\log R$ against $\log P$ using the values from your table and draw a line of best fit.

c Use your graph to estimate the values of a and n to two significant figures.

a

$\log R$	0.30	0.48	0.60	0.70	0.78
$\log P$	6	5.86	5.79	5.72	5.68

b



c $P = aR^n$

$\log P = \log a(R^n)$

$\log P = \log a + \log(R^n)$

$\log P = \log a + n \log R$

so the gradient is n and the intercept is $\log a$

Reading the gradient from the graph,

$$n = \frac{5.68 - 6.16}{0.77 - 0.05} = \frac{-0.48}{0.72} = -0.67$$

Reading the intercept from the graph,

$\log a = 6.2$

$a = 10^{6.2} = 1\,600\,000$ (2 s.f.).

Start with the formula given in the question. Take logs of both sides and use the laws of logarithms to rearrange it into a linear relationship between $\log P$ and $\log R$.

The gradient of the line of best fit will give you your value for n.

The vertical intercept will give you the value of $\log a$. You need to raise 10 to this power to find the value of a.

Case 2: $y = ab^x$

Start with a non-linear relationship — $y = ab^x$

Take logs of both sides ($\log = \log_{10}$) — $\log y = \log ab^x$

Use the multiplication law — $\log y = \log a + \log b^x$

Use the power law — $\log y = \log a + x \log b$

Compare this equation to the straight line equation, $Y = MX + C$.

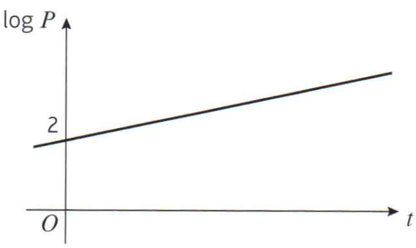
$\log y$ variable	=	$\log b$ constant (gradient)	x variable	+	$\log a$ constant (intercept)
Y variable	=	M constant (gradient)	X variable	+	C constant (intercept)

Watch out For $y = ab^x$ you need to plot $\log y$ against x to obtain a linear graph. If you plot $\log y$ against $\log x$ you will **not** get a linear relationship.

- **If $y = ab^x$ then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$.**

Example 20

The graph represents the growth of a population of bacteria, P, over t hours. The graph has a gradient of 0.6 and meets the vertical axis at (0, 2) as shown.



A scientist suggests that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

a Write down an equation for the line.

b Using your answer to part **a** or otherwise, find the values of a and b, giving them to 3 significant figures where necessary.

c Interpret the meaning of the constant a in this model.

a $\log P = 0.6t + 2$

> $\log P = (\text{gradient}) \times t + (y\text{-intercept})$

b $P = 10^{0.6t+2}$

$P = 10^{0.6t} \times 10^2$

$P = 10^2 \times (10^{0.6})^t$

$P = 100 \times 3.98^t$

$a = 100$, $b = 3.98$ (3 s.f.)

> Rewrite the logarithm as a power. An alternative method would be to start with $P = ab^t$ and take logs of both sides, as in Example 19.

> Rearrange the equation into the form ab^t. You can use $x^{mn} = (x^m)^n$ to write $10^{0.6t}$ in the form b^t.

c The value of a gives the initial size of the bacteria population.

Exercise 14H

1 Two variables, S and x satisfy the formula $S = 4 \times 7^x$.

a Show that $\log S = \log 4 + x \log 7$.

b The straight line graph of $\log S$ against x is plotted. Write down the gradient and the value of the intercept on the vertical axis.

2 Two variables A and x satisfy the formula $A = 6x^4$.

a Show that $\log A = \log 6 + 4 \log x$.

b The straight line graph of $\log A$ against $\log x$ is plotted. Write down the gradient and the value of the intercept on the vertical axis.

3 The data below follows a trend of the form $y = ax^n$, where a and n are constants.

x	3	5	8	10	15
y	16.3	33.3	64.3	87.9	155.1

a Copy and complete the table of values of $\log x$ and $\log y$, giving your answers to 2 decimal places.

$\log x$	0.48	0.70	0.90	1	1.18
$\log y$	1.21				2.19

b Plot a graph of $\log y$ against $\log x$ and draw in a line of best fit.

c Use your graph to estimate the values of a and n to one decimal place.

4 The data below follows a trend of the form $y = ab^x$, where a and b are constants.

x	2	3	5	6.5	9
y	124.8	424.4	4097.0	30 763.6	655 743.5

a Copy and complete the table of values of x and $\log y$, giving your answers to 2 decimal places.

x	2	3	5	6.5	9
$\log y$	2.10				

b Plot a graph of $\log y$ against x and draw in a line of best fit.

c Use your graph to estimate the values of a and b to one decimal place.

(E) **5** Kleiber's law is an empirical law in biology which connects the mass of an animal, m, to its resting metabolic rate, R. The law follows the form $R = am^b$, where a and b are constants.

The table below contains data on five animals.

Animal	Mouse	Guinea pig	Rabbit	Goat	Cow
Mass, *m* (kg)	0.030	0.408	4.19	34.6	650
Metabolic rate *R* (kcal per day)	4.2	32.3	195	760	7637

a Copy and complete this table giving values of $\log R$ and $\log m$ to 2 decimal places. **(1 mark)**

log *m*	−1.52				
log *R*	0.62	1.51	2.29	2.88	3.88

b Plot a graph of $\log R$ against $\log m$ using the values from your table and draw in a line of best fit. **(2 marks)**

c Use your graph to estimate the values of a and b to two significant figures. **(4 marks)**

d Using your values of a and b, estimate the resting metabolic rate of a human male with a mass of 80 kg. **(1 mark)**

6 Zipf's law is an empirical law which relates how frequently a word is used, f, to its ranking in a list of the most common words of a language, R. The law follows the form $f = AR^b$, where A and b are constants to be found.

The table below contains data on four words.

Word	'the'	'it'	'well'	'detail'
Rank, *R*	1	10	100	1000
Frequency per 100 000 words, *f*	4897	861	92	9

a Copy and complete this table giving values of $\log f$ to 2 decimal places.

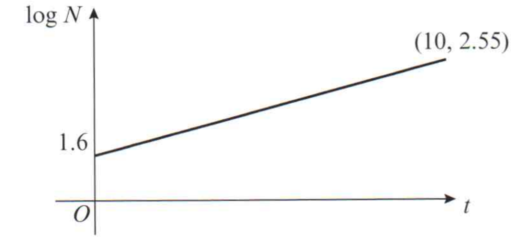
log *R*	0	1	2	3
log *f*	3.69			

b Plot a graph of $\log f$ against $\log R$ using the values from your table and draw in a line of best fit.

c Use your graph to estimate the value of A to two significant figures and the value of b to one significant figure.

d The word 'when' is the 57th most commonly used word in the English language. A trilogy of novels contains 455 125 words. Use your values of A and b to estimate the number of times the word 'when' appears in the trilogy.

7 A scientist is modelling the number of people, N, who have fallen sick with a virus after t days.



From looking at this graph, the scientist suggests that the number of sick people can be modelled by the equation $N = ab^t$, where a and b are constants to be found.

The graph passes through the points (0, 1.6) and (10, 2.55).

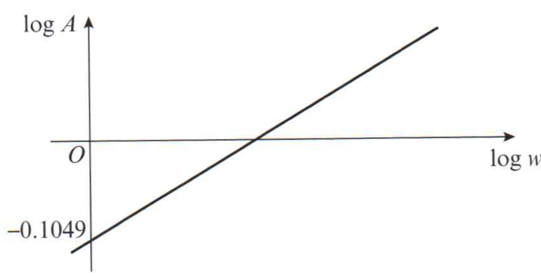
a Write down the equation of the line. **(2 marks)**

b Using your answer to part **a** or otherwise, find the values of a and b, giving them to 2 significant figures. **(4 marks)**

c Interpret the meaning of the constant a in this model. **(1 mark)**

d Use your model to predict the number of sick people to the nearest 100 after 30 days. Give one reason why this might be an overestimate. **(2 marks)**

8 A student is investigating a family of similar shapes. She measures the width, w, and the area, A, of each shape. She suspects there is a formula of the form $A = pw^q$, so she plots the logarithms of her results.



The graph has a gradient of 2 and passes through –0.1049 on the vertical axis.

a Write down an equation for the line.

b Starting with your answer to part **a**, or otherwise, find the exact value of q and the value of p to 4 decimal places.

c Suggest the name of the family of shapes that the student is investigating, and justify your answer.

Hint Multiply p by 4 and think about another name for 'half the width'.

Challenge

Find a formula to describe the relationship between the data in this table.

x	1	2	3	4
y	5.22	4.698	4.2282	3.805 38

Hint Sketch the graphs of $\log y$ against $\log x$ and $\log y$ against x. This will help you determine whether the relationship is of the form $y = ax^n$ or $y = ab^x$.

Mixed exercise 14

1 Sketch each of the following graphs, labelling all intersections and asymptotes.

a $y = 2^{-x}$ **b** $y = 5e^x - 1$ **c** $y = \ln x$

Hint Recall that $2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$

P 2 **a** Express $\log_a(p^2q)$ in terms of $\log_a p$ and $\log_a q$.

b Given that $\log_a(pq) = 5$ and $\log_a(p^2q) = 9$, find the values of $\log_a p$ and $\log_a q$.

P 3 Given that $p = \log_q 16$, express in terms of p,

a $\log_q 2$

b $\log_q(8q)$

4 Solve these equations, giving your answers to 3 significant figures.

a $4^x = 23$ **b** $7^{2x+1} = 1000$ **c** $10^x = 6^{x+2}$

E/P 5 **a** Using the substitution $u = 2^x$, show that the equation $4^x - 2^{x+1} - 15 = 0$ can be written in the form $u^2 - 2u - 15 = 0$. **(2 marks)**

b Hence solve the equation $4^x - 2^{x+1} - 15 = 0$, giving your answer to 2 decimal places. **(3 marks)**

E 6 Solve the equation $\log_2(x + 10) - \log_2(x - 5) = 4$. **(4 marks)**

7 Differentiate each of the following expressions with respect to x.

a e^{-x} **b** e^{11x} **c** $6e^{5x}$

8 Solve the following equations, giving exact solutions.

a $\ln(2x - 5) = 8$ **b** $e^{4x} = 5$ **c** $24 - e^{-2x} = 10$

d $\ln x + \ln(x - 3) = 0$ **e** $e^x + e^{-x} = 2$ **f** $\ln 2 + \ln x = 4$

P 9 The price of a computer system can be modelled by the formula

$$P = 100 + 850e^{-\frac{t}{2}}$$

where P is the price of the system in £s and t is the age of the computer in years after being purchased.

a Calculate the new price of the system.

b Calculate its price after 3 years, giving your answer to the nearest £.

c When will it be worth less than £200?

d Find its price as $t \to \infty$.

e Sketch the graph showing P against t.

f Comment on the appropriateness of this model.

P 10 The points P and Q lie on the curve with equation $y = e^{\frac{1}{2}x}$.
The x-coordinates of P and Q are $\ln 4$ and $\ln 16$ respectively.

a Find an equation for the line PQ.

b Show that this line passes through the origin O.

c Calculate the length, to 3 significant figures, of the line segment PQ.

P 11 The temperature, T °C, of a cup of tea is given by $T = 55e^{-\frac{t}{8}} + 20 \qquad t \geqslant 0$
where t is the time in minutes since measurements began.

a Briefly explain why $t \geqslant 0$. **(1 mark)**

b State the starting temperature of the cup of tea. **(1 mark)**

c Find the time at which the temperature of the tea is 50 °C, giving your answer to the nearest minute. **(3 marks)**

d By sketching a graph or otherwise, explain why the temperature of the tea will never fall below 20 °C. **(2 marks)**

E 12 The table below gives the surface area, S, and the volume, V of five different spheres, rounded to 1 decimal place.

S	18.1	50.3	113.1	221.7	314.2
V	7.2	33.5	113.1	310.3	523.6

Given that $V = aS^b$, where a and b are constants,

a show that $\log V = \log a + b \log S$. **(2 marks)**

b copy and complete the table of values of $\log S$ and $\log V$, giving your answers to 2 decimal places. **(1 mark)**

log S					
log V	0.86				

c plot a graph of $\log V$ against $\log S$ and draw in a line of best fit. **(2 marks)**

d use your graph to confirm that $b = 1.5$ and estimate the value of a to one significant figure. **(4 marks)**

P 13 The radioactive decay of a substance is modelled by the formula $R = 140e^{kt} \qquad t \geqslant 0$
where R is a measure of radioactivity (in counts per minute) at time t days, and k is a constant.

a Explain briefly why k must be negative. **(1 mark)**

b Sketch the graph of R against t. **(2 marks)**

After 30 days the radiation is measured at 70 counts per minute.

c Show that $k = c \ln 2$, stating the value of the constant c. **(3 marks)**

P 14 The total number of views (in millions) V of a viral video in x days is modelled by

$$V = e^{0.4x} - 1$$

a Find the total number of views after 5 days, giving your answer to 2 significant figures.

b Find $\frac{dV}{dx}$.

c Find the rate of increase of the number of views after 100 days, stating the units of your answer.

d Use your answer to part **c** to comment on the validity of the model after 100 days.

(P) **15** The moment magnitude scale is used by seismologists to express the sizes of earthquakes. The scale is calculated using the formula

$$M = \frac{2}{3}\log_{10}(S) - 10.7$$

where S is the seismic moment in dyne cm.

a Find the magnitude of an earthquake with a seismic moment of 2.24×10^{22} dyne cm.

b Find the seismic moment of an earthquake with
i magnitude 6 **ii** magnitude 7

c Using your answers to part **b** or otherwise, show that an earthquake of magnitude 7 is approximately 32 times as powerful as an earthquake of magnitude 6.

(E/P) **16** A student is asked to solve the equation

$$\log_2 x - \frac{1}{2}\log_2(x + 1) = 1$$

The student's attempt is shown

$$\log_2 x - \log_2\sqrt{x + 1} = 1$$
$$x - \sqrt{x + 1} = 2^1$$
$$x - 2 = \sqrt{x + 1}$$
$$(x - 2)^2 = x + 1$$
$$x^2 - 5x + 3 = 0$$
$$x = \frac{5 + \sqrt{13}}{2} \quad x = \frac{5 - \sqrt{13}}{2}$$

a Identify the error made by the student. **(1 mark)**

b Solve the equation correctly. **(3 marks)**

(E/P) **17** Solve the equation $9^x - 11(3^x) + 18 = 0$. **(5 marks)**

Challenge

a Given that $y = 9^x$, show that $\log_3 y = 2x$.

b Hence deduce that $\log_3 y = \log_9 y^2$.

c Use your answer to part **b** to solve the equation $\log_3(2 - 3x) = \log_9(6x^2 - 19x + 2)$

Summary of key points

1 For all real values of x:

- If $f(x) = e^x$ then $f'(x) = e^x$
- If $y = e^x$ then $\frac{dy}{dx} = e^x$

2 For all real values of x and for any constant k:

- If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$
- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

3 $\log_a n = x$ is equivalent to $a^x = n$ $\quad (a \neq 1)$

4 The laws of logarithms:

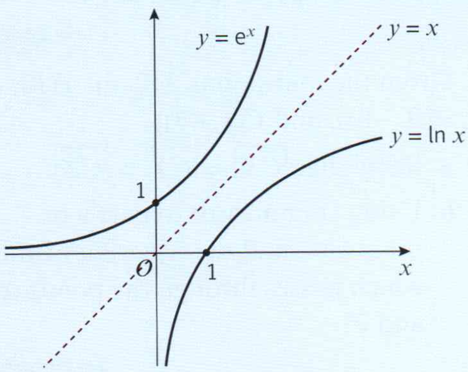
- $\log_a x + \log_a y = \log_a xy$ (the multiplication law)
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ (the division law)
- $\log_a (x^k) = k \log_a x$ (the power law)

5 You should also learn to recognise the following special cases:

- $\log_a \left(\frac{1}{x}\right) = \log_a (x^{-1}) = -\log_a x$ (the power law when $k = -1$)
- $\log_a a = 1$ $\quad (a > 0, a \neq 1)$
- $\log_a 1 = 0$ $\quad (a > 0, a \neq 1)$

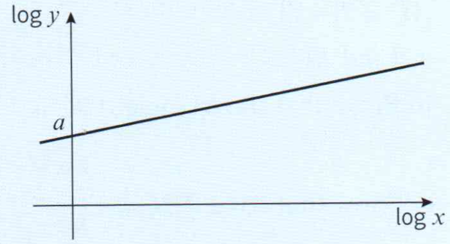
6 Whenever $f(x) = g(x)$, $\log_a f(x) = \log_a g(x)$

7 The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$.



8 $e^{\ln x} = \ln (e^x) = x$

9 If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$.



10 If $y = ab^x$ then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$.