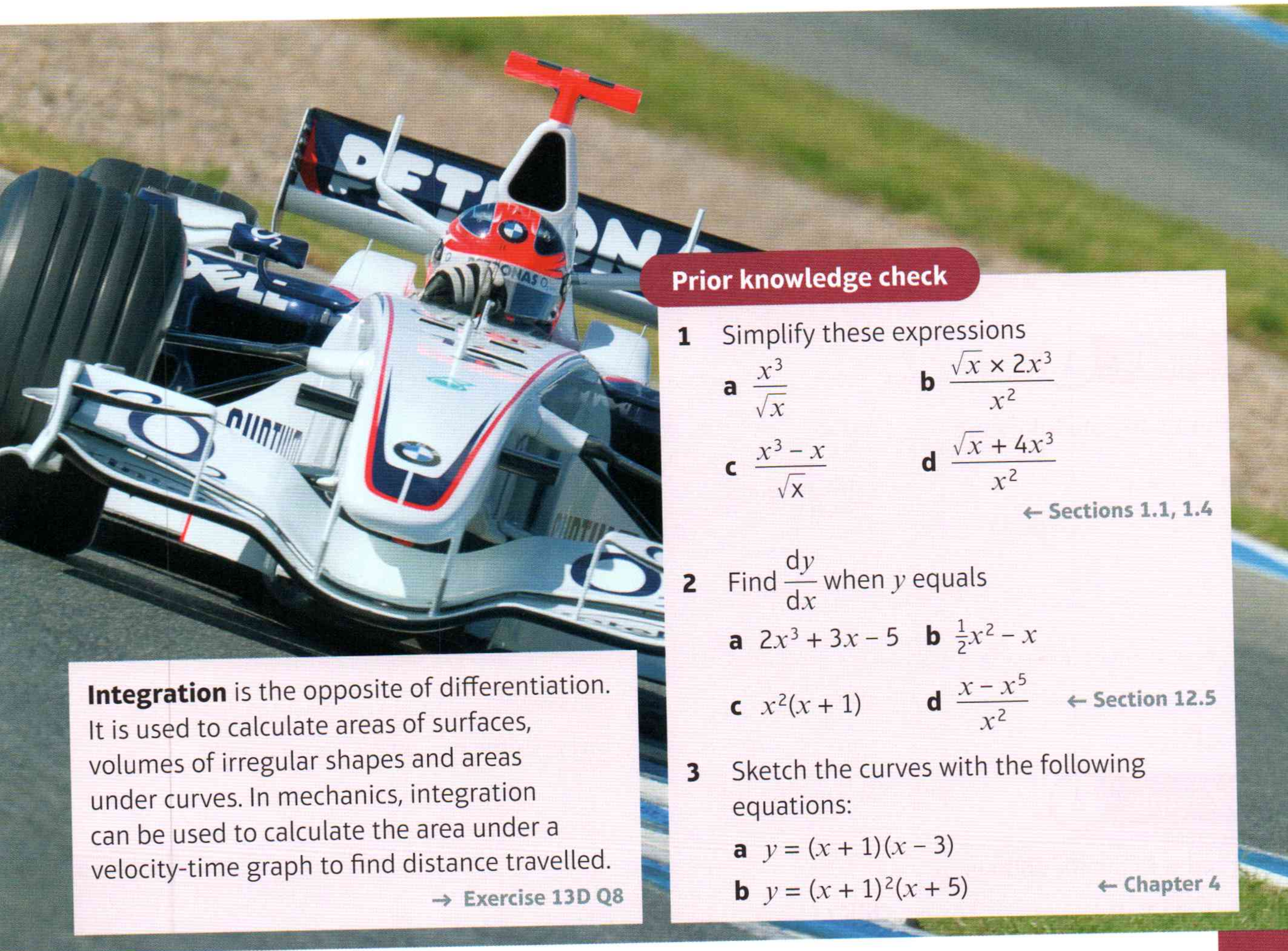
Integration

13

Objectives

After completing this unit you should be able to:

- Find y given $\frac{dy}{dx}$ for x^n → pages 288–290
- Integrate polynomials → pages 290–293
- Find $f(x)$, given $f'(x)$ and a point on the curve → pages 293–295
- Evaluate a definite integral → pages 295–297
- Find the area bounded by a curve and the x-axis → pages 297–302
- Find areas bounded by curves and straight lines → pages 302–306



Integration is the opposite of differentiation. It is used to calculate areas of surfaces, volumes of irregular shapes and areas under curves. In mechanics, integration can be used to calculate the area under a velocity-time graph to find distance travelled.

→ Exercise 13D Q8

Prior knowledge check

1 Simplify these expressions

a $\frac{x^3}{\sqrt{x}}$ **b** $\frac{\sqrt{x} \times 2x^3}{x^2}$

c $\frac{x^3 - x}{\sqrt{x}}$ **d** $\frac{\sqrt{x} + 4x^3}{x^2}$

← Sections 1.1, 1.4

2 Find $\frac{dy}{dx}$ when y equals

a $2x^3 + 3x - 5$ **b** $\frac{1}{2}x^2 - x$

c $x^2(x + 1)$ **d** $\frac{x - x^5}{x^2}$

← Section 12.5

3 Sketch the curves with the following equations:

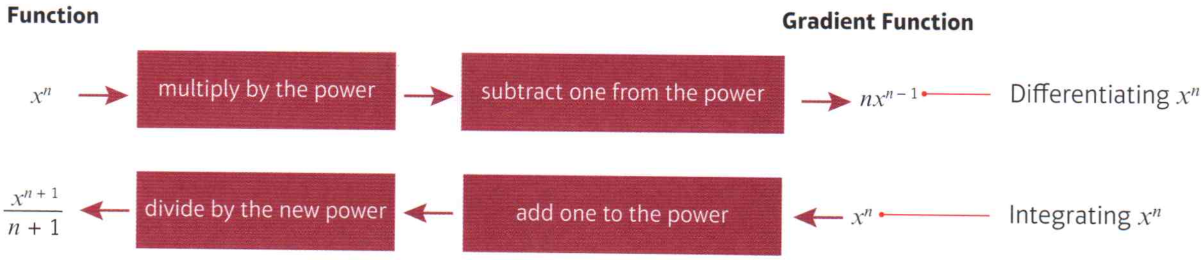
a $y = (x + 1)(x - 3)$

b $y = (x + 1)^2(x + 5)$

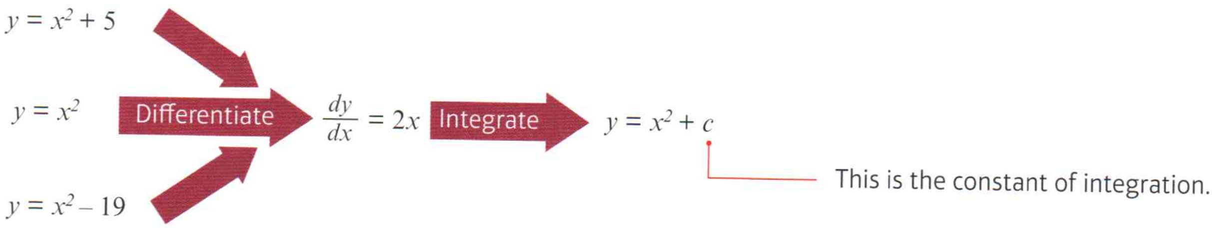
← Chapter 4

13.1 Integrating x^n

Integration is the reverse process of differentiation:



Constant terms disappear when you differentiate. This means that when you differentiate functions that only differ in the constant term, they will all differentiate to give the same function. To allow for this, you need to add a **constant of integration** at the end of a function when you integrate.



- **If $\frac{dy}{dx} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c$, $n \neq -1$.**
- **If $f'(x) = x^n$, then $f(x) = \frac{1}{n+1}x^{n+1} + c$, $n \neq -1$.**

Links You cannot use this rule if $n = -1$ because $\frac{1}{n+1} = \frac{1}{0}$ and so is not defined. You will learn how to integrate the function x^{-1} in Year 2. → Year 2, Section 11.2

Example 1

Find y for the following:

a $\frac{dy}{dx} = x^4$ **b** $\frac{dy}{dx} = x^{-5}$

a $y = \frac{x^5}{5} + c$

Use $y = \frac{1}{n+1}x^{n+1} + c$ with $n = 4$.
Don't forget to add c.

b $y = \frac{x^{-4}}{-4} + c = -\frac{1}{4}x^{-4} + c$

Remember, adding 1 to the power gives $-5 + 1 = -4$.
Divide by the new power (-4) and add c.

Example 2

Find $f(x)$ for the following:

a $f'(x) = 3x^{\frac{1}{2}}$ **b** $f'(x) = 3$

a $f(x) = 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^{\frac{3}{2}} + c$

Remember $3 \div \frac{3}{2} = 3 \times \frac{2}{3} = 2$

Simplify your answer.

b $f'(x) = 3 = 3x^0$

So $f(x) = 3 \times \frac{x^1}{1} + c = 3x + c$

$x^0 = 1$, so 3 can be written as $3x^0$.

You can integrate a function in the form kx^n by integrating x^n and multiplying the integral by k.

- **If $\frac{dy}{dx} = kx^n$, then $y = \frac{k}{n+1}x^{n+1} + c$, $n \neq -1$.**
- **Using function notation, if $f'(x) = kx^n$, then $f(x) = \frac{k}{n+1}x^{n+1} + c$, $n \neq -1$.**
- **When integrating polynomials, apply the rule of integration separately to each term.**

Watch out You don't need to multiply the constant term (c) by k.

Example 3

Given $\frac{dy}{dx} = 6x + 2x^{-3} - 3x^{\frac{1}{2}}$, find y.

$y = \frac{6x^2}{2} + \frac{2}{-2}x^{-2} - \frac{3}{\frac{3}{2}}x^{\frac{3}{2}} + c$

$= 3x^2 - x^{-2} - 2x^{\frac{3}{2}} + c$

Apply the rule of integration to each term of the expression and add c.

Now simplify each term and remember to add c.

Exercise 13A

1 Find an expression for y when $\frac{dy}{dx}$ is the following:

a x^5 b $10x^4$ c $-x^{-2}$ d $-4x^{-3}$ e $x^{\frac{2}{3}}$ f $4x^{\frac{1}{2}}$

g $-2x^6$ h $x^{-\frac{1}{2}}$ i $5x^{-\frac{3}{2}}$ j $6x^{\frac{1}{3}}$ k $36x^{11}$ l $-14x^{-8}$

m $-3x^{-\frac{2}{3}}$ n -5 o $6x$ p $2x^{-0.4}$

2 Find y when $\frac{dy}{dx}$ is given by the following expressions. In each case simplify your answer.

a $x^3 - \frac{3}{2}x^{-\frac{1}{2}} - 6x^{-2}$ b $4x^3 + x^{-\frac{2}{3}} - x^{-2}$ c $4 - 12x^{-4} + 2x^{-\frac{1}{2}}$

d $5x^{\frac{2}{3}} - 10x^4 + x^{-3}$ e $-\frac{4}{3}x^{-\frac{4}{3}} - 3 + 8x$ f $5x^4 - x^{-\frac{3}{2}} - 12x^{-5}$

3 Find $f(x)$ when $f'(x)$ is given by the following expressions. In each case simplify your answer.

a $12x + \frac{3}{2}x^{-\frac{3}{2}} + 5$ b $6x^5 + 6x^{-7} - \frac{1}{6}x^{-\frac{7}{6}}$ c $\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

d $10x^4 + 8x^{-3}$ e $2x^{-\frac{1}{3}} + 4x^{-\frac{5}{3}}$ f $9x^2 + 4x^{-3} + \frac{1}{4}x^{-\frac{1}{2}}$

E/P 4 Find y given that $\frac{dy}{dx} = (2x+3)^2$. **(4 marks)**

Problem-solving Start by expanding the brackets.

E 5 Find $f(x)$ given that $f'(x) = 3x^{-2} + 6x^{\frac{1}{2}} + x - 4$. **(4 marks)**

Challenge

Find y when $\frac{dy}{dx} = (2\sqrt{x} - x^2)\left(\frac{3 + x}{x^5}\right)$

13.2 Indefinite integrals

You can use the symbol $\int$ to represent the process of **integration**.

- $\int f'(x)dx = f(x) + c$

You can write the process of integrating x^n as follows:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

The elongated S means integrate.

The expression to be integrated.

The dx tells you to integrate with respect to x.

Notation This process is called **indefinite integration**. You will learn about **definite** integration later in this chapter.

When you are integrating a polynomial function, you can integrate the terms one at a time.

- $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

Example 4

Find:

a $\int (x^{\frac{1}{2}} + 2x^3)\,dx$ **b** $\int (x^{-\frac{3}{2}} + 2)\,dx$ **c** $\int (p^2x^{-2} + q)\,dx$ **d** $\int (4t^2 + 6)\,dt$

a $\int (x^{\frac{1}{2}} + 2x^3)dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^4}{4} + c$

First apply the rule term by term.

$= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^4 + c$

Simplify each term.

b $\int (x^{-\frac{3}{2}} + 2)dx = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 2x + c$

$= -2x^{-\frac{1}{2}} + 2x + c$

Remember $-\frac{3}{2} + 1 = -\frac{1}{2}$ and the integral of the constant 2 is $2x$.

c $\int (p^2x^{-2} + q)dx = \frac{p^2}{-1}x^{-1} + qx + c$

$= -p^2x^{-1} + qx + c$

The dx tells you to integrate with respect to the variable x, so any other letters must be treated as constants.

d $\int (4t^2 + 6)dt = \frac{4t^3}{3} + 6t + c$

The dt tells you that this time you must integrate with respect to t.

Use the rule for integrating x^n but replace x with t:

If $\frac{dy}{dt} = kt^n$, then $y = \frac{k}{n+1}t^{n+1} + c, n \neq -1$.

Before you integrate, you need to ensure that each term of the expression is in the form kx^n, where k and n are real numbers.

Example 5

Find:

a $\int\left(\frac{2}{x^3} - 3\sqrt{x}\right)dx$ **b** $\int x\left(x^2 + \frac{2}{x}\right)dx$ **c** $\int\left((2x)^2 + \frac{\sqrt{x}+5}{x^2}\right)dx$

a $\int\left(\frac{2}{x^3} - 3\sqrt{x}\right)dx$

$= \int(2x^{-3} - 3x^{\frac{1}{2}})dx$ — First write each term in the form x^n.

$= \frac{2}{-2}x^{-2} - \frac{3}{\frac{3}{2}}x^{\frac{3}{2}} + c$ — Apply the rule term by term.

$= -x^{-2} - 2x^{\frac{3}{2}} + c$ — Simplify each term.

$= -\frac{1}{x^2} - 2\sqrt{x^3} + c$ — Sometimes it is helpful to write the answer in the same form as the question.

b $\int x\left(x^2 + \frac{2}{x}\right)dx$

$= \int(x^3 + 2)dx$ — First multiply out the bracket.

$= \frac{x^4}{4} + 2x + c$ — Then apply the rule to each term.

c $\int\left((2x)^2 + \frac{\sqrt{x}+5}{x^2}\right)dx$

$= \int\left(4x^2 + \frac{x^{\frac{1}{2}}}{x^2} + \frac{5}{x^2}\right)dx$ — Simplify $(2x)^2$ and write $\sqrt{x}$ as $x^{\frac{1}{2}}$.

$= \int(4x^2 + x^{-\frac{3}{2}} + 5x^{-2})dx$ — Write each term in the form x^n.

$= \frac{4}{3}x^3 + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{5x^{-1}}{-1} + c$ — Apply the rule term by term.

$= \frac{4}{3}x^3 - 2x^{-\frac{1}{2}} - 5x^{-1} + c$

$= \frac{4}{3}x^3 - \frac{2}{\sqrt{x}} - \frac{5}{x} + c$ — Finally simplify the answer.

Exercise 13B

1 Find the following integrals:

a $\int x^3\,dx$ **b** $\int x^7\,dx$ **c** $\int 3x^{-4}\,dx$ **d** $\int 5x^2\,dx$

2 Find the following integrals:

a $\int(x^4 + 2x^3)dx$ **b** $\int(2x^3 - x^2 + 5x)dx$ **c** $\int(5x^{\frac{3}{2}} - 3x^2)dx$

3 Find the following integrals:

a $\int(4x^{-2} + 3x^{-\frac{1}{2}})dx$ **b** $\int(6x^{-2} - x^{\frac{1}{2}})dx$ **c** $\int(2x^{-\frac{3}{2}} + x^2 - x^{-\frac{1}{2}})dx$

4 Find the following integrals:

a $\int(4x^3 - 3x^{-4} + r)\,dx$ **b** $\int(x + x^{-\frac{1}{2}} + x^{-\frac{3}{2}})\,dx$

c $\int(px^4 + 2q + 3x^{-2})\,dx$

Hint In **Q4** part **c** you are integrating with respect to x, so treat p and q as constants.

5 Find the following integrals:

a $\int(3t^2 - t^{-2})\,dt$ **b** $\int(2t^2 - 3t^{-\frac{3}{2}} + 1)\,dt$ **c** $\int(pt^3 + q^2 + pr^3)\,dt$

6 Find the following integrals:

a $\int\frac{(2x^3 + 3)}{x^2}\,dx$ **b** $\int(2x + 3)^2\,dx$ **c** $\int(2x + 3)\sqrt{x}\,dx$

7 Find $\int f(x)\,dx$ when $f(x)$ is given by the following:

a $\left(x + \frac{1}{x}\right)^2$ **b** $(\sqrt{x} + 2)^2$ **c** $\left(\frac{1}{\sqrt{x}} + 2\sqrt{x}\right)$

8 Find the following integrals:

a $\int\left(x^{\frac{2}{3}} + \frac{4}{x^3}\right)dx$ **b** $\int\left(\frac{2 + x}{x^3} + 3\right)dx$ **c** $\int(x^2 + 3)(x - 1)\,dx$

d $\int\frac{(2x + 1)^2}{\sqrt{x}}\,dx$ **e** $\int\left(3 + \frac{\sqrt{x} + 6x^3}{x}\right)dx$ **f** $\int\sqrt{x}(\sqrt{x} + 3)^2\,dx$

9 Find the following integrals:

a $\int\left(\frac{A}{x^2} - 3\right)dx$ **b** $\int\left(\sqrt{Px} + \frac{2}{x^3}\right)dx$ **c** $\int\left(\frac{p}{x^2} + q\sqrt{x} + r\right)dx$

(E) **10** Given that $f(x) = \frac{6}{x^2} + 4\sqrt{x} - 3x + 2$, $x > 0$, find $\int f(x)\,dx$. **(5 marks)**

(E) **11** Find $\int\left(8x^3 + 6x - \frac{3}{\sqrt{x}}\right)dx$, giving each term in its simplest form. **(4 marks)**

(E/P) **12** **a** Show that $(2 + 5\sqrt{x})^2$ can be written as $4 + k\sqrt{x} + 25x$, where k is a constant to be found. **(2 marks)**

b Hence find $\int(2 + 5\sqrt{x})^2\,dx$. **(3 marks)**

(E) **13** Given that $y = 3x^5 - \frac{4}{\sqrt{x}}$, $x > 0$, find $\int y\,dx$ in its simplest form. **(3 marks)**

(E/P) **14** $\int\left(\frac{p}{2x^2} + pq\right)dx = \frac{2}{x} + 10x + c$ **(5 marks)**

Find the value of p and the value of q.

Problem-solving

Integrate the expression on the left-hand side, treating p and q as constants, then compare the result with the right-hand side.

P **15** $f(x) = (2 - x)^{10}$

Given that x is small, and so terms in x^3 and higher powers of x can be ignored:

a find an approximation for $f(x)$ in the form $A + Bx + Cx^2$ **(3 marks)**

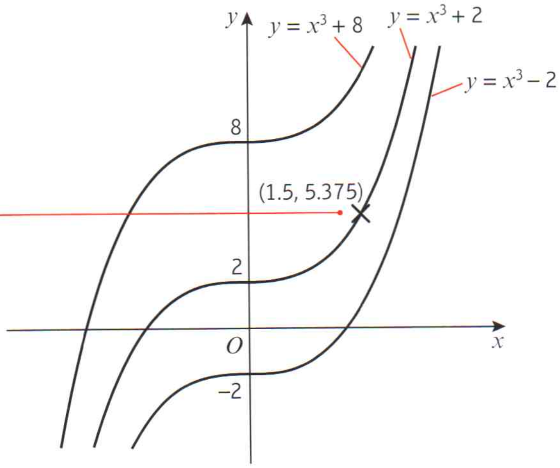
b find an approximation for $\int f(x)dx$. **(3 marks)**

Hint Find the first three terms of the binomial expansion of $(2 - x)^{10}$. ← Section 8.3

13.3 Finding functions

You can find the constant of integration, c, when you are given (i) any point (x, y) that the curve of the function passes through or (ii) the value that the function takes for a given value of x. For example, if $\frac{dy}{dx} = 3x^2$ then $y = x^3 + c$. There are infinitely many curves with this equation, depending on the value of c.

Only one of these curves passes through this point. Choosing a point on the curve determines the value of c.



- **To find the constant of integration, c**
  - **Integrate the function**
  - **Substitute the values (x, y) of a point on the curve, or the value of the function at a given point $f(x) = k$, into the integrated function**
  - **Solve the equation to find c**

Example 6

The curve C with equation $y = f(x)$ passes through the point (4, 5). Given that $f'(x) = \frac{x^2 - 2}{\sqrt{x}}$, find the equation of C.

$f'(x) = \frac{x^2 - 2}{\sqrt{x}} = x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$ — First write $f'(x)$ in a form suitable for integration.

So $f(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$ — Integrate as normal and don't forget the $+ c$.

$= \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + c$

But $f(4) = 5$ — Use the fact that the curve passes through (4, 5).

So $\quad 5 = \frac{2}{5} \times 2^5 - 4 \times 2 + c$

$5 = \frac{64}{5} - 8 + c$

$5 = \frac{24}{5} + c$

So $\quad c = \frac{1}{5}$

So $y = \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + \frac{1}{5}$

Remember $4^{\frac{5}{2}} = 2^5$.

Solve for c.

Finally write down the equation of the curve.

Online Explore the solution using technology.

Exercise 13C

1 Find the equation of the curve with the given derivative of y with respect to x that passes through the given point:

a $\frac{dy}{dx} = 3x^2 + 2x$; point (2, 10)

b $\frac{dy}{dx} = 4x^3 + \frac{2}{x^3} + 3$; point (1, 4)

c $\frac{dy}{dx} = \sqrt{x} + \frac{1}{4}x^2$; point (4, 11)

d $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - x$; point (4, 0)

e $\frac{dy}{dx} = (x + 2)^2$; point (1, 7)

f $\frac{dy}{dx} = \frac{x^2 + 3}{\sqrt{x}}$; point (0, 1)

2 The curve C, with equation $y = f(x)$, passes through the point (1, 2) and $f'(x) = 2x^3 - \frac{1}{x^2}$. Find the equation of C in the form $y = f(x)$.

3 The gradient of a particular curve is given by $\frac{dy}{dx} = \frac{\sqrt{x} + 3}{x^2}$. Given that the curve passes through the point (9, 0), find an equation of the curve.

E 4 The curve with equation $y = f(x)$ passes through the point $(-1, 0)$. Given that $f'(x) = 9x^2 + 4x - 3$, find $f(x)$. **(5 marks)**

E/P 5 $\frac{dy}{dx} = 3x^{-\frac{1}{2}} - 2x\sqrt{x}$, $x > 0$.

Given that $y = 10$ at $x = 4$, find y in terms of x, giving each term in its simplest form. **(7 marks)**

E/P 6 Given that $\frac{6x + 5x^{\frac{3}{2}}}{\sqrt{x}}$ can be written in the form $6x^p + 5x^q$,

a write down the value of p and the value of q. **(2 marks)**

Given that $\frac{dy}{dx} = \frac{6x + 5x^{\frac{3}{2}}}{\sqrt{x}}$ and that $y = 100$ when $x = 9$,

b find y in terms of x, simplifying the coefficient of each term. **(5 marks)**

(P) 7 The displacement of a particle at time t is given by the function $f(t)$, where $f(0) = 0$.
Given that the velocity of the particle is given by $f'(t) = 10 - 5t$,

a find $f(t)$

b determine the displacement of the particle when $t = 3$.

Problem-solving

You don't need any specific knowledge of mechanics to answer this question. You are told that the displacement of the particle at time t is given by $f(t)$.

(P) 8 The height, in metres, of an arrow fired horizontally from the top of a castle is modelled by the function $f(t)$, where $f(0) = 35$. Given that $f'(t) = -9.8t$,

a find $f(t)$.

b determine the height of the arrow when $t = 1.5$.

c write down the height of the castle according to this model.

d estimate the time it will take the arrow to hit the ground.

e state one assumption used in your calculation.

Challenge

1 A set of curves, where each curve passes through the origin, has equations $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x)$ … where $f_n'(x) = f_{n-1}(x)$ and $f_1(x) = x^2$.

a Find $f_2(x)$, $f_3(x)$.

b Suggest an expression for $f_n(x)$.

2 A set of curves, with equations $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x)$, … all pass through the point (0, 1) and they are related by the property $f_n'(x) = f_{n-1}(x)$ and $f_1(x) = 1$. Find $f_2(x)$, $f_3(x)$, $f_4(x)$.

13.4 Definite integrals

You can calculate an integral between two **limits**. This is called a **definite integral**. A definite integral usually produces a **value** whereas an indefinite integral always produces a **function**.

Here are the steps for integrating the function $3x^2$ between the limits $x = 1$ and $x = 2$.

$$\int_1^2 3x^2\,dx = [x^3]_1^2$$
$$= (2^3) - (1^3)$$
$$= 8 - 1$$
$$= 7$$

The limits of the integral are from $x = 1$ to $x = 2$.

Write the integral in [ ] brackets.

Write this step in ( ) brackets.

Evaluate the integral at the upper limit.

Evaluate the integral at the lower limit.

There are three stages when you work out a definite integral:

Write the definite integral **statement** with its limits, a and b.
$\int_a^b \ldots \mathrm{d}x$

Integrate, and write the integral in square brackets
$[\ldots]_a^b$

Evaluate the definite integral by working out $\mathrm{f}(b) - \mathrm{f}(a)$.
$(\ldots) - (\ldots)$

- **If $\mathrm{f}'(x)$ is the derivative of $\mathrm{f}(x)$ for all values of x in the interval $[a, b]$, then the definite integral is defined as $\int_a^b \mathrm{f}'(x)\mathrm{d}x = [\mathrm{f}(x)]_a^b = \mathrm{f}(b) - \mathrm{f}(a)$.**

Notation

The relationship between the derivative and the integral is called the **fundamental theorem of calculus**.

Example 7

Evaluate

$$\int_0^1 (x^{\frac{1}{3}} - 1)^2 \mathrm{d}x$$

$$\int_0^1 (x^{\frac{1}{3}} - 1)^2 \mathrm{d}x$$
$$= \int_0^1 (x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1)\mathrm{d}x$$
$$= \left[\frac{x^{\frac{5}{3}}}{\frac{5}{3}} - 2\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + x\right]_0^1$$
$$= \left[\frac{3}{5}x^{\frac{5}{3}} - \frac{3}{2}x^{\frac{4}{3}} + x\right]_0^1$$
$$= \left(\frac{3}{5} - \frac{3}{2} + 1\right) - (0 + 0 + 0)$$
$$= \frac{1}{10}$$

First multiply out the bracket to put the expression in a form ready to be integrated.

For definite integrals you don't need to include $+c$ in your square brackets.

Simplify each term.

Example 8

Given that P is a constant and $\int_1^5 (2Px + 7)\mathrm{d}x = 4P^2$, show that there are two possible values for P and find these values.

$$\int_1^5 (2Px + 7)\mathrm{d}x = [Px^2 + 7x]_1^5$$
$$= (25P + 35) - (P + 7)$$
$$= 24P + 28$$
$$24P + 28 = 4P^2$$
$$4P^2 - 24P - 28 = 0$$
$$P^2 - 6P - 7 = 0$$
$$(P + 1)(P - 7) = 0$$
$$P = -1 \text{ or } 7$$

Problem-solving

You are integrating with respect to x so treat P as a constant. Find the definite integral in terms of P then set it equal to $4P^2$. The fact that the question asks for 'two possible values' gives you a clue that the resulting equation will be quadratic.

Divide every term by 4 to simplify.

Exercise 13D

1 Evaluate the following definite integrals:

a $\int_2^5 x^3\,dx$ b $\int_1^3 x^4\,dx$

c $\int_0^4 \sqrt{x}\,dx$ d $\int_1^3 \frac{3}{x^2}\,dx$

Watch out You must not use a calculator to work out definite integrals in your exam. You need to use calculus and show clear algebraic working.

2 Evaluate the following definite integrals:

a $\int_1^2 \left(\frac{2}{x^3} + 3x\right)dx$ b $\int_0^2 (2x^3 - 4x + 5)dx$ c $\int_4^9 \left(\sqrt{x} - \frac{6}{x^2}\right)dx$ d $\int_1^8 (x^{-\frac{1}{3}} + 2x - 1)dx$

3 Evaluate the following definite integrals:

a $\int_1^3 \frac{x^3 + 2x^2}{x}\,dx$ b $\int_3^6 \left(x - \frac{3}{x}\right)^2 dx$ c $\int_0^1 x^2\left(\sqrt{x} + \frac{1}{x}\right)dx$ d $\int_1^4 \frac{2 + \sqrt{x}}{x^2}\,dx$

4 Given that A is a constant and $\int_1^4 (6\sqrt{x} - A)dx = A^2$, show that there are two possible values for A and find these values. **(5 marks)**

5 Use calculus to find the value of $\int_1^9 (2x - 3\sqrt{x})dx$. **(5 marks)**

6 Evaluate $\int_4^{12} \frac{2}{\sqrt{x}}\,dx$, giving your answer in the form $a + b\sqrt{3}$, where a and b are integers. **(4 marks)**

7 Given that $\int_1^k \frac{1}{\sqrt{x}}\,dx = 3$, calculate the value of k. **(4 marks)**

Problem-solving You might encounter a definite integral with an unknown in the limits. Here, you can find an expression for the definite integral in terms of k then set that expression equal to 3.

8 The speed, v ms^{-1}, of a train at time t seconds is given by $v = 20 + 5t$, $0 \leqslant t \leqslant 10$.

The distance, s metres, travelled by the train in 10 seconds is given by $s = \int_0^{10} (20 + 5t)dt$. Find the value of s.

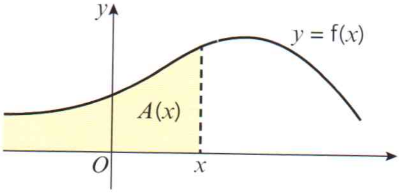
Challenge

Given that $\int_k^{3k} \frac{3x + 2}{8}\,dx = 7$ and $k > 0$, calculate the value of k.

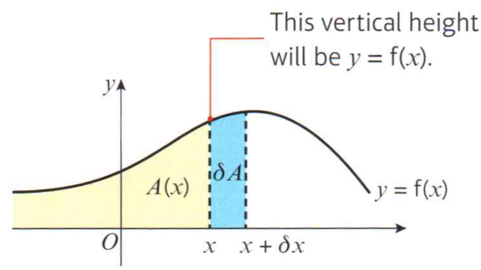
13.5 Areas under curves

Definite integration can be used to find the area under a curve.

For any curve with equation $y = f(x)$, you can define the area under the curve to the left of x as a function of x called $A(x)$. As x increases, this area $A(x)$ also increases (since x moves further to the right).



If you look at a small increase in x, say δx, then the area increases by an amount $\delta A = A(x + \delta x) - A(x)$.



This increase in the δA is approximately rectangular and of magnitude $y\delta x$. (As you make δx smaller any error between the actual area and this will be negligible.)

So you have $\quad \delta A \approx y\delta x$

or $\quad \dfrac{\delta A}{\delta x} \approx y$

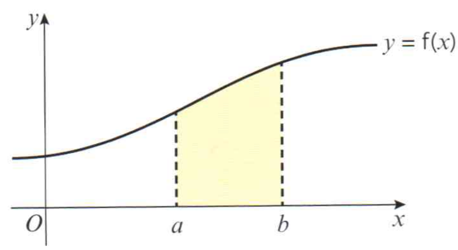
and if you take the limit $\lim\limits_{\delta x \to 0}\left(\dfrac{\delta A}{\delta x}\right)$ then you will see that $\dfrac{\mathrm{d}A}{\mathrm{d}x} = y$.

Now if you know that $\dfrac{\mathrm{d}A}{\mathrm{d}x} = y$, then to find A you have to integrate, giving $A = \int y\,\mathrm{d}x$.

- **The area between a positive curve, the x-axis and the lines $x = a$ and $x = b$ is given by**

  Area = $\displaystyle\int_a^b y\,\mathrm{d}x$

  where $y = \mathrm{f}(x)$ is the equation of the curve.

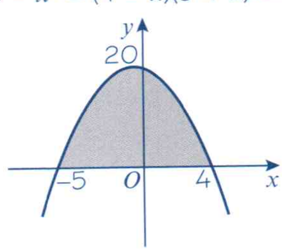


Example 9

Find the area of the finite region between the curve with equation $y = 20 - x - x^2$ and the x-axis.

$y = 20 - x - x^2 = (4 - x)(5 + x)$

Factorise the expression.



Draw a sketch of the graph. $x = 4$ and $x = -5$ are the points of intersection of the curve and the x-axis.

$$\text{Area} = \int_{-5}^{4}(20 - x - x^2)\,\mathrm{d}x$$

$$= \left[20x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-5}^{4}$$

$$= \left(80 - 8 - \frac{64}{3}\right) - \left(-100 - \frac{25}{2} + \frac{125}{3}\right)$$

$$= \frac{243}{2}$$

You don't normally need to give units when you are finding areas on graphs.

Exercise 13E

1 Find the area between the curve with equation $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ in each of the following cases:

a $f(x) = -3x^2 + 17x - 10;$ $a = 1,$ $b = 3$

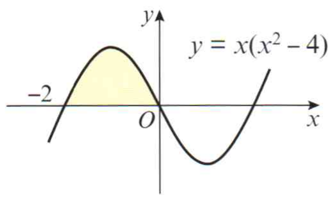
b $f(x) = 2x^3 + 7x^2 - 4x;$ $a = -3,$ $b = -1$

c $f(x) = -x^4 + 7x^3 - 11x^2 + 5x;$ $a = 0,$ $b = 4$

d $f(x) = \frac{8}{x^2};$ $a = -4,$ $b = -1$

Hint For part **c**, $f(x) = -x(x - 1)^2(x - 5)$

2 The sketch shows part of the curve with equation $y = x(x^2 - 4)$. Find the area of the shaded region.

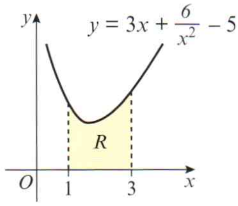


3 The diagram shows a sketch of the curve with equation

$y = 3x + \frac{6}{x^2} - 5, x > 0.$

The region R is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 3$.

Find the area of R.

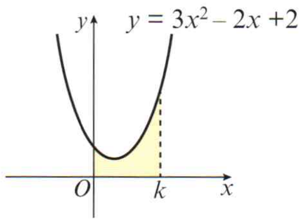


4 Find the area of the finite region between the curve with equation $y = (3 - x)(1 + x)$ and the x-axis.

5 Find the area of the finite region between the curve with equation $y = x(x - 4)^2$ and the x-axis.

6 Find the area of the finite region between the curve with equation $y = 2x^2 - 3x^3$ and the x-axis.

7 The shaded area under the graph of the function $f(x) = 3x^2 - 2x + 2$, bounded by the curve, the x-axis and the lines $x = 0$ and $x = k$, is 8. Work out the value of k.



Problem-solving

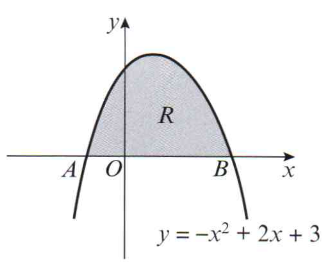
$\int_0^k (3x^2 - 2x + 2)\,dx = 8$

8 The finite region R is bounded by the x-axis and the curve with equation $y = -x^2 + 2x + 3, x \geqslant 0$.

The curve meets the x-axis at points A and B.

a Find the coordinates of point A and point B. **(2 marks)**

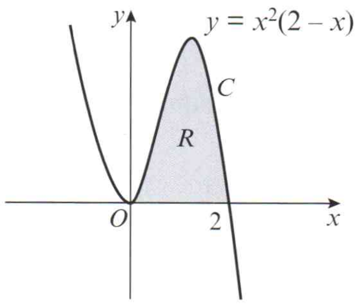
b Find the area of the region R. **(4 marks)**



E **9** The graph shows part of the curve C with equation $y = x^2(2 - x)$.

The region R, shown shaded, is bounded by C and the x-axis.

Use calculus to find the exact area of R. **(5 marks)**



Watch out If a question says "use calculus" then you need to use integration or differentiation, and show clear algebraic working.

13.6 Areas under the x-axis

You need to be careful when you are finding areas below the x-axis.

- **When the area bounded by a curve and the x-axis is below the x-axis, $\int y\,dx$ gives a negative answer.**

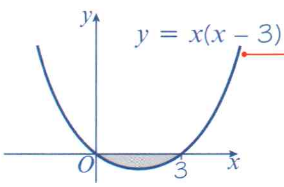
Example 10

Find the area of the finite region bounded by the curve $y = x(x - 3)$ and the x-axis.

Online Check your solution using your calculator.

When $x = 0$, $y = 0$

When $y = 0$, $x = 0$ or 3



First sketch the curve. It is ⋃-shaped and crosses the x-axis at 0 and 3.

$$\text{Area} = \int_0^3 x(x - 3)\,dx$$

The limits on the integral will therefore be 0 and 3.

$$= \int_0^3 (x^2 - 3x)\,dx$$

Multiply out the brackets.

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$$

Integrate as usual.

$$= \left(\frac{27}{3} - \frac{27}{2}\right) - (0 - 0)$$

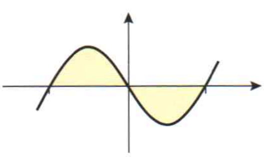
$$= -\frac{27}{6} \text{ or } -\frac{9}{2} \text{ or } -4.5$$

The area is below the x-axis so the definite integral is negative.

So the area is 4.5

State the area as a positive quantity.

The following example shows that great care must be taken if you are trying to find an area which straddles the x-axis such as the shaded region.



For examples of this type you need to draw a sketch, unless one is given in the question.

Example 11

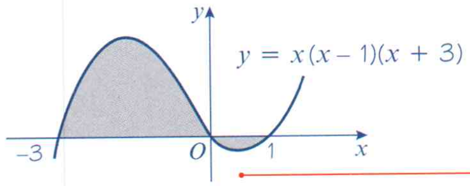
Sketch the curve with equation $y = x(x-1)(x+3)$ and find the area of the finite region bounded by the curve and the x-axis.

When $x = 0$, $y = 0$
When $y = 0$, $x = 0$, 1 or -3

Find out where the curve cuts the axes.

$x \to \infty$, $y \to \infty$
$x \to -\infty$, $y \to -\infty$

Find out what happens to y when x is large and positive or large and negative.



Problem-solving Always draw a sketch, and use the points of intersection with the x-axis as the limits for your integrals.

The area is given by $\int_{-3}^{0} y\,dx - \int_{0}^{1} y\,dx$

Since the area between $x = 0$ and 1 is below the axis the integral between these points will give a negative answer.

Now $\int y\,dx = \int (x^3 + 2x^2 - 3x)dx$

$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]$

Multiply out the brackets.

So $\int_{-3}^{0} y\,dx = (0) - \left(\frac{81}{4} - \frac{2}{3} \times 27 - \frac{3}{2} \times 9\right)$

$= \frac{45}{4}$

and $\int_{0}^{1} y\,dx = \left(\frac{1}{4} + \frac{2}{3} - \frac{3}{2}\right) - (0)$

$= -\frac{7}{12}$

So the area required is $\frac{45}{4} + \frac{7}{12} = \frac{71}{6}$

Watch out If you try to calculate the area as a single definite integral, the positive and negative areas will partly cancel each other out.

Exercise 13F

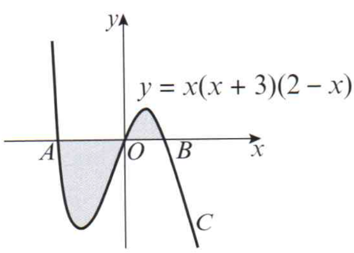
1 Sketch the following and find the total area of the finite region or regions bounded by the curves and the x-axis:

a $y = x(x+2)$ **b** $y = (x+1)(x-4)$ **c** $y = (x+3)x(x-3)$
d $y = x^2(x-2)$ **e** $y = x(x-2)(x-5)$

(E) 2 The graph shows a sketch of part of the curve C with equation $y = x(x+3)(2-x)$.

The curve C crosses the x-axis at the origin O and at points A and B.

a Write down the x-coordinates of A and B. **(1 mark)**

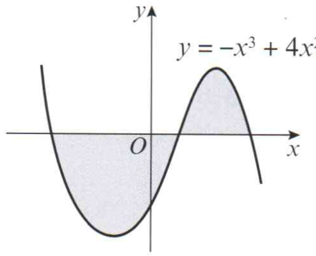


The finite region, shown shaded, is bounded by the curve C and the x-axis.

b Use integration to find the total area of the finite shaded region. **(7 marks)**

3 $f(x) = -x^3 + 4x^2 + 11x - 30$

The graph shows a sketch of part of the curve with equation $y = -x^3 + 4x^2 + 11x - 30$.



a Use the factor theorem to show that $(x + 3)$ is a factor of $f(x)$.

b Write $f(x)$ in the form $(x + 3)(Ax^2 + Bx + C)$.

c Hence, factorise $f(x)$ completely.

d Hence, determine the x-coordinates where the curve intersects the x-axis.

e Hence, determine the total shaded area shown on the sketch.

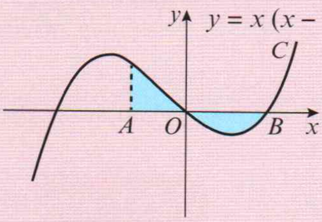
Challenge

1 Given that $f(x) = x(3 - x)$, find the area of the finite region bounded by the x-axis and the curve with equation

a $y = f(x)$ **b** $y = 2f(x)$ **c** $y = af(x)$

d $y = f(x + a)$ **e** $y = f(ax)$.

2 The graph shows a sketch of part of the curve C with equation $y = x(x - 1)(x + 2)$. The curve C crosses the x-axis at the origin O and at point B. The shaded areas above and below the x-axis are equal.



a Show that the x-coordinate of A satisfies the equation

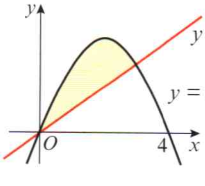
$$(x - 1)^2(3x^2 + 10x + 5) = 0$$

b Hence find the exact coordinates of A, and interpret geometrically the other roots of this equation.

13.7 Areas between curves and lines

- **You can use definite integration together with areas of trapeziums and triangles to find more complicated areas on graphs.**

Example 12



The diagram shows a sketch of part of the curve with equation $y = x(4 - x)$ and the line with equation $y = x$.

Find the area of the region bounded by the curve and the line.

$x(4 - x) = x$
$3x - x^2 = 0$
$x(3 - x) = 0$
$x = 0$ or 3

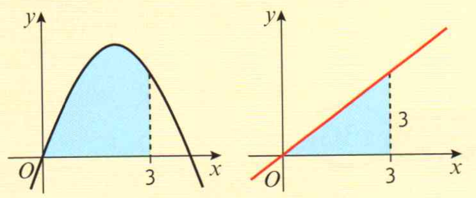
Area beneath curve $= \int_0^3 (4x - x^2)\,dx$
$= \left[2x^2 - \frac{x^3}{3}\right]_0^3$
$= 9$

Area beneath triangle $= \frac{1}{2} \times 3 \times 3$
$= \frac{9}{2}$

Shaded area $= 9 - \frac{9}{2} = \frac{9}{2}$

First, find the x-coordinate of the points of intersection of the curve $y = x(4 - x)$ and the line $y = x$.

Shaded area = area beneath curve − area beneath triangle

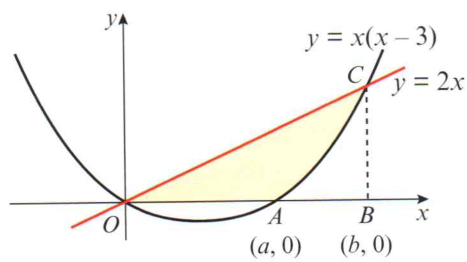


$\left[2x^2 - \frac{x^3}{3}\right]_0^3 = \left(18 - \frac{27}{3}\right) - (0 - 0) = 18 - 9$

Example 13

The diagram shows a sketch of the curve with equation $y = x(x - 3)$ and the line with equation $y = 2x$.

Find the area of the shaded region OAC.



The required area is given by:

Area of triangle $OBC - \int_a^b x(x - 3)\,dx$

The curve cuts the x-axis at $x = 3$ (and $x = 0$) so $a = 3$.

The curve meets the line $y = 2x$ when $2x = x(x - 3)$.

So $\quad 0 = x^2 - 5x$
$\quad 0 = x(x - 5)$
$\quad x = 0$ or 5, so $b = 5$

The point C is $(5, 10)$.

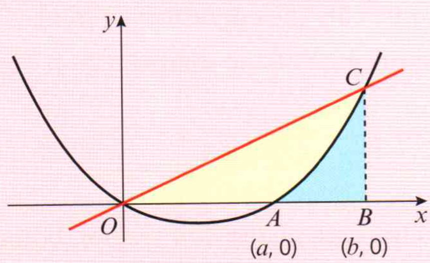
Area of triangle $OBC = \frac{1}{2} \times 5 \times 10 = 25$.

Area between curve, x-axis and the line $x = 5$ is

$\int_3^5 x(x - 3)\,dx = \int_3^5 (x^2 - 3x)\,dx$
$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_3^5$

Problem-solving

Look for ways of combining triangles, trapeziums and direct integrals to find the missing area.



Substituting $x = 5$ into the equation of the line gives $y = 2 \times 5 = 10$.

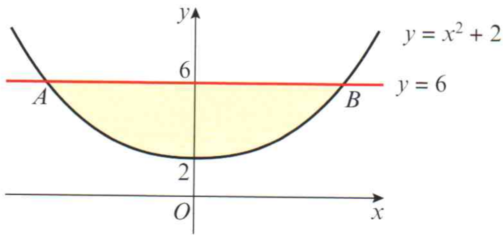
Work out the definite integral separately. This will help you avoid making errors in your working.

$$= \left(\frac{125}{3} - \frac{75}{2}\right) - \left(\frac{27}{3} - \frac{27}{2}\right)$$
$$= \left(\frac{25}{6}\right) - \left(-\frac{27}{6}\right)$$
$$= \frac{26}{3}$$

Shaded region is therefore $= 25 - \frac{26}{3} = \frac{49}{3}$

Exercise 13G

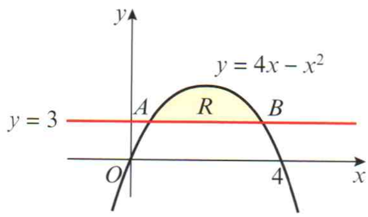
1 The diagram shows part of the curve with equation $y = x^2 + 2$ and the line with equation $y = 6$. The line cuts the curve at the points A and B.



a Find the coordinates of the points A and B.

b Find the area of the finite region bounded by line AB and the curve.

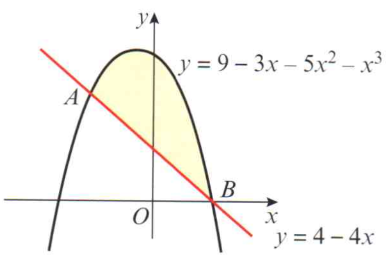
2 The diagram shows the finite region, R, bounded by the curve with equation $y = 4x - x^2$ and the line $y = 3$. The line cuts the curve at the points A and B.



a Find the coordinates of the points A and B.

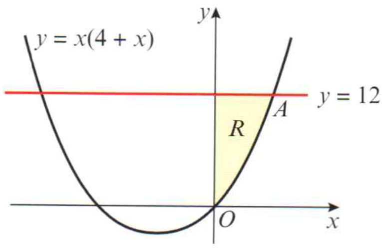
b Find the area of R.

P 3 The diagram shows a sketch of part of the curve with equation $y = 9 - 3x - 5x^2 - x^3$ and the line with equation $y = 4 - 4x$. The line cuts the curve at the points A $(-1, 8)$ and $B(1, 0)$. Find the area of the shaded region between AB and the curve.



P 4 Find the area of the finite region bounded by the curve with equation $y = (1 - x)(x + 3)$ and the line $y = x + 3$.

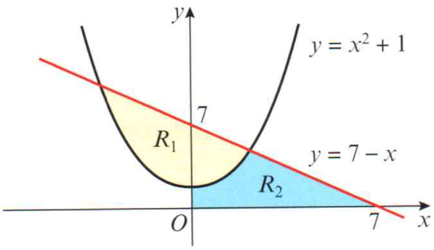
5 The diagram shows the finite region, R, bounded by the curve with equation $y = x(4 + x)$, the line with equation $y = 12$ and the y-axis.



a Find the coordinates of the point A where the line meets the curve.

b Find the area of R.

6 The diagram shows a sketch of part of the curve with equation $y = x^2 + 1$ and the line with equation $y = 7 - x$. The finite region, R_1 is bounded by the line and the curve. The finite region, R_2 is below the curve and the line and is bounded by the positive x- and y-axes as shown in the diagram.



a Find the area of R_1.

b Find the area of R_2.

7 The curve C has equation $y = x^{\frac{2}{3}} - \dfrac{2}{x^{\frac{1}{3}}} + 1$.

a Verify that C crosses the x-axis at the point (1, 0).

b Show that the point $A(8, 4)$ also lies on C.

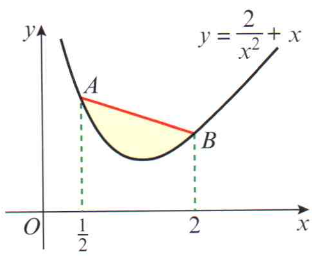
c The point B is (4, 0). Find the equation of the line through AB.

The finite region R is bounded by C, AB and the positive x-axis.

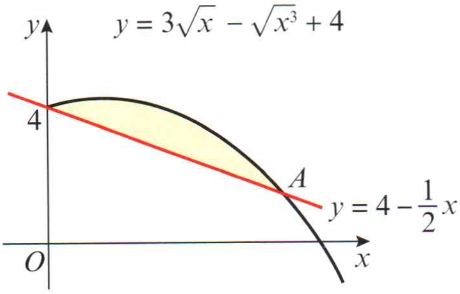
d Find the area of R.

8 The diagram shows part of a sketch of the curve with equation $y = \dfrac{2}{x^2} + x$. The points A and B have x-coordinates $\frac{1}{2}$ and 2 respectively.

Find the area of the finite region between AB and the curve.



9 The diagram shows part of the curve with equation $y = 3\sqrt{x} - \sqrt{x^3} + 4$ and the line with equation $y = 4 - \frac{1}{2}x$.



a Verify that the line and the curve cross at the point $A(4, 2)$.

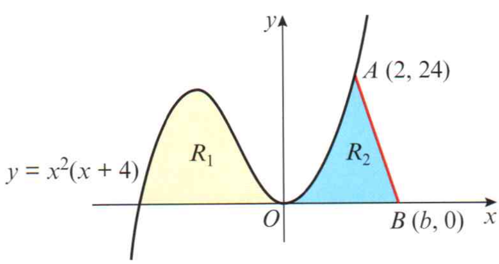
b Find the area of the finite region bounded by the curve and the line.

10 The sketch shows part of the curve with equation $y = x^2(x + 4)$. The finite region R_1 is bounded by the curve and the negative x-axis. The finite region R_2 is bounded by the curve, the positive x-axis and AB, where $A(2, 24)$ and $B(b, 0)$.

The area of R_1 = the area of R_2.

a Find the area of R_1.

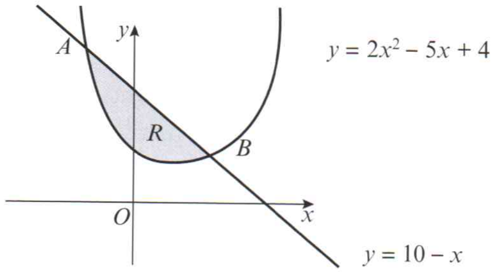
b Find the value of b.



Problem-solving

Split R_2 into two areas by drawing a vertical line at $x = 2$.

(E/P) **11** The line with equation $y = 10 - x$ cuts the curve with equation $y = 2x^2 - 5x + 4$ at the points A and B, as shown.



a Find the coordinates of A and the coordinates of B. **(5 marks)**

The shaded region R is bounded by the line and the curve as shown.

b Find the exact area of R. **(6 marks)**

Mixed exercise 13

1 Find:

a $\int(x+1)(2x-5)\,dx$ **b** $\int(x^{\frac{1}{3}} + x^{-\frac{1}{3}})\,dx$

2 The gradient of a curve is given by $f'(x) = x^2 - 3x - \frac{2}{x^2}$. Given that the curve passes through the point (1, 1), find the equation of the curve in the form $y = f(x)$.

3 Find:

a $\int(8x^3 - 6x^2 + 5)\,dx$ **b** $\int(5x+2)x^{\frac{1}{2}}\,dx$

(P) **4** Given $y = \frac{(x+1)(2x-3)}{\sqrt{x}}$, find $\int y\,dx$.

(P) **5** Given that $\frac{dx}{dt} = (t+1)^2$ and that $x = 0$ when $t = 2$, find the value of x when $t = 3$.

(E/P) **6** Given that $y^{\frac{1}{2}} = x^{\frac{1}{3}} + 3$:

a show that $y = x^{\frac{2}{3}} + Ax^{\frac{1}{3}} + B$, where A and B are constants to be found. **(2 marks)**

b hence find $\int y\,dx$. **(3 marks)**

(E/P) **7** Given that $y^{\frac{1}{2}} = 3x^{\frac{1}{4}} - 4x^{-\frac{1}{4}}$ $(x > 0)$:

a find $\frac{dy}{dx}$ **(2 marks)**

b find $\int y\,dx$. **(3 marks)**

(P) **8** $\int\left(\frac{a}{3x^3} - ab\right)dx = -\frac{2}{3x^2} + 14x + c$

Find the value of a and the value of b.

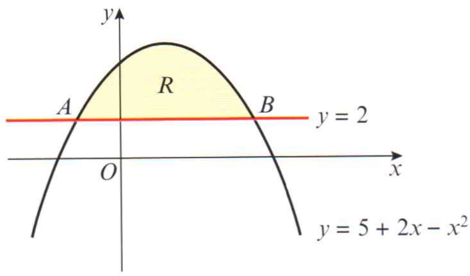
(P) **9** A rock is dropped off a cliff. The height in metres of the rock above the ground after t seconds is given by the function $f(t)$. Given that $f(0) = 70$ and $f'(t) = -9.8t$, find the height of the rock above the ground after 3 seconds.

10 A cyclist is travelling along a straight road. The distance in metres of the cyclist from a fixed point after t seconds is modelled by the function $f(t)$, where $f'(t) = 5 + 2t$ and $f(0) = 0$.

a Find an expression for $f(t)$.

b Calculate the time taken for the cyclist to travel 100 m.

11 The diagram shows the curve with equation $y = 5 + 2x - x^2$ and the line with equation $y = 2$. The curve and the line intersect at the points A and B.



a Find the x-coordinates of A and B.

b The shaded region R is bounded by the curve and the line. Find the area of R.

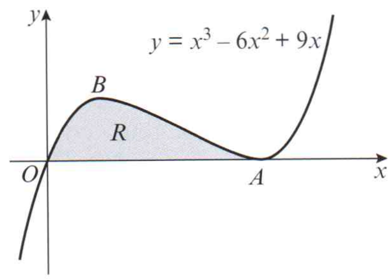
12 **a** Find $\int (x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 1)\,dx$. **(4 marks)**

b Use your answer to part **a** to evaluate

$$\int_1^4 (x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 1)\,dx$$

giving your answer as an exact fraction. **(2 marks)**

13 The diagram shows part of the curve with equation $y = x^3 - 6x^2 + 9x$. The curve touches the x-axis at A and has a local maximum at B.



a Show that the equation of the curve may be written as $y = x(x - 3)^2$, and hence write down the coordinates of A. **(2 marks)**

b Find the coordinates of B. **(2 marks)**

c The shaded region R is bounded by the curve and the x-axis. Find the area of R. **(6 marks)**

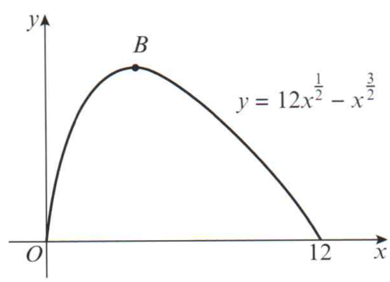
14 Consider the function $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$, $x > 0$.

a Find $\frac{dy}{dx}$. **(2 marks)**

b Find $\int y\,dx$. **(3 marks)**

c Hence show that $\int_1^3 y\,dx = A + B\sqrt{3}$, where A and B are integers to be found. **(2 marks)**

15 The diagram shows a sketch of the curve with equation $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$ for $0 \leqslant x \leqslant 12$.

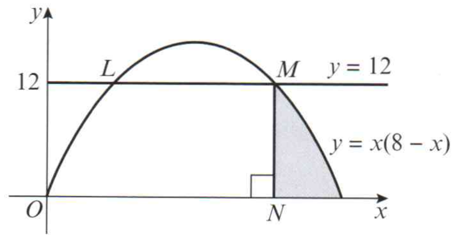


a Show that $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$. **(2 marks)**

b At the point B on the curve the tangent to the curve is parallel to the x-axis. Find the coordinates of the point B. **(2 marks)**

c Find, to 3 significant figures, the area of the finite region bounded by the curve and the x-axis. **(6 marks)**

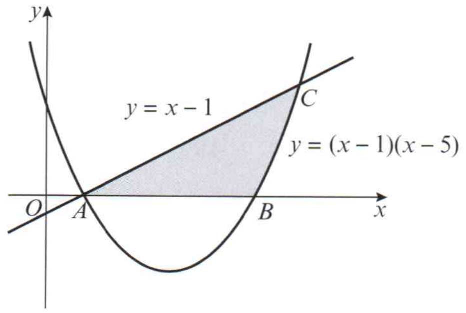
E/P **16** The diagram shows the curve C with equation $y = x(8 - x)$ and the line with equation $y = 12$ which meet at the points L and M.



a Determine the coordinates of the point M. **(2 marks)**

b Given that N is the foot of the perpendicular from M on to the x-axis, calculate the area of the shaded region which is bounded by NM, the curve C and the x-axis. **(6 marks)**

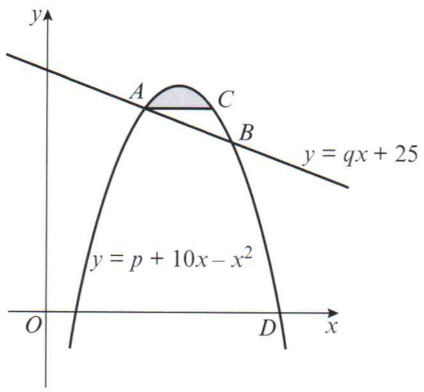
E/P **17** The diagram shows the line $y = x - 1$ meeting the curve with equation $y = (x - 1)(x - 5)$ at A and C. The curve meets the x-axis at A and B.



a Write down the coordinates of A and B and find the coordinates of C. **(4 marks)**

b Find the area of the shaded region bounded by the line, the curve and the x-axis. **(6 marks)**

E/P **18** The diagram shows part of the curve with equation $y = p + 10x - x^2$, where p is a constant, and part of the line l with equation $y = qx + 25$, where q is a constant. The line l cuts the curve at the points A and B. The x-coordinates of A and B are 4 and 8 respectively. The line through A parallel to the x-axis intersects the curve again at the point C.



a Show that $p = -7$ and calculate the value of q. **(3 marks)**

b Calculate the coordinates of C. **(2 marks)**

c The shaded region in the diagram is bounded by the curve and the line segment AC. Using integration and showing all your working, calculate the area of the shaded region. **(6 marks)**

E **19** Given that $f(x) = \frac{9}{x^2} - 8\sqrt{x} + 4x - 5$, $x > 0$, find $\int f(x)\,dx$. **(5 marks)**

E/P **20** Given that A is constant and $\int_4^9 \left(\frac{3}{\sqrt{x}} - A\right)dx = A^2$ show that there are two possible values for A and find these values. **(5 marks)**

E/P **21** $f'(x) = \frac{(2 - x^2)^3}{x^2}$, $x \neq 0$

a Show that $f'(x) = 8x^{-2} - 12 + Ax^2 + Bx^4$, where A and B are constants to be found. **(3 marks)**

b Find $f''(x)$.

Given that the point $(-2, 9)$ lies on the curve with equation $y = f(x)$,

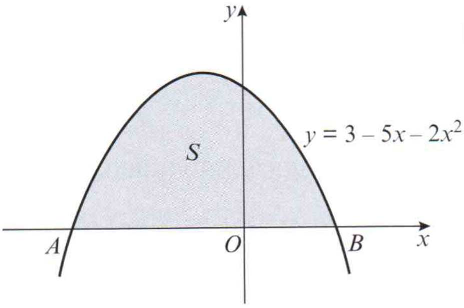
c find $f(x)$. **(5 marks)**

22 The finite region S, which is shown shaded, is bounded by the x-axis and the curve with equation $y = 3 - 5x - 2x^2$.

The curve meets the x-axis at points A and B.

a Find the coordinates of point A and point B. **(2 marks)**

b Find the area of the region S. **(4 marks)**



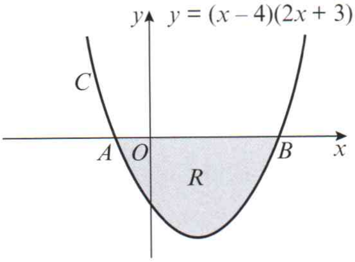
23 The graph shows a sketch of part of the curve C with equation $y = (x - 4)(2x + 3)$.

The curve C crosses the x-axis at the points A and B.

a Write down the x-coordinates of A and B. **(1 mark)**

The finite region R, shown shaded, is bounded by C and the x-axis.

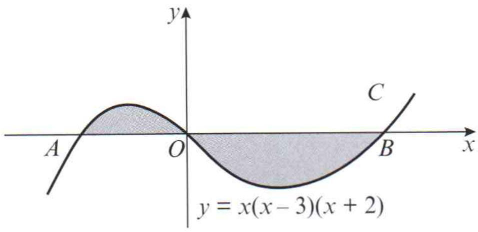
b Use integration to find the area of R. **(6 marks)**



24 The graph shows a sketch of part of the curve C with equation $y = x(x - 3)(x + 2)$.

The curve crosses the x-axis at the origin O and the points A and B.

a Write down the x-coordinates of the points A and B. **(1 mark)**



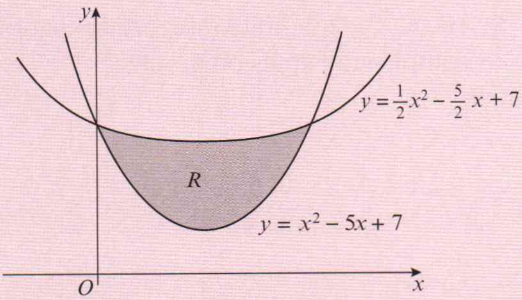
The finite region shown shaded is bounded by the curve C and the x-axis.

b Use integration to find the total area of this region. **(7 marks)**

Challenge

The curve with equation $y = x^2 - 5x + 7$ cuts the curve with equation $y = \frac{1}{2}x^2 - \frac{5}{2}x + 7$. The shaded region R is bounded by the curves as shown.

Find the exact area of R.



Summary of key points

1 If $\frac{dy}{dx} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c, n \neq -1$.

Using function notation, if $f'(x) = x^n$, then $f(x) = \frac{1}{n+1}x^{n+1} + c, n \neq -1$.

2 If $\frac{dy}{dx} = kx^n$, then $y = \frac{k}{n+1}x^{n+1} + c, n \neq -1$.

Using function notation, if $f'(x) = kx^n$, then $f(x) = \frac{k}{n+1}x^{n+1} + c, n \neq -1$.

When integrating polynomials, apply the rule of integration separately to each term.

3 $\int f'(x)dx = f(x) + c$

4 $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

5 To find the constant of integration, c

- Integrate the function
- Substitute the values (x, y) of a point on the curve, or the value of the function at a given point $f(x) = k$ into the integrated function
- Solve the equation to find c

6 If $f'(x)$ is the derivative of $f(x)$ for all values of x in the interval $[a, b]$, then the definite integral is defined as $\int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$

7 The area between a positive curve, the x-axis and the lines $x = a$ and $x = b$ is given by

Area = $\int_a^b y\, dx$

where $y = f(x)$ is the equation of the curve.

8 When the area bounded by a curve and the x-axis is below the x-axis, $\int y\, dx$ gives a negative answer.

9 You can use definite integration together with areas of trapeziums and triangles to find more complicated areas on graphs.