3 Review exercise

E/P **1** A curve has equation $y = \frac{1}{2}x^2 + 4\cos x$. Show that an equation of the normal to the curve at $x = \frac{\pi}{2}$ is

$8y(8 - \pi) - 16x + \pi(\pi^2 - 8\pi + 8) = 0$ **(7)**

← Section 9.1

E/P **2** A curve has equation $y = e^{3x} - \ln(x^2)$. Show that an equation of the tangent at $x = 2$ is $y - (3e^6 - 1)x - 2 + \ln 4 + 5e^6 = 0$ **(6)**

← Section 9.2

E/P **3** A curve has equation

$$y = -\frac{3}{(4 - 6x)^2}, x \neq \frac{2}{3}$$

Find an equation of the normal to the curve at $x = 1$ in the form $ax + by + c = 0$, where a, b and c are integers. **(7)**

← Section 9.3

E **4** A curve C has equation $y = (2x - 3)^2 e^{2x}$.

a Use the product rule to find $\frac{dy}{dx}$ **(3)**

b Hence find the coordinates of the stationary points of C. **(3)**

← Section 9.4

E **5** The curve C has equation $y = \frac{(x - 1)^2}{\sin x}$

a Use the quotient rule to find $\frac{dy}{dx}$ **(3)**

b Show that the equation of the tangent to the curve at $x = \frac{\pi}{2}$ is

$y = (\pi - 2)x + \left(1 - \frac{\pi^2}{4}\right)$ **(4)**

← Section 9.5

E/P **6** **a** Show that if $y = \operatorname{cosec} x$ then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$ **(4)**

b Given $x = \operatorname{cosec} 6y$, find $\frac{dy}{dx}$ in terms of x. **(6)**

← Section 9.6

E/P **7** Assuming standard results for $\sin x$ and $\cos x$, prove that the derivative of $\arcsin x$ is $\frac{1}{\sqrt{1 - x^2}}$ **(5)**

← Section 9.6

E **8** A curve has parametric equations

$$x = 2\cot t,\ y = 2\sin^2 t,\ 0 < t \leqslant \frac{\pi}{2}$$

a Find $\frac{dy}{dx}$ in terms of t. **(3)**

b Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$ **(3)**

c Find a Cartesian equation of the curve in the form $y = f(x)$. State the domain on which the curve is defined. **(3)**

← Section 9.7

E/P **9** The curve C has parametric equations

$$x = \frac{1}{1 + t},\ y = \frac{1}{1 - t},\ -1 < t < 1$$

The line l is a tangent to C at the point where $t = \frac{1}{2}$

a Find an equation for the line l. **(5)**

b Show that a Cartesian equation for the curve C is $y = \frac{x}{2x - 1}$ **(3)**

← Section 9.7

P **10** A curve C is described by the equation

$$3x^2 - 2y^2 + 2x - 3y + 5 = 0$$

Find an equation of the normal to C at the point (0, 1), giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. **(7)**

← Section 9.8

P **11** A set of curves is given by the equation

$$\sin x + \cos y = 0.5$$

a Use implicit differentiation to find an expression for $\frac{dy}{dx}$ **(4)**

For $-\pi < x < \pi$ and $-\pi < y < \pi$

b find the coordinates of the points where $\frac{dy}{dx} = 0$. **(3)**

← Section 9.8

P **12** A curve C has equation

$$y = x^2e^{-x}, \ x < 0$$

Show that C is convex for all $x < 0$. **(5)**

← Sections 9.4, 9.9

P **13** The volume of a spherical balloon of radius r cm is V cm³, where $V = \frac{4}{3}\pi r^3$.

a Find $\frac{dV}{dr}$ **(1)**

The volume of the balloon increases with time t seconds according to the formula

$$\frac{dV}{dt} = \frac{1000}{(2t+1)^2}, \ t \geqslant 0.$$

b Find an expression in terms of r and t for $\frac{dr}{dt}$ **(3)**

← Section 9.10

E **14** $g(x) = x^3 - x^2 - 1$

a Show that there is a root α of $g(x) = 0$ in the interval [1.4, 1.5]. **(2)**

b By considering a change of sign of $g(x)$ in a suitable interval, verify that $\alpha = 1.466$ correct to 3 decimal places. **(3)**

← Section 10.1

E **15** $p(x) = \cos x + e^{-x}$

a Show that there is a root α of $p(x) = 0$ in the interval [1.7, 1.8]. **(2)**

b By considering a change of sign of $p(x)$ in a suitable interval, verify that $\alpha = 1.746$ correct to 3 decimal places. **(3)**

← Section 10.1

E **16** $f(x) = e^{x-2} - 3x + 5$

a Show that the equation $f(x) = 0$ can be written as

$$x = \ln(3x - 5) + 2, x > \tfrac{5}{3}$$ **(2)**

The root of $f(x) = 0$ is α.

The iterative formula $x_{n+1} = \ln(3x_n - 5) + 2, x_0 = 4$ is used to find a value for α.

b Calculate the values of x_1, x_2 and x_3 to 4 decimal places. **(3)**

← Section 10.2

E **17** $f(x) = \frac{1}{(x-2)^3} + 4x^2, x \neq 2$

a Show that there is a root α of $f(x) = 0$ in the interval [0.2, 0.3]. **(2)**

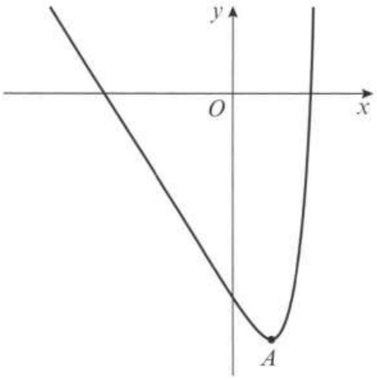
b Show that the equation $f(x) = 0$ can be written in the form $x = \sqrt[3]{\frac{-1}{4x^2}} + 2$. **(3)**

c Use the iterative formula $x_{n+1} = \sqrt[3]{\frac{-1}{4x_n^2}} + 2, x_0 = 1$ to calculate the values of x_1, x_2, x_3 and x_4 giving your answers to 4 decimal places. **(3)**

d By considering the change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.524$ correct to 3 decimal places. **(2)**

← Section 10.2

(E/P) **18** The diagram shows part of the curve with equation $y = f(x)$, where $f(x) = \frac{1}{10}x^2e^x - 2x - 10$. The point A, with x-coordinate a, is a stationary point on the curve. The equation $f(x) = 0$ has a root α in the interval [2.9, 3.0].



a Explain why $x_0 = a$ is not suitable to use as a first approximation if using the Newton–Raphson process to find an approximation for α. **(1)**

b Taking $x_0 = 2.9$ as a first approximation to α, apply the Newton–Raphson process once to find $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4)**

← Section 10.3

(E/P) **19** $f(x) = \frac{3}{10}x^3 - x^{\frac{2}{3}} + \frac{1}{x} - 4,\ x \neq 0$

a Show that there is a root α of $f(x) = 0$ in the intervals

i [0.2, 0.3] **(1)**

ii [2.6, 2.7] **(1)**

b Show that the equation $f(x) = 0$ can be written in the form

$$x = \sqrt[3]{\frac{10}{3}\left(4 + x^{\frac{2}{3}} - \frac{1}{x}\right)}$$ **(3)**

c Use the iterative formula,

$$x_{n+1} = \sqrt[3]{\frac{10}{3}\left(4 + x_n^{\frac{2}{3}} - \frac{1}{x_n}\right)},\ x_0 = 2.5$$ to

calculate the values of x_1, x_2, x_3 and x_4 giving your answers to 4 decimal places. **(3)**

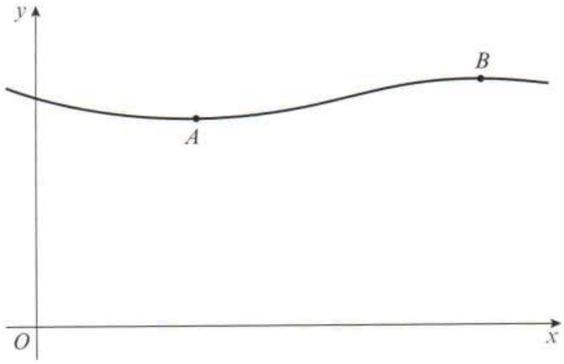
d Taking $x_0 = 0.3$ as a first approximation to α, apply the Newton–Raphson process once to find $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4)**

← Sections 10.2, 10.3

(E/P) **20** The value of a currency x hours into a 14-hour trading window can be modelled by the function

$$v(x) = 0.12\cos\left(\frac{2x}{5}\right) - 0.35\sin\left(\frac{2x}{5}\right) + 120$$

where $0 \leqslant x \leqslant 14$.



Given that $v(x)$ can be written in the form $R\cos\left(\frac{2x}{5} + \alpha\right) + 120$ where $R > 0$ and $0 \leqslant \alpha \leqslant \frac{\pi}{2}$,

a find the value of R and the value of α, correct to 4 decimal places. **(4)**

b Use your answer to part **a** to find $v'(x)$. **(3)**

c Show that the curve has a turning point in the interval [4.7, 4.8]. **(1)**

d Taking $x = 12.6$ as a first approximation, apply the Newton–Raphson method once to $v'(x)$ to obtain a second approximation for the time when the share index is a maximum. Give your answer to 3 decimal places. **(3)**

e By considering the change of sign of $v'(x)$ in a suitable interval, verify that the x-coordinate at point B is 12.6067, correct to 4 decimal places. **(2)**

← Sections 7.5, 9.3, 10.4

P 21 Given $\int_a^3 (12 - 3x)^2\,dx = 78$, find the value of a. **(4)**

← Section 11.2

P 22 **a** By expanding $\cos(5x + 2x)$ and $\cos(5x - 2x)$ using the double-angle formulae, or otherwise, show that $\cos 7x + \cos 3x \equiv 2\cos 5x \cos 2x$. **(4)**

b Hence find $\int 6\cos 5x \cos 2x\,dx$ **(3)**

← Sections 7.1, 11.3

P 23 Given that $\int_0^m mx^3 e^{x^4}\,dx = \frac{3}{4}(e^{81} - 1)$, find the value of m. **(3)**

← Section 11.4

E 24 Using the substitution $u^2 = 2x - 1$, or otherwise, find the exact value of

$$\int_1^5 \frac{3x}{\sqrt{2x-1}}\,dx$$ **(6)**

← Section 11.5

E 25 Use the substitution $u = 1 - x^2$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{x^3}{(1-x^2)^{\frac{1}{2}}}\,dx$$ **(6)**

← Section 11.5

P 26 $f(x) = (x^2 + 1)\ln x$

Find the exact value of $\int_1^e f(x)\,dx$. **(7)**

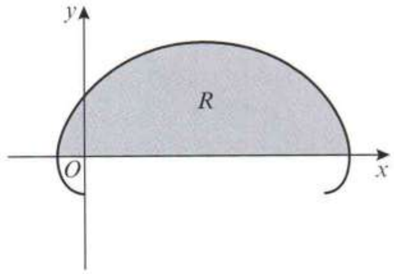
← Section 11.6

P 27 **a** Express $\frac{5x+3}{(2x-3)(x+2)}$ in partial fractions. **(3)**

b Hence find the exact value of $\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,dx$, giving your answer as a single logarithm. **(4)**

← Section 11.7

E/P 28 The curve shown in the diagram has parametric equations
$x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \leqslant t \leqslant 2\pi$



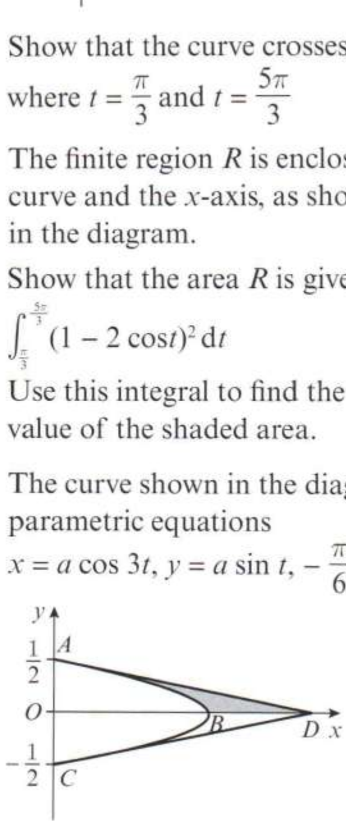
a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ **(3)**

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the diagram.

b Show that the area R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2\,dt$ **(3)**

c Use this integral to find the exact value of the shaded area. **(4)**

E/P 29 The curve shown in the diagram has parametric equations
$x = a\cos 3t,\ y = a\sin t,\ -\frac{\pi}{6} \leqslant t \leqslant \frac{\pi}{6}$.



The curve meets the axes at points A, B and C, as shown.

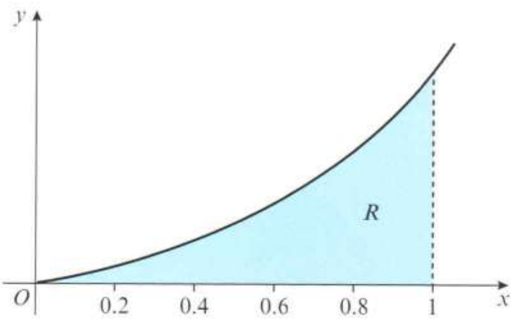
The straight lines shown are tangents to the curve at the points A and C and meet the x-axis at point D.

a Find, in terms of a, the area of the finite region between the curve, the tangent at A and the x-axis, shown shaded in the diagram.

Given that the total area of the finite region between the two tangents and the curve is 10 cm^2

b find the value of a.

E/P **30**



The diagram shows the graph of the curve with equation

$y = xe^{2x}, x \geqslant 0.$

The finite region R bounded by the lines $x = 1$, the x-axis and the curve is shown shaded in the diagram.

a Use integration to find the exact area of R. **(4)**

The table shows values of x and y between 0 and 1.

x	0	0.2	0.4	0.6	0.8	1
$y = xe^{2x}$	0	0.29836		1.99207		7.38906

b Find the missing values in the table. **(1)**

c Using the trapezium rule, with all the values for y in the completed table, find an approximation for the area of R, giving your answer to 4 significant figures. **(4)**

d Calculate the percentage error in your answer in part **c**. **(2)**

← Sections, 11.6, 11.9

E/P **31** **a** Express $\dfrac{2x-1}{(x-1)(2x-3)}$ in partial fractions. **(4)**

b Given that $x \geqslant 2$, find the general solution of the differential equation

$$(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y \qquad \textbf{(4)}$$

c Hence find the particular solution of this differential equation that satisfies $y = 10$ at $x = 2$, giving your answer in the form $y = f(x)$. **(2)**

← Sections 11.7, 11.10

E/P **32** A spherical balloon is being inflated in such a way that the rate of increase of its volume, V cm³, with respect to time t seconds is given by $\dfrac{dV}{dt} = \dfrac{k}{V}$, where k is a positive constant.

Given that the radius of the balloon is r cm, and that $V = \frac{4}{3}\pi r^3$,

a prove that r satisfies the differential equation

$$\frac{dr}{dt} = \frac{B}{r^5}$$

where B is a constant. **(4)**

b Find a general solution of the differential equation obtained in part **a**. **(5)**

← Sections 9.10, 11.10, 11.11

E/P **33** Liquid is pouring into a container at a constant rate of 20 cm³ s⁻¹ and is leaking out at a rate proportional to the volume of the liquid already in the container.

a Explain why, at time t seconds, the volume, V cm³, of liquid in the container satisfies the differential equation

$$\frac{dV}{dt} = 20 - kV$$

where k is a positive constant. **(2)**

The container is initially empty.

b By solving the differential equation, show that

$$V = A + Be^{-kt}$$

giving the values of A and B in terms of k. **(5)**

Given also that $\dfrac{dV}{dt} = 10$ when $t = 5$,

c find the volume of liquid in the container at 10 s after the start. **(3)**

← Sections 11.10, 11.11

P **34** The rate of decrease of the concentration of a drug in the blood stream is proportional to the concentration C of that drug which is present at that time. The time t is measured in hours from the administration of the drug and C is measured in micrograms per litre.

a Show that this process is described by the differential equation $\frac{dC}{dt} = -kC$, explaining why k is a positive constant. **(2)**

b Find the general solution of the differential equation, in the form $C = f(t)$. **(4)**

After 4 hours, the concentration of the drug in the bloodstream is reduced to 10% of its starting value C_0.

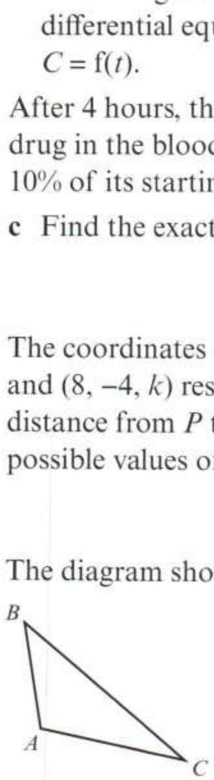
c Find the exact value of k. **(3)**

← Sections 11.10, 11.11

P **35** The coordinates of P and Q are $(-1, 4, 6)$ and $(8, -4, k)$ respectively. Given that the distance from P to Q is $7\sqrt{5}$ units, find the possible values of k. **(3)**

← Section 12.1

P **36** The diagram shows the triangle ABC.



Given that $\overrightarrow{AB} = -\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{AC} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, find the size of $\angle BAC$ to one decimal place. **(5)**

← Section 12.2

37 P is the point $(-6, 3, 2)$ and Q is the point $(4, -2, 0)$. Find:

a the vector $\overrightarrow{PQ}$ **(1)**

b the unit vector in the direction of $\overrightarrow{PQ}$ **(2)**

c the angle $\overrightarrow{PQ}$ makes with the positive z-axis. **(2)**

The vector $\overrightarrow{AB} = 30\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$.

d Explain, with a reason, whether the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are parallel. **(2)**

← Section 12.2

E/P **38** The vertices of triangle MNP have coordinates $M(-2, 0, 5)$, $N(8, -5, 1)$ and $P(k, -2, -6)$. Given that triangle MNP is isosceles and k is a positive integer, find the value of k. **(4)**

← Section 12.3

E/P **39** Given that

$$-6\mathbf{i} + 40\mathbf{j} + 16\mathbf{k} = 3p\mathbf{i} + (8 + qr)\mathbf{j} + 2pr\mathbf{k}$$

find the values of p, q and r. **(3)**

← Section 12.3

Challenge

1 The curve C has implicit equation

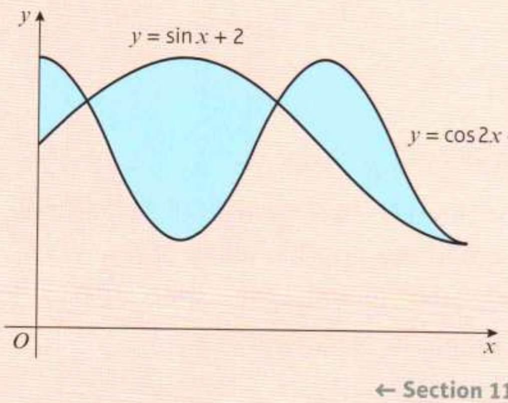
$$ay + x^2 + 4xy = y^2$$

a Find, in terms of a where necessary, the coordinates of the points such that $\frac{dy}{dx} = 0$.

b Given that $a \neq 0$, show that there does not exist a point where $\frac{dx}{dy} = 0$. ← Section 9.8

2 The diagram shows the curves $y = \sin x + 2$ and $y = \cos 2x + 2$, $0 \leqslant x \leqslant \frac{3\pi}{2}$

Find the exact value of the total shaded area on the diagram.



← Section 11.8