11 Vectors

Objectives

After completing this chapter you should be able to:

- Use vectors in two dimensions → pages 231–235
- Use column vectors and carry out arithmetic operations on vectors → pages 235–238
- Calculate the magnitude and direction of a vector → pages 239–242
- Understand and use position vectors → pages 242–244
- Use vectors to solve geometric problems → pages 244–247
- Understand vector magnitude and use vectors in speed and distance calculations → pages 248–251
- Use vectors to solve problems in context → pages 248–251

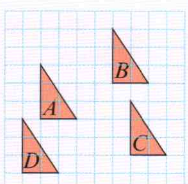
Prior knowledge check

1 Write the column vector for the translation of shape

a A to B

b A to C

c A to D



← GCSE Mathematics

2 P divides the line AB in the ratio $AP:PB = 7:2$.

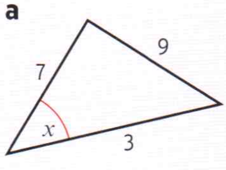
Find:

a $\frac{AP}{AB}$ **b** $\frac{PB}{AB}$ **c** $\frac{AP}{PB}$

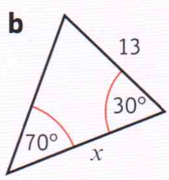
← GCSE Mathematics

3 Find x to one decimal place.

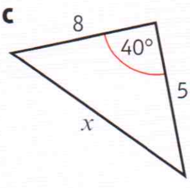
a



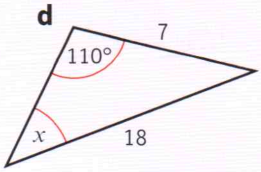
b



c



d



← Sections 9.1, 9.2

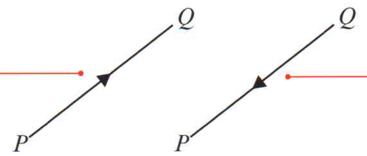
Pilots use **vector addition** to work out the resultant vector for their speed and heading when a plane encounters a strong cross-wind. Engineers also use vectors to work out the resultant forces acting on structures in construction.

11.1 Vectors

A **vector** has both **magnitude** and **direction**.

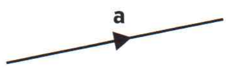
You can represent a vector using a **directed line segment**.

This is vector $\overrightarrow{PQ}$. It starts at P and finishes at Q.

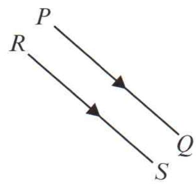


This is vector $\overrightarrow{QP}$. It starts at Q and finishes at P.

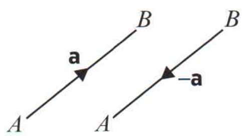
The direction of the arrow shows the direction of the vector. Small (lower case) letters are also used to represent vectors. In print, the small letter will be in bold type. In writing, you should underline the small letter to show it is a vector: $\underline{a}$ or $\underset{\sim}{a}$



- **If $\overrightarrow{PQ} = \overrightarrow{RS}$ then the line segments PQ and RS are equal in length and are parallel.**

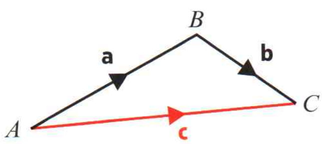


- **$\overrightarrow{AB} = -\overrightarrow{BA}$ as the line segment AB is equal in length, parallel and in the opposite direction to BA.**



You can add two vectors together using the **triangle law** for vector addition.

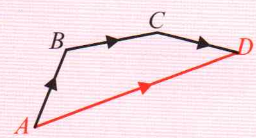
- **Triangle law for vector addition:**
  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



If $\overrightarrow{AB}$ = a, $\overrightarrow{BC}$ = b and $\overrightarrow{AC}$ = c, then a + b = c

Notation The **resultant** is the **vector sum** of two or more vectors.

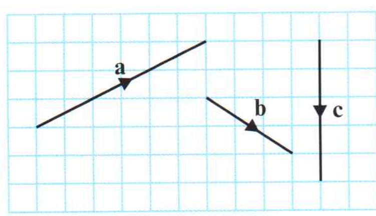
$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$

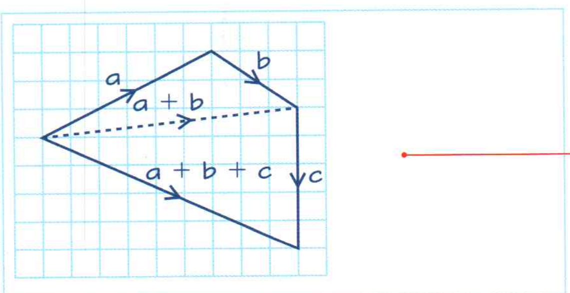


Example 1

The diagram shows vectors **a**, **b** and **c**.

Draw a diagram to illustrate the vector addition **a** + **b** + **c**.

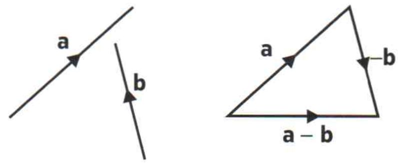




First use the triangle law for **a** + **b**, then use it again for (**a** + **b**) + **c**.
The resultant goes from the start of **a** to the end of **c**.

Online Explore vector addition using GeoGebra.

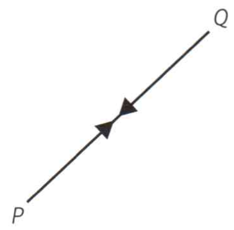
- **Subtracting a vector is equivalent to 'adding a negative vector':**
  $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$



Hint To subtract **b**, you reverse the direction of **b** then add.

If you travel from P to Q, then back from Q to P, you are back where you started, so your displacement is zero.

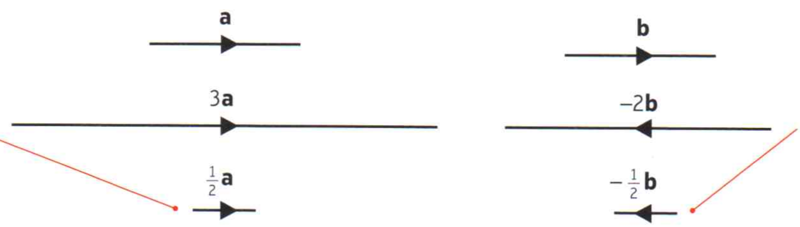
- **Adding the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ gives the zero vector 0: $\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$**



Hint $\overrightarrow{QP} = -\overrightarrow{PQ}$.
So $\overrightarrow{PQ} + \overrightarrow{QP} = \overrightarrow{PQ} - \overrightarrow{PQ} = \mathbf{0}$.

You can multiply a vector by a scalar (or number).

If the number is positive ($\neq 1$) the new vector has a different length but the **same** direction.



If the number is negative ($\neq -1$) the new vector has a different length and the **opposite** direction.

- **Any vector parallel to the vector a may be written as $\lambda\mathbf{a}$, where λ is a non-zero scalar.**

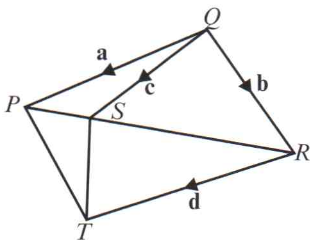
Notation Real numbers are examples of **scalars**. They have magnitude but no direction.

Example 2

In the diagram, $\overrightarrow{QP} = \mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$, $\overrightarrow{QS} = \mathbf{c}$ and $\overrightarrow{RT} = \mathbf{d}$.

Find in terms of **a**, **b**, **c** and **d**:

a $\overrightarrow{PS}$ **b** $\overrightarrow{RP}$
c $\overrightarrow{PT}$ **d** $\overrightarrow{TS}$



a $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS} = -\mathbf{a} + \mathbf{c}$
$= \mathbf{c} - \mathbf{a}$

Add vectors using $\triangle PQS$.

b $\overrightarrow{RP} = \overrightarrow{RQ} + \overrightarrow{QP} = -\mathbf{b} + \mathbf{a}$
$= \mathbf{a} - \mathbf{b}$

Add vectors using $\triangle RQP$.

c $\overrightarrow{PT} = \overrightarrow{PR} + \overrightarrow{RT} = (\mathbf{b} - \mathbf{a}) + \mathbf{d}$
$= \mathbf{b} + \mathbf{d} - \mathbf{a}$

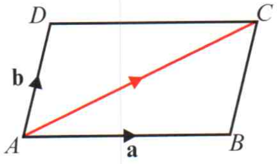
Add vectors using $\triangle PRT$.
Use $\overrightarrow{PR} = -\overrightarrow{RP} = -(\mathbf{a} - \mathbf{b}) = \mathbf{b} - \mathbf{a}$.

d $\overrightarrow{TS} = \overrightarrow{TR} + \overrightarrow{RS} = -\mathbf{d} + (\overrightarrow{RQ} + \overrightarrow{QS})$
$= -\mathbf{d} + (-\mathbf{b} + \mathbf{c})$
$= \mathbf{c} - \mathbf{b} - \mathbf{d}$

Add vectors using $\triangle TRS$ and $\triangle RQS$.

Example 3

$ABCD$ is a parallelogram. $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AD} = \mathbf{b}$. Find $\overrightarrow{AC}$.



Notation This is called the **parallelogram law** for vector addition.

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{BC} = \overrightarrow{AD} = \mathbf{b}$

So $\overrightarrow{AC} = \mathbf{a} + \mathbf{b}$

Using the triangle law for addition of vectors.

AD and BC are opposite sides of a parallelogram so they are parallel and equal in magnitude.

Example 4

Show that the vectors $6\mathbf{a} + 8\mathbf{b}$ and $9\mathbf{a} + 12\mathbf{b}$ are parallel.

$9\mathbf{a} + 12\mathbf{b} = \frac{3}{2}(6\mathbf{a} + 8\mathbf{b})$

∴ the vectors are parallel.

Here $\lambda = \frac{3}{2}$

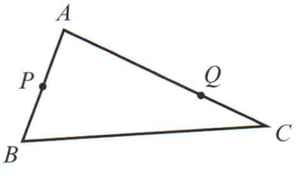
Example 5

In triangle ABC, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$.

P is the midpoint of AB.

Q divides AC in the ratio $3:2$.

Write in terms of $\mathbf{a}$ and $\mathbf{b}$:



a $\overrightarrow{BC}$ **b** $\overrightarrow{AP}$ **c** $\overrightarrow{AQ}$ **d** $\overrightarrow{PQ}$

a $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= -\overrightarrow{AB} + \overrightarrow{AC}$

$\overrightarrow{BC} = \mathbf{b} - \mathbf{a}$

b $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\mathbf{a}$

c $\overrightarrow{AQ} = \frac{3}{5}\overrightarrow{AC} = \frac{3}{5}\mathbf{b}$

d $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$

$= -\overrightarrow{AP} + \overrightarrow{AQ}$

$= \frac{3}{5}\mathbf{b} - \frac{1}{2}\mathbf{a}$

$\overrightarrow{BA} = -\overrightarrow{AB}$

$AP = \frac{1}{2}AB$ so $\overrightarrow{AP} = \frac{1}{2}\mathbf{a}$

Watch out AP is the line segment between A and P, whereas $\overrightarrow{AP}$ is the vector from A to P.

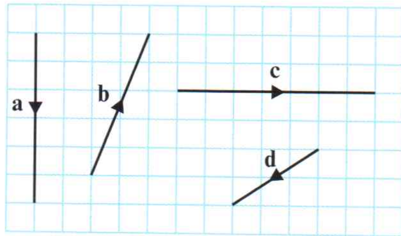
Q divides AC in the ratio $3:2$ so $AQ = \frac{3}{5}AC$.

Going from P to Q is the same as going from P to A, then from A to Q.

Exercise 11A

1 The diagram shows the vectors **a**, **b**, **c** and **d**.

Draw a diagram to illustrate these vectors:



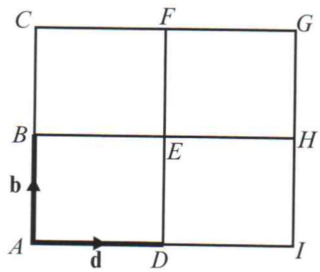
a $\mathbf{a} + \mathbf{c}$ **b** $-\mathbf{b}$

c $\mathbf{c} - \mathbf{d}$ **d** $\mathbf{b} + \mathbf{c} + \mathbf{d}$

e $2\mathbf{c} + 3\mathbf{d}$ **f** $\mathbf{a} - 2\mathbf{b}$

g $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$

2 *ACGI* is a square, *B* is the midpoint of *AC*, *F* is the midpoint of *CG*, *H* is the midpoint of *GI*, *D* is the midpoint of *AI*. $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$. Find, in terms of **b** and **d**:



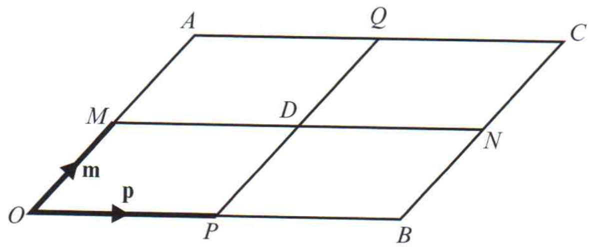
a $\overrightarrow{AC}$ **b** $\overrightarrow{BE}$ **c** $\overrightarrow{HG}$ **d** $\overrightarrow{DF}$

e $\overrightarrow{AE}$ **f** $\overrightarrow{DH}$ **g** $\overrightarrow{HB}$ **h** $\overrightarrow{FE}$

i $\overrightarrow{AH}$ **j** $\overrightarrow{BI}$ **k** $\overrightarrow{EI}$ **l** $\overrightarrow{FB}$

3 *OACB* is a parallelogram. *M*, *Q*, *N* and *P* are the midpoints of *OA*, *AC*, *BC* and *OB* respectively.

Vectors **p** and **m** are equal to $\overrightarrow{OP}$ and $\overrightarrow{OM}$ respectively. Express in terms of **p** and **m**.

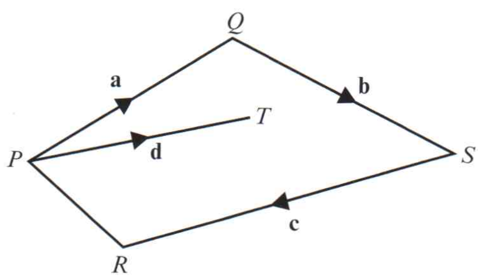


a $\overrightarrow{OA}$ **b** $\overrightarrow{OB}$ **c** $\overrightarrow{BN}$ **d** $\overrightarrow{DQ}$

e $\overrightarrow{OD}$ **f** $\overrightarrow{MQ}$ **g** $\overrightarrow{OQ}$ **h** $\overrightarrow{AD}$

i $\overrightarrow{CD}$ **j** $\overrightarrow{AP}$ **k** $\overrightarrow{BM}$ **l** $\overrightarrow{NO}$

4 In the diagram, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{QS} = \mathbf{b}$, $\overrightarrow{SR} = \mathbf{c}$ and $\overrightarrow{PT} = \mathbf{d}$. Find in terms of **a**, **b**, **c** and **d**:



a $\overrightarrow{QT}$ **b** $\overrightarrow{PR}$

c $\overrightarrow{TS}$ **d** $\overrightarrow{TR}$

5 In the triangle *PQR*, $PQ = 2\mathbf{a}$ and $QR = 2\mathbf{b}$. The midpoint of *PR* is *M*. Find, in terms of **a** and **b**:

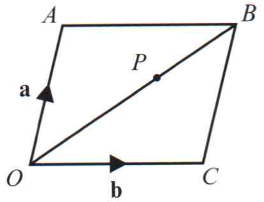
a $\overrightarrow{PR}$ **b** $\overrightarrow{PM}$ **c** $\overrightarrow{QM}$

(P) **6** *ABCD* is a trapezium with *AB* parallel to *DC* and $DC = 3AB$. *M* divides *DC* such that $DM : MC = 2 : 1$. $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$. Find, in terms of **a** and **b**:

a $\overrightarrow{AM}$ **b** $\overrightarrow{BD}$ **c** $\overrightarrow{MB}$ **d** $\overrightarrow{DA}$

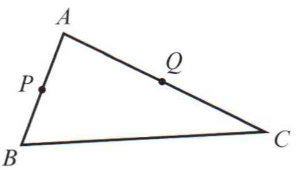
> **Problem-solving**
>
> Draw a sketch to show the information given in the question.

7 $OABC$ is a parallelogram. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{b}$.
The point P divides OB in the ratio 5:3.
Find, in terms of **a** and **b**:
a $\overrightarrow{OB}$ **b** $\overrightarrow{OP}$ **c** $\overrightarrow{AP}$

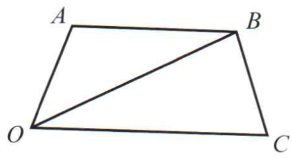


8 State with a reason whether each of these vectors is parallel to the vector $\mathbf{a} - 3\mathbf{b}$:
a $2\mathbf{a} - 6\mathbf{b}$ **b** $4\mathbf{a} - 12\mathbf{b}$ **c** $\mathbf{a} + 3\mathbf{b}$ **d** $3\mathbf{b} - \mathbf{a}$ **e** $9\mathbf{b} - 3\mathbf{a}$ **f** $\frac{1}{2}\mathbf{a} - \frac{2}{3}\mathbf{b}$

P 9 In triangle ABC, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$.
P is the midpoint of AB and Q is the midpoint of AC.
a Write in terms of **a** and **b**:
i $\overrightarrow{BC}$ **ii** $\overrightarrow{AP}$ **iii** $\overrightarrow{AQ}$ **iv** $\overrightarrow{PQ}$
b Show that PQ is parallel to BC.



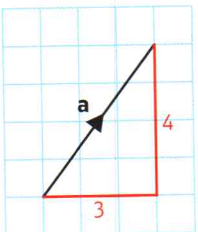
P 10 $OABC$ is a quadrilateral. $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = 3\mathbf{b}$ and $\overrightarrow{OB} = \mathbf{a} + 2\mathbf{b}$.
a Find, in terms of **a** and **b**:
i $\overrightarrow{AB}$ **ii** $\overrightarrow{CB}$
b Show that AB is parallel to OC.



P 11 The vectors $2\mathbf{a} + k\mathbf{b}$ and $5\mathbf{a} + 3\mathbf{b}$ are parallel. Find the value of k.

11.2 Representing vectors

A vector can be described by its change in position or **displacement** relative to the x- and y-axes.



$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ where 3 is the change in the x-direction and 4 is the change in the y-direction.

This is called **column vector** form.

Notation The top number is the x-component and the bottom number is the y-component.

- **To multiply a column vector by a scalar, multiply each component by the scalar:** $\lambda\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda p \\ \lambda q \end{pmatrix}$
- **To add two column vectors, add the x-components and the y-components:** $\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p + r \\ q + s \end{pmatrix}$

Example 6

$\mathbf{a} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Find **a** $\frac{1}{3}\mathbf{a}$ **b** $\mathbf{a} + \mathbf{b}$ **c** $2\mathbf{a} - 3\mathbf{b}$

a $\frac{1}{3}\mathbf{a} = \begin{pmatrix} \frac{2}{3} \\ 2 \end{pmatrix}$ — **Both** of the components are divided by 3.

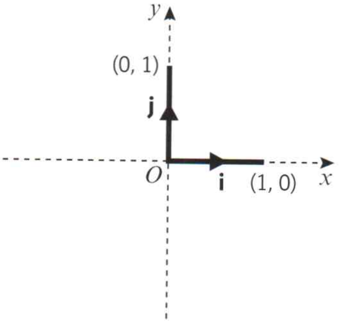
b $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ — Add the x-components and the y-components.

c $2\mathbf{a} - 3\mathbf{b} = 2\begin{pmatrix} 2 \\ 6 \end{pmatrix} - 3\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ — Multiply each of the vectors by the scalars then subtract the x- and y-components.

$= \begin{pmatrix} 4 \\ 12 \end{pmatrix} - \begin{pmatrix} 9 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 - 9 \\ 12 + 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 15 \end{pmatrix}$

You can use **unit vectors** to represent vectors in two dimensions.

- **A unit vector is a vector of length 1. The unit vectors along the x- and y-axes are usually denoted by i and j respectively.**
  - $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

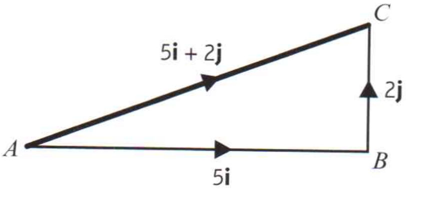


- **You can write any two-dimensional vector in the form $p\mathbf{i} + q\mathbf{j}$.**

By the triangle law of addition:

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= 5\mathbf{i} + 2\mathbf{j}$



You can also write this as a column vector: $5\mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

- **For any two-dimensional vector: $\begin{pmatrix} p \\ q \end{pmatrix} = p\mathbf{i} + q\mathbf{j}$**

Example 7

$\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + 7\mathbf{j}$

Find **a** $\frac{1}{2}\mathbf{a}$ **b** $\mathbf{a} + \mathbf{b}$ **c** $3\mathbf{a} - 2\mathbf{b}$

a $\frac{1}{2}\mathbf{a} = \frac{1}{2}(3\mathbf{i} - 4\mathbf{j}) = 1.5\mathbf{i} - 2\mathbf{j}$ — Divide the **i** component and the **j** component by 2.

b $\mathbf{a} + \mathbf{b} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{i} + 7\mathbf{j}$

$= (3 + 2)\mathbf{i} + (-4 + 7)\mathbf{j} = 5\mathbf{i} + 3\mathbf{j}$ — Add the **i** components and the **j** components.

c $3\mathbf{a} - 2\mathbf{b} = 3(3\mathbf{i} - 4\mathbf{j}) - 2(2\mathbf{i} + 7\mathbf{j})$

$= 9\mathbf{i} - 12\mathbf{j} - (4\mathbf{i} + 14\mathbf{j})$

$= (9 - 4)\mathbf{i} + (-12 - 14)\mathbf{j}$

$= 5\mathbf{i} - 26\mathbf{j}$

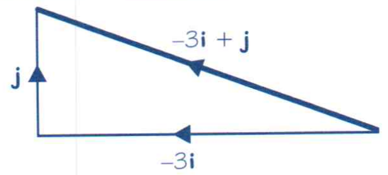
Multiply each of the vectors by the scalar then subtract the **i** and **j** components.

Example 8

a Draw a diagram to represent the vector $-3\mathbf{i} + \mathbf{j}$

b Write this as a column vector.

a



b $-3\mathbf{i} + \mathbf{j} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

3 units in the direction of the unit vector $-\mathbf{i}$ and 1 unit in the direction of the unit vector $\mathbf{j}$.

Example 9

Given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = 12\mathbf{i} - 10\mathbf{j}$ and $\mathbf{c} = -3\mathbf{i} + 9\mathbf{j}$, find $\mathbf{a} + \mathbf{b} + \mathbf{c}$, using column vector notation in your working.

$\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 12 \\ -10 \end{pmatrix} + \begin{pmatrix} -3 \\ 9 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix}$

Add the numbers in the top line to get 11 (the x-component), and the bottom line to get 4 (the y-component). This is $11\mathbf{i} + 4\mathbf{j}$.

Example 10

Given $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$, find $2\mathbf{a} - \mathbf{b}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

Online Explore this solution as a vector diagram on a coordinate grid using GeoGebra.

$2\mathbf{a} = 2\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$

To find the column vector for vector $2\mathbf{a}$ multiply the $\mathbf{i}$ and $\mathbf{j}$ components of vector $\mathbf{a}$ by 2.

$2\mathbf{a} - \mathbf{b} = \begin{pmatrix} 10 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 10 - 3 \\ 4 - (-4) \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$

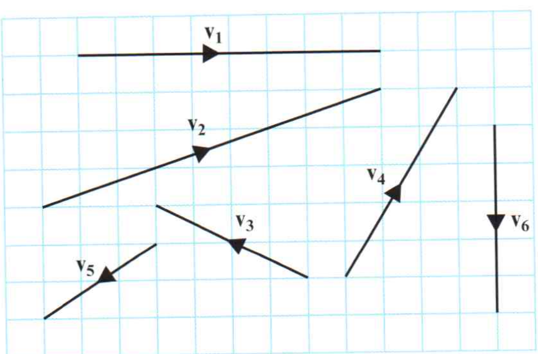
To find the column vector for $2\mathbf{a} - \mathbf{b}$ subtract the components of vector $\mathbf{b}$ from those of vector $2\mathbf{a}$.

$2\mathbf{a} - \mathbf{b} = 7\mathbf{i} + 8\mathbf{j}$

Remember to give your answer in terms of $\mathbf{i}$ and $\mathbf{j}$.

Exercise 11B

1 These vectors are drawn on a grid of unit squares. Express the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$ and $\mathbf{v}_6$ in $\mathbf{i}$, $\mathbf{j}$ notation and column vector form.



2 Given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - \mathbf{j}$, find these vectors in terms of $\mathbf{i}$ and $\mathbf{j}$.

a $4\mathbf{a}$ **b** $\frac{1}{2}\mathbf{a}$ **c** $-\mathbf{b}$ **d** $2\mathbf{b} + \mathbf{a}$

e $3\mathbf{a} - 2\mathbf{b}$ **f** $\mathbf{b} - 3\mathbf{a}$ **g** $4\mathbf{b} - \mathbf{a}$ **h** $2\mathbf{a} - 3\mathbf{b}$

3 Given that $\mathbf{a} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 11 \\ -3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -8 \\ -1 \end{pmatrix}$ find:

a $5\mathbf{a}$ **b** $-\frac{1}{2}\mathbf{c}$ **c** $\mathbf{a} + \mathbf{b} + \mathbf{c}$ **d** $2\mathbf{a} - \mathbf{b} + \mathbf{c}$

e $2\mathbf{b} + 2\mathbf{c} - 3\mathbf{a}$ **f** $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

(P) **4** Given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$, find:

a λ if $\mathbf{a} + \lambda\mathbf{b}$ is parallel to the vector $\mathbf{i}$ **b** μ if $\mu\mathbf{a} + \mathbf{b}$ is parallel to the vector $\mathbf{j}$

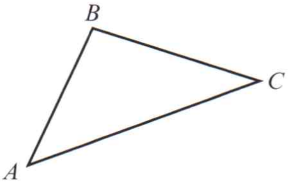
(P) **5** Given that $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j}$, find:

a λ if $\mathbf{c} + \lambda\mathbf{d}$ is parallel to $\mathbf{i} + \mathbf{j}$ **b** μ if $\mu\mathbf{c} + \mathbf{d}$ is parallel to $\mathbf{i} + 3\mathbf{j}$

c s if $\mathbf{c} - s\mathbf{d}$ is parallel to $2\mathbf{i} + \mathbf{j}$ **d** t if $\mathbf{d} - t\mathbf{c}$ is parallel to $-2\mathbf{i} + 3\mathbf{j}$

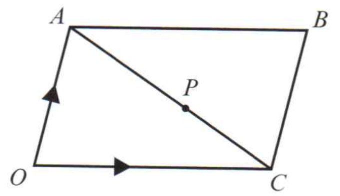
(E) **6** In triangle ABC, $\overrightarrow{AB} = 4\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{AC} = 5\mathbf{i} + 2\mathbf{j}$.
Find $\overrightarrow{BC}$.

(2 marks)



(P) **7** $OABC$ is a parallelogram.
P divides AC in the ratio 3 : 2. $\overrightarrow{OA} = 2\mathbf{i} + 4\mathbf{j}$, $\overrightarrow{OC} = 7\mathbf{i}$.
Find in $\mathbf{i}$, $\mathbf{j}$ format and column vector format:

a $\overrightarrow{AC}$ **b** $\overrightarrow{AP}$ **c** $\overrightarrow{OP}$



(E/P) **8** $\mathbf{a} = \begin{pmatrix} j \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 10 \\ k \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

Given that $\mathbf{b} - 2\mathbf{a} = \mathbf{c}$, find the values of j and k.

(2 marks)

Problem-solving

You can consider $\mathbf{b} - 2\mathbf{a} = \mathbf{c}$ as two linear equations. One for the x-components and one for the y-components.

(E/P) **9** $\mathbf{a} = \begin{pmatrix} p \\ -q \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} q \\ p \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

Given that $\mathbf{a} + 2\mathbf{b} = \mathbf{c}$, find the values of p and q. **(2 marks)**

(E/P) **10** The resultant of the vectors $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = p\mathbf{i} - 2p\mathbf{j}$ is parallel to the vector $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$.
Find:

a the value of p **(4 marks)**

b the resultant of vectors $\mathbf{a}$ and $\mathbf{b}$. **(1 mark)**

11.3 Magnitude and direction

You can use Pythagoras' theorem to calculate the **magnitude** of a vector.

- **For the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} = \begin{pmatrix} x \\ y \end{pmatrix}$, the magnitude of the vector is given by:**

  $$|\mathbf{a}| = \sqrt{x^2 + y^2}$$

Notation You use straight lines on either side of the vector:

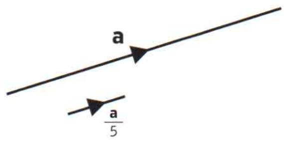
$|\mathbf{a}| = |x\mathbf{i} + y\mathbf{j}| = \left|\begin{pmatrix} x \\ y \end{pmatrix}\right|$

You need to be able to find a **unit vector** in the direction of a given vector.

- **A unit vector in the direction of a is $\frac{\mathbf{a}}{|\mathbf{a}|}$**

If $|\mathbf{a}| = 5$ then a unit vector in the direction of **a** is $\frac{\mathbf{a}}{5}$.

Notation A unit vector is any vector with magnitude 1.
A unit vector in the direction of **a** is sometimes written as $\hat{\mathbf{a}}$.



Example 11

Given that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} - 4\mathbf{j}$:

a find $|\mathbf{a}|$

b find a unit vector in the direction of **a**

c find the exact value of $|2\mathbf{a} + \mathbf{b}|$

Online Explore the magnitude of a vector using GeoGebra.

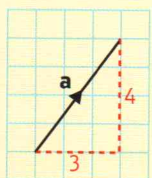
a $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

It is often quicker and easier to convert from **i**, **j** form to column vector form for calculations.

$|\mathbf{a}| = \sqrt{3^2 + 4^2}$

$|\mathbf{a}| = \sqrt{25} = 5$

Using Pythagoras.



b a unit vector is $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3\mathbf{i} + 4\mathbf{j}}{5}$

$= \frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$ or $\begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$

Unless specified in the question it is acceptable to give your answer in **i**, **j** form or column vector form.

c $2\mathbf{a} + \mathbf{b} = 2\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 - 2 \\ 8 - 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$|2\mathbf{a} + \mathbf{b}| = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$

You need to give an exact answer, so leave your answer in surd form:

$\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$

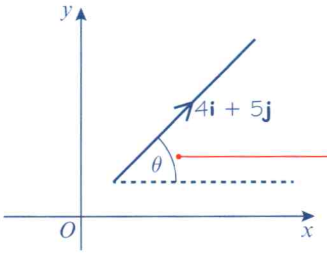
← Section 1.5

You can define a vector by giving its magnitude, and the angle between the vector and one of the coordinate axes. This is called **magnitude-direction form**.

Example 12

Find the angle between the vector $4\mathbf{i} + 5\mathbf{j}$ and the positive x-axis.

This might be referred to as the angle between the vector and **i**.



Identify the angle that you need to find. A diagram always helps.

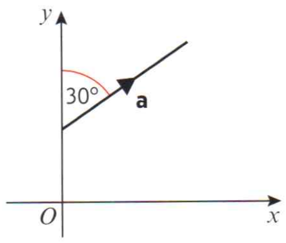
$\tan\theta = \frac{5}{4}$

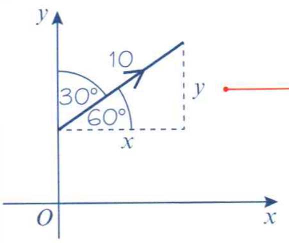
You have a right-angled triangle with base 4 units and height 5 units, so use trigonometry.

$\theta = \tan^{-1}\left(\frac{5}{4}\right) = 51.3°$ (3 s.f.)

Example 13

Vector **a** has magnitude 10 and makes an angle of 30° with **j**.
Find **a** in **i**, **j** and column vector format.





Use trigonometry to find the lengths of the x- and y-components for vector **a**.

$\cos 60° = \frac{x}{10}$ $\quad x = 10\cos 60° = 5$

$\sin 60° = \frac{y}{10}$ $\quad y = 10\sin 60° = 5\sqrt{3}$

$\mathbf{a} = 5\mathbf{i} + 5\sqrt{3}\mathbf{j}$ or $\mathbf{a} = \begin{pmatrix} 5 \\ 5\sqrt{3} \end{pmatrix}$

Watch out The direction of a vector can be given relative to either the positive x-axis (the **i** direction) or the positive y-axis (or the **j** direction).

Exercise 11C

1 Find the magnitude of each of these vectors.

a $3\mathbf{i} + 4\mathbf{j}$ b $6\mathbf{i} - 8\mathbf{j}$ c $5\mathbf{i} + 12\mathbf{j}$ d $2\mathbf{i} + 4\mathbf{j}$

e $3\mathbf{i} - 5\mathbf{j}$ f $4\mathbf{i} + 7\mathbf{j}$ g $-3\mathbf{i} + 5\mathbf{j}$ h $-4\mathbf{i} - \mathbf{j}$

2 $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{c} = 5\mathbf{i} - \mathbf{j}$. Find the exact value of the magnitude of:

a $\mathbf{a} + \mathbf{b}$ **b** $2\mathbf{a} - \mathbf{c}$ **c** $3\mathbf{b} - 2\mathbf{c}$

3 For each of the following vectors, find the unit vector in the same direction.

a $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ **b** $\mathbf{b} = 5\mathbf{i} - 12\mathbf{j}$ **c** $\mathbf{c} = -7\mathbf{i} + 24\mathbf{j}$ **d** $\mathbf{d} = \mathbf{i} - 3\mathbf{j}$

4 Find the angle that each of these vectors makes with the positive x-axis.

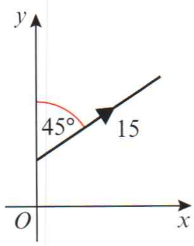
a $3\mathbf{i} + 4\mathbf{j}$ **b** $6\mathbf{i} - 8\mathbf{j}$ **c** $5\mathbf{i} + 12\mathbf{j}$ **d** $2\mathbf{i} + 4\mathbf{j}$

5 Find the angle that each of these vectors makes with $\mathbf{j}$.

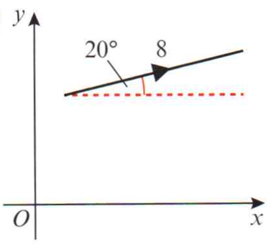
a $3\mathbf{i} - 5\mathbf{j}$ **b** $4\mathbf{i} + 7\mathbf{j}$ **c** $-3\mathbf{i} + 5\mathbf{j}$ **d** $-4\mathbf{i} - \mathbf{j}$

6 Write these vectors in **i**, **j** and column vector form.

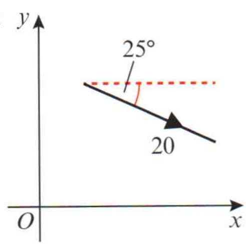
a



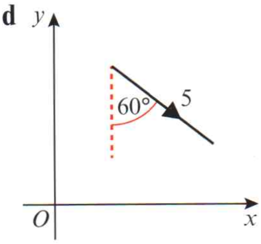
b



c



d



7 Draw a sketch for each vector and work out the exact value of its magnitude and the angle it makes with the positive x-axis to one decimal place.

a $3\mathbf{i} + 4\mathbf{j}$ **b** $2\mathbf{i} - \mathbf{j}$ **c** $-5\mathbf{i} + 2\mathbf{j}$

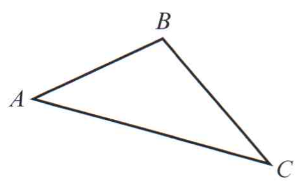
/P **8** Given that $|2\mathbf{i} - k\mathbf{j}| = 2\sqrt{10}$, find the exact value of k. **(3 marks)**

/P **9** Vector $\mathbf{a} = p\mathbf{i} + q\mathbf{j}$ has magnitude 10 and makes an angle θ with the positive x-axis where $\sin\theta = \frac{3}{5}$. Find the possible values of p and q. **(4 marks)**

Problem-solving

Make sure you consider all the possible cases.

10 In triangle ABC, $\overrightarrow{AB} = 4\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{AC} = 6\mathbf{i} - 4\mathbf{j}$.



a Find the angle between $\overrightarrow{AB}$ and $\mathbf{i}$.

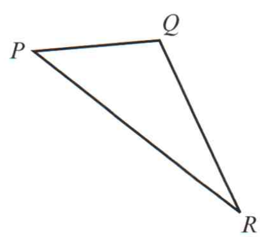
b Find the angle between $\overrightarrow{AC}$ and $\mathbf{i}$.

c Hence find the size of $\angle BAC$, in degrees, to one decimal place.

/P **11** In triangle PQR, $\overrightarrow{PQ} = 4\mathbf{i} + \mathbf{j}$, $\overrightarrow{PR} = 6\mathbf{i} - 8\mathbf{j}$.

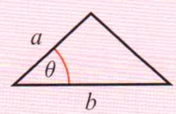
a Find the size of $\angle QPR$, in degrees, to one decimal place. **(5 marks)**

b Find the area of triangle PQR. **(2 marks)**



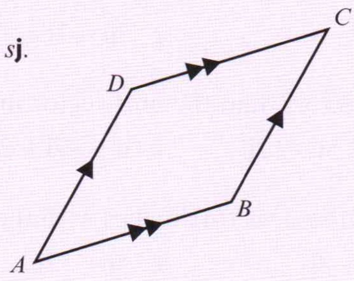
Hint The area of a triangle is $\frac{1}{2}ab\sin\theta$.

← Section 9.3



Challenge

In the diagram $\overrightarrow{AB} = p\mathbf{i} + q\mathbf{j}$ and $\overrightarrow{AD} = r\mathbf{i} + s\mathbf{j}$.
$ABCD$ is a parallelogram.
Prove that the area of $ABCD$ is $ps - qr$.



Problem-solving

Draw the parallelogram on a coordinate grid, and choose a position for the origin that will simplify your calculations.

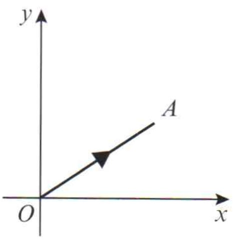
11.4 Position vectors

You need to be able to use vectors to describe the position of a point in two dimensions.

Position vectors are vectors giving the position of a point, relative to a fixed origin.

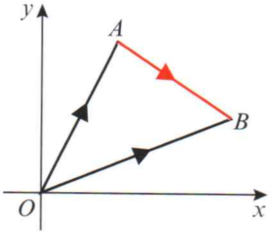
The position vector of a point A is the vector $\overrightarrow{OA}$, where O is the origin.

If $\overrightarrow{OA} = a\mathbf{i} + b\mathbf{j}$ then the position vector of A is $\begin{pmatrix} a \\ b \end{pmatrix}$.



- **In general, a point P with coordinates (p, q) has a position vector $\overrightarrow{OP} = p\mathbf{i} + q\mathbf{j} = \begin{pmatrix} p \\ q \end{pmatrix}$.**

- **$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, where $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors of A and B respectively.**



Link Use the triangle law:
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB}$
So $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

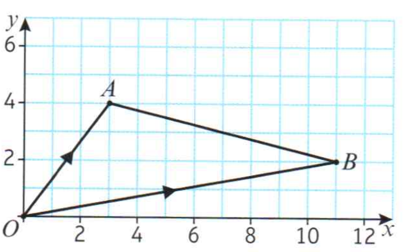
← Section 11.1

Example 14

The points A and B in the diagram have coordinates (3, 4) and (11, 2) respectively.
Find, in terms of **i** and **j**:

a the position vector of A **b** the position vector of B

c the vector $\overrightarrow{AB}$



a $\overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{j}$ — In column vector form this is $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

b $\overrightarrow{OB} = 11\mathbf{i} + 2\mathbf{j}$ — In column vector form this is $\begin{pmatrix} 11 \\ 2 \end{pmatrix}$.

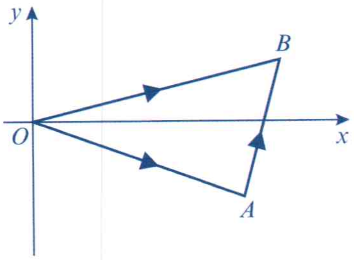
c $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= (11\mathbf{i} + 2\mathbf{j}) - (3\mathbf{i} + 4\mathbf{j}) = 8\mathbf{i} - 2\mathbf{j}$ — In column vector form this is $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$.

Example 15

$\overrightarrow{OA} = 5\mathbf{i} - 2\mathbf{j}$ and $\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j}$. Find:

a the position vector of B

b the exact value of $|\overrightarrow{OB}|$ in simplified surd form.



a $\overrightarrow{OA} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

It is usually quicker to use column vector form for calculations.

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

In **i**, **j** form the answer is 8**i** + 2**j**.

b $|\overrightarrow{OB}| = \sqrt{8^2 + 2^2} = \sqrt{64 + 4} = \sqrt{68} = 2\sqrt{17}$

$\sqrt{68} = \sqrt{4 \times 17} = 2\sqrt{17}$ in simplified surd form.

Exercise 11D

1 The points A, B and C have coordinates (3, –1), (4, 5) and (–2, 6) respectively, and O is the origin.

Find, in terms of **i** and **j**:

a **i** the position vectors of A, B and C **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{AC}$

b Find, in surd form: **i** $|\overrightarrow{OC}|$ **ii** $|\overrightarrow{AB}|$ **iii** $|\overrightarrow{AC}|$

2 $\overrightarrow{OP} = 4\mathbf{i} - 3\mathbf{j}$, $\overrightarrow{OQ} = 3\mathbf{i} + 2\mathbf{j}$

a Find $\overrightarrow{PQ}$

b Find, in surd form: **i** $|\overrightarrow{OP}|$ **ii** $|\overrightarrow{OQ}|$ **iii** $|\overrightarrow{PQ}|$

3 $\overrightarrow{OQ} = 4\mathbf{i} - 3\mathbf{j}$, $\overrightarrow{PQ} = 5\mathbf{i} + 6\mathbf{j}$

a Find $\overrightarrow{OP}$

b Find, in surd form: **i** $|\overrightarrow{OP}|$ **ii** $|\overrightarrow{OQ}|$ **iii** $|\overrightarrow{PQ}|$

(P) **4** $OABCDE$ is a regular hexagon. The points A and B have position vectors **a** and **b** respectively, where O is the origin.

Find, in terms of **a** and **b**, the position vectors of

a C **b** D **c** E.

(P) **5** The position vectors of 3 vertices of a parallelogram are $\begin{pmatrix}4\\2\end{pmatrix}$, $\begin{pmatrix}3\\5\end{pmatrix}$ and $\begin{pmatrix}8\\6\end{pmatrix}$.
Find the possible position vectors of the fourth vertex.

Problem-solving

Use a sketch to check that you have considered all the possible positions for the fourth vertex.

(E) **6** Given that the point A has position vector $4\mathbf{i} - 5\mathbf{j}$ and the point B has position vector $6\mathbf{i} + 3\mathbf{j}$,

a find the vector $\overrightarrow{AB}$. **(2 marks)**

b find $|\overrightarrow{AB}|$ giving your answer as a simplified surd. **(2 marks)**

(E/P) **7** The point A lies on the circle with equation $x^2 + y^2 = 9$. Given that $\overrightarrow{OA} = 2k\mathbf{i} + k\mathbf{j}$, find the exact value of k. **(3 marks)**

Challenge

The point B lies on the line with equation $2y = 12 - 3x$. Given that $|OB| = \sqrt{13}$, find possible expressions for $\overrightarrow{OB}$ in the form $p\mathbf{i} + q\mathbf{j}$.

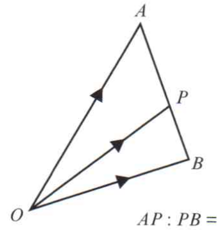
11.5 Solving geometric problems

You need to be able to use vectors to solve geometric problems and to find the position vector of a point that divides a line segment in a given ratio.

- **If the point P divides the line segment AB in the ratio $\lambda : \mu$, then**

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{\lambda}{\lambda + \mu}\overrightarrow{AB}$$

$$= \overrightarrow{OA} + \frac{\lambda}{\lambda + \mu}(\overrightarrow{OB} - \overrightarrow{OA})$$

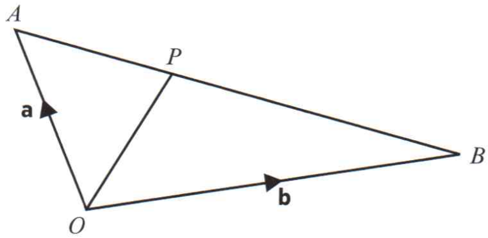


$AP : PB = \lambda : \mu$

Example 16

In the diagram the points A and B have position vectors **a** and **b** respectively (referred to the origin O). The point P divides AB in the ratio $1 : 2$.

Find the position vector of P.



$\overrightarrow{OP} = \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$

There are 3 parts in the ratio in total, so P is $\frac{1}{3}$ of the way along the line segment AB.

$= \overrightarrow{OA} + \frac{1}{3}(\overrightarrow{OB} - \overrightarrow{OA})$

Rewrite $\overrightarrow{AB}$ in terms of the position vectors for A and B.

$= \frac{2}{3}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB}$

$= \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

Give your final answer in terms of **a** and **b**.

You can solve geometric problems by comparing coefficients on both sides of an equation:

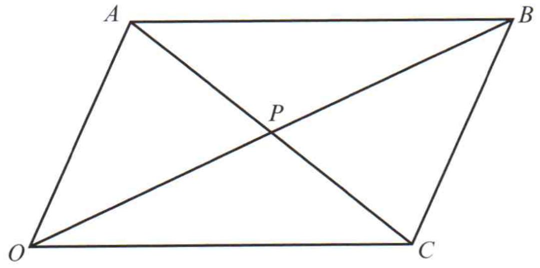
- **If a and b are two non-parallel vectors and $p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b}$ then $p = r$ and $q = s$.**

Example 17

$OABC$ is a parallelogram. P is the point where the diagonals OB and AC intersect.

The vectors **a** and **c** are equal to $\overrightarrow{OA}$ and $\overrightarrow{OC}$ respectively.

Prove that the diagonals bisect each other.



From the diagram,

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \mathbf{c} + \mathbf{a}$

and $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$.

$= -\overrightarrow{OA} + \overrightarrow{OC} = -\mathbf{a} + \mathbf{c}$

Express $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of **a** and **c**.

P lies on OB $\Rightarrow$ $\overrightarrow{OP} = \lambda(\mathbf{c} + \mathbf{a})$

P lies on AC $\Rightarrow$ $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \mathbf{a} + \mu(-\mathbf{a} + \mathbf{c})$

Use the fact that P lies on both diagonals to find two different routes from O to P, giving two different forms of $\overrightarrow{OP}$.

$\Rightarrow$ $\lambda(\mathbf{c} + \mathbf{a}) = \mathbf{a} + \mu(-\mathbf{a} + \mathbf{c})$

The two expressions for $\overrightarrow{OP}$ must be equal.

$\Rightarrow$ $\lambda = 1 - \mu$ and $\lambda = \mu$

Form and solve a pair of simultaneous equations by equating the coefficients of **a** and **c**.

$\Rightarrow$ $\lambda = \mu = \frac{1}{2}$, so P is the midpoint of both diagonals, so the diagonals bisect each other.

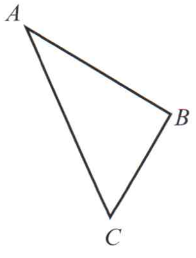
If P is halfway along the line segment then it must be the midpoint.

Online Use technology to show that diagonals of a parallelogram bisect each other.

Example 18

In triangle ABC, $\overrightarrow{AB} = 3\mathbf{i} - 2\mathbf{j}$ and $\overrightarrow{AC} = \mathbf{i} - 5\mathbf{j}$.

Find the exact size of $\angle BAC$ in degrees.



Problem-solving

Work out what information you would need to find the angle. You could:

- find the lengths of all three sides then use the cosine rule
- convert $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to magnitude-direction form

The working here shows the first method.

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ — Use the triangle law to find $\overrightarrow{BC}$.

$|\overrightarrow{AB}| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

$|\overrightarrow{AC}| = \sqrt{1^2 + (-5)^2} = \sqrt{26}$

$|\overrightarrow{BC}| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$

Leave your answers in surd form.

$$\cos \angle BAC = \frac{|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 - |\overrightarrow{BC}|^2}{2 \times |\overrightarrow{AB}| \times |\overrightarrow{AC}|}$$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ ← Section 9.1

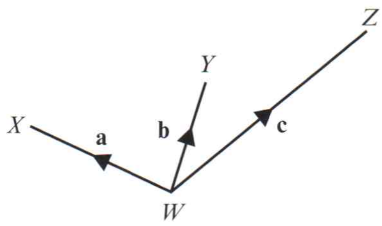
$$= \frac{13 + 26 - 13}{2 \times \sqrt{13} \times \sqrt{26}} = \frac{26}{26\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\angle BAC = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

Online Check your answer by entering the vectors directly into your calculator.

Exercise 11E

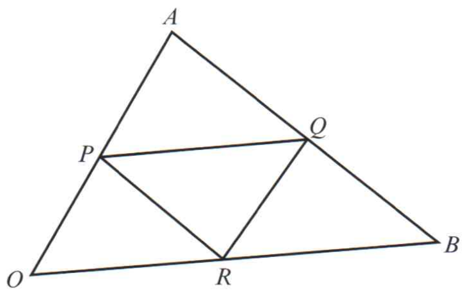
(P) **1** In the diagram, $\overrightarrow{WX} = \mathbf{a}$, $\overrightarrow{WY} = \mathbf{b}$ and $\overrightarrow{WZ} = \mathbf{c}$. It is given that $\overrightarrow{XY} = \overrightarrow{YZ}$.
Prove that $\mathbf{a} + \mathbf{c} = 2\mathbf{b}$.



(P) **2** OAB is a triangle. P, Q and R are the midpoints of OA, AB and OB respectively.
OP and OR are equal to $\mathbf{p}$ and $\mathbf{r}$ respectively.

a Find **i** $\overrightarrow{OB}$ **ii** $\overrightarrow{PQ}$

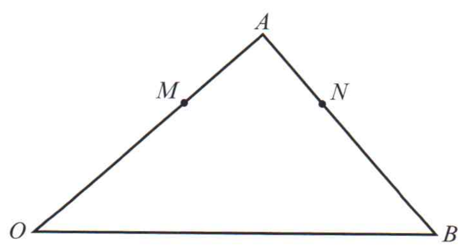
b Hence prove that triangle PAQ is similar to triangle OAB.



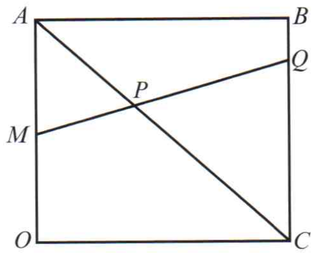
(P) **3** OAB is a triangle. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
The point M divides OA in the ratio $2:1$.
MN is parallel to OB.

a Express the vector $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

b Show that $AN : NB = 1 : 2$



4 $OABC$ is a square. M is the midpoint of OA, and Q divides BC in the ratio 1 : 3.
AC and MQ meet at P.



a If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$, express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.

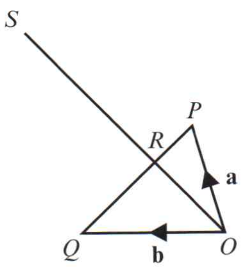
b Show that P divides AC in the ratio 2 : 3.

5 In triangle ABC the position vectors of the vertices A, B and C are $\begin{pmatrix}5\\8\end{pmatrix}$, $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}7\\6\end{pmatrix}$. Find:

a $|\overrightarrow{AB}|$ **b** $|\overrightarrow{AC}|$ **c** $|\overrightarrow{BC}|$

d the size of $\angle BAC$, $\angle ABC$ and $\angle ACB$ to the nearest degree.

6 OPQ is a triangle.
$2\overrightarrow{PR} = \overrightarrow{RQ}$ and $3\overrightarrow{OR} = \overrightarrow{OS}$
$\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OQ} = \mathbf{b}$.



a Show that $\overrightarrow{OS} = 2\mathbf{a} + \mathbf{b}$.

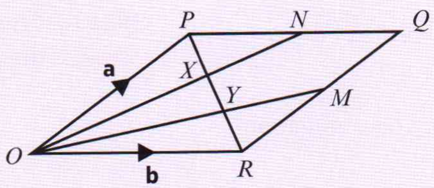
b Point T is added to the diagram such that $\overrightarrow{OT} = -\mathbf{b}$.
Prove that points T, P and S lie on a straight line.

Problem-solving

To show that T, P and S lie on the same straight line you need to show that any **two** of the vectors $\overrightarrow{TP}$, $\overrightarrow{TS}$ or $\overrightarrow{PS}$ are parallel.

Challenge

$OPQR$ is a parallelogram.



N is the midpoint of PQ and M is the midpoint of QR.
$\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OR} = \mathbf{b}$. The lines ON and OM intersect the diagonal PR at points X and Y respectively.

a Explain why $\overrightarrow{PX} = -j\mathbf{a} + j\mathbf{b}$, where j is a constant.

b Show that $\overrightarrow{PX} = (k - 1)\mathbf{a} + \frac{1}{2}k\mathbf{b}$, where k is a constant.

c Explain why the values of j and k must satisfy these simultaneous equations:
$k - 1 = -j$
$\frac{1}{2}k = j$

d Hence find the values of j and k.

e Deduce that the lines ON and OM divide the diagonal PR into 3 equal parts.

11.6 Modelling with vectors

You need to be able to use vectors to solve problems in context.

In mechanics, **vector quantities** have both magnitude and direction. Here are three examples:

- velocity
- displacement
- force

You can also refer to the **magnitude** of these vectors. The magnitude of a vector is a **scalar quantity** – it has size but no direction:

- **speed** is the magnitude of the velocity vector
- **distance** in a straight line between A and B is the magnitude of the displacement vector $\overrightarrow{AB}$

When modelling with vectors in mechanics, it is common to use the unit vector **j** to represent north and the unit vector **i** to represent east.

Example 19

A girl walks 2 km due east from a fixed point O to A, and then 3 km due south from A to B. Find:

a the total distance travelled

b the position vector of B relative to O

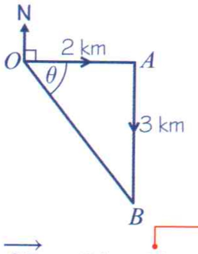
c $|\overrightarrow{OB}|$

d the bearing of B from O.

a The distance the girl has walked is
2 km + 3 km = 5 km

Note that the distance of B from O is not the same as the distance the girl has walked.

b Representing the girl's journey on a diagram:



$\overrightarrow{OB} = (2\mathbf{i} - 3\mathbf{j})$ km

j represents north, so 3 km south is written as $-3\mathbf{j}$ km.

Remember to include the units with your answer.

c $|\overrightarrow{OB}| = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.61$ km (3 s.f.)

$|\overrightarrow{OB}|$ is the length of the line segment OB in the diagram and represents the girl's distance from the starting point.

d $\tan\theta = \frac{3}{2}$

$\theta = 56.3°$

The bearing of B from O is
$56.3° + 90° = 146.3° = 146°$

A three-figure bearing is always measured clockwise from north.

Example 20

In an orienteering exercise, a cadet leaves the starting point O and walks 15 km on a bearing of 120° to reach A, the first checkpoint. From A he walks 9 km on a bearing of 240° to the second checkpoint, at B. From B he returns directly to O.

Find:

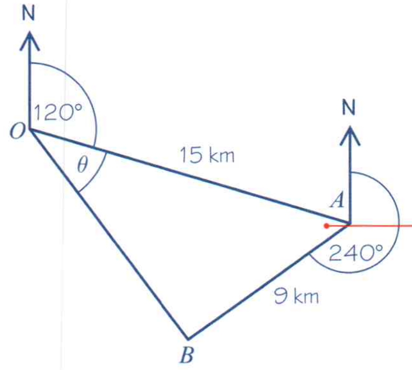
a the position vector of A relative to O

b $|\overrightarrow{OB}|$

c the bearing of B from O

d the position vector of B relative to O.

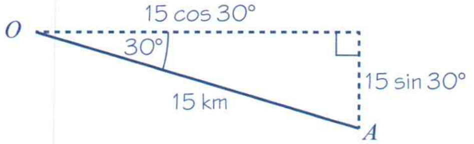
a



Start by drawing a diagram.

$\angle OAB = 360° - (240° + 60°) = 60°$

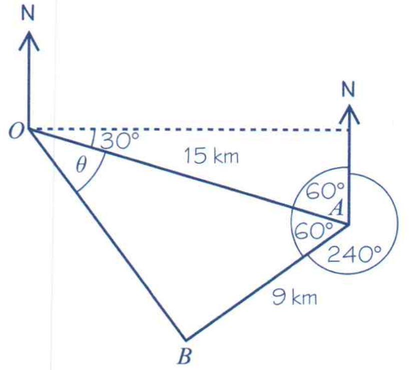
The position vector of A relative to O is $\overrightarrow{OA}$.



Draw a right angled triangle to work out the lengths of the **i** and **j** components for the position vector of A relative to O.

$\overrightarrow{OA} = (15\cos 30°\mathbf{i} - 15\sin 30°\mathbf{j})$ km
$= (13.0\mathbf{i} - 7.5\mathbf{j})$ km

b



$|\overrightarrow{OB}|^2 = 15^2 + 9^2 - 2 \times 15 \times 9 \times \cos 60°$
$= 171$

$|\overrightarrow{OB}| = \sqrt{171} = 13.1$ km (3 s.f.)

$|\overrightarrow{OB}|$ is the length of OB in triangle OAB.
Use the cosine rule in triangle OAB.

c $\frac{\sin\theta}{9} = \frac{\sin 60°}{\sqrt{171}}$

Use the sine rule to work out θ.

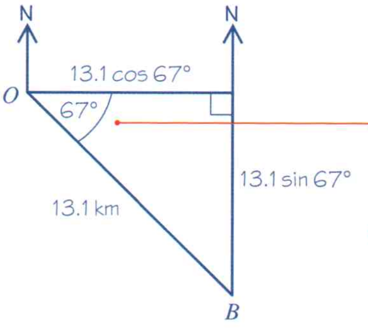
$\sin\theta = \frac{9 \times \sin 60°}{\sqrt{171}} = 0.596...$

$\theta = 36.6...°$

The bearing of B from $O = 120 + 36.6...$

$= 157°$ (3 s.f.)

d



$157° - 90° = 67°$

Draw a right angled triangle to work out the lengths of the **i** and **j** components for the position vector of B relative to O.

$\overrightarrow{OB} = (5.1\mathbf{i} - 12.1\mathbf{j})$ km

Exercise 11F

1 Find the speed of a particle moving with these velocities:

a $(3\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$ **b** $(24\mathbf{i} - 7\mathbf{j})\,\text{km h}^{-1}$

c $(5\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$ **d** $(-7\mathbf{i} + 4\mathbf{j})\,\text{cm s}^{-1}$

Hint Speed is the magnitude of the velocity vector.

2 Find the distance moved by a particle which travels for:

a 5 hours at velocity $(8\mathbf{i} + 6\mathbf{j})\,\text{km h}^{-1}$

b 10 seconds at velocity $(5\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$

c 45 minutes at velocity $(6\mathbf{i} + 2\mathbf{j})\,\text{km h}^{-1}$

d 2 minutes at velocity $(-4\mathbf{i} - 7\mathbf{j})\,\text{cm s}^{-1}$.

Hint Find the speed in each case then use:
Distance travelled = speed × time

3 Find the speed and the distance travelled by a particle moving in a straight line with:

a velocity $(-3\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$ for 15 seconds **b** velocity $(2\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$ for 3 seconds

c velocity $(5\mathbf{i} - 2\mathbf{j})\,\text{km h}^{-1}$ for 3 hours **d** velocity $(12\mathbf{i} - 5\mathbf{j})\,\text{km h}^{-1}$ for 30 minutes.

4 A particle P is accelerating at a constant rate.
When $t = 0$, P has velocity $\mathbf{u} = (2\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$
and at time $t = 5$ s, P has velocity $\mathbf{v} = (16\mathbf{i} - 5\mathbf{j})\,\text{m s}^{-1}$.

The acceleration vector of the particle is given by the formula: $\mathbf{a} = \frac{\mathbf{v} - \mathbf{u}}{t}$

Find the acceleration of P in terms of **i** and **j**.

Hint The units of acceleration will be m/s² or m s⁻².

5 A particle P of mass $m = 0.3$ kg moves under the action of a single constant force $\mathbf{F}$ newtons. The acceleration of P is $\mathbf{a} = (5\mathbf{i} + 7\mathbf{j})\ \text{m s}^{-2}$.

a Find the angle between the acceleration and $\mathbf{i}$. **(2 marks)**

Force, mass and acceleration are related by the formula $\mathbf{F} = m\mathbf{a}$.

b Find the magnitude of $\mathbf{F}$. **(3 marks)**

6 Two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, are given by the vectors $\mathbf{F}_1 = (3\mathbf{i} - 4\mathbf{j})$ N and $\mathbf{F}_2 = (p\mathbf{i} + q\mathbf{j})$ N. The resultant force, $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$ acts in a direction which is parallel to the vector $(2\mathbf{i} - \mathbf{j})$.

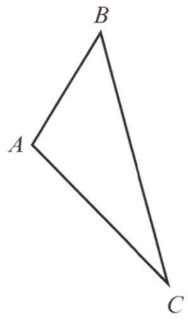
a Find the angle between $\mathbf{R}$ and the vector $\mathbf{i}$. **(2 marks)**

b Show that $p + 2q = 5$. **(3 marks)**

c Given that $p = 1$, find the magnitude of $\mathbf{R}$. **(3 marks)**

7 The diagram shows a sketch of a field in the shape of a triangle ABC.

Given $\overrightarrow{AB} = 30\mathbf{i} + 40\mathbf{j}$ metres and $\overrightarrow{AC} = 40\mathbf{i} - 60\mathbf{j}$ metres,



a find $\overrightarrow{BC}$ **(2 marks)**

b find the size of $\angle BAC$, in degrees, to one decimal place **(4 marks)**

c find the area of the field in square metres. **(3 marks)**

8 A boat A has a position vector of $(2\mathbf{i} + \mathbf{j})$ km and a buoy B has a position vector of $(6\mathbf{i} - 4\mathbf{j})$ km, relative to a fixed origin O.

a Find the distance of the boat from the buoy.

b Find the bearing of the boat from the buoy.

The boat travels with constant velocity $(8\mathbf{i} - 10\mathbf{j})$ km/h.

c Verify that the boat is travelling directly towards the buoy

d Find the speed of the boat.

e Work out how long it will take the boat to reach the buoy.

Problem-solving

Draw a sketch showing the initial positions of the boat, the buoy and the origin.

Mixed exercise 11

1 Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a particle.

$\mathbf{F}_1 = -3\mathbf{i} + 7\mathbf{j}$ newtons

$\mathbf{F}_2 = \mathbf{i} - \mathbf{j}$ newtons

The resultant force $\mathbf{R}$ acting on the particle is given by $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$.

a Calculate the magnitude of $\mathbf{R}$ in newtons. **(3 marks)**

b Calculate, to the nearest degree, the angle between the line of action of $\mathbf{R}$ and the vector $\mathbf{j}$. **(2 marks)**

(P) **2** A small boat S, drifting in the sea, is modelled as a particle moving in a straight line at constant speed. When first sighted at 09:00, S is at a point with position vector $(-2\mathbf{i} - 4\mathbf{j})$ km relative to a fixed origin O, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north respectively. At 09:40, S is at the point with position vector $(4\mathbf{i} - 6\mathbf{j})$ km.

a Calculate the bearing on which S is drifting.

b Find the speed of S.

(P) **3** A football player kicks a ball from point A on a flat football field. The motion of the ball is modelled as that of a particle travelling with constant velocity $(4\mathbf{i} + 9\mathbf{j})\,\text{m s}^{-1}$.

a Find the speed of the ball.

b Find the distance of the ball from A after 6 seconds.

c Comment on the validity of this model for large values of t.

(P) **4** $ABCD$ is a trapezium with AB parallel to DC and $DC = 4AB$.
M divides DC such that $DM : MC = 3 : 2$, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$.
Find, in terms of $\mathbf{a}$ and $\mathbf{b}$:

a $\overrightarrow{AM}$ **b** $\overrightarrow{BD}$ **c** $\overrightarrow{MB}$ **d** $\overrightarrow{DA}$

(E/P) **5** The vectors $5\mathbf{a} + k\mathbf{b}$ and $8\mathbf{a} + 2\mathbf{b}$ are parallel. Find the value of k. **(3 marks)**

6 Given that $\mathbf{a} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 10 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$ find:

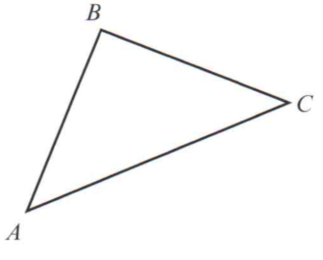
a $\mathbf{a} + \mathbf{b} + \mathbf{c}$ **b** $\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ **c** $2\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}$

(E) **7** In triangle ABC, $\overrightarrow{AB} = 3\mathbf{i} + 5\mathbf{j}$ and $\overrightarrow{AC} = 6\mathbf{i} + 3\mathbf{j}$, find:

a $\overrightarrow{BC}$ **(2 marks)**

b $\angle BAC$ **(4 marks)**

c the area of the triangle. **(2 marks)**



(E/P) **8** The resultant of the vectors $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = 2p\mathbf{i} - p\mathbf{j}$ is parallel to the vector $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$. Find:

a the value of p **(3 marks)**

b the resultant of vectors $\mathbf{a}$ and $\mathbf{b}$. **(1 mark)**

9 For each of the following vectors, find
i a unit vector in the same direction **ii** the angle the vector makes with $\mathbf{i}$

a $\mathbf{a} = 8\mathbf{i} + 15\mathbf{j}$ **b** $\mathbf{b} = 24\mathbf{i} - 7\mathbf{j}$ **c** $\mathbf{c} = -9\mathbf{i} + 40\mathbf{j}$ **d** $\mathbf{d} = 3\mathbf{i} - 2\mathbf{j}$

P **10** The vector $\mathbf{a} = p\mathbf{i} + q\mathbf{j}$, where p and q are positive constants, is such that $|\mathbf{a}| = 15$. Given that $\mathbf{a}$ makes an angle of 55° with $\mathbf{i}$, find the values of p and q.

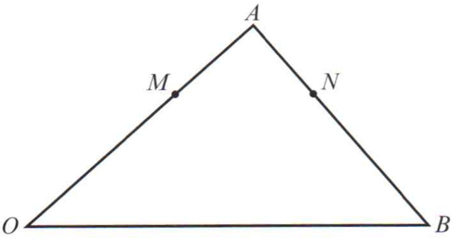
P **11** Given that $|3\mathbf{i} - k\mathbf{j}| = 3\sqrt{5}$, find the value of k. **(3 marks)**

P **12** OAB is a triangle. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point M divides OA in the ratio $3:2$. MN is parallel to OB.

a Express the vector $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. **(4 marks)**

b Find vector $\overrightarrow{MN}$. **(2 marks)**

c Show that $AN : NB = 2:3$. **(2 marks)**



P **13** Two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, are given by the vectors $\mathbf{F}_1 = (4\mathbf{i} - 5\mathbf{j})$ N and $\mathbf{F}_2 = (p\mathbf{i} + q\mathbf{j})$ N. The resultant force, $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$ acts in a direction which is parallel to the vector $(3\mathbf{i} - \mathbf{j})$

a Find the angle between $\mathbf{R}$ and the vector $\mathbf{i}$. **(3 marks)**

b Show that $p + 3q = 11$. **(4 marks)**

c Given that $p = 2$, find the magnitude of $\mathbf{R}$. **(2 marks)**

E **14** A particle P is accelerating at a constant rate. When $t = 0$, P has velocity $\mathbf{u} = (3\mathbf{i} + 4\mathbf{j})\text{ m s}^{-1}$ and at time $t = 2$ s, P has velocity $\mathbf{v} = (15\mathbf{i} - 3\mathbf{j})\text{ m s}^{-1}$.

The acceleration vector of the particle is given by the formula: $\mathbf{a} = \dfrac{\mathbf{v} - \mathbf{u}}{t}$

Find the magnitude of the acceleration of P. **(3 marks)**

Challenge

The point B lies on the line with equation $3y = 15 - 5x$.

Given that $|\overrightarrow{OB}| = \dfrac{\sqrt{34}}{2}$, find two possible expressions for $\overrightarrow{OB}$ In the form $p\mathbf{i} + q\mathbf{j}$.

Summary of key points

1 If $\overrightarrow{PQ} = \overrightarrow{RS}$ then the line segments PQ and RS are equal in length and are parallel.

2 $\overrightarrow{AB} = -\overrightarrow{BA}$ as the line segment AB is equal in length, parallel and in the opposite direction to BA.

3 **Triangle law for vector addition:** $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

If $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$, then $\mathbf{a} + \mathbf{b} = \mathbf{c}$

4 Subtracting a vector is equivalent to 'adding a negative vector': $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

5 Adding the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ gives the zero vector $\mathbf{0}$: $\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$.

6 Any vector parallel to the vector $\mathbf{a}$ may be written as $\lambda\mathbf{a}$, where λ is a non-zero scalar.

7 To multiply a column vector by a scalar, multiply each component by the scalar: $\lambda\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda p \\ \lambda q \end{pmatrix}$

8 To add two column vectors, add the x-components and the y-components $\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix}$

9 A unit vector is a vector of length 1. The unit vectors along the x- and y-axes are usually denoted by $\mathbf{i}$ and $\mathbf{j}$ respectively. $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

10 For any two-dimensional vector: $\begin{pmatrix} p \\ q \end{pmatrix} = p\mathbf{i} + q\mathbf{j}$

11 For the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} = \begin{pmatrix} x \\ y \end{pmatrix}$, the magnitude of the vector is given by: $|\mathbf{a}| = \sqrt{x^2 + y^2}$

12 A unit vector in the direction of $\mathbf{a}$ is $\dfrac{\mathbf{a}}{|\mathbf{a}|}$

13 In general, a point P with coordinates (p, q) has position vector:

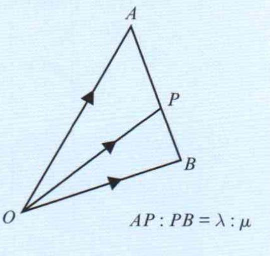
$$\overrightarrow{OP} = p\mathbf{i} + q\mathbf{j} = \begin{pmatrix} p \\ q \end{pmatrix}$$

14 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, where $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors of A and B respectively.

15 If the point P divides the line segment AB in the ratio $\lambda : \mu$, then

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{\lambda}{\lambda + \mu}\overrightarrow{AB}$$

$$= \overrightarrow{OA} + \frac{\lambda}{\lambda + \mu}(\overrightarrow{OB} - \overrightarrow{OA})$$



16 If $\mathbf{a}$ and $\mathbf{b}$ are two non-parallel vectors and $p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b}$ then $p = r$ and $q = s$