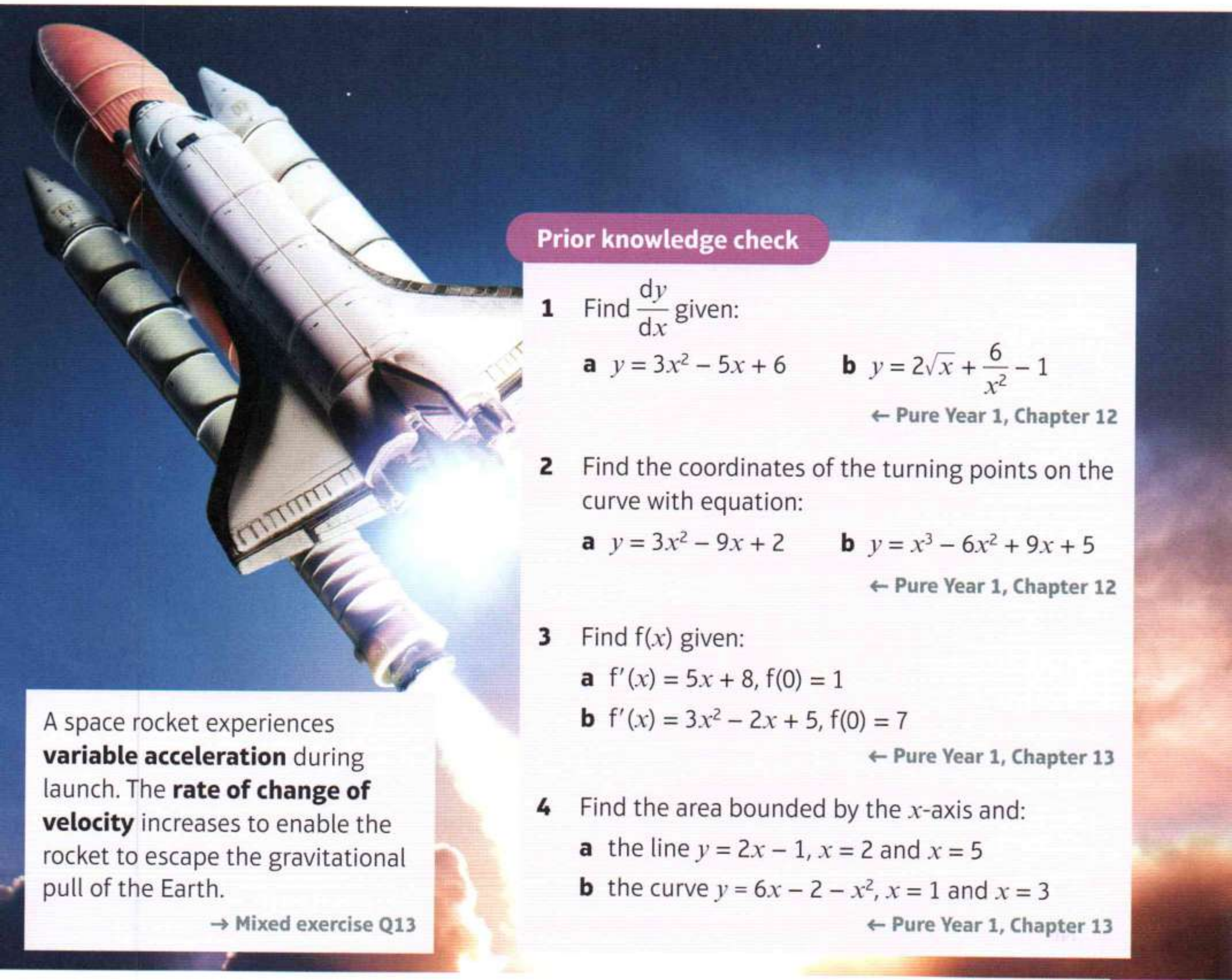
Variable acceleration

11

Objectives

After completing this chapter you should be able to:

- Understand that displacement, velocity and acceleration may be given as functions of time → pages 182–184
- Use differentiation to solve kinematics problems → pages 185–186
- Use calculus to solve problems involving maxima and minima → pages 186–188
- Use integration to solve kinematics problems → pages 188–191
- Use calculus to derive constant acceleration formulae → pages 191–193



A space rocket experiences **variable acceleration** during launch. The **rate of change of velocity** increases to enable the rocket to escape the gravitational pull of the Earth.

→ Mixed exercise Q13

Prior knowledge check

1 Find $\frac{dy}{dx}$ given:

a $y = 3x^2 - 5x + 6$ **b** $y = 2\sqrt{x} + \frac{6}{x^2} - 1$

← Pure Year 1, Chapter 12

2 Find the coordinates of the turning points on the curve with equation:

a $y = 3x^2 - 9x + 2$ **b** $y = x^3 - 6x^2 + 9x + 5$

← Pure Year 1, Chapter 12

3 Find f(x) given:

a $f'(x) = 5x + 8$, $f(0) = 1$

b $f'(x) = 3x^2 - 2x + 5$, $f(0) = 7$

← Pure Year 1, Chapter 13

4 Find the area bounded by the x-axis and:

a the line $y = 2x - 1$, $x = 2$ and $x = 5$

b the curve $y = 6x - 2 - x^2$, $x = 1$ and $x = 3$

← Pure Year 1, Chapter 13

11.1 Functions of time

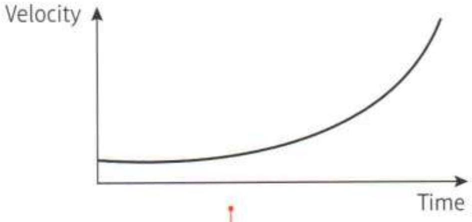
If acceleration of a moving particle is variable, it changes with time and can be expressed as a function of time.

In the same way, velocity and displacement can also be expressed as functions of time.

Links Acceleration is the gradient of a velocity–time graph. ← **Section 9.2**

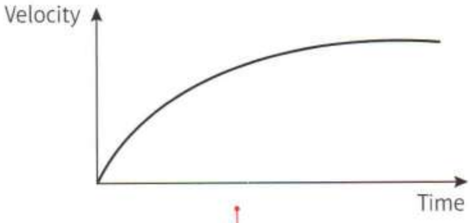
These velocity–time graphs represent the motion of a particle travelling in a straight line. They show examples of increasing and decreasing acceleration.

Increasing acceleration



The rate of increase of velocity is increasing with time and the gradient of the curve is increasing.

Decreasing acceleration



The rate of increase of velocity is decreasing with time and the gradient of the curve is decreasing.

Example 1

A body moves in a straight line, such that its displacement, s metres, from a point O at time t seconds is given by $s = 2t^3 - 3t$ for $t > 0$. Find:

a s when $t = 2$ **b** the time taken for the particle to return to O.

a $s = 2 \times 2^3 - 3 \times 2$

Substitute $t = 2$ into the equation for s.

$= 16 - 6 = 10$ metres

b $2t^3 - 3t = 0$

When the particle returns to the starting point, the displacement is equal to zero.

$t(2t^2 - 3) = 0$ either $t = 0$ or $2t^2 = 3$

$\Rightarrow t^2 = \frac{3}{2}$ so $t = \pm\sqrt{\frac{3}{2}}$ seconds

Time taken to return to $O = \sqrt{\frac{3}{2}}$ seconds

Answer is $+\sqrt{\frac{3}{2}}$ as equation is only valid for $t > 0$.

Example 2

A toy train travels along a straight track, leaving the start of the track at time $t = 0$. It then returns to the start of the track. The distance, s metres, from the start of the track at time t seconds is modelled by:

$s = 4t^2 - t^3$, where $0 \leqslant t \leqslant 4$

Explain the restriction $0 \leqslant t \leqslant 4$.

s is distance from start of track so $s \geqslant 0$.

So $4t^2 - t^3 \geqslant 0$

$t^2(4 - t) \geqslant 0$

$t^2 \geqslant 0$ for all t, and $(4 - t) < 0$ for all $t > 4$, so $t^2(4 - t)$ is only non-negative for $t \leqslant 4$

Motion begins at $t = 0$ hence $t \geqslant 0$

Hence $0 \leqslant t \leqslant 4$

Use the initial conditions given.

Distance is a scalar quantity and must be $\geqslant 0$.

The restriction $t \geqslant 0$ is due to the motion beginning at $t = 0$, not due to the function.

Problem-solving

You could also sketch the graph of $s = 4t^2 - t^3$ to show the values of t for which the model is valid.

Example 3

A body moves in a straight line such that its velocity, $v\,\text{m s}^{-1}$, at time t seconds is given by:

$$v = 2t^2 - 16t + 24, \text{ for } t \geqslant 0$$

Find:

a the initial velocity

b the values of t when the body is instantaneously at rest

c the value of t when the velocity is $64\,\text{m s}^{-1}$

d the greatest speed of the body in the interval $0 \leqslant t \leqslant 5$.

a $v = 0 - 0 + 24$

$v = 24\,\text{m s}^{-1}$

The initial velocity means the velocity at $t = 0$.

b $2t^2 - 16t + 24 = 0$

$t^2 - 8t + 12 = 0$

$(t - 6)(t - 2) = 0$

Body at rest when $t = 2$ seconds and $t = 6$ seconds

The body is at rest when $v = 0$, so solve the quadratic equation when $v = 0$.

c $2t^2 - 16t + 24 = 64$

$2t^2 - 16t - 40 = 0$

$t^2 - 8t - 20 = 0$

$(t - 10)(t + 2) = 0$

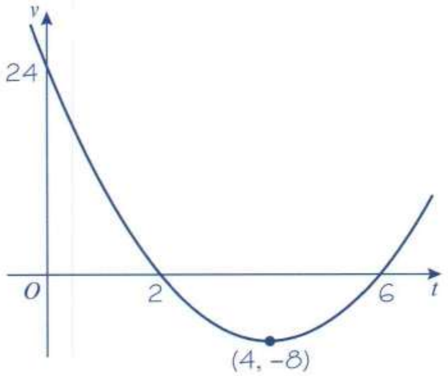
Either $t = 10$ or $t = -2$

Velocity $= 64\,\text{m s}^{-1}$ when $t = 10$ seconds

Rearrange the quadratic equation to make it equal to zero and factorise.

The equation for velocity is valid for $t \geqslant 0$, so $t = -2$ is not a valid solution.

d



When $t = 4$, $v = 2(4)^2 - 16(4) + 24 = -8$

So in $0 \leqslant t \leqslant 5$ range of v is $-8 \leqslant v \leqslant 24$

Greatest speed is $24\,\text{m s}^{-1}$

Sketch a velocity–time graph for the motion of the body. You can use the symmetry of the quadratic curve to determine the position of the turning point. ← **Pure Year 1, Chapter 2**

Watch out You need to find the greatest **speed**. This could occur when the velocity is positive or negative, so find the range of values taken by v in the interval $0 \leqslant t \leqslant 5$.

Online Explore the solution using technology.

Exercise 11A

1 A body moves in a straight line such that its displacement, s metres, at time t seconds is given by $s = 9t - t^3$. Find:

a s when $t = 1$ **b** the values of t when $s = 0$.

2 A particle P moves on the x-axis. At time t seconds the displacement s metres is given by $s = 5t^2 - t^3$. Find:

a the change in displacement between $t = 2$ and $t = 4$

b the change in displacement in the third second.

Hint The third second is the time between $t = 2$ and $t = 3$.

3 A particle moves in a straight line such that its velocity, $v\,\text{m s}^{-1}$, at time t seconds is given by $v = 3 + 5t - t^2$ for $t \geqslant 0$. Find:

a the velocity of the particle when $t = 1$

b the greatest speed of the particle in the interval $0 \leqslant t \leqslant 4$

c the velocity of the particle when $t = 7$ and describe the direction of motion of the particle at this time.

4 At time $t = 0$, a toy car is at point P. It moves in a straight line from point P and then returns to P. Its distance from P, s m, at time t seconds can be modelled by $s = \frac{1}{5}(4t - t^2)$. Find:

a the maximum displacement **b** the time taken for the toy car to return to P

c the total distance travelled **d** the values of t for which the model is valid.

5 A body moves in a straight line such that its velocity, $v\,\text{m s}^{-1}$, at time t seconds is given by $v = 3t^2 - 10t + 8$, for $t \geqslant 0$. Find:

a the initial velocity

b the values of t when the body is instantaneously at rest

c the values of t when the velocity is $5\,\text{m s}^{-1}$

d the greatest speed of the body in the interval $0 \leqslant t \leqslant 2$.

Ⓔ 6 A particle P moves on the x-axis. At time t seconds the velocity of P is $v\,\text{m s}^{-1}$ in the direction of x increasing, where $v = 8t - 2t^2$. When $t = 0$, P is at the origin O. Find:

a the time taken for the particle to come to instantaneous rest **(2 marks)**

b the greatest speed of the particle in the interval $0 \leqslant t \leqslant 4$. **(3 marks)**

Ⓔ 7 At time $t = 0$, a particle moves in a straight horizontal line from a point O, then returns to the starting point. The distance, s metres, from the point O at time t seconds is given by:

$$s = 3t^2 - t^3, \quad 0 \leqslant t \leqslant T$$

Given that the model is valid when $s \geqslant 0$, find the value of T. Explain your answer. **(3 marks)**

Ⓔ 8 A particle P moves on the x-axis. At time t seconds the velocity of P is $v\,\text{m s}^{-1}$ in the direction of x increasing, where:

$$v = \frac{1}{5}(3t^2 - 10t + 3), \qquad t \geqslant 0$$

a Find the values of t when P is instantaneously at rest. **(3 marks)**

b Determine the greatest speed of P in the interval $0 \leqslant t \leqslant 3$. **(4 marks)**

11.2 Using differentiation

Velocity is the rate of change of displacement.

- **If the displacement, s, is expressed as a function of t, then the velocity, v, can be expressed as $v = \frac{ds}{dt}$**

Links The gradient of a displacement–time graph represents the velocity. ← Section 9.1

In the same way, acceleration is the rate of change of velocity.

- **If the velocity, v, is expressed as a function of t, then the acceleration, a, can be expressed as $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$**

Links The gradient of a velocity–time graph represents the acceleration. ← Section 9.2
$\frac{d^2s}{dt^2}$ is the second derivative (or second-order derivative) of s with respect to t.
← Pure Year 1, Chapter 12

Example 4

A particle P is moving on the x-axis. At time t seconds, the displacement x metres from O is given by $x = t^4 - 32t + 12$. Find:

a the velocity of P when $t = 3$

b the value of t for which P is instantaneously at rest

c the acceleration of P when $t = 1.5$.

a $x = t^4 - 32t + 12$

$v = \frac{dx}{dt} = 4t^3 - 32$

You find the velocity by differentiating the displacement.

When $t = 3$,

$v = 4 \times 3^3 - 32 = 76$

To find the velocity when $t = 3$, you substitute $t = 3$ into the expression.

The velocity of P when $t = 3$ is $76\,\text{m s}^{-1}$ in the direction of x increasing.

b $v = 4t^3 - 32 = 0$

The particle is at rest when $v = 0$. You substitute $v = 0$ into your expression for v and solve the resulting equation to find t.

$t^3 = \frac{32}{4} = 8$

$t = 2$

c $v = 4t^3 - 32$

$a = \frac{dv}{dt} = 12t^2$

You find the acceleration by differentiating the velocity.

When $t = 1.5$,

$a = 12 \times 1.5^2 = 27$

The acceleration of P when $t = 1.5$ is $27\,\text{m s}^{-2}$.

Exercise 11B

1 Find an expression for **i** the velocity and **ii** the acceleration of a particle given that the displacement is given by:

a $s = 4t^4 - \frac{1}{t}$ **b** $x = \frac{2}{3}t^3 + \frac{1}{t^2}$ **c** $s = (3t^2 - 1)(2t + 5)$ **d** $x = \frac{3t^4 - 2t^3 + 5}{2t}$

2 A particle is moving in a straight line. At time t seconds, its displacement, x m, from a fixed point O on the line is given by $x = 2t^3 - 8t$. Find:

a the velocity of the particle when $t = 3$ **b** the acceleration of the particle when $t = 2$.

(P) 3 A particle P is moving on the x-axis. At time t seconds (where $t \geqslant 0$) , the velocity of P is $v\,\text{m s}^{-1}$ in the direction of x increasing, where $v = 12 - t - t^2$.

Find the acceleration of P when P is instantaneously at rest.

(P) 4 A particle is moving in a straight line. At time t seconds, its displacement, x m, from a fixed point O on the line is given by $x = 4t^3 - 39t^2 + 120t$.

Find the distance between the two points where P is instantaneously at rest.

E/P 5 A particle P moves in a straight line. At time t seconds the acceleration of P is $a\,\text{m s}^{-2}$ and the velocity $v\,\text{m s}^{-1}$ is given by $v = kt - 3t^2$, where k is a constant.

The initial acceleration of P is $4\,\text{m s}^{-2}$.

a Find the value of k. **(3 marks)**

b Using the value of k found in part **a**, find the acceleration when P is instantaneously at rest. **(3 marks)**

E/P 6 The print head on a printer moves such that its displacement s cm from the side of the printer at time t seconds is given by:

$$\tfrac{1}{4}(4t^3 - 15t^2 + 12t + 30),\ 0 \leqslant t \leqslant 3$$

Find the distance between the points when the print head is instantaneously at rest, in cm to 1 decimal place. **(6 marks)**

11.3 Maxima and minima problems

You can use calculus to determine maximum and minimum values of displacement, velocity and acceleration.

Example 5

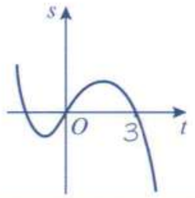
A child is playing with a yo-yo. The yo-yo leaves the child's hand at time $t = 0$ and travels vertically in a straight line before returning to the child's hand. The distance, s m, of the yo-yo from the child's hand after time t seconds is given by:

$$s = 0.6t + 0.4t^2 - 0.2t^3,\ 0 \leqslant t \leqslant 3$$

a Justify the restriction $0 \leqslant t \leqslant 3$.

b Find the maximum distance of the yo-yo from the child's hand, correct to 3 s.f.

a $s = \frac{1}{5}t(3 + 2t - t^2) = \frac{1}{5}t(3 - t)(1 + t)$



s is a cubic function with a negative coefficient of t^3, and roots at $t = -1$, $t = 0$ and $t = 3$.

← Pure Year 1, Section 4.1

$s = 0$ at $t = 0$ and $t = 3$

s is positive for all values of t between 0 and 3.

$s < 0$ for $t > 3$. Since s is a distance the model is not valid for $t > 3$.

Comment on the value of s at the limits of the range **and** the behaviour of s within the range.

b $\frac{ds}{dt} = 0.6 + 0.8t - 0.6t^2$

You know from your answer to part **a** that the maximum value of s must occur at the turning point. To find the turning point differentiate and set $\frac{ds}{dt} = 0$. ← **Pure Year 1, Section 12.9**

$$\frac{ds}{dt} = 0$$

$$0.6 + 0.8t - 0.6t^2 = 0$$

$$3t^2 - 4t - 3 = 0$$

Multiply each term by −5 to obtain an equation with a positive t^2 term and integer coefficients. This makes your working easier.

$$t = \frac{4 \pm \sqrt{52}}{6} = 1.8685\ldots \text{ or } -0.5351\ldots$$

Use the quadratic formula to solve the equation and take the positive value of t.

$$s = 0.6(1.8685\ldots) + 0.4(1.8685\ldots)^2 - 0.2(1.8685\ldots)^3$$

$$= 1.2129\ldots = 1.21\text{ m (3 s.f.)}$$

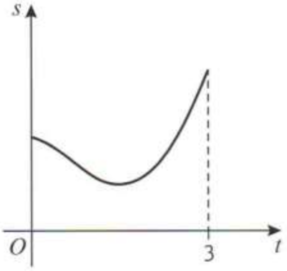
Substitute this value of t back into the original equation to find the corresponding value of s. Remember to use unrounded values in your calculation, and check that your answer makes sense in the context of the question.

Exercise 11C

1 A particle P moves in a straight line such that its distance, s m, from a fixed point O at time t is given by:

$$s = 0.4t^3 - 0.3t^2 - 1.8t + 5,\ 0 \leqslant t \leqslant 3$$

The diagram shows the displacement–time graph of the motion of P.



a Determine the time at which P is moving with minimum velocity.

b Find the displacement of P from O at this time.

c Find the velocity of P at this time.

Ⓟ **2** A body starts at rest and moves in a straight line. At time t seconds the displacement of the body from its starting point, s m, is given by:

$$s = 4t^3 - t^4,\ 0 \leqslant t \leqslant 4.$$

a Show that the body returns to its starting position at $t = 4$.

b Explain why s is always non-negative.

c Find the maximum displacement of the body from its starting point.

Hint Write $s = t^3(4 - t)$ and consider the sign of each factor in the range $0 \leqslant t \leqslant 4$.

3 At time $t = 0$ a particle P leaves the origin O and moves along the x-axis. At time t seconds the velocity of P is $v\text{ m s}^{-1}$, where:

$$v = t^2(6 - t)^2,\ 0 \leq t \leq 6$$

a Sketch a velocity–time graph for the motion of P.

b Find the maximum value of v and the time at which it occurs.

(P) **4** A particle P moves along the x-axis. Its velocity, $v\,\text{m s}^{-1}$ in the positive x-direction, at time t seconds is given by:

$$v = 2t^2 - 3t + 5,\ t \geqslant 0$$

a Show that P never comes to rest.

b Find the minimum velocity of P.

(E/P) **5** A particle P starts at the origin O at time $t = 0$ and moves along the x-axis. At time t seconds the distance of the particle, s m, from the origin is given by:

$$s = \frac{9t^2}{2} - t^3,\ 0 \leqslant t \leqslant 4.5$$

a Sketch a displacement–time graph for the motion of P. **(2 marks)**

b Hence justify the restriction $0 \leqslant t \leqslant 4.5$. **(2 marks)**

c Find the maximum distance of the particle from O. **(5 marks)**

d Find the magnitude of the acceleration of the particle at this point. **(3 marks)**

(E/P) **6** A train moves in a straight line along a 4 km test track. The motion of the train is modelled as a particle travelling in a straight line, and the distance, s m, of the train from the start of the track after time t seconds is given by $s = 3.6t + 1.76t^2 - 0.02t^3$, $0 \leqslant t \leqslant 90$. Show that the train never reaches the end of the track. **(7 marks)**

11.4 Using integration

Integration is the reverse process to differentiation. You can integrate acceleration with respect to time to find velocity, and you can integrate velocity with respect to time to find displacement.

Links The area under a velocity–time graph represents the displacement. ← **Section 9.2**

- **Differentiate** ↓ | **Integrate** ↑

  displacement $= s = \int v\,\mathrm{d}t$

  $\frac{\mathrm{d}s}{\mathrm{d}t} =$ **velocity** $= v = \int a\,\mathrm{d}t$

  $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2} =$ **acceleration** $= a$

Example 6

A particle is moving on the x-axis. At time $t = 0$, the particle is at the point where $x = 5$. The velocity of the particle at time t seconds (where $t \geqslant 0$) is $(6t - t^2)\,\text{m s}^{-1}$. Find:

a an expression for the displacement of the particle from O at time t seconds

b the distance of the particle from its starting point when $t = 6$.

a $x = \int v \, dt$

$= 3t^2 - \dfrac{t^3}{3} + c$, where c is a constant of integration.

When $t = 0$, $x = 5$

$5 = 3 \times 0^2 - \dfrac{0^3}{3} + c = c \Rightarrow c = 5$

The displacement of the particle from O after t seconds is $\left(3t^2 - \dfrac{t^3}{3} + 5\right)$ m.

b Using the result in **a**, when $t = 6$

$x = 3 \times 6^2 - \dfrac{6^3}{3} + 5 = 41$

The distance from the starting point is $(41 - 5)$ m $= 36$ m.

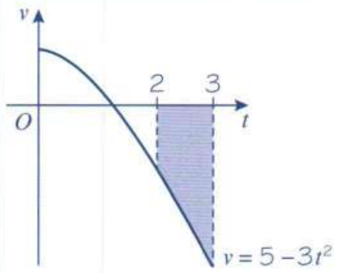
You integrate the velocity to find the displacement. You must remember to add the constant of integration. **← Pure Year 1, Section 13.1**

This information enables you to find the value of the constant of integration. **← Pure Year 1, Section 13.3**

This calculation shows you that, when $t = 6$, the particle is 41 m from O. When the particle started, it was 5 m from O. So the distance from its starting point is $(41 - 5)$ m.

Example 7

A particle travels in a straight line. After t seconds its velocity, v m s^{-1}, is given by $v = 5 - 3t^2$, $t \geqslant 0$. Find the distance travelled by the particle in the third second of its motion.



$s = \int_2^3 (5 - 3t^2) dt$

$s = [5t - t^3]_2^3 = (15 - 27) - (10 - 8)$

$= -12 - 2 = -14$

Distance travelled = 14 metres

Watch out Before using definite integration to find the distance travelled, check that v doesn't **change sign** in the interval you are considering. A sketch of the velocity–time graph can help.

The distance travelled is the area under the velocity–time graph. Use definite integration to find it. **← Pure Year 1, Section 13.4**

The velocity is negative between $t = 2$ and $t = 3$ so the **displacement** will be negative. You are asked to find the distance travelled so give the positive numerical value of the displacement.

Exercise 11D

1 A particle is moving in a straight line. Given that $s = 0$ when $t = 0$, find an expression for the displacement of the particle if the velocity is given by:

a $v = 3t^2 - 1$ **b** $v = 2t^3 - \dfrac{3t^2}{2}$ **c** $v = 2\sqrt{t} + 4t^2$

2 A particle is moving in a straight line. Given that $v = 0$ when $t = 0$, find an expression for the velocity of the particle if the acceleration is given by:

a $a = 8t - 2t^2$ **b** $a = 6 + \frac{t^2}{3}$

3 A particle P is moving on the x-axis. At time t seconds, the velocity of P is $(8 + 2t - 3t^2)\,\text{m s}^{-1}$ in the direction of x increasing. At time $t = 0$, P is at the point where $x = 4$. Find the distance of P from O when $t = 1$.

4 A particle P is moving on the x-axis. At time t seconds, the acceleration of P is $(16 - 2t)\,\text{m s}^{-2}$ in the direction of x increasing. The velocity of P at time t seconds is $v\,\text{m s}^{-1}$. When $t = 0$, $v = 6$ and when $t = 3$, $x = 75$. Find:

a v in terms of t **b** the value of x when $t = 0$.

Ⓟ **5** A particle is moving in a straight line. At time t seconds, its velocity, $v\,\text{m s}^{-1}$, is given by $v = 6t^2 - 51t + 90$. When $t = 0$ the displacement is 0. Find the distance between the two points where P is instantaneously at rest.

Ⓟ **6** At time t seconds, where $t \geqslant 0$, the velocity $v\,\text{m s}^{-1}$ of a particle moving in a straight line is given by $v = 12 + t - 6t^2$. When $t = 0$, P is at a point O on the line. Find the distance of P from O when $v = 0$.

Ⓟ **7** A particle P is moving on the x-axis. At time t seconds, the velocity of P is $(4t - t^2)\,\text{m s}^{-1}$ in the direction of x increasing. At time $t = 0$, P is at the origin O. Find:

a the value of x at the instant when $t > 0$ and P is at rest

b the total distance moved by P in the interval $0 \leqslant t \leqslant 5$.

Problem-solving

You will need to consider the motion when v is positive and negative separately.

Ⓟ **8** A particle P is moving on the x-axis. At time t seconds, the velocity of P is $(6t^2 - 26t + 15)\,\text{m s}^{-1}$ in the direction of x increasing. At time $t = 0$, P is at the origin O. In the subsequent motion P passes through O twice. Find the two non-zero values of t when P passes through O.

Ⓟ **9** A particle P moves along the x-axis. At time t seconds (where $t \geqslant 0$) the velocity of P is $(3t^2 - 12t + 5)\,\text{m s}^{-1}$ in the direction of x increasing. When $t = 0$, P is at the origin O. Find:

a the values of t when P is again at O

b the distance travelled by P in the interval $2 \leqslant t \leqslant 3$.

Ⓟ **10** A particle P moves on the x-axis. The acceleration of P at time t seconds, $t \geqslant 0$, is $(4t - 3)\,\text{m s}^{-2}$ in the positive x-direction. When $t = 0$, the velocity of P is $4\,\text{m s}^{-1}$ in the positive x-direction. When $t = T$ $(T \neq 0)$, the velocity of P is $4\,\text{m s}^{-1}$ in the positive x-direction. Find the value of T. **(6 marks)**

11 A particle P travels in a straight line such that its acceleration at time t seconds is $(t - 3)\,\text{m s}^{-2}$. The velocity of P at time t seconds is $v\,\text{m s}^{-1}$. When $t = 0$, $v = 4$. Find:

a v in terms of t **(4 marks)**

b the values of t when P is instantaneously at rest **(3 marks)**

c the distance between the two points at which P is instantaneously at rest. **(4 marks)**

12 A particle travels in a straight line such that its acceleration, $a\,\text{m s}^{-2}$, at time t seconds is given by $a = 6t + 2$. When $t = 2$ seconds, the displacement, s, is 10 metres and when $t = 3$ seconds the displacement is 38 metres. Find:

a the displacement when $t = 4$ seconds **(6 marks)**

b the velocity when $t = 4$ seconds. **(2 marks)**

Problem-solving

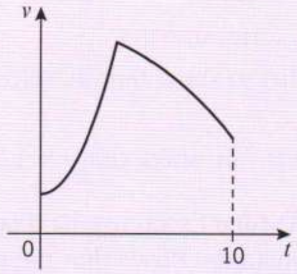
You need to use integration to find expressions for the velocity and displacement then substitute in the given values. Use simultaneous equations to find the values of the constants of integration.

Challenge

The motion of a robotic arm moving along a straight track is modelled using the equations:

$$v = \frac{t^2}{2} + 2,\ 0 \leqslant t \leqslant k \text{ and } v = 10 + \frac{t}{3} - \frac{t^2}{12},\ k \leqslant t \leqslant 10$$

The diagram shows a sketch of the velocity–time graph of the motion of the arm.



Work out the total distance travelled by the robotic arm.

11.5 Constant acceleration formulae

You can use calculus to derive the formulae for motion with constant acceleration.

Example 8

A particle moves in a straight line with constant acceleration, $a\,\text{m s}^{-2}$. Given that its initial velocity is $u\,\text{m s}^{-1}$ and its initial displacement is 0 m, prove that:

a its velocity, $v\,\text{m s}^{-1}$ at time t s is given by $v = u + at$

b its displacement, s m, at time t is given by $s = ut + \frac{1}{2}at^2$

a $v = \int a \, \mathrm{d}t$

$= at + c$

When $t = 0$, $v = u$,

so $u = a \times 0 + c = c$

So $v = u + at$

b $s = \int v \, \mathrm{d}t$

$= \int (u + at) \, \mathrm{d}t$

$= ut + \frac{1}{2}at^2 + c$

When $t = 0$, $s = 0$

so $0 = u \times 0 + \frac{1}{2} \times a \times 0^2 + c$

$c = 0$

So $s = ut + \frac{1}{2}at^2$

Use the initial condition you are given for the velocity to work out the value of c.

Use the equation for velocity you have just proved.

Use the initial condition you are given for the displacement to work out the value of c.

Watch out The *suvat* equations can only be used when the acceleration is constant.

Exercise 11E

(P) **1** A particle moves on the x-axis with constant acceleration a m s^{-2}. The particle has initial velocity 0 and initial displacement x m. After time t seconds the particle has velocity v m s^{-1} and displacement s m.

Prove that $s = \frac{1}{2}at^2 + x$.

2 A particle moves in a straight line with constant acceleration 5 m s^{-2}.

a Given that its initial velocity is 12 m s^{-1}, use calculus to show that its velocity at time t s is given by $v = 12 + 5t$.

b Given that the initial displacement of the particle is 7 m, show that $s = 12t + 2.5t^2 + 7$.

(P) **3** A particle moves in a straight line from a point O. At time t seconds, its displacement, s m, from P is given by $s = ut + \frac{1}{2}at^2$ where u and a are constants. Prove that the particle moves with constant acceleration a.

4 Which of these equations for displacement describe constant acceleration? Explain your answers.

A $s = 2t^2 - t^3$ B $s = 4t + 7$ C $s = \frac{t^2}{4}$ D $s = 3t - \frac{2}{t^2}$ E $s = 6$

(E/P) **5** A particle moves in a straight line with constant acceleration. The initial velocity of the particle is 5 m s^{-1} and after 2 seconds it is moving with velocity 13 m s^{-1}.

a Find the acceleration of the particle. **(3 marks)**

b Without making use of the kinematics formulae, show that the displacement, s m, of the particle from its starting position is given by:

$$s = pt^2 + qt + r, \ t \geqslant 0$$

where p, q and r are constants to be found. **(5 marks)**

Watch out An exam question might specify that you cannot use certain formulae or techniques. In this case you need to use calculus to find the answer to part **b**.

6 A train travels along a straight track, passing point A at time $t = 0$ and passing point B 40 seconds later. Its distance from A at time t seconds is given by:

$$s = 25t - 0.2t^2, 0 \leqslant t \leqslant 40$$

a Find the distance AB. **(1 mark)**

b Show that the train travels with constant acceleration. **(3 marks)**

A bird passes point B at time $t = 0$ at an initial velocity towards A of $7\,\text{m}\,\text{s}^{-1}$. It flies in a straight line towards point A with constant acceleration $0.6\,\text{m}\,\text{s}^{-2}$.

c Find the distance from A at which the bird is directly above the train. **(6 marks)**

Mixed exercise 11

1 A particle P moves in a horizontal straight line. At time t seconds (where $t \geqslant 0$) the velocity $v\,\text{m}\,\text{s}^{-1}$ of P is given by $v = 15 - 3t$. Find:

a the value of t when P is instantaneously at rest

b the distance travelled by P between the time when $t = 0$ and the time when P is instantaneously at rest.

2 A particle P moves along the x-axis so that, at time t seconds, the displacement of P from O is x metres and the velocity of P is $v\,\text{m}\,\text{s}^{-1}$, where:

$$v = 6t + \tfrac{1}{2}t^3$$

a Find the acceleration of P when $t = 4$.

b Given also that $x = -5$ when $t = 0$, find the distance OP when $t = 4$.

3 A particle P is moving along a straight line. At time $t = 0$, the particle is at a point A and is moving with velocity $8\,\text{m}\,\text{s}^{-1}$ towards a point B on the line, where $AB = 30\,\text{m}$. At time t seconds (where $t \geqslant 0$), the acceleration of P is $(2 - 2t)\,\text{m}\,\text{s}^{-2}$ in the direction $\overrightarrow{AB}$.

a Find an expression, in terms of t, for the displacement of P from A at time t seconds.

b Show that P does not reach B.

c Find the value of t when P returns to A, giving your answer to 3 significant figures.

d Find the total distance travelled by P in the interval between the two instants when it passes through A.

4 A particle starts from rest at a point O and moves along a straight line OP with an acceleration, a, after t seconds given by $a = (8 - 2t^2)\,\text{m}\,\text{s}^{-2}$.

Find:

a the greatest speed of the particle in the direction OP **(5 marks)**

b the distance covered by the particle in the first two seconds of its motion. **(4 marks)**

5 A particle P passes through a point O and moves in a straight line. The displacement, s metres, of P from O, t seconds after passing through O is given by:

$$s = -t^3 + 11t^2 - 24t$$

a Find an expression for the velocity, $v\,\text{m}\,\text{s}^{-1}$, of P at time t seconds. **(2 marks)**

b Calculate the values of t at which P is instantaneously at rest. **(3 marks)**

c Find the value of t at which the acceleration is zero. **(2 marks)**

d Sketch a velocity–time graph to illustrate the motion of P in the interval $0 \leqslant t \leqslant 6$, showing on your sketch the coordinates of the points at which the graph crosses the axes. **(3 marks)**

e Calculate the values of t in the interval $0 \leqslant t \leqslant 6$ between which the speed of P is greater than $16\,\text{m}\,\text{s}^{-1}$. **(6 marks)**

E **6** A body moves in a straight line. Its velocity, $v\,\text{m}\,\text{s}^{-1}$, at time t seconds is given by $v = 3t^2 - 11t + 10$. Find:

a the values of t when the body is instantaneously at rest **(3 marks)**

b the acceleration of the body when $t = 4$ **(3 marks)**

c the total distance travelled by the body in the interval $0 \leqslant t \leqslant 4$. **(4 marks)**

E **7** A particle moves along the positive x-axis. At time $t = 0$ the particle passes through the origin with velocity $6\,\text{m}\,\text{s}^{-1}$. The acceleration, $a\,\text{m}\,\text{s}^{-2}$, of the particle at time t seconds is given by $a = 2t^3 - 8t$ for $t \geqslant 0$. Find:

a the velocity of the particle at time t seconds **(3 marks)**

b the displacement of the particle from the origin at time t seconds **(2 marks)**

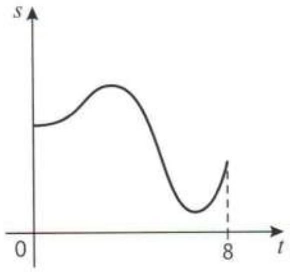
c the values of t at which the particle is instantaneously at rest. **(3 marks)**

E **8** A remote control drone hovers such that its vertical height, s m, above ground level at time t seconds is given by the equation:

$$s = \frac{t^4 - 12t^3 + 28t^2 + 400}{50}, \quad 0 \leqslant t \leqslant 8$$

The diagram shows a sketch of a displacement–time graph of the drone's motion.

Determine the maximum and minimum height of the drone.



(7 marks)

E/P **9** A rocket sled is used to test a parachute braking mechanism for a space capsule.

At the moment the parachute is deployed, the sled is 1.5 km from its launch site and is travelling away from it at a speed of $800\,\text{m}\,\text{s}^{-1}$. The sled comes to rest 25 seconds after the parachute is deployed.

The rocket sled is modelled as a particle moving in a straight horizontal line with constant acceleration. At a time t seconds after the parachute is deployed, its distance, s m, from the launch site is given by:

$s = a + bt + ct^2, 0 \leqslant t \leqslant 25$

Find the values of a, b and c in this model. **(6 marks)**

E **10** A particle P moves along the x-axis. It passes through the origin O at time $t = 0$ with speed $7\,\text{m}\,\text{s}^{-1}$ in the direction of x increasing.

At time t seconds the acceleration of P in the direction of x increasing is $(20 - 6t)\,\text{m}\,\text{s}^{-2}$.

a Show that the velocity $v\,\text{m}\,\text{s}^{-1}$ of P at time t seconds is given by:

$v = 7 + 20t - 3t^2$ **(3 marks)**

b Show that $v = 0$ when $t = 7$ and find the greatest speed of P in the interval $0 \leqslant t \leqslant 7$. **(4 marks)**

c Find the distance travelled by P in the interval $0 \leqslant t \leqslant 7$. **(4 marks)**

11 A particle P moves along a straight line. Initially, P is at rest at a point O on the line. At time t seconds (where $t \geqslant 0$) the acceleration of P is proportional to $(7 - t^2)$ and the displacement of P from O is s metres. When $t = 3$, the velocity of P is $6\,\text{m s}^{-1}$.

Show that $s = \frac{1}{24}t^2(42 - t^2)$. **(7 marks)**

12 A mouse leaves its hole and makes a short journey along a straight wall before returning to its hole. The mouse is modelled as a particle moving in a straight line. The distance of the mouse, s m, from its hole at time t minutes is given by:

$$s = t^4 - 10t^3 + 25t^2, \; 0 \leqslant t \leqslant 5$$

a Explain the restriction $0 \leqslant t \leqslant 5$. **(3 marks)**

b Find the greatest distance of the mouse from its hole. **(6 marks)**

13 At a time t seconds after launch, the space shuttle can be modelled as a particle moving in a straight line with acceleration, $a\,\text{m s}^{-2}$, given by the equation:

$$a = (6.77 \times 10^{-7})t^3 - (3.98 \times 10^{-4})t^2 + 0.105t + 0.859, \quad 124 \leqslant t \leqslant 446$$

a Suggest two reasons why the space shuttle might experience variable acceleration during its launch phase.

Given that the velocity of the space shuttle at time $t = 124$ is $974\,\text{m s}^{-1}$:

b find an expression for the velocity $v\,\text{m s}^{-1}$ of the space shuttle at time t. Give your coefficients to 3 significant figures.

c Hence find the velocity of the space shuttle at time $t = 446$, correct to 3 s.f.

From $t = 446$, the space shuttle maintains a constant acceleration of $28.6\,\text{m s}^{-2}$ until it reaches its escape velocity of $7.85\,\text{km s}^{-1}$. It then cuts its main engines.

d Calculate the time at which the space shuttle cuts its main engines.

Challenge

1 A particle starts at rest and moves in a straight line. At time t seconds after the beginning of its motion, the acceleration of the particle, $a\,\text{m s}^{-2}$, is given by:

$$a = 3t^2 - 18t + 20, \; t \geqslant 0$$

Find the distance travelled by the particle in the first 5 seconds of its motion.

2 A particle travels in a straight line with an acceleration, $a\,\text{m s}^{-2}$, given by $a = 6t + 2$.

The particle travels 50 metres in the fourth second. Find the velocity of the particle when $t = 5$ seconds.

Summary of key points

1 If the displacement, s, is expressed as a function of t, then the velocity, v, can be expressed as

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}$$

2 If the velocity, v, is expressed as a function of t, then the acceleration, a, can be expressed as

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}$$

3

$$\text{Differentiate}\ \Bigg\downarrow \quad \begin{array}{rl} \text{displacement} & = s = \int v\,\mathrm{d}t \\ \dfrac{\mathrm{d}s}{\mathrm{d}t} = \text{velocity} & = v = \int a\,\mathrm{d}t \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}^2s}{\mathrm{d}t^2} = \text{acceleration} & = a \end{array} \quad \Bigg\uparrow\ \text{Integrate}$$