Integration

11

Objectives

After completing this chapter you should be able to:

- Integrate standard mathematical functions including trigonometric and exponential functions and use the reverse of the chain rule to integrate functions of the form $f(ax + b)$ → pages 294–298
- Use trigonometric identities in integration → pages 298–300
- Use the reverse of the chain rule to integrate more complex functions → pages 300–303
- Integrate functions by making a substitution, using integration by parts and using partial fractions → pages 303–313
- Use integration to find the area under a curve → pages 313–317
- Use the trapezium rule to approximate the area under a curve. → pages 317–322
- Solve simple differential equations and model real-life situations with differential equations → pages 322–329

Prior knowledge check

1 Differentiate:

a $(2x - 7)^6$ **b** $\sin 5x$

c $e^{\frac{x}{3}}$ ← Sections 9.1, 9.2, 9.3

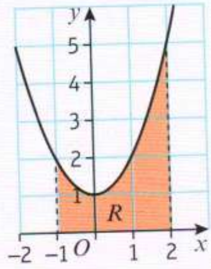
2 Given $f(x) = 8x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$

a find $\int f(x)\,dx$

b find $\int_4^9 f(x)\,dx$ ← Year 1, Chapter 13

3 Write $\dfrac{3x + 22}{(4x - 1)(x + 3)}$ as partial fractions. ← Section 1.3

4 Find the area of the region R bounded by the curve $y = x^2 + 1$, the x-axis and the lines $x = -1$ and $x = 2$.



← Year 1, Chapter 13

Integration can be used to solve differential equations. Archaeologists use differential equations to estimate the age of fossilised plants and animals. → Exercise 11K Q9

11.1 Integrating standard functions

Integration is the inverse of differentiation. You can use your knowledge of derivatives to integrate familiar functions.

① $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

② $\int e^x \, dx = e^x + c$

③ $\int \frac{1}{x} \, dx = \ln|x| + c$

④ $\int \cos x \, dx = \sin x + c$

⑤ $\int \sin x \, dx = -\cos x + c$

⑥ $\int \sec^2 x \, dx = \tan x + c$

⑦ $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

⑧ $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

⑨ $\int \sec x \tan x \, dx = \sec x + c$

Watch out This is true for all values of n except -1.

Notation When finding $\int \frac{1}{x}\,dx$ it is usual to write the answer as $\ln|x| + c$. The modulus sign removes difficulties that could arise when evaluating the integral for negative values of x.

Links For example, if $y = \cos x$ then $\frac{dy}{dx} = -\sin x$. This means that $\int (-\sin x)\,dx = \cos x + c$ and hence $\int \sin x \, dx = -\cos x + c$. ← Section 9.1

Example 1

Find the following integrals.

a $\int \left(2\cos x + \frac{3}{x} - \sqrt{x}\right) dx$ **b** $\int \left(\frac{\cos x}{\sin^2 x} - 2e^x\right) dx$

a $\int 2\cos x \, dx = 2\sin x + c$ — Integrate each term separately. Use ④.

$\int \frac{3}{x} \, dx = 3\ln|x| + c$ — Use ③.

$\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{2}{3}x^{\frac{3}{2}} + c$ — Use ①.

So $\int \left(2\cos x + \frac{3}{x} - \sqrt{x}\right) dx$

$= 2\sin x + 3\ln|x| - \frac{2}{3}x^{\frac{3}{2}} + c$ — This is an indefinite integral so don't forget the $+c$.

b $\frac{\cos x}{\sin^2 x} = \frac{\cos x}{\sin x} \times \frac{1}{\sin x} = \cot x \operatorname{cosec} x$ — Look at the list of integrals of standard functions and express the integrand in terms of these standard functions.

$\int (\cot x \operatorname{cosec} x) \, dx = -\operatorname{cosec} x + c$ — Remember the minus sign.

$\int 2e^x \, dx = 2e^x + c$

So $\int \left(\frac{\cos x}{\sin^2 x} - 2e^x \, dx\right)$

$= -\operatorname{cosec} x - 2e^x + c$

Example 2

Given that a is a positive constant and $\int_a^{3a} \frac{2x+1}{x}\,dx = \ln 12$, find the exact value of a.

Problem-solving

Integrate as normal and write the limits as a and $3a$. Substitute these limits into your integral to get an expression in a and set this equal to ln12. Solve the resulting equation to find the value of a.

$\int_a^{3a} \frac{2x+1}{x}\,dx$

$= \int_a^{3a} \left(2 + \frac{1}{x}\right) dx$ — Separate the terms by dividing by x, then integrate term by term.

$= [2x + \ln x]_a^{3a}$ — Remember the limits are a and $3a$.

$= (6a + \ln 3a) - (2a + \ln a)$ — Substitute $3a$ and a into the integrated expression.

$= 4a + \ln\left(\frac{3a}{a}\right)$ — Use the laws of logarithms: $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$= 4a + \ln 3$

So, $4a + \ln 3 = \ln 12$

$4a = \ln 12 - \ln 3$

$4a = \ln 4$ — $\ln 12 - \ln 3 = \ln\left(\frac{12}{3}\right) = \ln 4$

$a = \frac{1}{4}\ln 4$

Online Use your calculator to check your value of a using numerical integration.

Exercise 11A

1 Integrate the following with respect to x.

a $3\sec^2 x + \frac{5}{x} + \frac{2}{x^2}$

b $5e^x - 4\sin x + 2x^3$

c $2(\sin x - \cos x + x)$

d $3\sec x \tan x - \frac{2}{x}$

e $5e^x + 4\cos x - \frac{2}{x^2}$

f $\frac{1}{2x} + 2\operatorname{cosec}^2 x$

g $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

h $e^x + \sin x + \cos x$

i $2\operatorname{cosec} x \cot x - \sec^2 x$

j $e^x + \frac{1}{x} - \operatorname{cosec}^2 x$

2 Find the following integrals.

a $\int\left(\frac{1}{\cos^2 x} + \frac{1}{x^2}\right) dx$

b $\int\left(\frac{\sin x}{\cos^2 x} + 2e^x\right) dx$

c $\int\left(\frac{1+\cos x}{\sin^2 x} + \frac{1+x}{x^2}\right) dx$

d $\int\left(\frac{1}{\sin^2 x} + \frac{1}{x}\right) dx$

e $\int \sin x(1 + \sec^2 x)\,dx$

f $\int \cos x(1 + \operatorname{cosec}^2 x)\,dx$

g $\int \operatorname{cosec}^2 x(1 + \tan^2 x)\,dx$

h $\int \sec^2 x(1 - \cot^2 x)\,dx$

i $\int \sec^2 x(1 + e^x\cos^2 x)\,dx$

j $\int\left(\frac{1+\sin x}{\cos^2 x} + \cos^2 x \sec x\right) dx$

3 Evaluate the following. Give your answers as exact values.

a $\int_3^7 2e^x\,dx$

b $\int_1^6 \frac{1+x}{x^3}\,dx$

c $\int_{\frac{\pi}{2}}^{\pi} -5\sin x\,dx$

d $\int_{-\frac{\pi}{4}}^{0} \sec x(\sec x + \tan x)\,dx$

Watch out When applying limits to integrated trigonometric functions, always work in radians.

E/P **4** Given that a is a positive constant and $\int_a^{2a} \frac{3x-1}{x}\,dx = 6 + \ln\left(\frac{1}{2}\right)$, find the exact value of a. **(4 marks)**

E/P **5** Given that a is a positive constant and $\int_{\ln 1}^{\ln a} e^x + e^{-x}\,dx = \frac{48}{7}$, find the exact value of a. **(4 marks)**

E/P **6** Given $\int_2^b (3e^x + 6e^{-2x})\,dx = 0$, find the value of b. **(4 marks)**

E/P **7** $f(x) = \frac{1}{8}x^{\frac{3}{2}} - \frac{4}{x}, x > 0$

a Solve the equation $f(x) = 0$. **(2 marks)**

b Find $\int f(x)\,dx$. **(2 marks)**

c Evaluate $\int_1^4 f(x)\,dx$, giving your answer in the form $p + q\ln r$, where p, q and r are rational numbers. **(3 marks)**

11.2 Integrating $f(ax + b)$

If you know the integral of a function $f(x)$ you can integrate a function of the form $f(ax + b)$ using the reverse of the chain rule for differentiation.

Example 3

Find the following integrals.

a $\int \cos(2x + 3)\,dx$ **b** $\int e^{4x+1}\,dx$ **c** $\int \sec^2 3x\,dx$

a Consider $y = \sin(2x + 3)$:

$$\frac{dy}{dx} = \cos(2x + 3) \times 2$$

So $\int \cos(2x + 3)\,dx = \frac{1}{2}\sin(2x + 3) + c$

b Consider $y = e^{4x+1}$:

$$\frac{dy}{dx} = e^{4x+1} \times 4$$

So $\int e^{4x+1}\,dx = \frac{1}{4}e^{4x+1} + c$

c Consider $y = \tan 3x$:

$$\frac{dy}{dx} = \sec^2 3x \times 3$$

So $\int \sec^2 3x\,dx = \frac{1}{3}\tan 3x + c$

Integrating $\cos x$ gives $\sin x$, so try $\sin(2x + 3)$.

Use the chain rule. Remember to multiply by the derivative of $2x + 3$ which is 2.

This is 2 times the required expression so you need to divide $\sin(2x + 3)$ by 2.

The integral of e^x is e^x, so try e^{4x+1}.

This is 4 times the required expression so you divide by 4.

Recall (6). Let $y = \tan 3x$ and differentiate using the chain rule. This is 3 times the required expression so you divide by 3.

In general:

- $\int f'(ax+b)\,dx = \frac{1}{a}f(ax+b) + c$

Watch out You cannot use this method to integrate an expression such as $\cos(2x^2+3)$ since it is not in the form $f(ax+b)$.

Example 4

Find the following integrals:

a $\int \frac{1}{3x+2}\,dx$ **b** $\int (2x+3)^4\,dx$

a Consider $y = \ln(3x+2)$

So $\frac{dy}{dx} = \frac{1}{3x+2} \times 3$

So $\int \frac{1}{3x+2}\,dx = \frac{1}{3}\ln|3x+2| + c$

b Consider $y = (2x+3)^5$

So $\frac{dy}{dx} = 5 \times (2x+3)^4 \times 2$

$= 10 \times (2x+3)^4$

So $\int (2x+3)^4\,dx = \frac{1}{10}(2x+3)^5 + c$

Integrating $\frac{1}{x}$ gives $\ln|x|$ so try $\ln(3x+2)$.

The 3 comes from the chain rule. It is 3 times the required expression, so divide by 3.

To integrate $(ax+b)^n$ try $(ax+b)^{n+1}$.

The 5 comes from the exponent and the 2 comes from the chain rule.

This answer is 10 times the required expression, so divide by 10.

Exercise 11B

1 Integrate the following:

Hint For part **a** consider $y = \cos(2x+1)$. You do not need to write out this step once you are confident with using this method.

a $\sin(2x+1)$ **b** $3e^{2x}$

c $4e^{x+5}$ **d** $\cos(1-2x)$

e $\operatorname{cosec}^2 3x$ **f** $\sec 4x \tan 4x$

g $3\sin\left(\frac{1}{2}x+1\right)$ **h** $\sec^2(2-x)$ **i** $\operatorname{cosec} 2x \cot 2x$ **j** $\cos 3x - \sin 3x$

2 Find the following integrals.

a $\int \left(e^{2x} - \frac{1}{2}\sin(2x-1)\right)dx$ **b** $\int (e^x+1)^2\,dx$

c $\int \sec^2 2x(1+\sin 2x)\,dx$ **d** $\int \frac{3 - 2\cos\frac{1}{2}x}{\sin^2\frac{1}{2}x}\,dx$

e $\int \left(e^{3-x} + \sin(3-x) + \cos(3-x)\right)dx$

3 Integrate the following:

a $\frac{1}{2x+1}$ **b** $\frac{1}{(2x+1)^2}$ **c** $(2x+1)^2$ **d** $\frac{3}{4x-1}$

e $\frac{3}{1-4x}$ **f** $\frac{3}{(1-4x)^2}$ **g** $(3x+2)^5$ **h** $\frac{3}{(1-2x)^3}$

4 Find the following integrals.

a $\int\left(3\sin(2x+1)+\dfrac{4}{2x+1}\right)\mathrm{d}x$

b $\int(\mathrm{e}^{5x}+(1-x)^5)\,\mathrm{d}x$

c $\int\left(\dfrac{1}{\sin^2 2x}+\dfrac{1}{1+2x}+\dfrac{1}{(1+2x)^2}\right)\mathrm{d}x$

d $\int\left((3x+2)^2+\dfrac{1}{(3x+2)^2}\right)\mathrm{d}x$

5 Evaluate:

a $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos(\pi-2x)\,\mathrm{d}x$

b $\int_{\frac{1}{2}}^{1}\dfrac{12}{(3-2x)^4}\,\mathrm{d}x$

c $\int_{\frac{2\pi}{9}}^{\frac{5\pi}{18}}\sec^2(\pi-3x)\,\mathrm{d}x$

d $\int_{2}^{3}\dfrac{5}{7-2x}\,\mathrm{d}x$

E/P **6** Given $\int_3^b(2x-6)^2\,\mathrm{d}x=36$, find the value of b. **(4 marks)**

E/P **7** Given $\int_{e^2}^{e^8}\dfrac{1}{kx}\,\mathrm{d}x=\dfrac{1}{4}$, find the value of k. **(4 marks)**

E/P **8** Given $\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}}(1-\pi\sin kx)\,\mathrm{d}x=\pi(7-6\sqrt{2})$, find the exact value of k. **(7 marks)**

Problem-solving

Calculate the value of the indefinite integral in terms of k and solve the resulting equation.

Challenge

Given $\int_5^{11}\dfrac{1}{ax+b}\,\mathrm{d}x=\dfrac{1}{a}\ln\left(\frac{41}{17}\right)$, and that a and b are integers with $0 < a < 10$, find two different pairs of values for a and b.

11.3 Using trigonometric identities

- **Trigonometric identities can be used to integrate expressions. This allows an expression that cannot be integrated to be replaced by an identical expression that can be integrated.**

Links Make sure you are familiar with the standard trigonometric identities. The list of identities in the summary of Chapter 7 will be useful. ← page 196

Example 5

Find $\int\tan^2 x\,\mathrm{d}x$

Since $\sec^2 x\equiv 1+\tan^2 x$

$\tan^2 x\equiv\sec^2 x-1$

So $\int\tan^2 x\,\mathrm{d}x=\int(\sec^2 x-1)\,\mathrm{d}x$

$=\int\sec^2 x\,\mathrm{d}x-\int 1\,\mathrm{d}x$

$=\tan x-x+c$

You cannot integrate $\tan^2 x$ but you can integrate $\sec^2 x$ directly.

Using (6).

Example 6

Show that $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sin^2 x \, dx = \frac{\pi}{48} + \frac{1-\sqrt{2}}{8}$

Recall $\cos 2x \equiv 1 - 2\sin^2 x$

So $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$

So $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sin^2 x \, dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{8}}$

$= \left(\frac{\pi}{16} - \frac{1}{4}\sin\left(\frac{\pi}{4}\right)\right) - \left(\frac{\pi}{24} - \frac{1}{4}\sin\left(\frac{\pi}{6}\right)\right)$

$= \left(\frac{\pi}{16} - \frac{1}{4}\left(\frac{\sqrt{2}}{2}\right)\right) - \left(\frac{\pi}{24} - \frac{1}{4}\left(\frac{1}{2}\right)\right)$

$= \left(\frac{\pi}{16} - \frac{\pi}{24}\right) + \frac{1}{4}\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)$

$= \left(\frac{3\pi}{48} - \frac{2\pi}{48}\right) + \frac{1-\sqrt{2}}{8}$

$= \frac{\pi}{48} + \frac{1-\sqrt{2}}{8}$

You cannot integrate $\sin^2 x$ directly. Use the trigonometric identity to write it in terms of $\cos 2x$.

Use the reverse chain rule. If $y = \sin 2x$, $\frac{dy}{dx} = 2\cos 2x$. Adjust for the constant.

Substitute the limits into the integrated expression.

Problem-solving

You will save lots of time in your exam if you are familiar with the exact values for trigonometric functions given in radians.

Write $\sin\left(\frac{\pi}{4}\right)$ in its rationalised denominator form, as $\frac{\sqrt{2}}{2}$ rather than $\frac{1}{\sqrt{2}}$. This will make it easier to simplify your fractions.

Watch out This is a 'show that' question so don't use your calculator to simplify the fractions. Show each line of your working carefully.

Example 7

Find:

a $\int \sin 3x \cos 3x \, dx$

b $\int (\sec x + \tan x)^2 \, dx$

a $\int \sin 3x \cos 3x \, dx = \int \frac{1}{2}\sin 6x \, dx$

$= -\frac{1}{2} \times \frac{1}{6}\cos 6x + c$

$= -\frac{1}{12}\cos 6x + c$

b $(\sec x + \tan x)^2$

$\equiv \sec^2 x + 2\sec x \tan x + \tan^2 x$

$\equiv \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$

$\equiv 2\sec^2 x + 2\sec x \tan x - 1$

So $\int (\sec x + \tan x)^2 \, dx$

$= \int (2\sec^2 x + 2\sec x \tan x - 1) \, dx$

$= 2\tan x + 2\sec x - x + c$

Remember $\sin 2A \equiv 2\sin A \cos A$, so $\sin 6x \equiv 2\sin 3x \cos 3x$.

Use the reverse chain rule.

Simplify $\frac{1}{2} \times \frac{1}{6}$ to $\frac{1}{12}$

Multiply out the bracket.

Write $\tan^2 x$ as $\sec^2 x - 1$. Then all the terms are standard integrals.

Integrate each term using (6) and (9).

Exercise 11C

1 Integrate the following:

a $\cot^2 x$
b $\cos^2 x$
c $\sin 2x \cos 2x$
d $(1 + \sin x)^2$
e $\tan^2 3x$
f $(\cot x - \operatorname{cosec} x)^2$
g $(\sin x + \cos x)^2$
h $\sin^2 x \cos^2 x$
i $\frac{1}{\sin^2 x \cos^2 x}$
j $(\cos 2x - 1)^2$

Hint For part **a**, use $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$. For part **c**, use $\sin 2A \equiv 2 \sin A \cos A$, making a suitable substitution for A.

2 Find the following integrals.

a $\int \frac{1 - \sin x}{\cos^2 x} \, dx$
b $\int \frac{1 + \cos x}{\sin^2 x} \, dx$
c $\int \frac{\cos 2x}{\cos^2 x} \, dx$
d $\int \frac{\cos^2 x}{\sin^2 x} \, dx$
e $\int \frac{(1 + \cos x)^2}{\sin^2 x} \, dx$
f $\int (\cot x - \tan x)^2 \, dx$
g $\int (\cos x - \sin x)^2 \, dx$
h $\int (\cos x - \sec x)^2 \, dx$
i $\int \frac{\cos 2x}{1 - \cos^2 2x} \, dx$

E/P 3 Show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{2 + \pi}{8}$ **(4 marks)**

4 Find the exact value of each of the following:

a $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x \cos^2 x} \, dx$
b $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x - \operatorname{cosec} x)^2 \, dx$
c $\int_0^{\frac{\pi}{4}} \frac{(1 + \sin x)^2}{\cos^2 x} \, dx$
d $\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{\sin 2x}{1 - \sin^2 2x} \, dx$

E/P 5 a By expanding $\sin(3x + 2x)$ and $\sin(3x - 2x)$ using the double-angle formulae, or otherwise, show that $\sin 5x + \sin x \equiv 2 \sin 3x \cos 2x$. **(4 marks)**

b Hence find $\int \sin 3x \cos 2x \, dx$ **(3 marks)**

E/P 6 $f(x) = 5 \sin^2 x + 7 \cos^2 x$

a Show that $f(x) = \cos 2x + 6$. **(3 marks)**

b Hence, find the exact value of $\int_0^{\frac{\pi}{4}} f(x) \, dx$. **(4 marks)**

E/P 7 a Show that $\cos^4 x \equiv \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8}$ **(4 marks)**

b Hence find $\int \cos^4 x \, dx$. **(4 marks)**

11.4 Reverse chain rule

If a function can be written in the form $k\frac{f'(x)}{f(x)}$, you can integrate it using the reverse of the chain rule for differentiation.

Example 8

Find

a $\int \frac{2x}{x^2 + 1} \, dx$
b $\int \frac{\cos x}{3 + 2 \sin x} \, dx$

Problem-solving

If $f(x) = 3 + 2 \sin x$, then $f'(x) = 2 \cos x$.
By adjusting for the constant, the numerator is the derivative of the denominator.

a Let $I = \int \frac{2x}{x^2 + 1}\,dx$

Consider $y = \ln|x^2 + 1|$

Then $\frac{dy}{dx} = \frac{1}{x^2 + 1} \times 2x$

So $I = \ln|x^2 + 1| + c$

b Let $I = \int \frac{\cos x}{3 + 2\sin x}\,dx$

Let $y = \ln|3 + 2\sin x|$

$\frac{dy}{dx} = \frac{1}{3 + 2\sin x} \times 2\cos x$

So $I = \frac{1}{2}\ln|3 + 2\sin x| + c$

This is equal to the original integrand, so you don't need to adjust it.

Since integration is the reverse of differentiation.

Try differentiating $y = \ln|3 + 2\sin x|$.

The derivative of $\ln|3 + 2\sin x|$ is twice the original integrand, so you need to divide it by 2.

- **To integrate expressions of the form $\int k\frac{f'(x)}{f(x)}\,dx$, try $\ln|f(x)|$ and differentiate to check, and then adjust any constant.**

Watch out You can't use this method to integrate a function such as $\frac{1}{x^2 + 3}$ because the derivative of $x^2 + 3$ is $2x$, and the top of the fraction does not contain an x term.

You can use a similar method with functions of the form $kf'(x)(f(x))^n$.

Example 9

Find:

a $\int 3\cos x \sin^2 x\,dx$ **b** $\int x(x^2 + 5)^3\,dx$

a Let $I = \int 3\cos x \sin^2 x\,dx$

Consider $y = \sin^3 x$

$\frac{dy}{dx} = 3\sin^2 x \cos x$

So $I = \sin^3 x + c$

b Let $I = \int x(x^2 + 5)^3\,dx$

Then let $y = (x^2 + 5)^4$

$\frac{dy}{dx} = 4(x^2 + 5)^3 \times 2x$

$= 8x(x^2 + 5)^3$

So $I = \frac{1}{8}(x^2 + 5)^4 + c$

Try differentiating $\sin^3 x$.

This is equal to the original integrand, so you don't need to adjust it.

Try differentiating $(x^2 + 5)^4$.

The $2x$ comes from differentiating $x^2 + 5$.

This is 8 times the required expression so you divide by 8.

- **To integrate an expression of the form $\int k\,f'(x)(f(x))^n\,dx$, try $(f(x))^{n+1}$ and differentiate to check, and then adjust any constant.**

Example 10

Use integration to find $\int \frac{\operatorname{cosec}^2 x}{(2+\cot x)^3}\,dx$

Let $I = \int \frac{\operatorname{cosec}^2 x}{(2+\cot x)^3}\,dx$ — This is in the form $\int k f'(x)(f(x))^n\,dx$ with $f(x) = 2 + \cot x$ and $n = -3$.

Let $y = (2+\cot x)^{-2}$

$\frac{dy}{dx} = -2(2+\cot x)^{-3} \times (-\operatorname{cosec}^2 x)$ — Use the chain rule.

$= 2(2+\cot x)^{-3}\operatorname{cosec}^2 x$ — This is 2 times the required answer so you need to divide by 2.

So $I = \frac{1}{2}(2+\cot x)^{-2} + c$

Example 11

Given that $\int_0^\theta 5\tan x \sec^4 x\,dx = \frac{15}{4}$ where $0 < \theta < \frac{\pi}{2}$, find the exact value of θ.

Let $I = \int_0^\theta 5\tan x\sec^4 x\,dx$ — This is in the form $\int k f'(x)(f(x))^n\,dx$ with $f(x) = \sec x$ and $n = 4$.

Let $y = \sec^4 x$

$\frac{dy}{dx} = 4\sec^3 x \times \sec x\tan x$

$= 4\sec^4 x\tan x$ — This is $\frac{4}{5}$ times the required answer so you need to divide by $\frac{4}{5}$

So $I = \left[\frac{5}{4}\sec^4 x\right]_0^\theta = \frac{15}{4}$

$\left(\frac{5}{4}\sec^4\theta\right) - \left(\frac{5}{4}\sec^4 0\right) = \frac{15}{4}$ — Substitute the limits into the integrated expression.

$\frac{5}{4}\sec^4\theta - \frac{5}{4} = \frac{15}{4}$ — $\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

$\frac{5}{4}\sec^4\theta = \frac{20}{4}$

$\sec^4\theta = 4$

$\sec\theta = \pm\sqrt{2}$ — Take the 4th root of both sides.

$\theta = \frac{\pi}{4}$ — The solutions to $\cos\theta = \pm\frac{1}{\sqrt{2}}$ are $\theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}\ldots$ The only solution within the given range for θ is $\frac{\pi}{4}$

Online Check your solution by using your calculator.

Exercise 11D

Hint Decide carefully whether each expression is in the form $k\frac{f'(x)}{f(x)}$ or $kf'(x)(f(x))^n$.

1 Integrate the following functions.

a $\frac{x}{x^2+4}$

b $\frac{e^{2x}}{e^{2x}+1}$

c $\frac{x}{(x^2+4)^3}$

d $\frac{e^{2x}}{(e^{2x}+1)^3}$

e $\frac{\cos 2x}{3+\sin 2x}$

f $\frac{\sin 2x}{(3+\cos 2x)^3}$

g xe^{x^2}

h $\cos 2x(1+\sin 2x)^4$

i $\sec^2 x\tan^2 x$

j $\sec^2 x(1+\tan^2 x)$

2 Find the following integrals.

a $\int (x+1)(x^2+2x+3)^4\,dx$

b $\int \operatorname{cosec}^2 2x \cot 2x\,dx$

c $\int \sin^5 3x \cos 3x\,dx$

d $\int \cos x\, e^{\sin x}\,dx$

e $\int \frac{e^{2x}}{e^{2x}+3}\,dx$

f $\int x(x^2+1)^{\frac{3}{2}}\,dx$

g $\int (2x+1)\sqrt{x^2+x+5}\,dx$

h $\int \frac{2x+1}{\sqrt{x^2+x+5}}\,dx$

i $\int \frac{\sin x \cos x}{\sqrt{\cos 2x+3}}\,dx$

j $\int \frac{\sin x \cos x}{\cos 2x+3}\,dx$

3 Find the exact value of each of the following:

a $\int_0^3 (3x^2+10x)\sqrt{x^3+5x^2+9}\,dx$

b $\int_{\frac{\pi}{9}}^{\frac{2\pi}{9}} \frac{6\sin 3x}{1-\cos 3x}\,dx$

c $\int_4^7 \frac{x}{x^2-1}\,dx$

d $\int_0^{\frac{\pi}{4}} \sec^2 x\, e^{4\tan x}\,dx$

E/P **4** Given that $\int_0^k kx^2 e^{x^3}\,dx = \frac{2}{3}(e^8-1)$, find the value of k. **(3 marks)**

P **5** Given that $\int_0^\theta 4\sin 2x \cos^4 2x\,dx = \frac{4}{5}$ where $0 < \theta < \pi$, find the exact value of θ.

E/P **6** **a** By writing $\cot x = \frac{\cos x}{\sin x}$, find $\int \cot x\,dx$. **(2 marks)**

b Show that $\int \tan x\,dx \equiv \ln|\sec x| + c$. **(3 marks)**

11.5 Integration by substitution

- **Sometimes you can simplify an integral by changing the variable. The process is similar to using the chain rule in differentiation and is called integration by substitution.**

In your exam you will often be told which substitution to use.

Example 12

Find $\int x\sqrt{2x+5}\,dx$ using the substitutions:

a $u = 2x+5$ **b** $u^2 = 2x+5$

a Let $I = \int x\sqrt{2x+5}\,dx$

Let $u = 2x+5$

So $\frac{du}{dx} = 2$

So dx can be replaced by $\frac{1}{2}du$.

$\sqrt{2x+5} = \sqrt{u} = u^{\frac{1}{2}}$

$x = \frac{u-5}{2}$

You need to replace each 'x' term with a corresponding 'u' term. Start by finding the relationship between dx and du.

So $dx = \frac{1}{2}du$.

Next rewrite the function in terms of $u = 2x+5$.

Rearrange $u = 2x+5$ to get $2x = u-5$ and hence $x = \frac{u-5}{2}$

So $I = \int \left(\frac{u-5}{2}\right)u^{\frac{1}{2}} \times \frac{1}{2}\,du$

$= \int \frac{1}{4}(u-5)u^{\frac{1}{2}}\,du$

$= \int \frac{1}{4}(u^{\frac{3}{2}} - 5u^{\frac{1}{2}})\,du$

$= \frac{1}{4} \times \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{5u^{\frac{3}{2}}}{4 \times \frac{3}{2}} + c$

$= \frac{u^{\frac{5}{2}}}{10} - \frac{5u^{\frac{3}{2}}}{6} + c$

So $I = \frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + c$

Rewrite I in terms of u and simplify.

Multiply out the brackets and integrate using rules from your Year 1 book. ← Year 1, Chapter 13

Simplify.

Finally rewrite the answer in terms of x.

b Let $I = \int x\sqrt{2x+5}\,dx$

$u^2 = 2x + 5$

$2u\frac{du}{dx} = 2$

So replace dx with $u\,du$.

$\sqrt{2x+5} = u$

and $x = \frac{u^2 - 5}{2}$

So $I = \int \left(\frac{u^2-5}{2}\right)u \times u\,du$

$= \int \left(\frac{u^4}{2} - \frac{5u^2}{2}\right)du$

$= \frac{u^5}{10} - \frac{5u^3}{6} + c$

So $I = \frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + c$

First find the relationship between dx and du.

Using implicit differentiation, cancel 2 and rearrange to get dx = udu.

Rewrite the integrand in terms of u. You will need to make x the subject of $u^2 = 2x + 5$.

Multiply out the brackets and integrate.

Rewrite answer in terms of x.

Example 13

Use the substitution $u = \sin x + 1$ to find

$$\int \cos x \sin x (1 + \sin x)^3\,dx$$

Let $I = \int \cos x \sin x (1 + \sin x)^3\,dx$

Let $u = \sin x + 1$

$\frac{du}{dx} = \cos x$

So substitute $\cos x\,dx$ with du.

First replace the dx.

$\cos x$ appears in the integrand, so you can write this as du = $\cos x$ dx and substitute.

$(\sin x + 1)^3 = u^3$

$\sin x = u - 1$

So $I = \int (u - 1)u^3\, du$

$= \int (u^4 - u^3)\, du$

$= \frac{u^5}{5} - \frac{u^4}{4} + c$

So $I = \frac{(\sin x + 1)^5}{5} - \frac{(\sin x + 1)^4}{4} + c$

Use $u = \sin x + 1$ to substitute for the remaining terms, rearranging where required to get $\sin x = u - 1$.

Rewrite I in terms of u.

Multiply out the brackets and integrate in the usual way.

Problem-solving

Although it looks different, $\int \sin 2x(1 + \sin x)^3\, dx$ can be integrated in exactly the same way. Remember $\sin 2x \equiv 2\sin x \cos x$, so the above integral would just need adjusting by a factor of 2.

Example 14

Prove that $\int \frac{1}{\sqrt{1 - x^2}}\, dx = \arcsin x + c$.

Let $I = \int \frac{1}{\sqrt{1 - x^2}}\, dx$

Let $x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

So replace dx with $\cos\theta\, d\theta$.

$I = \int \frac{1}{\sqrt{1 - \sin^2\theta}} \cos\theta\, d\theta$

$= \int \frac{1}{\sqrt{\cos^2\theta}} \cos\theta\, d\theta$

$= \int \frac{1}{\cos\theta} \cos\theta\, d\theta$

$= \int 1\, d\theta = \theta + c$

$x = \sin\theta \Rightarrow \theta = \arcsin x$

So $I = \arcsin x + c$

This substitution is not obvious at first. Think how the integral will be transformed by using trigonometric identities. ← Chapter 7

The substitution is in the form $x = f(\theta)$, so find $\frac{dx}{d\theta}$ to work out the relationship between dx and $d\theta$.

Make the substitution, and replace dx with $\cos\theta\, d\theta$.

Remember $\sin^2 A + \cos^2 A \equiv 1$

Remember to use your substitution to write the final answer in terms of x, not θ.

Example 15

Use integration by substitution to evaluate:

a $\int_0^2 x(x + 1)^3\, dx$ **b** $\int_0^{\frac{\pi}{2}} \cos x\sqrt{1 + \sin x}\, dx$

Watch out If you use integration by substitution to evaluate a definite integral, you have to be careful of whether your limits are x values or u values. You can use a table to keep track.

a Let $I = \int_0^2 x(x + 1)^3\, dx$

Let $u = x + 1$

$\frac{du}{dx} = 1$

so replace dx with du and replace $(x + 1)^3$ with u^3, and x with $u - 1$.

Replace each term in x with a term in u in the usual way.

x	u
2	3
0	1

Change the limits. When $x = 2$, $u = 2 + 1 = 3$ and when $x = 0$, $u = 1$.

So $I = \int_1^3 (u - 1)u^3\,du$

Note that the new u limits replace their corresponding x limits.

$= \int_1^3 (u^4 - u^3)\,du$

$= \left[\frac{u^5}{5} - \frac{u^4}{4}\right]_1^3$

Multiply out and integrate. Remember there is no need for a constant of integration with definite integrals.

$= \left(\frac{243}{5} - \frac{81}{4}\right) - \left(\frac{1}{5} - \frac{1}{4}\right)$

The integral can now be evaluated using the limits for u without having to change back into x.

$= 48.4 - 20 = 28.4$

b $\int_0^{\frac{\pi}{2}} \cos x\sqrt{1 + \sin x}\,dx$

$u = 1 + \sin x \Rightarrow \frac{du}{dx} = \cos x$, so replace $\cos x\,dx$ with du and replace $\sqrt{1 + \sin x}$ with $u^{\frac{1}{2}}$.

Use $u = 1 + \sin x$.

x	u
$\frac{\pi}{2}$	2
0	1

Remember that limits for integrals involving trigonometric functions will always be in radians. $x = \frac{\pi}{2}$ means $u = 1 + 1 = 2$ and $x = 0$ means $u = 1 + 0 = 1$.

So $I = \int_1^2 u^{\frac{1}{2}}\,du$

Rewrite the integral in terms of u.

$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2$

$= \left(\frac{2}{3}2^{\frac{3}{2}}\right) - \left(\frac{2}{3}\right)$

Remember that $2^{\frac{3}{2}} = \sqrt{8} = 2\sqrt{2}$.

So $I = \frac{2}{3}(2\sqrt{2} - 1)$

Problem-solving

You could also convert the integral back into a function of x and use the original limits.

Exercise 11E

1 Use the substitutions given to find:

a $\int x\sqrt{1 + x}\,dx$; $u = 1 + x$

b $\int \frac{1 + \sin x}{\cos x}\,dx$; $u = \sin x$

c $\int \sin^3 x\,dx$; $u = \cos x$

d $\int \frac{2}{\sqrt{x}(x - 4)}\,dx$; $u = \sqrt{x}$

e $\int \sec^2 x \tan x\sqrt{1 + \tan x}\,dx$; $u^2 = 1 + \tan x$

f $\int \sec^4 x\,dx$; $u = \tan x$

2 Use the substitutions given to find the exact values of:

a $\int_0^5 x\sqrt{x + 4}\,dx$; $u = x + 4$

b $\int_0^2 x(2 + x)^3\,dx$; $u = 2 + x$

c $\int_0^{\frac{\pi}{2}} \sin x\sqrt{3\cos x + 1}\,dx$; $u = \cos x$

d $\int_0^{\frac{\pi}{3}} \sec x \tan x\sqrt{\sec x + 2}\,dx$; $u = \sec x$

e $\int_1^4 \frac{1}{\sqrt{x}(4x - 1)}\,dx$; $u = \sqrt{x}$

P **3** By choosing a suitable substitution, find:

a $\int x(3+2x)^5\,\mathrm{d}x$ **b** $\int \frac{x}{\sqrt{1+x}}\,\mathrm{d}x$ **c** $\int \frac{\sqrt{x^2+4}}{x}\,\mathrm{d}x$

P **4** By choosing a suitable substitution, find the exact values of:

a $\int_2^7 x\sqrt{2+x}\,\mathrm{d}x$ **b** $\int_2^5 \frac{1}{1+\sqrt{x-1}}\,\mathrm{d}x$ **c** $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,\mathrm{d}\theta$

E **5** Using the substitution $u^2 = 4x+1$, or otherwise, find the exact value of $\int_6^{20} \frac{8x}{\sqrt{4x+1}}\,\mathrm{d}x$. **(8 marks)**

E/P **6** Use the substitution $u^2 = \mathrm{e}^x - 2$ to show that $\int_{\ln 3}^{\ln 4} \frac{\mathrm{e}^{4x}}{\mathrm{e}^x - 2}\,\mathrm{d}x = \frac{a}{b} + c\ln d$, where a, b, c and d are integers to be found. **(7 marks)**

E/P **7** Prove that $-\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arccos x + c$. **(5 marks)**

E/P **8** Use the substitution $u = \cos x$ to show

$\int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x\,\mathrm{d}x = \frac{47}{480}$ **(7 marks)**

Hint Use exact trigonometric values to change the limits in x to limits in u.

E/P **9** Using a suitable trigonometric substitution for x, find $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x^2\sqrt{1-x^2}\,\mathrm{d}x$. **(8 marks)**

Challenge

By using a substitution of the form $x = k\sin u$, show that

$$\int \frac{1}{x^2\sqrt{9-x^2}}\,\mathrm{d}x = -\frac{\sqrt{9-x^2}}{9x} + c$$

11.6 Integration by parts

You can rearrange the product rule for differentiation:

Links u and v are both functions of x. ← Section 9.4

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - v\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Differentiating a function and then integrating it leaves the original function unchanged.

So, $\int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x = uv$.

- **This method is called integration by parts.** $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

To use integration by parts you need to write the function you are integrating in the form $u\frac{\mathrm{d}v}{\mathrm{d}x}$

You will have to choose what to set as u and what to set as $\frac{\mathrm{d}v}{\mathrm{d}x}$

Example 16

Find $\int x\cos x\,dx$

Let $I = \int x\cos x\,dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

Using the integration by parts formula:

$I = x\sin x - \int \sin x \times 1\,dx$

$= x\sin x + \cos x + c$

Problem-solving

For expressions like $x\cos x$, $x^2\sin x$ and x^3e^x let u equal the x^n term. When the expression involves $\ln x$, for example $x^2\ln x$, let u equal the $\ln x$ term.

Let $u = x$ and $\frac{dv}{dx} = \cos x$.

Find expressions for u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

Take care to differentiate u but integrate $\frac{dv}{dx}$

Notice that $\int v\frac{du}{dx}\,dx$ is a simpler integral than $\int u\frac{dv}{dx}\,dx$.

Example 17

Find $\int x^2\ln x\,dx$

Let $I = \int x^2\ln x\,dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \Rightarrow v = \frac{x^3}{3}$

$I = \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \times \frac{1}{x}\,dx$

$= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx$

$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$

Since there is a $\ln x$ term, let $u = \ln x$ and $\frac{dv}{dx} = x^2$.

Find expressions for u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

Take care to differentiate u but integrate $\frac{dv}{dx}$

Apply the integration by parts formula.

Simplify the $v\frac{du}{dx}$ term.

It is sometimes necessary to use integration by parts twice, as shown in the following example.

Example 18

Find $\int x^2e^x\,dx$

Let $I = \int x^2e^x\,dx$

$u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = e^x \Rightarrow v = e^x$

So $I = x^2e^x - \int 2xe^x\,dx$

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$\frac{dv}{dx} = e^x \Rightarrow v = e^x$

There is no $\ln x$ term, so let $u = x^2$ and $\frac{dv}{dx} = e^x$.

Find expressions for u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

Take care to differentiate u but integrate $\frac{dv}{dx}$

Apply the integration by parts formula.

Notice that this integral is simpler than I but still not one you can write down. It has a similar structure to I and so you can use integration by parts again with $u = 2x$ and $\frac{dv}{dx} = e^x$.

So $I = x^2e^x - \left(2xe^x - \int 2e^x dx\right)$
$= x^2e^x - 2xe^x + \int 2e^x dx$
$= x^2e^x - 2xe^x + 2e^x + c$

Apply the integration by parts formula for a second time.

Example 19

Evaluate $\int_1^2 \ln x \, dx$, leaving your answer in terms of natural logarithms.

Let $I = \int_1^2 \ln x \, dx = \int_1^2 \ln x \times 1 \, dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$I = [x \ln x]_1^2 - \int_1^2 x \times \frac{1}{x} dx$

$= (2 \ln 2) - (1 \ln 1) - \int_1^2 1 \, dx$

$= 2\ln 2 - [x]_1^2$

$= 2 \ln 2 - (2 - 1)$

$= 2 \ln 2 - 1$

Write the expression to be integrated as $\ln x \times 1$, then $u = \ln x$ and $\frac{dv}{dx} = 1$.
Remember if an expression involves $\ln x$ you should always set $u = \ln x$.

Problem-solving

Apply limits to the uv term and the $\int v \frac{du}{dx} dx$ term separately.

Evaluate the limits on uv and remember $\ln 1 = 0$.

Exercise 11F

1 Find the following integrals.

a $\int x \sin x \, dx$ **b** $\int xe^x \, dx$ **c** $\int x \sec^2 x \, dx$

d $\int x \sec x \tan x \, dx$ **e** $\int \frac{x}{\sin^2 x} dx$

2 Find the following integrals.

a $\int 3 \ln x \, dx$ **b** $\int x \ln x \, dx$ **c** $\int \frac{\ln x}{x^3} dx$

d $\int (\ln x)^2 dx$ **e** $\int (x^2 + 1) \ln x \, dx$

Hint You will need to use these standard results. In your exam they will be given in the formulae booklet:

- $\int \tan x \, dx = \ln|\sec x| + c$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + c$
- $\int \cot x \, dx = \ln|\sin x| + c$
- $\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + c$

3 Find the following integrals.

a $\int x^2e^{-x} dx$ **b** $\int x^2 \cos x \, dx$ **c** $\int 12x^2(3 + 2x)^5 dx$ **d** $\int 2x^2 \sin 2x \, dx$ **e** $\int 2x^2 \sec^2 x \tan x \, dx$

4 Evaluate the following:

a $\int_0^{\ln 2} xe^{2x} dx$ **b** $\int_0^{\frac{\pi}{2}} x \sin x \, dx$ **c** $\int_0^{\frac{\pi}{2}} x \cos x \, dx$ **d** $\int_1^2 \frac{\ln x}{x^2} dx$

e $\int_0^1 4x(1 + x)^3 dx$ **f** $\int_0^{\pi} x \cos \frac{1}{4} x \, dx$ **g** $\int_0^{\frac{\pi}{3}} \sin x \ln(\sec x) \, dx$

(E) **5** **a** Use integration by parts to find $\int x\cos 4x\,dx$. **(3 marks)**

b Use your answer to part **a** to find $\int x^2\sin 4x\,dx$. **(3 marks)**

(E/P) **6** **a** Find $\int\sqrt{8-x}\,dx$. **(2 marks)**

b Using integration by parts, or otherwise, show that

$$\int(x-2)\sqrt{8-x}\,dx = -\tfrac{2}{5}(8-x)^{\frac{3}{2}}(x+2)+c$$ **(6 marks)**

c Hence find $\int_4^7(x-2)\sqrt{8-x}\,dx$. **(2 marks)**

(E/P) **7** **a** Find $\int\sec^2 3x\,dx$. **(3 marks)**

b Using integration by parts, or otherwise, find $\int x\sec^2 3x\,dx$. **(6 marks)**

c Hence show that $\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} x\sec^2 3x\,dx = p\pi - q\ln 3$, finding the exact values of the constants p and q. **(4 marks)**

11.7 Partial fractions

- **Partial fractions can be used to integrate algebraic fractions.**

Using partial fractions enables an expression that looks hard to integrate to be transformed into two or more expressions that are easier to integrate.

Links Make sure you are confident expressing algebraic fractions as partial fractions ← Chapter 1

Example 20

Use partial fractions to find the following integrals.

a $\int\frac{x-5}{(x+1)(x-2)}dx$ **b** $\int\frac{8x^2-19x+1}{(2x+1)(x-2)^2}dx$ **c** $\int\frac{2}{1-x^2}dx$

a $\frac{x-5}{(x+1)(x-2)} \equiv \frac{A}{x+1}+\frac{B}{x-2}$

Split the expression to be integrated into partial fractions.

So $x-5 \equiv A(x-2)+B(x+1)$

Let $x=-1$: $-6 = A(-3)$ so $A = 2$

Let $x=2$: $-3 = B(3)$ so $B = -1$

Let $x = -1$ and 2.

So $\int\frac{x-5}{(x+1)(x-2)}dx$

$= \int\left(\frac{2}{x+1}-\frac{1}{x-2}\right)dx$

Rewrite the integral and integrate each term as in ← Section 11.2

$= 2\ln|x+1| - \ln|x-2| + c$

Remember to use the modulus when using ln in integration.

$= \ln\left|\frac{(x+1)^2}{x-2}\right| + c$

The answer could be left in this form, but sometimes you may be asked to combine the ln terms using the rules of logarithms. ← Year 1, Chapter 14

b Let $I = \int \frac{8x^2 - 19x + 1}{(2x+1)(x-2)^2} dx$

It is sometimes useful to label the integral as I.

$\frac{8x^2 - 19x + 1}{(2x+1)(x-2)^2} \equiv \frac{A}{2x+1} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$

Remember the partial fraction form for a repeated factor in the denominator.

$8x^2 - 19x + 1 \equiv A(x-2)^2 + B(2x+1) + C(2x+1)(x-2)$

Let $x = 2$: $-5 = 0 + 5B + 0$ so $B = -1$

Let $x = -\frac{1}{2}$: $12\frac{1}{2} = \frac{25}{4}A + 0 + 0$ so $A = 2$

Let $x = 0$: Then $1 = 4A + B - 2C$

So $1 = 8 - 1 - 2C$ so $C = 3$

$I = \int \left(\frac{2}{2x+1} - \frac{1}{(x-2)^2} + \frac{3}{x-2}\right) dx$

Rewrite the intergral using the partial fractions. Note that using I saves copying the question again.

$= \frac{2}{2}\ln|2x+1| + \frac{1}{x-2} + 3\ln|x-2| + c$

Don't forget to divide by 2 when integrating $\frac{1}{2x+1}$ and remember that the integral of $\frac{1}{(x-2)^2}$ does not involve ln.

$= \ln|2x+1| + \frac{1}{x-2} + \ln|x-2|^3 + c$

$= \ln|(2x+1)(x-2)^3| + \frac{1}{x-2} + c$

Simplify using the laws of logarithms.

c Let $I = \int \frac{2}{1-x^2} dx$

$\frac{2}{1-x^2} = \frac{2}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$

Remember that $1 - x^2$ can be factorised using the difference of two squares.

$2 = A(1+x) + B(1-x)$

Let $x = -1$ then $2 = 2B$ so $B = 1$

Let $x = 1$ then $2 = 2A$ so $A = 1$

So $I = \int \left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx$

Rewrite the integral using the partial fractions.

$= \ln|1+x| - \ln|1-x| + c$

$= \ln\left|\frac{1+x}{1-x}\right| + c$

Notice the minus sign that comes from integrating $\frac{1}{1-x}$

When the degree of the polynomial in the numerator is greater than or equal to the degree of the denominator, it is necessary to first divide the numerator by the denominator.

Example 21

Find $\int \frac{9x^2 - 3x + 2}{9x^2 - 4} dx$

Let $I = \int \frac{9x^2 - 3x + 2}{9x^2 - 4} dx$

First divide the numerator by $9x^2 - 4$.

$$\begin{array}{r} 1 \\ 9x^2 - 4 \overline{)9x^2 - 3x + 2} \\ \underline{9x^2 \qquad -4} \\ -3x + 6 \end{array}$$

$9x^2 \div 9x^2$ gives 1, so put this on top and subtract $1 \times (9x^2 - 4)$. This leaves a remainder of $-3x + 6$.

so $I = \int\left(1 + \frac{6 - 3x}{9x^2 - 4}\right)dx$

$\frac{6 - 3x}{9x^2 - 4} \equiv \frac{A}{3x - 2} + \frac{B}{3x + 2}$ — Factorise $9x^2 - 4$ and then split into partial fractions.

Let $x = -\frac{2}{3}$ then $8 = -4B$ so $B = -2$

Let $x = \frac{2}{3}$ then $4 = 4A$ so $A = 1$

So $I = \int\left(1 + \frac{1}{3x - 2} - \frac{2}{3x + 2}\right)dx$ — Rewrite the integral using the partial fractions.

$= x + \frac{1}{3}\ln|3x - 2| - \frac{2}{3}\ln|3x + 2| + c$ — Integrate and don't forget the $\frac{1}{3}$

$= x + \frac{1}{3}\ln\left|\frac{3x - 2}{(3x + 2)^2}\right| + c$ — Simplify using the laws of logarithms.

Exercise 11G

1 Use partial fractions to integrate the following:

a $\frac{3x + 5}{(x + 1)(x + 2)}$ **b** $\frac{3x - 1}{(2x + 1)(x - 2)}$ **c** $\frac{2x - 6}{(x + 3)(x - 1)}$ **d** $\frac{3}{(2 + x)(1 - x)}$

2 Find the following integrals.

a $\int\frac{2(x^2 + 3x - 1)}{(x + 1)(2x - 1)}dx$ **b** $\int\frac{x^3 + 2x^2 + 2}{x(x + 1)}dx$ **c** $\int\frac{x^2}{x^2 - 4}dx$ **d** $\int\frac{x^2 + x + 2}{3 - 2x - x^2}dx$

E/P **3** $f(x) = \frac{4}{(2x + 1)(1 - 2x)}, x \neq \pm\frac{1}{2}$

a Given that $f(x) = \frac{A}{2x + 1} + \frac{B}{1 - 2x}$, find the value of the constants A and B. **(3 marks)**

b Hence find $\int f(x)\,dx$, writing your answer as a single logarithm. **(4 marks)**

c Find $\int_1^2 f(x)\,dx$, giving your answer in the form $\ln k$ where k is a rational constant. **(2 marks)**

E/P **4** $f(x) = \frac{17 - 5x}{(3 + 2x)(2 - x)^2}, -\frac{3}{2} < x < 2.$

a Express $f(x)$ in partial fractions. **(4 marks)**

b Hence find the exact value of $\int_0^1 \frac{17 - 5x}{(3 + 2x)(2 - x)^2}dx$, writing your answer in the form $a + \ln b$, where a and b are constants to be found. **(5 marks)**

E/P **5** $f(x) = \frac{9x^2 + 4}{9x^2 - 4}, x \neq \pm\frac{2}{3}$

a Given that $f(x) = A + \frac{B}{3x - 2} + \frac{C}{3x + 2}$, find the values of the constants A, B and C. **(4 marks)**

b Hence find the exact value of $\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{9x^2 + 4}{9x^2 - 4}dx$, writing your answer in the form $a + b\ln c$, where a, b and c are rational numbers to be found. **(5 marks)**

Problem-solving

Simplify the integral as much as possible before substituting your limits.

6 $f(x) = \dfrac{6 + 3x - x^2}{x^3 + 2x^2}, x > 0$

a Express f(x) in partial fractions. **(4 marks)**

b Hence find the exact value of $\int_2^4 \dfrac{6 + 3x - x^2}{x^3 + 2x^2}\,dx$, writing your answer in the form $a + \ln b$, where a and b are rational numbers to be found. **(5 marks)**

7 $\dfrac{32x^2 + 4}{(4x + 1)(4x - 1)} \equiv A + \dfrac{B}{4x + 1} + \dfrac{C}{4x - 1}$

a Find the value of the constants A, B and C. **(4 marks)**

b Hence find the exact value of $\int_1^2 \dfrac{32x^2 + 4}{(4x + 1)(4x - 1)}\,dx$ writing your answer in the form $2 + k\ln m$, giving the values of the rational constants k and m. **(5 marks)**

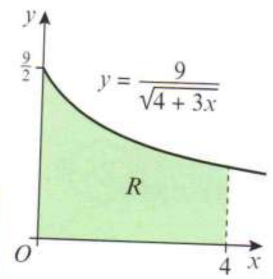
11.8 Finding areas

You need to be able to use the integration techniques from this chapter to find areas under curves.

Example 22

The diagram shows part of the curve $y = \dfrac{9}{\sqrt{4 + 3x}}$

The region R is bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$, as shown in the diagram. Use integration to find the area of R.



$$\text{Area} = \int_0^4 \frac{9}{\sqrt{4 + 3x}}\,dx$$
$$= 9\int_0^4 (4 + 3x)^{-\frac{1}{2}}\,dx$$
$$= 6\left[(4 + 3x)^{\frac{1}{2}}\right]_0^4$$
$$= 6\left((4 + 3 \times 4)^{\frac{1}{2}} - (4 + 3 \times 0)^{\frac{1}{2}}\right)$$
$$= 6(\sqrt{16} - \sqrt{4})$$
$$= 12$$

Remember $\dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$

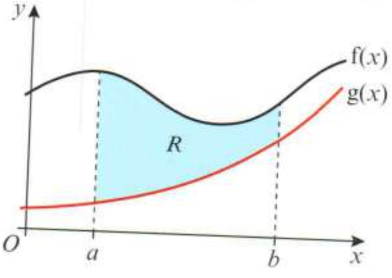
Use the chain rule in reverse. If $y = (4 + 3x)^{\frac{1}{2}}$, $\dfrac{dy}{dx} = \dfrac{3}{2}(4 + 3x)^{-\frac{1}{2}}$. Adjust for the constant.

Substitute the limits.

You don't need to give units when finding areas under graphs in pure maths.

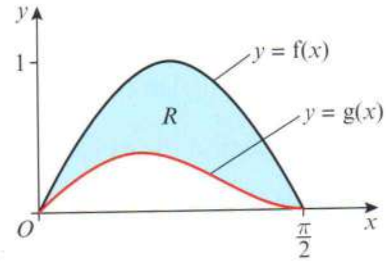
- **The area bounded by two curves can be found using integration:**

  Area of $R = \int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$



Watch out You can only use this formula if the two curves do not intersect between a and b.

Example 23



The diagram shows part of the curves $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$, where $\mathrm{f}(x) = \sin 2x$ and $\mathrm{g}(x) = \sin x \cos^2 x$, $0 \leqslant x \leqslant \frac{\pi}{2}$

The region R is bounded by the two curves. Use integration to find the area of R.

$$\text{Area} = \int_a^b (\mathrm{f}(x) - \mathrm{g}(x))\,\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} (\sin 2x - \sin x \cos^2 x)\,\mathrm{d}x$$

$$= \left[-\tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos^3 x\right]_0^{\frac{\pi}{2}}$$

$$= \left(-\tfrac{1}{2}(-1) + \tfrac{1}{3}(0)\right) - \left(-\tfrac{1}{2} + \tfrac{1}{3}\right) = \tfrac{2}{3}$$

The region R is bounded by two curves.

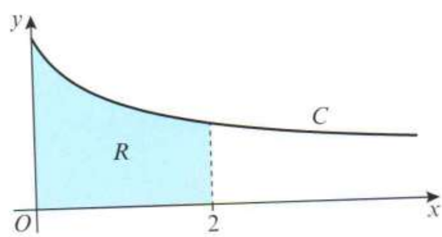
Substitute the limits and functions given in the question.

Online Explore the area between two curves using technology.

You can use integration to find the area under a curve defined by parametric equations. It is often easier to integrate with respect to the parameter.

Links For a parametric curve, x and y are given as functions of a parameter, t. ← Chapter 8

Example 24



The curve C has parametric equations

$x = t(1 + t)$, $y = \frac{1}{1 + t}$, $t \geqslant 0$

Find the exact area of the region R, bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.

$$\text{Area} = \int y\,\mathrm{d}x = \int y\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

$x = t(1 + t)$, so $\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + 2t$

When $x = 0$, $t(1 + t) = 0$, so $\underline{t = 0}$ or $t = -1$

When $x = 2$, $t(1 + t) = 2$

$$t^2 + t - 2 = 0$$

$(t + 2)(t - 1) = 0$, so $t = -2$ or $\underline{t = 1}$

$$\text{So Area} = \int_0^1 y\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_0^1 \frac{1}{1 + t}(1 + 2t)\,\mathrm{d}t$$

$$= \int_0^1 \left(2 - \frac{1}{1 + t}\right)\mathrm{d}t$$

$$= [2t - \ln|1 + t|]_0^1$$

$$= (2 - \ln 2) - (0 - \ln 1)$$

$$= 2 - \ln 2$$

Use a change of variable to write the integral in terms of the parameter, t. Using the chain rule, you can replace $\mathrm{d}x$ with $\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$.

$x = t + t^2$

Watch out You will be integrating **with respect to t** so you need to convert the limits from values of x to values of t. Use the parametric equation for x, and choose solutions that are within the domain of the parameter, $t \geqslant 0$.

Write $\frac{1 + 2t}{1 + t}$ in the form $A + \frac{B}{1 + t}$

Dividing $2t + 1$ by $t + 1$ gives 2 with remainder -1.

Exercise 11H

1 Find the area of the finite region R bounded by the curve with equation $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

a $f(x) = \frac{2}{1+x}$; $a = 0$, $b = 1$ **b** $f(x) = \sec x$; $a = 0$, $b = \frac{\pi}{3}$ **c** $f(x) = \ln x$; $a = 1$, $b = 2$

d $f(x) = \sec x \tan x$; $a = 0$, $b = \frac{\pi}{4}$ **e** $f(x) = x\sqrt{4 - x^2}$; $a = 0$, $b = 2$

2 Find the exact area of the finite region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ where:

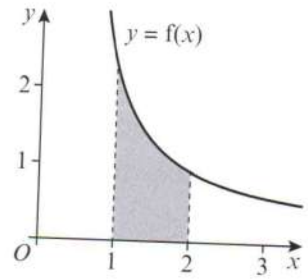
a $f(x) = \frac{4x - 1}{(x+2)(2x+1)}$; $a = 0$, $b = 2$ **b** $f(x) = \frac{x}{(x+1)^2}$; $a = 0$, $b = 2$

c $f(x) = x \sin x$; $a = 0$, $b = \frac{\pi}{2}$ **d** $f(x) = \cos x\sqrt{2\sin x + 1}$; $a = 0$, $b = \frac{\pi}{6}$

e $f(x) = xe^{-x}$; $a = 0$, $b = \ln 2$

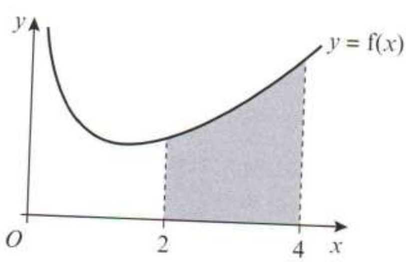
3 The diagram shows a sketch of the curve with equation, $y = f(x)$, where $f(x) = \frac{4x + 3}{(x+2)(2x-1)}$, $x > \frac{1}{2}$

Find the area of the shaded region bounded by the curve, the x-axis and the lines $x = 1$ and $x = 2$. **(7 marks)**



4 The diagram shows a sketch of the curve with equation $y = f(x)$, where $f(x) = e^{0.5x} + \frac{1}{x}$, $x > 0$.

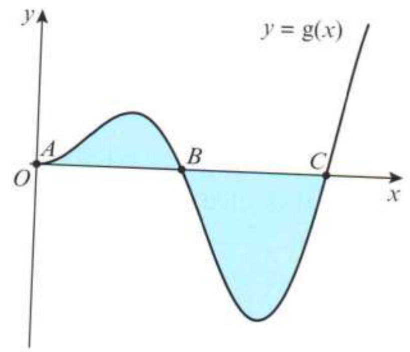
Find the area of the shaded region bounded by the curve, the x-axis and the lines $x = 2$ and $x = 4$. **(7 marks)**



5 The diagram shows a sketch of the curve with equation $y = g(x)$, where $g(x) = x \sin x$.

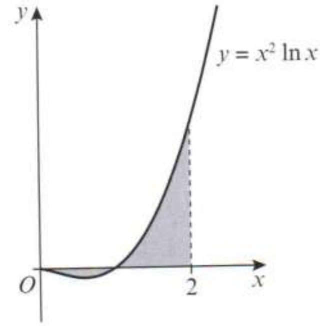
a Write down the coordinates of points A, B and C.

b Find the area of the shaded region.



Watch out Find the area of each region separately and then add the answers. Remember areas cannot be negative, so take the absolute value of any negative area.

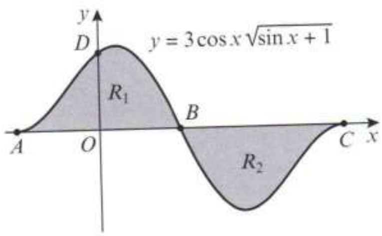
E/P **6** The diagram shows a sketch of the curve with equation $y = x^2 \ln x$. The shaded region is bounded by the curve, the x-axis and the line $x = 2$.



a Use integration by parts to find $\int x^2 \ln x \, dx$. **(3 marks)**

b Hence find the exact area of the shaded region, giving your answer in the form $\frac{2}{3}(a \ln 2 + b)$, where a and b are integers. **(5 marks)**

E/P **7** The diagram shows a sketch of the curve with equation $y = 3\cos x\sqrt{\sin x + 1}$.



a Find the coordinates of the points A, B, C and D. **(3 marks)**

b Use a suitable substitution to find

$\int 3\cos x\sqrt{\sin x + 1} \, dx$ **(5 marks)**

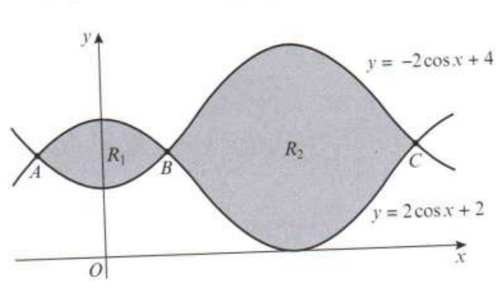
c Show that the regions R_1 and R_2 have the same area, and find the exact value of this area in the form $\sqrt{a}$, where a is a positive integer to be found. **(3 marks)**

P **8** $f(x) = x^2$ and $g(x) = 3x - x^2$

a On the same axes, sketch the graphs of $y = f(x)$ and $y = g(x)$, and find the coordinates of any points of intersection of the two curves.

b Find the area of the finite region bounded by the two curves.

E/P **9** The diagram shows a sketch of part of the curves with equations $y = 2\cos x + 2$ and $y = -2\cos x + 4$.

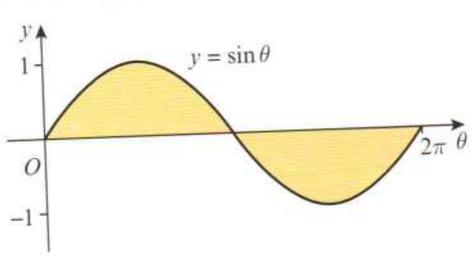


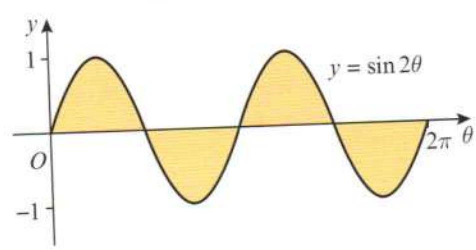
a Find the coordinates of the points A, B and C. **(2 marks)**

b Find the area of region R_1 in the form $a\sqrt{3} + \frac{b\pi}{c}$, where a, b and c are integers to be found. **(4 marks)**

c Show that the ratio of $R_2 : R_1$ can be expressed as $(3\sqrt{3} + 2\pi) : (3\sqrt{3} - \pi)$. **(5 marks)**

P **10** The diagrams show the curves $y = \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$ and $y = \sin 2\theta$, $0 \leqslant \theta \leqslant 2\pi$.
By choosing suitable limits, show that the total shaded area in the first diagram is equal to the total shaded area in the second diagram, and state the exact value of this shaded area.





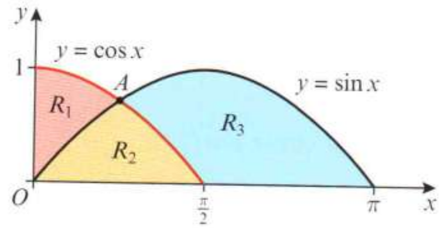
11 The diagram shows parts of the graphs of $y = \sin x$ and $y = \cos x$.

a Find the coordinates of point A.

b Find the areas of:

i R_1 **ii** R_2 **iii** R_3

c Show that the ratio of areas $R_1 : R_2$ can be written as $\sqrt{2} : 2$.

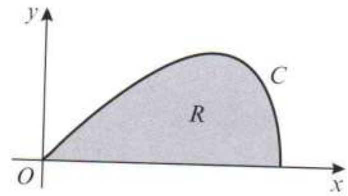


12 The curve C has parametric equations $x = t^3$, $y = t^2$, $t \geqslant 0$. Show that the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is $k\sqrt[3]{2}$, where k is a rational constant to be found.

13 The curve C has parametric equations

$$x = \sin t,\ y = \sin 2t,\ 0 \leqslant t \leqslant \frac{\pi}{2}$$

The finite region R is bounded by the curve and the x-axis. Find the exact area of R. **(6 marks)**



14 This graph shows part of the curve C with parametric equations $x = (t + 1)^2$, $y = \frac{1}{2}t^3 + 3$, $t \geqslant -1$

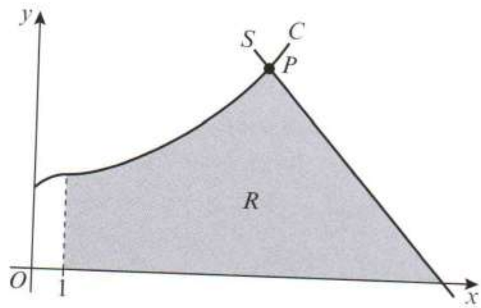
P is the point on the curve where $t = 2$.

The line S is the normal to C at P.

a Find an equation of S. **(5 marks)**

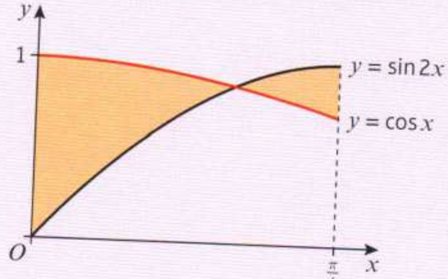
The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration, find the area of R. **(5 marks)**



Challenge

The diagram shows the curves $y = \sin 2x$ and $y = \cos x$, $0 \leqslant x \leqslant \frac{\pi}{4}$

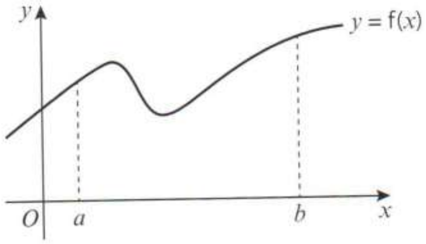


Find the exact value of the total shaded area on the diagram.

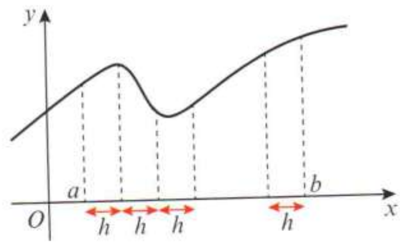
11.9 The trapezium rule

If you cannot integrate a function algebraically, you can use a numerical method to approximate the area beneath a curve.

Consider the curve $y = \mathrm{f}(x)$:

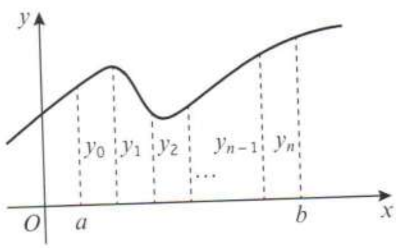


To approximate the area given by $\int_a^b y\,\mathrm{d}x$, you can divide the area up into n equal strips. Each strip will be of width h, where $h = \frac{b-a}{n}$



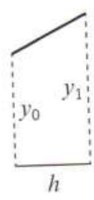
Next you calculate the value of y for each value of x that forms a boundary of one of the strips. So you find y for $x = a$, $x = a + h$, $x = a + 2h$, $x = a + 3h$ and so on up to $x = b$.
You can label these values $y_0, y_1, y_2, y_3, \dots, y_n$.

Hint Notice that for n strips there will be $n + 1$ values of x and $n + 1$ values of y.



Finally you join adjacent points to form n trapezia and approximate the original area by the sum of the areas of these n trapeziums.

You may recall from GCSE maths that the area of a trapezium like this:



is given by $\frac{1}{2}(y_0 + y_1)h$. The required area under the curve is therefore given by:

$$\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h(y_0 + y_1) + \tfrac{1}{2}h(y_1 + y_2) + \dots + \tfrac{1}{2}h(y_{n-1} + y_n)$$

Factorising gives:

$$\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h(y_0 + y_1 + y_1 + y_2 + y_2 \dots + y_{n-1} + y_{n-1} + y_n)$$

or $$\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 \dots + y_{n-1}) + y_n)$$

This formula is given in the formula booklet but you will need to know how to use it.

- **The trapezium rule:**

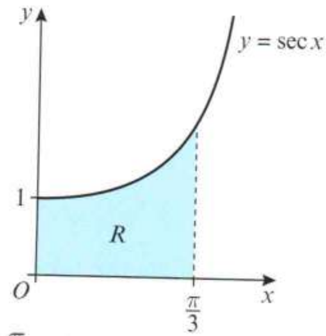
  $$\int_a^b y\,dx \approx \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 \ldots + y_{n-1}) + y_n)$$

  where $h = \dfrac{b-a}{n}$ **and** $y_i = f(a + ih)$

Example 25

The diagram shows a sketch of the curve $y = \sec x$. The finite region R is bounded by the curve, the x-axis, the y-axis and the line $x = \frac{\pi}{3}$

The table shows the corresponding values of x and y for $y = \sec x$.



x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1	1.035			2

a Complete the table with the values of y corresponding to $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$, giving your answers to 3 decimal places.

b Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 2 decimal places.

c Explain with a reason whether your estimate in part **b** will be an underestimate or an overestimate.

a $\sec\frac{\pi}{6} = \dfrac{1}{\cos\frac{\pi}{6}} \approx 1.155$

$\sec\frac{\pi}{4} = \dfrac{1}{\cos\frac{\pi}{4}} \approx 1.414$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1	1.035	1.155	1.414	2

b $I = \int_a^b y\,dx \approx \frac{1}{2}h(y_0 + 2(y_1 + y_2 \ldots + y_{n-1}) + y_n)$

$I \approx \frac{1}{2}\left(\frac{\pi}{12}\right)(1 + 2(1.035 + 1.155 + 1.414) + 2)$

$= \frac{\pi}{24} \times 10.208$

$= 1.336\,224\,075\ldots = 1.34$ (2 d.p.)

Substitute $h = \frac{\pi}{12}$ and the five y-values into the formula.

c The answer would be an overestimate. The graph is convex so the lines connecting two endpoints would be above the curve, giving a greater answer than the real answer.

Online Explore under- and over-estimation when using the trapezium rule, using GeoGebra.

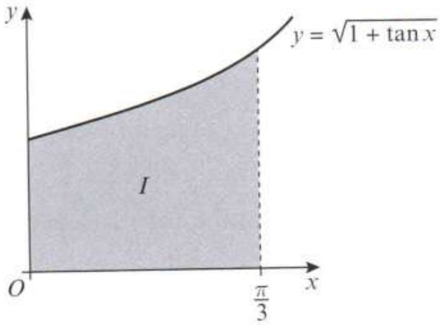
Problem-solving

If $f(x)$ is convex on the interval $[a, b]$ then the trapezium rule will give an overestimate for $\int_a^b f(x)\,dx$. If it is concave then it will give an underestimate.

← Section 9.9

Exercise 11I

(E) **1** The diagram shows a sketch of the curve with equation $y = \sqrt{1 + \tan x}$, $0 \leqslant x \leqslant \frac{\pi}{3}$



a Complete the table with the values for y corresponding to $x = \frac{\pi}{12}$ and $x = \frac{\pi}{4}$ **(1 mark)**

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1		1.2559		1.6529

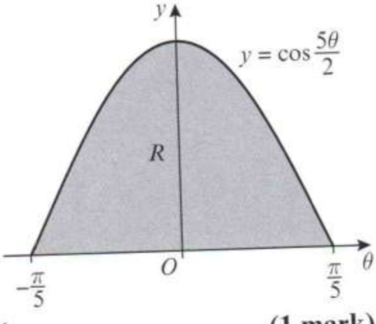
Given that $I = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan x}\, dx$,

b use the trapezium rule:

i with the values of y at $x = 0, \frac{\pi}{6}$ and $\frac{\pi}{3}$ to find an approximate value for I, giving your answer to 4 significant figures; **(3 marks)**

ii with the values of y at $x = 0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$ to find an approximate value for I, giving your answer to 4 significant figures. **(3 marks)**

(E/P) **2** The diagram shows the region R bounded by the x-axis and the curve with equation $y = \cos\frac{5\theta}{2}$, $-\frac{\pi}{5} \leqslant \theta \leqslant \frac{\pi}{5}$



The table shows corresponding values of θ and y for $y = \cos\frac{5\theta}{2}$

θ	$-\frac{\pi}{5}$	$-\frac{\pi}{10}$	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$
y	0		1		0

a Complete the table giving the missing values for y to 4 decimal places. **(1 mark)**

b Using the trapezium rule, with all the values for y in the completed table, find an approximation for the area of R, giving your answer to 3 decimal places. **(4 marks)**

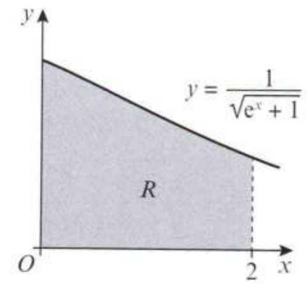
c State, with a reason, whether your approximation in part **b** is an underestimate or an overestimate. **(1 mark)**

d Use integration to find the exact area of R. **(3 marks)**

e Calculate the percentage error in your answer in part **b**. **(2 marks)**

3 The diagram shows a sketch of the curve with equation $y = \dfrac{1}{\sqrt{e^x + 1}}$

The shaded region R is bounded by the curve, the x-axis, the y-axis and the line $x = 2$.

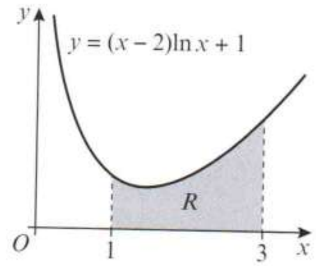


a Complete the table giving values of y to 3 decimal places. **(2 marks)**

x	0	0.5	1	1.5	2
y	0.707	0.614	0.519		0.345

b Use the trapezium rule, with all the values from your table, to estimate the area of the region R, giving your answer to 2 decimal places. **(4 marks)**

4 The diagram shows the curve with equation $y = (x - 2)\ln x + 1$, $x > 0$.



a Complete the table with the values of y corresponding to $x = 2$ and $x = 2.5$. **(1 mark)**

x	1	1.5	2	2.5	3
y	1	0.7973			2.0986

Given that $I = \int_1^3 ((x - 2)\ln x + 1)\,dx$,

b use the trapezium rule

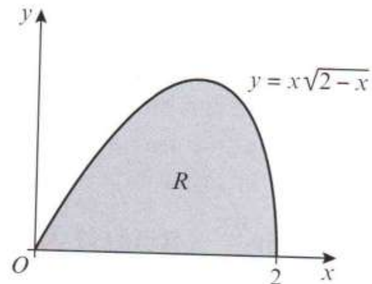
i with values of y at $x = 1$, 2 and 3 to find an approximate value for I, giving your answer to 4 significant figures. **(3 marks)**

ii with values of y at $x = 1$, 1.5, 2, 2.5 and 3 to find another approximate value for I, giving your answer to 4 significant figures. **(3 marks)**

c Use the diagram to explain why an increase in the number of values improves the accuracy of the approximation. **(1 mark)**

d Show by integration, that the exact value of $\int_1^3 ((x - 2)\ln x + 1)\,dx$ is $-\frac{3}{2}\ln 3 + 4$. **(6 marks)**

5 The diagram shows the curve with equation $y = x\sqrt{2 - x}$, $0 \leqslant x \leqslant 2$.



a Complete the table with the value of y corresponding to $x = 1.5$. **(1 mark)**

x	0	0.5	1	1.5	2
y	0	0.6124	1		0

Given that $I = \int_0^2 x\sqrt{2 - x}\,dx$,

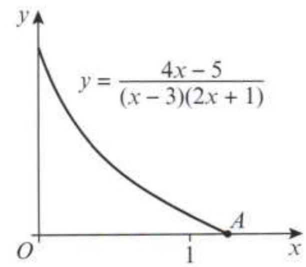
b use the trapezium rule with four strips to find an approximate value for I, giving your answer to 4 significant figures. **(5 marks)**

c By using an appropriate substitution, or otherwise, find the exact value of $\int_0^2 x\sqrt{2 - x}\,dx$, leaving your answer in the form $2^q p$, where p and q are rational constants. **(4 marks)**

d Calculate the percentage error of the approximation in part **b**. **(2 marks)**

E/P **6** The diagram shows part of the curve with equation

$$y = \frac{4x - 5}{(x - 3)(2x + 1)}$$



a Show that the coordinates of point A are $\left(\frac{5}{4}, 0\right)$. **(1 mark)**

b Complete the table with the value of y corresponding to $x = 0.75$. Give your answer to 4 decimal places. **(1 mark)**

x	0	0.25	0.5	0.75	1	1.25
y	1.6667	0.9697	0.6		0.1667	0

Given that $I = \int_0^{\frac{5}{4}} \frac{4x - 5}{(x - 3)(2x + 1)}\,dx$,

c use the trapezium rule with values of y at $x = 0, 0.25, 0.5, 0.75, 1$ and 1.25 to find an approximate value for I, giving your answer to 4 significant figures. **(3 marks)**

d Find the exact value of $\int_0^{\frac{5}{4}} \frac{4x - 5}{(x - 3)(2x + 1)}\,dx$, giving your answer in the form $\ln\left(\frac{a}{b}\right)$. **(4 marks)**

e Calculate the percentage error of the approximation in part **c**. **(2 marks)**

E/P **7** $I = \int_0^3 e^{\sqrt{2x+1}}\,dx$

a Given that $y = e^{\sqrt{2x+1}}$, complete the table of values of y corresponding to $x = 0.5$, 1 and 1.5. **(2 marks)**

x	0	0.5	1	1.5	2	2.5	3
y	2.7183				9.3565	11.5824	14.0940

b Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the original integral, I, giving your answer to 4 significant figures. **(3 marks)**

c Use the substitution $t = \sqrt{2x + 1}$ to show that I may be expressed as $\int_a^b kte^t\,dt$, giving the values of the constants a, b and k. **(5 marks)**

d Use integration by parts to evaluate this integral, and hence find the value of I correct to 4 significant figures. **(4 marks)**

11.10 Solving differential equations

Integration can be used to solve differential equations. In this chapter you will solve first order differential equations by **separating the variables**.

- **When $\frac{dy}{dx} = f(x)g(y)$ you can write**

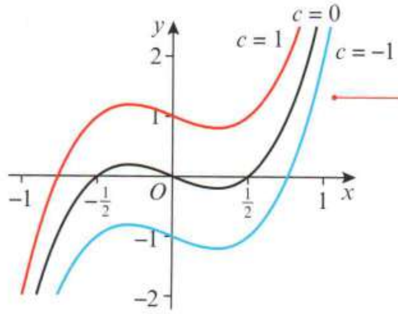
$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Notation A first order differential equation contains nothing higher than a first order derivative, for example $\frac{dy}{dx}$. A second order differential equation would have a term that contains a second order derivative, for example $\frac{d^2y}{dx^2}$

The solution to a differential equation will be a function.

When you integrate to solve a differential equation you still need to include a constant of integration. This gives the **general solution** to the differential equation. It represents a **family** of solutions, all with different constants. Each of these solutions satisfies the original differential equation.

For the first order differential equation $\frac{dy}{dx} = 12x^2 - 1$, the general solution is $y = 4x^3 - x + c$, or $y = x(2x - 1)(2x + 1) + c$.



Each of these curves represents a **particular solution** of the differential equation, for different values of the constant c. Together, the curves form a **family of solutions**.

Online Explore families of solutions using technology.

Example 26

Find a general solution to the differential equation $(1 + x^2)\frac{dy}{dx} = x \tan y$.

$$\frac{dy}{dx} = \frac{x}{1 + x^2}\tan y$$

Write the equation in the form $\frac{dy}{dx} = f(x)g(y)$.

$$\int \frac{1}{\tan y}\,dy = \int \frac{x}{1 + x^2}\,dx$$

Now **separate the variables**: $\frac{1}{g(y)}\,dy = f(x)\,dx$

$$\int \cot y\,dy = \int \frac{x}{1 + x^2}\,dx$$

Use $\cot y = \frac{1}{\tan y}$

$$\ln|\sin y| = \tfrac{1}{2}\ln|1 + x^2| + c$$

$\int \cot x\,dx = \ln|\sin x| + c$

or $\ln|\sin y| = \tfrac{1}{2}\ln|1 + x^2| + \ln k$

Don't forget the $+c$ which can be written as $\ln k$.

$$\ln|\sin y| = \ln|k\sqrt{1 + x^2}|$$

Combining logs.

so $\sin y = k\sqrt{1 + x^2}$

Finally remove the ln. Sometimes you might be asked to give your answer in the form $y = f(x)$. This question did not specify that so it is acceptable to give the answer in this form.

Sometimes you are interested in one specific solution to a differential equation. You can find a **particular solution** to a first-order differential equation if you know **one point** on the curve. This is sometimes called a **boundary condition**.

Example 27

Find the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{-3(y - 2)}{(2x + 1)(x + 2)}$$

given that $x = 1$ when $y = 4$. Leave your answer in the form $y = f(x)$.

Hint The boundary condition in this question is that $x = 1$ when $y = 4$.

$$\int \frac{1}{y-2}\,dy = \int \frac{-3}{(2x+1)(x+2)}\,dx$$

First separate the variables. Make sure the function on the left-hand side is in terms of y only, and the function on the right-hand side is in terms of x only.

$$\frac{-3}{(2x+1)(x+2)} \equiv \frac{A}{(2x+1)} + \frac{B}{(x+2)}$$

Convert the fraction on the RHS to partial fractions.

$$-3 = A(x+2) + B(2x+1)$$

Let $x = -2$: $\quad -3 = -3B$ so $B = 1$

Let $x = -\frac{1}{2}$: $\quad -3 = \frac{3}{2}A$ so $A = -2$

So

$$\int \frac{1}{y-2}\,dy = \int \left(\frac{1}{x+2} - \frac{2}{2x+1}\right) dx$$

Rewrite the integral using the partial fractions.

$$\ln|y-2| = \ln|x+2| - \ln|2x+1| + \ln k$$

Integrate and use $+\ln k$ instead of $+c$.

$$\ln|y-2| = \ln\left|\frac{k(x+2)}{2x+1}\right|$$

Combine ln terms.

$$y - 2 = k\left(\frac{x+2}{2x+1}\right)$$

Remove ln.

$$4 - 2 = k\left(\frac{1+2}{2+1}\right) \Rightarrow k = 2$$

Use the condition $x = 1$ when $y = 4$ by substituting these values into the general solution and solving to find k.

So $\quad y = 2 + 2\left(\frac{x+2}{2x+1}\right)$

$$y = 3 + \frac{3}{2x+1}$$

Substitute $k = 2$ and write the answer in the form $y = \text{f}(x)$ as requested.

Exercise 11J

1 Find general solutions to the following differential equations. Give your answers in the form $y = \text{f}(x)$.

a $\dfrac{dy}{dx} = (1+y)(1-2x)$

b $\dfrac{dy}{dx} = y\tan x$

c $\cos^2 x \dfrac{dy}{dx} = y^2 \sin^2 x$

d $\dfrac{dy}{dx} = 2e^{x-y}$

2 Find particular solutions to the following differential equations using the given boundary conditions.

a $\dfrac{dy}{dx} = \sin x \cos^2 x$; $y = 0$, $x = \dfrac{\pi}{3}$

b $\dfrac{dy}{dx} = \sec^2 x \sec^2 y$; $y = 0$, $x = \dfrac{\pi}{4}$

c $\dfrac{dy}{dx} = 2\cos^2 y \cos^2 x$; $y = \dfrac{\pi}{4}$, $x = 0$

d $\sin y \cos x \dfrac{dy}{dx} = \dfrac{\cos y}{\cos x}$, $y = 0$, $x = 0$

3 a Find the general solution to the differential equation $x^2\dfrac{dy}{dx} = y + xy$, giving your answer in the form $y = \text{g}(x)$.

b Find the particular solution to the differential equation that satisfies the boundary condition $y = e^4$ at $x = -1$.

Hint Begin by factorising the right-hand side of the equation.

4 Given that $x = 0$ when $y = 0$, find the particular solution to the differential equation $(2y + 2yx)\frac{dy}{dx} = 1 - y^2$, giving your answer in the form $y = g(x)$. **(6 marks)**

5 Find the general solution to the differential equation $e^{x+y}\frac{dy}{dx} = 2x + xe^y$, giving your answer in the form $\ln|g(y)| = f(x)$. **(6 marks)**

6 Find the particular solution to the differential equation $(1 - x^2)\frac{dy}{dx} = xy + y$, with boundary condition $y = 6$ at $x = 0.5$. Give your answer in the form $y = f(x)$. **(8 marks)**

7 Find the particular solution to the differential equation $(1 + x^2)\frac{dy}{dx} = x - xy^2$, with boundary condition $y = 2$ at $x = 0$. Give your answer in the form $y = f(x)$. **(8 marks)**

8 Find the particular solution to the differential equation $\frac{dy}{dx} = xe^{-y}$, with boundary condition $y = \ln 2$ at $x = 4$. Give your answer in the form $y = f(x)$. **(8 marks)**

9 Find the particular solution to the differential equation $\frac{dy}{dx} = \cos^2 y + \cos 2x \cos^2 y$, with boundary condition $y = \frac{\pi}{4}$ at $x = \frac{\pi}{4}$. Give your answer in the form $\tan y = f(x)$. **(8 marks)**

10 Given that $y = 1$ at $x = \frac{\pi}{2}$, solve the differential equation $\frac{dy}{dx} = xy \sin x$. **(6 marks)**

11 a Find $\int \frac{3x + 4}{x}\,dx$, $x > 0$. **(2 marks)**

b Given that $y = 16$ at $x = 1$, solve the differential equation $\frac{dy}{dx} = \frac{3x\sqrt{y} + 4\sqrt{y}}{x}$ giving your answer in the form $y = f(x)$. **(6 marks)**

12 a Express $\frac{8x - 18}{(3x - 8)(x - 2)}$ in partial fractions. **(3 marks)**

b Given that $x \geqslant 3$, find the general solution to the differential equation $(x - 2)(3x - 8)\frac{dy}{dx} = (8x - 18)y$ **(5 marks)**

c Hence find the particular solution to this differential equation that satisfies $y = 8$ at $x = 3$, giving your answer in the form $y = f(x)$. **(4 marks)**

13 a Find the general solution of $\frac{dy}{dx} = 2x - 4$.

b On the same axes, sketch three different particular solutions to this differential equation.

14 a Find the general solution to the differential equation $\frac{dy}{dx} = -\frac{1}{(x + 2)^2}$ **(3 marks)**

b On the same axes, sketch three different particular solutions to this differential equation. **(3 marks)**

c Write down the particular solution that passes through the point (8, 3.1). **(1 mark)**

E/P **15** **a** Show that the general solution to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ can be written in the form $x^2 + y^2 = c$. **(3 marks)**

b On the same axes, sketch three different particular solutions to this differential equation. **(3 marks)**

c Write down the particular solution that passes through the point (0, 7). **(1 mark)**

11.11 Modelling with differential equations

Differential equations can be used to model real-life situations.

Example 28

The rate of increase of a population P of microorganisms at time t, in hours, is given by

$$\frac{dP}{dt} = 3P$$

Initially the population was of size 8.

a Find a model for P in the form $P = Ae^{3t}$, stating the value of A.

b Find, to the nearest hundred, the size of the population at time $t = 2$.

c Find the time at which the population will be 1000 times its starting value.

d State one limitation of this model for large values of t.

a $\frac{dP}{dt} = 3P$

$\int \frac{1}{P}\,dP = \int 3\,dt$ — Integrate this function by separating the variables.

$\ln P = 3t + c$

$P = e^{3t+c} = e^{3t} \times e^{c}$ — Apply the laws of indices.

$P = Ae^{3t}$ — e^c is a constant so write it as A.

$8 = Ae^0 \Rightarrow A = 8$ — You are told that the initial population was 8. This gives you the boundary condition $P = 8$ when $t = 0$.

$P = 8e^{3t}$

b $P = 8e^{3t}$

$P = 8e^{3\times 2} = 8e^6$ — Substitute $t = 2$.

$= 3227.4\ldots \approx 3200$

c $P = 1000 \times 8 = 8000$

$8000 = 8e^{3t}$

$1000 = e^{3t}$

$\ln 1000 = 3t$ — Solve by taking the natural log of both sides of the equation.

$t = \frac{1}{3}\ln 1000$

≈ 2.3 hours $= 2$ h 18 mins

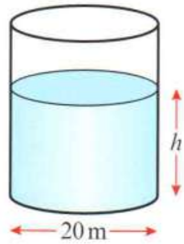
d The population could not increase in size in this way forever due to limitations such as available food or space.

Online Explore the solution to this example graphically using technology.

Watch out When commenting on a model you should always refer to the context of the question.

Example 29

Water in a manufacturing plant is held in a large cylindrical tank of diameter 20 m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.



a Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

b Show that the general solution to this differential equation may be written as $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 27 m. 10 minutes later, the height is 8 m.

c Find the values of the constants P and Q.

d Find the time in minutes when the water is at a depth of 1 m.

a $V = \pi r^2 h = 100\pi h$

Use the formula for the volume of a cylinder. The diameter is 20, so the radius is 10.

$$\frac{dV}{dh} = 100\pi$$

$$\frac{dV}{dt} = -c\sqrt[3]{V} = -c\sqrt[3]{100\pi h}$$

Problem-solving

You need to use the information given in the question to construct a mathematical model. Water flows out at a rate proportional to the cube root of the volume. $\frac{dV}{dt}$ is negative as the water is flowing out of the tank, so the volume is decreasing.

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

Use the chain rule to find $\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{1}{100\pi} \times \left(-c\sqrt[3]{100\pi h}\right)$$

Substitute for $\frac{dh}{dV}$ and $\frac{dV}{dt}$

$\frac{dh}{dV} = \frac{1}{\frac{dV}{dh}} = \frac{1}{100\pi}$

$$= \left(\frac{-c\sqrt[3]{100\pi}}{100\pi}\right)\sqrt[3]{h}$$

So $\frac{dh}{dt} = -k\sqrt[3]{h}$, where $k = \frac{c\sqrt[3]{100\pi}}{100\pi}$

c was the constant of proportionality and π is constant so $\frac{c \times \sqrt[3]{100\pi}}{100\pi} = k$ is a constant.

b $\int h^{-\frac{1}{3}}\,dh = -\int k\,dt$

Integrate this function by separating the variables.

$$\tfrac{3}{2}h^{\frac{2}{3}} = -kt + c$$

$$h^{\frac{2}{3}} = -\tfrac{2}{3}kt + \tfrac{2}{3}c$$

$$h^{\frac{2}{3}} = -Qt + P$$

Let $Q = \frac{2}{3}k$ and $P = \frac{2}{3}c$

$$h = (P - Qt)^{\frac{3}{2}}$$

c $t = 0$, $h = 27$

$27 = P^{\frac{3}{2}} \Rightarrow P = 9$

$t = 10$, $h = 8$

$8 = (9 - 10Q)^{\frac{3}{2}}$

$4 = 9 - 10Q$

$Q = \frac{1}{2}$

Use the boundary conditions to find the values of P and Q. If there are two boundary conditions then you should consider the initial condition (when $t = 0$) first.

d $h = \left(9 - \frac{1}{2}t\right)^{\frac{3}{2}}$

$1 = \left(9 - \frac{1}{2}t\right)^{\frac{3}{2}}$

$1 = 9 - \frac{1}{2}t$

$t = 16$ minutes

Set $h = 1$ and solve the resulting equation to find the corresponding value of t.

Exercise 11K

E/P **1** The rate of increase of a population P of rabbits at time t, in years, is given by $\frac{dP}{dt} = kP, k > 0$. Initially the population was of size 200.

a Solve the differential equations giving P in terms of k and t. **(3 marks)**

b Given that $k = 3$, find the time taken for the population to reach 4000. **(4 marks)**

c State a limitation of this model for large values of t. **(1 mark)**

E/P **2** The mass M at time t of the leaves of a certain plant varies according to the differential equation

$$\frac{dM}{dt} = M - M^2$$

a Given that at time $t = 0$, $M = 0.5$, find an expression for M in terms of t. **(5 marks)**

b Find a value of M when $t = \ln 2$. **(2 marks)**

c Explain what happens to the value of M as t increases. **(1 mark)**

E/P **3** The thickness of ice x, in cm, on a pond is increasing at a rate that is inversely proportional to the square of the existing thickness of ice. Initially, the thickness is 1 cm. After 20 days, the thickness is 2 cm.

a Show that the thickness of ice can be modelled by the equation $x = \sqrt[3]{\frac{7}{20}t + 1}$. **(7 marks)**

b Find the time taken for the ice to increase in thickness from 2 cm to 3 cm. **(2 marks)**

E/P **4** A mug of tea, with a temperature T °C is made and left to cool in a room with a temperature of 25 °C. The rate at which the tea cools is proportional to the difference in temperature between the tea and the room.

a Show that this process can be described by the differential equation $\frac{dT}{dt} = -k(T - 25)$, explaining why k is a positive constant. **(3 marks)**

Initially the tea is at a temperature of 85 °C. 10 minutes later the tea is at 55 °C.

b Find the temperature, to 1 decimal place, of the tea after 15 minutes. **(7 marks)**

E/P **5** The rate of change of the surface area of a drop of oil, A mm^2, at time t minutes can be modelled by the equation $\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{10t^2}$

Given that the surface area of the drop is 1 mm^2 at $t = 1$,

a find an expression for A in terms of t **(7 marks)**

b show that the surface area of the drop cannot exceed $\frac{400}{361}$ mm^2. **(2 marks)**

6 A bath tub is modelled as a cuboid with a base area of $6000\,\text{cm}^2$. Water flows into the bath tub from a tap at a rate of $12\,000\,\text{cm}^3/\text{min}$. At time t minutes, the depth of water in the bath tub is h cm. Water leaves the bottom of the bath through an open plughole at a rate of $500h\,\text{cm}^3/\text{min}$.

a Show that t minutes after the tap has been opened, $60\dfrac{dh}{dt} = 120 - 5h$. **(3 marks)**

When $t = 0$, $h = 6\,\text{cm}$.

b Find the value of t when $h = 10\,\text{cm}$. **(5 marks)**

7 a Express $\dfrac{1}{P(10\,000 - P)}$ using partial fractions. **(3 marks)**

The deer population, P, in a reservation can be modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{20\,000}P(10\,000 - P)$$

where t is the time in years since the study began. Given that the initial deer population is 2500,

b solve the differential equation giving your answer in the form $P = \dfrac{a}{b + ce^{-0.5t}}$ **(6 marks)**

c Find the maximum deer population according to the model. **(2 marks)**

8 Liquid is pouring into a container at a constant rate of $40\,\text{cm}^3\,\text{s}^{-1}$ and is leaking from the container at a rate of $\frac{1}{4}V\,\text{cm}^3\,\text{s}^{-1}$, where $V\,\text{cm}^3$ is the volume of liquid in the container.

a Show that $-4\dfrac{dV}{dt} = V - 160$. **(2 marks)**

Given that $V = 5000$ when $t = 0$,

b find the solution to the differential equation in the form $V = a + be^{-\frac{1}{4}t}$, where a and b are constants to be found **(7 marks)**

c write down the limiting value of V as $t \to \infty$. **(1 mark)**

9 Fossils are aged using a process called carbon dating. The amount of carbon remaining in a fossil, R, decreases over time, t, measured in years. The rate of decrease of carbon is proportional to the remaining carbon.

a Given that initially the amount of carbon is R_0, show that $R = R_0e^{-kt}$ **(4 marks)**

It is known that the half-life of carbon is 5730 years. This means that after 5730 years the amount of carbon remaining has reduced by half.

b Find the exact value of k. **(3 marks)**

c A fossil is found with 10% of its expected carbon remaining. Determine the age of the fossil to the nearest year. **(3 marks)**

Mixed exercise 11

1 By choosing a suitable method of integration, find:

a $\int(2x - 3)^7\,dx$

b $\int x\sqrt{4x - 1}\,dx$

c $\int \sin^2 x \cos x\,dx$

d $\int x \ln x\,dx$

e $\int \dfrac{4\sin x \cos x}{4 - 8\sin^2 x}\,dx$

f $\int \dfrac{1}{3 - 4x}\,dx$

2 By choosing a suitable method, evaluate the following definite integrals. Write your answers as exact values.

a $\int_{-3}^{0} x(x^2+3)^5\,dx$

b $\int_{0}^{\frac{\pi}{4}} x\sec^2 x\,dx$

c $\int_{1}^{4}\left(16x^{\frac{3}{2}}-\frac{2}{x}\right)dx$

d $\int_{\frac{\pi}{12}}^{\frac{\pi}{3}}(\cos x+\sin x)(\cos x-\sin x)\,dx$

e $\int_{1}^{4}\frac{4}{16x^2+8x-3}\,dx$

f $\int_{0}^{\ln 2}\frac{1}{1+e^x}\,dx$

E/P **3** **a** Show that $\int_{1}^{e}\frac{1}{x^2}\ln x\,dx = 1-\frac{2}{e}$ **(5 marks)**

b Given that $p>1$, show that $\int_{1}^{p}\frac{1}{(x+1)(2x-1)}\,dx=\frac{1}{3}\ln\frac{4p-2}{p+1}$ **(5 marks)**

E/P **4** Given $\int_{\frac{1}{2}}^{b}\left(\frac{2}{x^3}-\frac{1}{x^2}\right)dx=\frac{9}{4}$, find the value of b. **(4 marks)**

E/P **5** Given $\int_{0}^{\theta}\cos x\sin^3 x\,dx=\frac{9}{64}$, where $\theta>0$, find the smallest possible value of θ. **(4 marks)**

E **6** Using the substitution $t^2=x+1$, where $x>-1$,

a find $\int\frac{x}{\sqrt{x+1}}\,dx$. **(5 marks)**

b Hence evaluate $\int_{0}^{3}\frac{x}{\sqrt{x+1}}\,dx$. **(2 marks)**

E **7** **a** Use integration by parts to find $\int x\sin 8x\,dx$. **(4 marks)**

b Use your answer to part **a** to find $\int x^2\cos 8x\,dx$. **(4 marks)**

E/P **8** $f(x)=\frac{5x^2-8x+1}{2x(x-1)^2}$

a Given that $f(x)=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$, find the values of the constants A, B and C. **(4 marks)**

b Hence find $\int f(x)\,dx$. **(4 marks)**

c Hence show that $\int_{4}^{9}f(x)\,dx=\ln\left(\frac{32}{3}\right)-\frac{5}{24}$ **(4 marks)**

E/P **9** Given that $y=x^{\frac{3}{2}}+\frac{48}{x}$, $x>0$,

a find the value of x and the value of y when $\frac{dy}{dx}=0$. **(3 marks)**

b Show that the value of y which you found is a minimum. **(2 marks)**

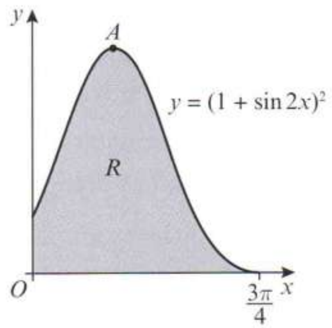
The finite region R is bounded by the curve with equation $y=x^{\frac{3}{2}}+\frac{48}{x}$, the lines $x=1$, $x=4$ and the x-axis.

c Find, by integration, the area of R giving your answer in the form $p+q\ln r$, where the numbers p, q and r are constants to be found. **(4 marks)**

E/P **10** **a** Find $\int x^2\ln 2x\,dx$. **(6 marks)**

b Hence show that the exact value of $\int_{\frac{1}{2}}^{3}x^2\ln 2x\,dx$ is $9\ln 6-\frac{215}{72}$ **(4 marks)**

E/P **11** The diagram shows the graph of $y = (1 + \sin 2x)^2, 0 \leqslant x \leqslant \frac{3\pi}{4}$



a Show that $(1 + \sin 2x)^2 \equiv \frac{1}{2}(3 + 4\sin 2x - \cos 4x)$. **(4 marks)**

b Hence find the area of the shaded region R. **(4 marks)**

c Find the coordinates of A, the turning point on the graph. **(3 marks)**

E **12** **a** Find $\int xe^{-x}\,dx$. **(4 marks)**

b Given that $y = \frac{\pi}{4}$ at $x = 0$, solve the differential equation

$$e^x\frac{dy}{dx} = \frac{x}{\sin 2y}$$ **(4 marks)**

E **13** **a** Find $\int x\sin 2x\,dx$. **(5 marks)**

b Given that $y = 0$ at $x = \frac{\pi}{4}$, solve the differential equation $\frac{dy}{dx} = x\sin 2x\cos^2 y$. **(5 marks)**

E/P **14** **a** Obtain the general solution to the differential equation

$$\frac{dy}{dx} = xy^2, y > 0$$ **(3 marks)**

b Given also that $y = 1$ at $x = 1$, show that

$$y = \frac{2}{3 - x^2}, -\sqrt{3} < x < \sqrt{3}$$

is a particular solution to the differential equation. **(3 marks)**

The curve C has equation $y = \frac{2}{3 - x^2}, x \neq \pm\sqrt{3}$

c Write down the gradient of C at the point $(1, 1)$. **(1 mark)**

d Hence write down an equation of the tangent to C at the points $(1, 1)$, and find the coordinates of the point where it again meets the curve. **(4 marks)**

E **15** **a** Using the substitution $u = 1 + 2x$, or otherwise, find

$$\int\frac{4x}{(1 + 2x)^2}\,dx, x \neq -\tfrac{1}{2}$$ **(5 marks)**

b Given that $y = \frac{\pi}{4}$ when $x = 0$, solve the differential equation

$$(1 + 2x)^2\frac{dy}{dx} = \frac{x}{\sin^2 y}$$ **(5 marks)**

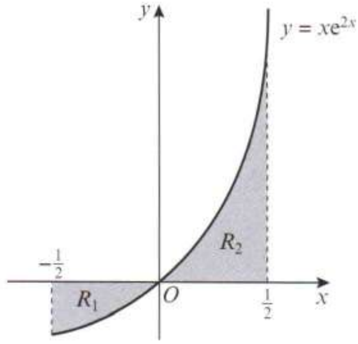
E/P **16** The diagram shows the curve with equation $y = xe^{2x}$, $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$

The finite region R_1 bounded by the curve, the x-axis and the line $x = -\frac{1}{2}$ has area A_1.

The finite region R_2 bounded by the curve, the x-axis and the line $x = \frac{1}{2}$ has area A_2.

a Find the exact values of A_1 and A_2 by integration. **(6 marks)**

b Show that $A_1 : A_2 = (e - 2) : e$. **(4 marks)**



E **17** **a** Find $\int x^2 e^{-x}\,dx$. **(5 marks)**

b Use your answer to part **a** to find the solution to the differential equation $\frac{dy}{dx} = x^2 e^{3y - x}$, given that $y = 0$ when $x = 0$. Express your answer in the form $y = f(x)$. **(7 marks)**

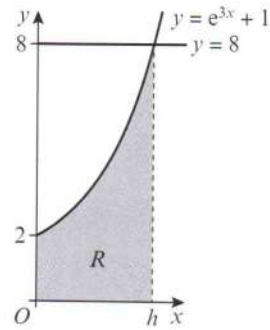
E/P **18** The diagram shows part of the curve $y = e^{3x} + 1$ and the line $y = 8$.

The curve and the line intersect at the point $(h, 8)$.

a Find h, giving your answer in terms of natural logarithms. **(3 marks)**

The region R is bounded by the curve, the x-axis, the y-axis and the line $x = h$.

b Use integration to show the area of R is $2 + \frac{1}{3}\ln 7$. **(5 marks)**



E **19** **a** Given that

$$\frac{x^2}{x^2 - 1} \equiv A + \frac{B}{x - 1} + \frac{C}{x + 1}$$

find the values of the constants A, B and C. **(4 marks)**

b Given that $x = 2$ at $t = 1$, solve the differential equation

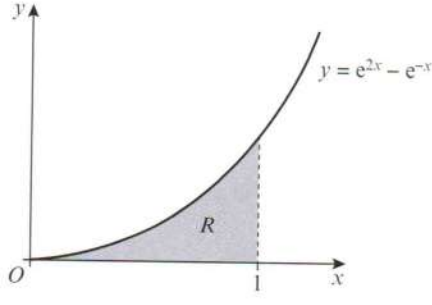
$$\frac{dx}{dt} = 2 - \frac{2}{x^2},\ x > 1$$

You do not need to simplify your final answer. **(7 marks)**

E/P **20** The curve with equation $y = e^{2x} - e^{-x}$, $0 \leqslant x \leqslant 1$, is shown in the diagram. The finite region enclosed by the curve, the x-axis and the line $x = 1$ is shaded.

The table below shows the corresponding values of x and y with the y values given to 5 decimal places as appropriate.

x	0	0.25	0.5	0.75	1
y	0	0.86992	2.11175		7.02118



a Complete the table with the missing value for y. Give your answer to 5 decimal places. **(1 mark)**

b Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R, giving your answer to 4 decimal places. **(3 marks)**

c State, with a reason, whether your answer to part **b** is an overestimate or an underestimate. **(1 mark)**

d Use integration to find the exact value of R. Write your answer in the form $\frac{e^3 + Pe + Q}{2e}$ where P and Q are constants to be found. **(6 marks)**

e Find the percentage error in the answer to part **b**. **(2 marks)**

(P) **21** The rate, in $cm^3 s^{-1}$, at which oil is leaking from an engine sump at any time t seconds is proportional to the volume of oil, $V cm^3$, in the sump at that instant. At time $t = 0$, $V = A$.

a By forming and integrating a differential equation, show that

$$V = Ae^{-kt}$$

where k is a positive constant. **(5 marks)**

b Sketch a graph to show the relation between V and t. **(2 marks)**

Given further that $V = \frac{1}{2}A$ at $t = T$,

c show that $kT = \ln 2$. **(3 marks)**

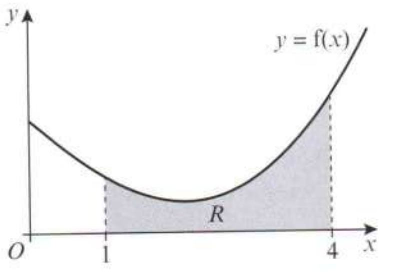
(P) **22** **a** Show that the general solution to the differential equation $\frac{dy}{dx} = \frac{x}{k - y}$ can be written in the form $x^2 + (y - k)^2 = c$. **(4 marks)**

b Describe the family of curves that satisfy this differential equation when $k = 2$. **(2 marks)**

(P) **23** The diagram shows a sketch of the curve $y = f(x)$, where $f(x) = \frac{1}{5}x^2 \ln x - x + 2$, $x > 0$.

The region R, shown in the diagram, is bounded by the curve, the x-axis and the lines with equations $x = 1$ and $x = 4$.

The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate.



x	1	1.5	2	2.5	3	3.5	4
y	1	0.6825	0.5545	0.6454		1.5693	2.4361

a Complete the table with the missing value of y. **(1 mark)**

b Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R, giving your answer to 3 decimal places. **(3 marks)**

c Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of R. **(1 mark)**

d Show that the exact area of R can be written in the form $\frac{a}{b} + \frac{c}{d}\ln e$, where a, b, c, d and e are integers. **(6 marks)**

e Find the percentage error in the answer in part **b**. **(2 marks)**

(E) **24** **a** Find $\int x(1 + 2x^2)^5 \, dx$. **(3 marks)**

b Given that $y = \frac{\pi}{8}$ at $x = 0$, solve the differential equation

$$\frac{dy}{dx} = x(1 + 2x^2)^5 \cos^2 2y$$ **(5 marks)**

E/P **25** By using an appropriate trigonometric substitution, find $\int \frac{1}{1+x^2}\,dx$. **(5 marks)**

E/P **26** Obtain the solution to

$$x(x+2)\frac{dy}{dx} = y,\ y > 0,\ x > 0$$

for which $y = 2$ at $x = 2$, giving your answer in the form $y^2 = f(x)$. **(7 marks)**

E/P **27** An oil spill is modelled as a circular disc with radius r km and area $A\ \text{km}^2$. The rate of increase of the area of the oil spill, in km^2/day at time t days after it occurs is modelled as:

$$\frac{dA}{dt} = k\sin\left(\frac{t}{3\pi}\right),\ 0 \leqslant t \leqslant 12$$

a Show that $\frac{dr}{dt} = \frac{k}{2\pi r}\sin\left(\frac{t}{3\pi}\right)$ **(2 marks)**

Given that the radius of the spill at time $t = 0$ is 1 km, and the radius of the spill at time $t = \pi^2$ is 2 km:

b find an expression for r^2 in terms of t **(7 marks)**

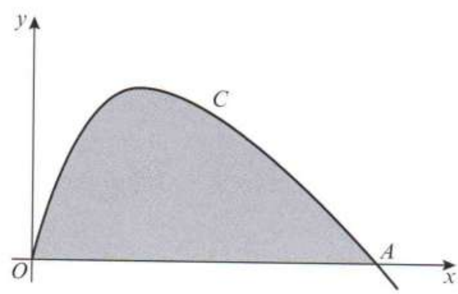
c find the time, in days and hours to the nearest hour, after which the radius of the spill is 1.5 km. **(3 marks)**

E/P **28** The diagram shows the curve C with parametric equations

$$x = 3t^2,\ y = \sin 2t,\ t \geqslant 0.$$

a Write down the value of t at the point A where the curve crosses the x-axis. **(1 mark)**

b Find, in terms of π, the exact area of the shaded region bounded by C and the x-axis. **(6 marks)**

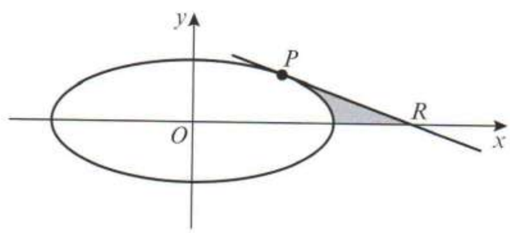


E/P **29** The curve shown has parametric equations

$$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leqslant \theta \leqslant 2\pi$$

a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$ **(3 marks)**

b Find an equation of the tangent to the curve at the point P. **(3 marks)**

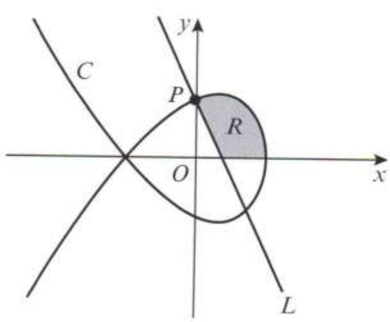


c Find the exact area of the shaded region bounded by the tangent PR, the curve and the x-axis. **(6 marks)**

E/P **30** The curve C has parametric equations

$$x = 1 - t^2,\ y = 2t - t^3,\ t \in \mathbb{R}$$

The line L is a normal to the curve at the point P where the curve intersects the positive y-axis. Find the exact area of the region R bounded by the curve C, the line L and the x-axis, as shown on the diagram. **(7 marks)**



Challenge

Given $f(x) = x^2 - x - 2$, find:

a $\int_{-3}^{3} |f(x)|\,dx$ **b** $\int_{-3}^{3} f(|x|)\,dx$

Hint Draw a sketch of each function.

Summary of key points

1 $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ $\int e^x\,dx = e^x + c$ $\int \frac{1}{x}\,dx = \ln|x| + c$

$\int \cos x\,dx = \sin x + c$ $\int \sin x\,dx = -\cos x + c$ $\int \sec^2 x = \tan x + c$

$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$ $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$ $\int \sec x \tan x\,dx = \sec x + c$

2 $\int f'(ax + b)\,dx = \frac{1}{a} f(ax + b) + c$

3 Trigonometric identities can be used to integrate expressions. This allows an expression that cannot be integrated to be replaced by an identical expression that can be integrated.

4 To integrate expressions of the form $\int k\frac{f'(x)}{f(x)}\,dx$, try $\ln|f(x)|$ and differentiate to check, and then adjust any constant.

5 To integrate an expression of the form $\int k f'(x)(f(x))^n\,dx$, try $(f(x))^{n+1}$ and differentiate to check, and then adjust any constant.

6 Sometimes you can simplify an integral by changing the variable. This process is similar to using the chain rule in differentiation and is called **integration by substitution**.

7 The **integration by parts** formula is given by: $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

8 Partial fractions can be used to integrate algebraic fractions.

9 The area bounded by two curves can be found using integration:

Area of $R = \int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

10 The **trapezium rule** is:

$$\int_a^b y\,dx \approx \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 \ldots + y_{n-1}) + y_n)$$

where $h = \frac{b-a}{n}$ and $y_i = f(a + ih)$.

11 When $\frac{dy}{dx} = f(x)g(y)$ you can write

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$