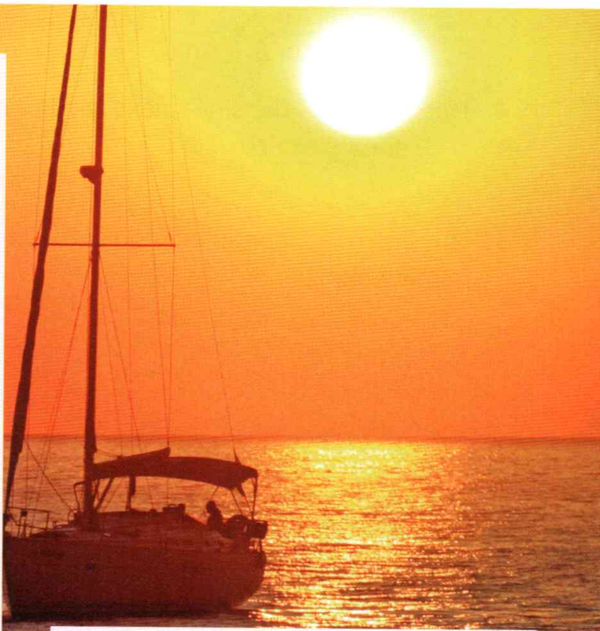
10 Trigonometric identities and equations

Objectives

After completing this chapter you should be able to:

- Calculate the sine, cosine and tangent of any angle → pages 203–208
- Know the exact trigonometric ratios for 30°, 45° and 60° → pages 208–209
- Know and use the relationships $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$ → pages 209–213
- Solve simple trigonometric equations of the forms $\sin\theta = k$, $\cos\theta = k$ and $\tan\theta = k$ → pages 213–217
- Solve more complicated trigonometric equations of the forms $\sin n\theta = k$ and $\sin(\theta \pm \alpha) = k$ and equivalent equations involving cos and tan → pages 217–219
- Solve trigonometric equations that produce quadratics → pages 219–222



Prior knowledge check

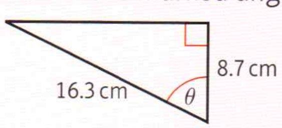
1 **a** Sketch the graph of $y = \sin x$ for $0 \leqslant x \leqslant 540°$.

b How many solutions are there to the equation $\sin x = 0.6$ in the range $0 \leqslant x \leqslant 540°$?

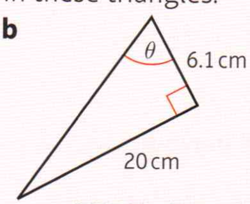
c Given that $\sin^{-1}(0.6) = 36.9°$ (to 3 s.f.), write down three other solutions to the equation $\sin x = 0.6$. ← Section 9.5

2 Work out the marked angles in these triangles.

a



b



← GCSE Mathematics

3 Solve the following equations.

a $2x - 7 = 15$ **b** $3x + 5 = 7x - 4$

c $\sin x = -0.7$ ← GCSE Mathematics

4 Solve the following equations.

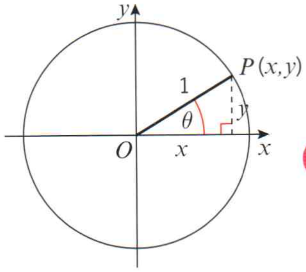
a $x^2 - 4x + 3 = 0$ **b** $x^2 + 8x - 9 = 0$

c $2x^2 - 3x - 7 = 0$ ← Section 2.1

Trigonometric equations can be used to model many real-life situations such as the rise and fall of the tides or the angle of elevation of the sun at different times of the day.

10.1 Angles in all four quadrants

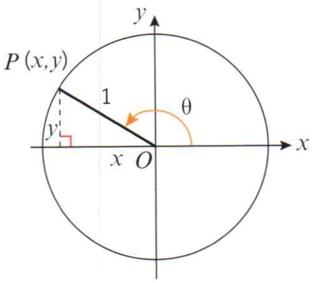
You can use a unit circle with its centre at the origin to help you understand the trigonometric ratios.



- **For a point $P(x, y)$ on a unit circle such that OP makes an angle θ with the positive x-axis:**
  - **$\cos\theta = x =$ x-coordinate of P**
  - **$\sin\theta = y =$ y-coordinate of P**
  - **$\tan\theta = \frac{y}{x} =$ gradient of OP**

Notation A **unit circle** is a circle with a radius of 1 unit.

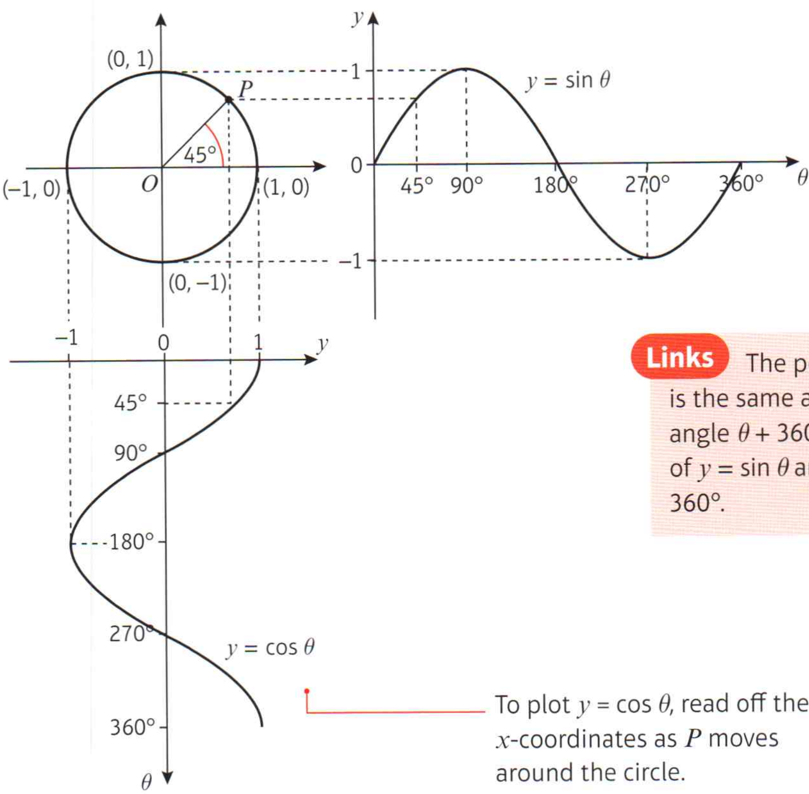
You can use these definitions to find the values of sine, cosine and tangent for any angle θ. You always measure positive angles θ **anticlockwise** from the **positive x-axis**.



When θ is **obtuse**, $\cos\theta$ is negative because the x-coordinate of P is negative.

Online Use GeoGebra to explore the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for any angle θ in a unit circle.

You can also use these definitions to generate the graphs of $y = \sin\theta$ and $y = \cos\theta$.



To plot $y = \sin\theta$, read off the y-coordinates as P moves around the circle.

To plot $y = \cos\theta$, read off the x-coordinates as P moves around the circle.

Links The point P corresponding to an angle θ is the same as the point P corresponding to an angle $\theta + 360°$. This shows you that the graphs of $y = \sin\theta$ and $y = \cos\theta$ are periodic with period 360°. ← Section 9.5

Example 1

Write down the values of:

a sin 90° **b** sin 180° **c** sin 270°

d cos 180° **e** cos (−90)° **f** cos 450°

a $\sin 90° = 1$

b $\sin 180° = 0$

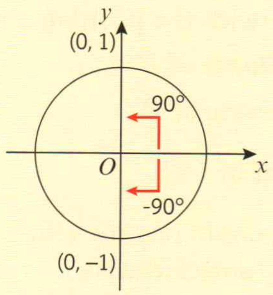
c $\sin 270° = -1$

d $\cos 180° = -1$

e $\cos(-90°) = 0$

f $\cos 450° = 0$

The y-coordinate is 1 when $\theta = 90°$.



If θ is negative, then measure **clockwise** from the positive x-axis.

An angle of −90° is equivalent to a positive angle of 270°. The x-coordinate is 0 when $\theta = -90°$ or 270°.

Example 2

Write down the values of:

a tan 45° **b** tan 135° **c** tan 225°

d tan (−45°) **e** tan 180° **f** tan 90°

a $\tan 45° = 1$

b $\tan 135° = -1$

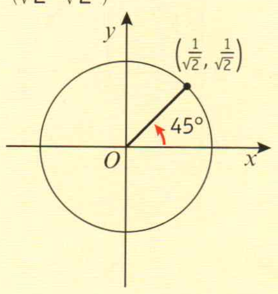
c $\tan 225° = 1$

d $\tan(-45°) = \tan 315° = -1$

e $\tan 180° = 0$

f $\tan 90° =$ undefined

When $\theta = 45°$, the coordinates of OP are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ so the gradient of OP is 1.



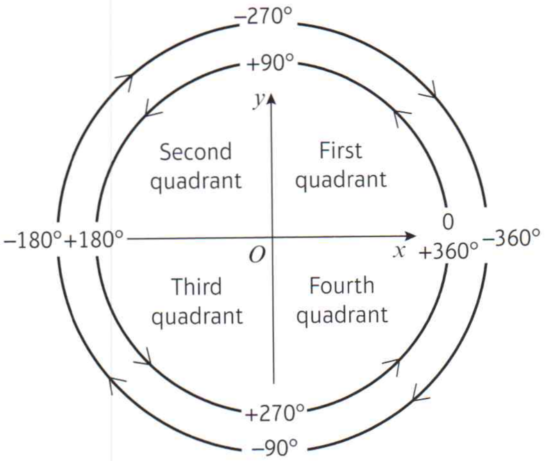
When $\theta = -45°$ the gradient of OP is −1.

When $\theta = 180°$, P has coordinates (−1, 0) so the gradient of $OP = \frac{0}{1} = 0$.

When $\theta = 90°$, P has coordinates (0, 1) so the gradient of $OP = \frac{1}{0}$. This is undefined since you cannot divide by zero.

Links $\tan \theta$ is undefined when $\theta = 270°$ or any other odd multiple of 90°. These values of θ correspond to the asymptotes on the graph of $y = \tan \theta$. ←Section 9.5

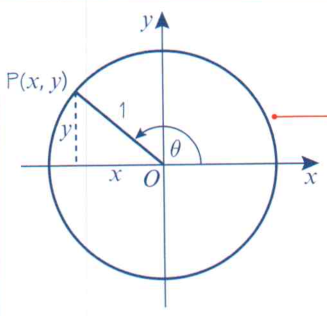
The x-y plane is divided into **quadrants**:



Angles may lie outside the range 0–360°, but they will always lie in one of the four quadrants.
For example, an angle of 600° would be equivalent to 600° − 360° = 240°, so it would lie in the third quadrant.

Example 3

Find the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in the second quadrant.



In the second quadrant, θ is obtuse, or $90° < \theta < 180°$.

Draw a circle, centre O and radius 1, with $P(x, y)$ on the circle in the second quadrant.

As x is −ve and y is +ve in this quadrant

$\sin\theta$ = +ve

$\cos\theta$ = −ve

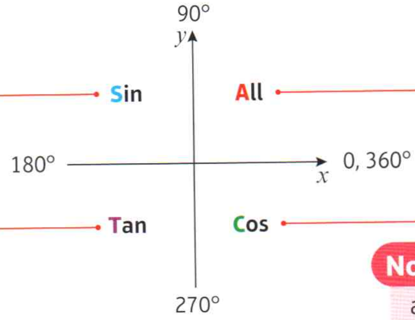
$\tan\theta = \frac{+ve}{-ve} = -ve$

So only $\sin\theta$ is positive.

- **You can use the quadrants to determine whether each of the trigonometric ratios is positive or negative.**

For an angle θ in the second quadrant, only $\sin\theta$ is positive.

For an angle θ in the third quadrant, only $\tan\theta$ is positive.



For an angle θ in the first quadrant, $\sin\theta$, $\cos\theta$ and $\tan\theta$ are all positive.

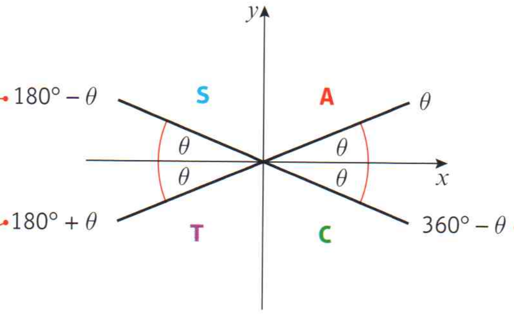
For an angle θ in the fourth quadrant, only $\cos\theta$ is positive.

Notation This diagram is often referred to as a CAST diagram since the word is spelled out from the bottom right going anti-clockwise.

- **You can use these rules to find sin, cos or tan of any positive or negative angle using the corresponding acute angle made with the x-axis, θ.**

$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$



$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$

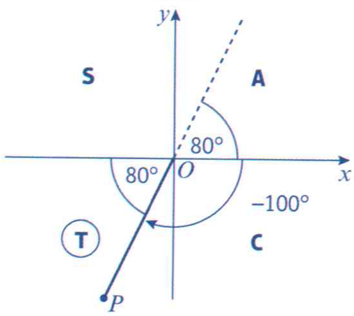
Example 4

Express in terms of trigonometric ratios of acute angles:

a $\sin(-100°)$ **b** $\cos 330°$ **c** $\tan 500°$

For each part, draw diagrams showing the position of OP for the given angle and insert the acute angle that OP makes with the x-axis.

a

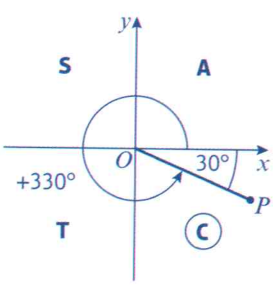


The acute angle made with the x-axis is 80°.

In the third quadrant only tan is +ve, so sin is −ve.

So $\sin(-100)° = -\sin 80°$

b

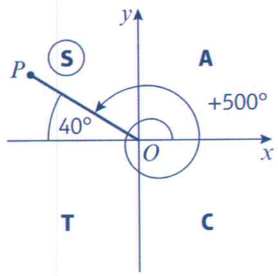


The acute angle made with the x-axis is 30°.

In the fourth quadrant only cos is +ve.

So $\cos 330° = +\cos 30°$

c



The acute angle made with the x-axis is 40°.

In the second quadrant only sin is +ve.

So $\tan 500° = -\tan 40°$

Exercise 10A

1 Draw diagrams to show the following angles. Mark in the acute angle that OP makes with the x-axis.

a −80° **b** 100° **c** 200° **d** 165° **e** −145°
f 225° **g** 280° **h** 330° **i** −160° **j** −280°

2 State the quadrant that OP lies in when the angle that OP makes with the positive x-axis is:

a 400° **b** 115° **c** −210° **d** 255° **e** −100°

3 Without using a calculator, write down the values of:

a $\sin(-90°)$ **b** $\sin 450°$ **c** $\sin 540°$ **d** $\sin(-450°)$ **e** $\cos(-180°)$
f $\cos(-270°)$ **g** $\cos 270°$ **h** $\cos 810°$ **i** $\tan 360°$ **j** $\tan(-180°)$

4 Express the following in terms of trigonometric ratios of acute angles:

a $\sin 240°$ **b** $\sin(-80°)$ **c** $\sin(-200°)$ **d** $\sin 300°$ **e** $\sin 460°$
f $\cos 110°$ **g** $\cos 260°$ **h** $\cos(-50°)$ **i** $\cos(-200°)$ **j** $\cos 545°$
k $\tan 100°$ **l** $\tan 325°$ **m** $\tan(-30°)$ **n** $\tan(-175°)$ **o** $\tan 600°$

5 Given that θ is an acute angle, express in terms of $\sin\theta$:

a $\sin(-\theta)$ **b** $\sin(180° + \theta)$ **c** $\sin(360° - \theta)$
d $\sin(-(180° + \theta))$ **e** $\sin(-180° + \theta)$ **f** $\sin(-360° + \theta)$
g $\sin(540° + \theta)$ **h** $\sin(720° - \theta)$ **i** $\sin(\theta + 720°)$

Hint The results obtained in questions **5** and **6** are true for all values of θ.

6 Given that θ is an acute angle, express in terms of $\cos\theta$ or $\tan\theta$:

a $\cos(180° - \theta)$ **b** $\cos(180° + \theta)$ **c** $\cos(-\theta)$ **d** $\cos(-(180° - \theta))$
e $\cos(\theta - 360°)$ **f** $\cos(\theta - 540°)$ **g** $\tan(-\theta)$ **h** $\tan(180° - \theta)$
i $\tan(180° + \theta)$ **j** $\tan(-180° + \theta)$ **k** $\tan(540° - \theta)$ **l** $\tan(\theta - 360°)$

Challenge

a Prove that $\sin(180° - \theta) = \sin\theta$

b Prove that $\cos(-\theta) = \cos\theta$

c Prove that $\tan(180° - \theta) = -\tan\theta$

Problem-solving

Draw a diagram showing the positions of θ and $180° - \theta$ on the unit circle.

10.2 Exact values of trigonometric ratios

You can find sin, cos and tan of 30°, 45° and 60° exactly using triangles.

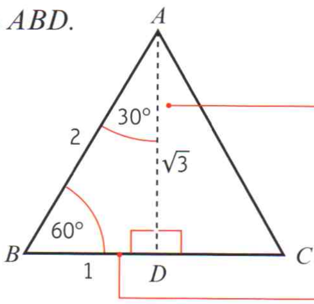
Consider an **equilateral** triangle ABC of side 2 units.

Draw a perpendicular from A to meet BC at D.

Apply the trigonometric ratios in the right-angled triangle ABD.

- $\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

  $\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \sqrt{3}$

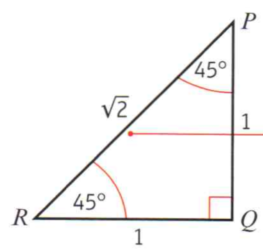


$AD = \sqrt{2^2 - 1^2} = \sqrt{3}$

$BD = 1$ unit

Consider an **isosceles right-angled** triangle PQR with $PQ = RQ = 1$ unit.

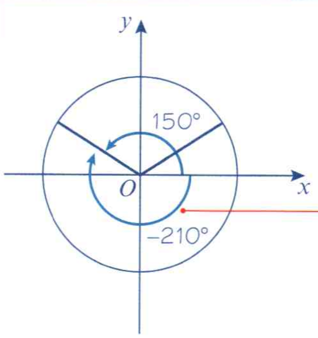
- $\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\tan 45° = 1$



$PR = \sqrt{1^2 + 1^2} = \sqrt{2}$

Example 5

Find the exact value of $\sin(-210°)$.



$\sin(-210°) = \sin(150°)$

$\sin(-210°) = \sin(150°) = \sin(30°) = \frac{1}{2}$

Use $\sin(180° - \theta) = \sin\theta$

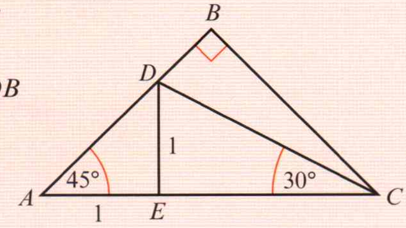
Exercise 10B

1 Express the following as trigonometric ratios of either 30°, 45° or 60°, and hence find their exact values.

a sin 135°	**b** sin (−60°)	**c** sin 330°	**d** sin 420°	**e** sin (−300°)
f cos 120°	**g** cos 300°	**h** cos 225°	**i** cos (−210°)	**j** cos 495°
k tan 135°	**l** tan (−225°)	**m** tan 210°	**n** tan 300°	**o** tan (−120°)

Challenge

The diagram shows an isosceles right-angled triangle ABC. $AE = DE = 1$ unit. Angle $ACD = 30°$.



a Calculate the exact lengths of

i CE **ii** DC **iii** BC **iv** DB

b State the size of angle BCD.

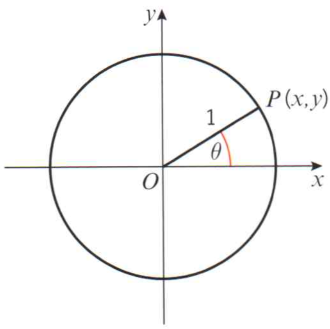
c Hence find exact values for

i sin 15° **ii** cos 15°

10.3 Trigonometric identities

You can use the definitions of sin, cos and tan, together with Pythagoras' theorem, to find two useful identities.

The unit circle has equation $x^2 + y^2 = 1$.



Links The equation of a circle with radius r and centre at the origin is $x^2 + y^2 = r^2$. ← Section 6.2

Since $\cos\theta = x$ and $\sin\theta = y$, it follows that $\cos^2\theta + \sin^2\theta = 1$.

- **For all values of θ, $\sin^2\theta + \cos^2\theta \equiv 1$.**

Since $\tan\theta = \frac{y}{x}$ it follows that $\tan\theta = \frac{\sin\theta}{\cos\theta}$

- **For all values of θ such that $\cos\theta \neq 0$, $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$**

You can use these two **identities** to simplify trigonometrical expressions and complete proofs.

Notation These results are called trigonometric identities. You use the $\equiv$ symbol instead of $=$ to show that they are always true for all values of θ (subject to any conditions given).

Watch out $\tan\theta$ is undefined when the denominator $= 0$. This occurs when $\cos\theta = 0$, so when $\theta = \ldots -90°, 90°, 270°, 450°, \ldots$

Example 6

Simplify the following expressions:

a $\sin^2 3\theta + \cos^2 3\theta$ **b** $5 - 5\sin^2\theta$ **c** $\dfrac{\sin 2\theta}{\sqrt{1-\sin^2 2\theta}}$

a $\sin^2 3\theta + \cos^2 3\theta = 1$

$\sin^2\theta + \cos^2\theta = 1$, with θ replaced by 3θ.

b $5 - 5\sin^2\theta = 5(1 - \sin^2\theta)$

Always look for factors.

$= 5\cos^2\theta$

$\sin^2\theta + \cos^2\theta = 1$, so $1 - \sin^2\theta = \cos^2\theta$.

c $\dfrac{\sin 2\theta}{\sqrt{1-\sin^2 2\theta}} = \dfrac{\sin 2\theta}{\sqrt{\cos^2 2\theta}}$

$\sin^2 2\theta + \cos^2 2\theta = 1$, so $1 - \sin^2 2\theta = \cos^2 2\theta$.

$= \dfrac{\sin 2\theta}{\cos 2\theta}$

$= \tan 2\theta$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, so $\dfrac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$.

Example 7

Prove that $\dfrac{\cos^4\theta - \sin^4\theta}{\cos^2\theta} \equiv 1 - \tan^2\theta$

Problem-solving

When you have to prove an identity like this you may quote the basic identities like '$\sin^2 + \cos^2 \equiv 1$'.

$\text{LHS} \equiv \dfrac{\cos^4\theta - \sin^4\theta}{\cos^2\theta}$

To prove an identity, start from the left-hand side, and manipulate the expression until it matches the right-hand side. ← Sections 7.4, 7.5

$\equiv \dfrac{(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)}{\cos^2\theta}$

The numerator can be factorised as the 'difference of two squares'.

$\sin^2\theta + \cos^2\theta \equiv 1$.

$\equiv \dfrac{(\cos^2\theta - \sin^2\theta)}{\cos^2\theta}$

$\equiv \dfrac{\cos^2\theta}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta}$

$\equiv 1 - \tan^2\theta = \text{RHS}$

Divide through by $\cos^2\theta$ and note that $\dfrac{\sin^2\theta}{\cos^2\theta} \equiv \left(\dfrac{\sin\theta}{\cos\theta}\right)^2 \equiv \tan^2\theta$.

Example 8

a Given that $\cos\theta = -\dfrac{3}{5}$ and that θ is reflex, find the value of $\sin\theta$.

b Given that $\sin\alpha = \dfrac{2}{5}$ and that α is obtuse, find the exact value of $\cos\alpha$.

a Since $\sin^2\theta + \cos^2\theta \equiv 1$,

$$\sin^2\theta = 1 - \left(-\frac{3}{5}\right)^2$$

$$= 1 - \frac{9}{25}$$

$$= \frac{16}{25}$$

So $\sin\theta = -\frac{4}{5}$

'θ is reflex' means θ is in the 3rd or 4th quadrants, but as $\cos\theta$ is negative, θ must be in the 3rd quadrant. $\sin\theta = \pm\frac{4}{5}$ but in the third quadrant $\sin\theta$ is negative.

b Using $\sin^2\alpha + \cos^2\alpha \equiv 1$,

$\cos^2\alpha = 1 - \frac{4}{25} = \frac{21}{25}$

As α is obtuse, $\cos\alpha$ is negative

Obtuse angles lie in the second quadrant, and have a negative cosine.

so $\cos\alpha = -\frac{\sqrt{21}}{5}$

The question asks for the exact value so leave your answer as a surd.

Watch out If you use your calculator to find $\cos^{-1}(-\frac{3}{5})$, then the sine of the result, you will get an incorrect answer. This is because the $\cos^{-1}$ function on your calculator gives results between 0 and 180°.

Example 9

Given that $p = 3\cos\theta$, and that $q = 2\sin\theta$, show that $4p^2 + 9q^2 = 36$.

As $p = 3\cos\theta$, and $q = 2\sin\theta$,

$\cos\theta = \frac{p}{3}$ and $\sin\theta = \frac{q}{2}$

Using $\sin^2\theta + \cos^2\theta \equiv 1$,

$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2 = 1$$

so $$\frac{q^2}{4} + \frac{p^2}{9} = 1$$

$\therefore$ $$4p^2 + 9q^2 = 36$$

Multiply both sides by 36.

Problem-solving

You need to eliminate θ from the equations. As you can find $\sin\theta$ and $\cos\theta$ in terms of p and q, use the identity $\sin^2\theta + \cos^2\theta \equiv 1$.

Exercise 10C

1 Simplify each of the following expressions:

a $1 - \cos^2\frac{1}{2}\theta$ **b** $5\sin^2 3\theta + 5\cos^2 3\theta$ **c** $\sin^2 A - 1$

d $\dfrac{\sin\theta}{\tan\theta}$ **e** $\dfrac{\sqrt{1-\cos^2 x}}{\cos x}$ **f** $\dfrac{\sqrt{1-\cos^2 3A}}{\sqrt{1-\sin^2 3A}}$

g $(1 + \sin x)^2 + (1 - \sin x)^2 + 2\cos^2 x$ **h** $\sin^4\theta + \sin^2\theta\cos^2\theta$

i $\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta$

2 Given that $2\sin\theta = 3\cos\theta$, find the value of $\tan\theta$.

3 Given that $\sin x\cos y = 3\cos x\sin y$, express $\tan x$ in terms of $\tan y$.

4 Express in terms of $\sin\theta$ only:

a $\cos^2\theta$ **b** $\tan^2\theta$ **c** $\cos\theta\tan\theta$

d $\frac{\cos\theta}{\tan\theta}$ **e** $(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$

(P) **5** Using the identities $\sin^2 A + \cos^2 A \equiv 1$ and/or $\tan A = \frac{\sin A}{\cos A}$ $(\cos A \neq 0)$, prove that:

a $(\sin\theta + \cos\theta)^2 \equiv 1 + 2\sin\theta\cos\theta$ **b** $\frac{1}{\cos\theta} - \cos\theta \equiv \sin\theta\tan\theta$

c $\tan x + \frac{1}{\tan x} \equiv \frac{1}{\sin x \cos x}$ **d** $\cos^2 A - \sin^2 A \equiv 1 - 2\sin^2 A$

e $(2\sin\theta - \cos\theta)^2 + (\sin\theta + 2\cos\theta)^2 \equiv 5$ **f** $2 - (\sin\theta - \cos\theta)^2 \equiv (\sin\theta + \cos\theta)^2$

g $\sin^2 x\cos^2 y - \cos^2 x\sin^2 y \equiv \sin^2 x - \sin^2 y$

6 Find, without using your calculator, the values of:

a $\sin\theta$ and $\cos\theta$, given that $\tan\theta = \frac{5}{12}$ and θ is acute.

b $\sin\theta$ and $\tan\theta$, given that $\cos\theta = -\frac{3}{5}$ and θ is obtuse.

c $\cos\theta$ and $\tan\theta$, given that $\sin\theta = -\frac{7}{25}$ and $270° < \theta < 360°$.

7 Given that $\sin\theta = \frac{2}{3}$ and that θ is obtuse, find the exact value of: **a** $\cos\theta$ **b** $\tan\theta$

8 Given that $\tan\theta = -\sqrt{3}$ and that θ is reflex, find the exact value of: **a** $\sin\theta$ **b** $\cos\theta$

9 Given that $\cos\theta = \frac{3}{4}$ and that θ is reflex, find the exact value of: **a** $\sin\theta$ **b** $\tan\theta$

(P) **10** In each of the following, eliminate θ to give an equation relating x and y:

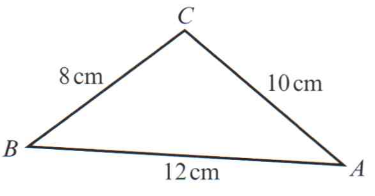
a $x = \sin\theta,\ y = \cos\theta$ **b** $x = \sin\theta,\ y = 2\cos\theta$

c $x = \sin\theta,\ y = \cos^2\theta$ **d** $x = \sin\theta,\ y = \tan\theta$

e $x = \sin\theta + \cos\theta,\ y = \cos\theta - \sin\theta$

Problem-solving
In part **e** find expressions for $x + y$ and $x - y$.

(E/P) **11** The diagram shows the triangle ABC with $AB = 12$ cm, $BC = 8$ cm and $AC = 10$ cm.

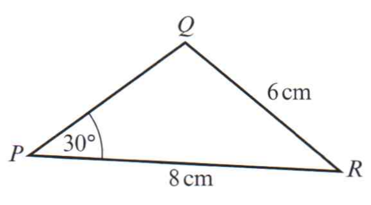


a Show that $\cos B = \frac{9}{16}$ **(3 marks)**

b Hence find the exact value of $\sin B$. **(2 marks)**

Hint Use the cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$ ← Section 9.1

(E/P) **12** The diagram shows triangle PQR with $PR = 8$ cm, $QR = 6$ cm and angle $QPR = 30°$.



a Show that $\sin Q = \frac{2}{3}$ **(3 marks)**

b Given that Q is obtuse, find the exact value of $\cos Q$ **(2 marks)**

10.4 Simple trigonometric equations

You need to be able to solve simple trigonometric equations of the form $\sin\theta = k$ and $\cos\theta = k$ (where $-1 \leqslant k \leqslant 1$) and $\tan\theta = p$ (where $p \in \mathbb{R}$) for given intervals of θ.

- **Solutions to $\sin\theta = k$ and $\cos\theta = k$ only exist when $-1 \leqslant k \leqslant 1$.**
- **Solutions to $\tan\theta = p$ exist for all values of p.**

Links The graphs of $y = \sin\theta$ and $y = \cos\theta$ have a maximum value of 1 and a minimum value of –1.
The graph of $y = \tan\theta$ has no maximum or minimum value. ← Section 9.5

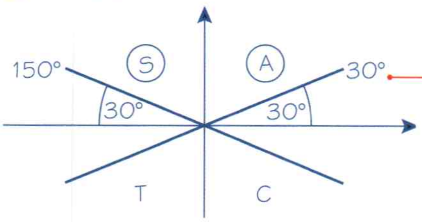
Example 10

Find the solutions of the equation $\sin\theta = \frac{1}{2}$ in the interval $0 \leqslant \theta \leqslant 360°$.

Method 1

$\sin\theta = \frac{1}{2}$

So $\theta = 30°$

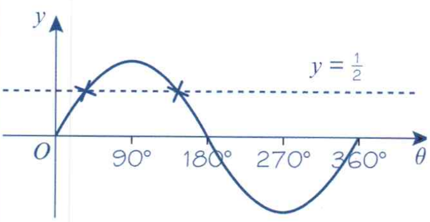


Putting 30° in the four positions shown gives the angles 30°, 150°, 210° and 330° but sine is only positive in the 1st and 2nd quadrants.

So $x = 30°$
or $x = 180° - 30° = 150°$

You can check this by putting sin 150° in your calculator.

Method 2



Draw the graph of $y = \sin\theta$ for the given interval.

$\sin\theta = \frac{1}{2}$ where the line $y = \frac{1}{2}$ cuts the curve.
Hence $\theta = 30°$ or $150°$

Use the symmetry properties of the $y = \sin\theta$ graph. ← Sections 9.5

- **When you use the inverse trigonometric functions on your calculator, the angle you get is called the principal value.**

Your calculator will give principal values in the following ranges:

$\cos^{-1}$ in the range $0 \leqslant \theta \leqslant 180°$
$\sin^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$
$\tan^{-1}$ in the range $-90° < \theta < 90°$

Notation The inverse trigonometric functions are also called **arccos**, **arcsin** and **arctan**.

Example 11

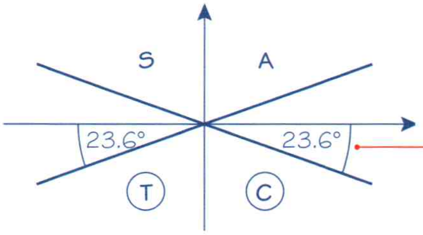
Solve, in the interval $0 \leqslant x \leqslant 360°$, $5 \sin x = -2$.

Method 1

$5 \sin x = -2$

$\sin x = -0.4$

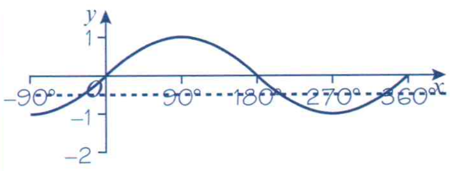
Principal value is $x = -23.6°$ (3 s.f.)



$x = 203.6°$ (204° to 3 s.f.)

or $x = 336.4°$ (336° to 3 s.f.)

Method 2



$\sin^{-1}(-0.4) = -23.578...°$

$x = 203.578...°$ (204° to 3 s.f.)

or $x = 336.421...°$ (336° to 3 s.f.)

First rewrite in the form $\sin x = \ldots$

Watch out The principal value will not always be a solution to the equation.

Sine is negative so you need to look in the 3rd and 4th quadrants for your solutions.

You can now find the solutions in the given interval. Note that in this case, if $\alpha = \sin^{-1}(-0.4)$, the solutions are $180 - \alpha$ and $360 + \alpha$.

Draw the graph of $y = \sin x$ starting from −90° since the principal solution given by $\sin^{-1}(-0.4)$ is negative.

Use the symmetry properties of the $y = \sin \theta$ graph.

Example 12

Solve, in the interval $0 < x \leqslant 360°$, $\cos x = \frac{\sqrt{3}}{2}$

A student writes down the following working:

> $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$
>
> So $x = 30°$ or $x = 180° - 30° = 150°$

a Identify the error made by the student.

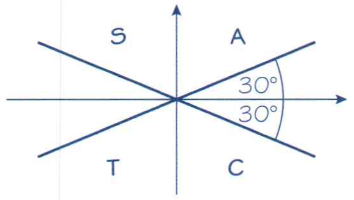
b Write down the correct answer.

a The principal solution is correct but the student has found a second solution in the second quadrant where cos is negative.

Problem-solving

In your exam you might have to analyse student working and identify errors. One strategy is to solve the problem yourself, then compare your working with the incorrect working that has been given.

b $x = 30°$ from the calculator



$x = 30°$ or $330°$

$\cos x$ is positive so you need to look in the 1st and 4th quadrants.

Find the solutions, in $0 < x \leqslant 360°$, from your diagram.

Note that these results are α and $360° - \alpha$ where $\alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

You can use the identity $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ to solve equations.

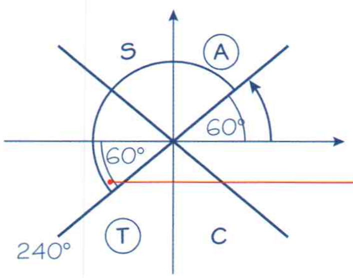
Example 13

Find the values of θ in the interval $0 < \theta \leqslant 360°$ that satisfy the equation $\sin\theta = \sqrt{3}\cos\theta$.

$\sin\theta = \sqrt{3}\cos\theta$

So $\tan\theta = \sqrt{3}$

$\tan^{-1}(\sqrt{3}) = 60°$



$\theta = 60°$ or $240°$

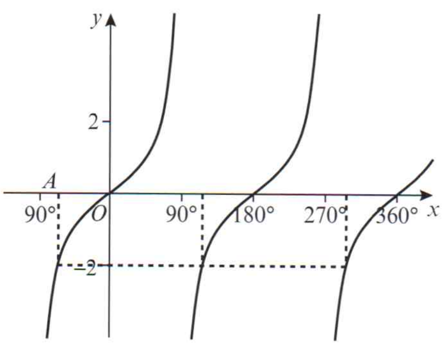
Since $\cos\theta = 0$ does not satisfy the equation, divide both sides by $\cos\theta$ and use the identity $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$

This is the principal solution.

Tangent is positive in the 1st and 3rd quadrants, so insert the angle in the correct positions.

Exercise 10D

1 The diagram shows a sketch of $y = \tan x$.

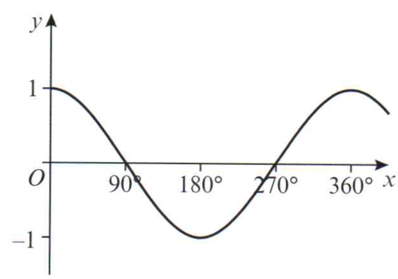


a Use your calculator to find the principal solution to the equation $\tan x = -2$.

Hint The principal solution is marked A on the diagram.

b Use the graph and your answer to part **a** to find solutions to the equation $\tan x = -2$ in the range $0 \leqslant x \leqslant 360°$.

2 The diagram shows a sketch of $y = \cos x$.



a Use your calculator to find the principal solution to the equation $\cos x = 0.4$.

b Use the graph and your answer to part **a** to find solutions to the equation $\cos x = \pm 0.4$ in the range $0 \leqslant x \leqslant 360°$.

3 Solve the following equations for θ, in the interval $0 < \theta \leqslant 360°$:

a $\sin\theta = -1$ b $\tan\theta = \sqrt{3}$ c $\cos\theta = \frac{1}{2}$

d $\sin\theta = \sin 15°$ e $\cos\theta = -\cos 40°$ f $\tan\theta = -1$

g $\cos\theta = 0$ h $\sin\theta = -0.766$

Hint Give your answers exactly where possible, or round to 3 significant figures.

4 Solve the following equations for θ, in the interval $0 < \theta \leqslant 360°$:

a $7\sin\theta = 5$ b $2\cos\theta = -\sqrt{2}$ c $3\cos\theta = -2$ d $4\sin\theta = -3$

e $7\tan\theta = 1$ f $8\tan\theta = 15$ g $3\tan\theta = -11$ h $3\cos\theta = \sqrt{5}$

5 Solve the following equations for θ, in the interval $0 < \theta \leqslant 360°$:

a $\sqrt{3}\sin\theta = \cos\theta$ b $\sin\theta + \cos\theta = 0$ c $3\sin\theta = 4\cos\theta$

d $2\sin\theta - 3\cos\theta = 0$ e $\sqrt{2}\sin\theta = 2\cos\theta$ f $\sqrt{5}\sin\theta + \sqrt{2}\cos\theta = 0$

6 Solve the following equations for x, giving your answers to 3 significant figures where appropriate, in the intervals indicated:

a $\sin x = -\frac{\sqrt{3}}{2}$, $-180° \leqslant x \leqslant 540°$ b $2\sin x = -0.3$, $-180° \leqslant x \leqslant 180°$

c $\cos x = -0.809$, $-180° \leqslant x \leqslant 180°$ d $\cos x = 0.84$, $-360° < x < 0°$

e $\tan x = -\frac{\sqrt{3}}{3}$, $0 \leqslant x \leqslant 720°$ f $\tan x = 2.90$, $80° \leqslant x \leqslant 440°$

E/P 7 A teacher asks two students to solve the equation $2\cos x = 3\sin x$ for $-180° \leqslant x \leqslant 180°$. The attempts are shown:

Student A:

$\tan x = \frac{3}{2}$

$x = 56.3°$ or $x = -123.7°$

Student B:

$4\cos^2 x = 9\sin^2 x$

$4(1 - \sin^2 x) = 9\sin^2 x$

$4 = 13\sin^2 x$

$\sin x = \pm\sqrt{\frac{4}{13}}$, $x = \pm 33.7°$ or $x = \pm 146.3°$

a Identify the mistake made by Student A. **(1 mark)**

b Identify the mistake made by Student B and explain the effect it has on their solution. **(2 marks)**

c Write down the correct answers to the question. **(1 mark)**

8 a Sketch the graphs of $y = 2\sin x$ and $y = \cos x$ on the same set of axes ($0 \leqslant x \leqslant 360°$).

b Write down how many solutions there are in the given range for the equation $2\sin x = \cos x$.

c Solve the equation $2\sin x = \cos x$ algebraically, giving your answers to 1 d.p.

E/P 9 Find all the values of θ, to 1 decimal place, in the interval $0 < \theta < 360°$ for which $\tan^2\theta = 9$. **(5 marks)**

Problem-solving When you take square roots of both sides of an equation you need to consider both the positive and the negative square roots.

E/P 10 a Show that $4\sin^2 x - 3\cos^2 x = 2$ can be written as $7\sin^2 x = 5$. **(2 marks)**

b Hence solve, for $0 \leqslant x \leqslant 360°$, the equation $4\sin^2 x - 3\cos^2 x = 2$. Give your answers to 1 decimal place. **(7 marks)**

E/P 11 a Show that the equation $2\sin^2 x + 5\cos^2 x = 1$ can be written as $3\sin^2 x = 4$. **(2 marks)**

b Use your result in part **a** to explain why the equation $2\sin^2 x + 5\cos^2 x = 1$ has no solutions. **(1 marks)**

10.5 Harder trigonometric equations

You need to be able to solve equations of the form $\sin n\theta = k$, $\cos n\theta = k$ and $\tan n\theta = p$.

Example 14

a Solve the equation $\cos 3\theta = 0.766$, in the interval $0 \leqslant \theta \leqslant 360°$.

b Solve the equation $2\sin 2\theta = \cos 2\theta$, in the interval $0 \leqslant \theta \leqslant 360°$.

a Let $X = 3\theta$

Replace 3θ by X and solve.

So $\cos X° = 0.766$

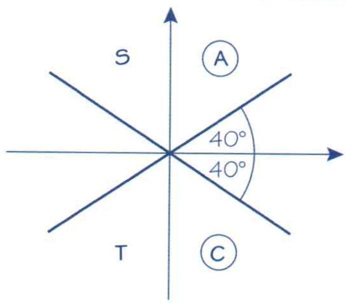
As $X = 3\theta$,

then as $0 \leqslant \theta \leqslant 360°$

So $3 \times 0 \leqslant X \leqslant 3 \times 360°$

So the interval for X is

$0 \leqslant X \leqslant 1080°$



Watch out If the range of values for θ is $0 \leqslant \theta \leqslant 360°$, then the range of values for 3θ is $0 \leqslant 3\theta \leqslant 1080°$.

$X = 40.0°, 320°, 400°, 680°, 760°, 1040°$

The value of X from your calculator is 40.0. You need to list all values in the 1st and 4th quadrants for three complete revolutions.

i.e. $3\theta = 40.0°, 320°, 400°, 680°, 760°, 1040°$

Remember $X = 3\theta$.

So $\theta = 13.3°, 107°, 133°, 227°, 253°, 347°$

b $\frac{\sin 2\theta}{\cos 2\theta} = \frac{1}{2}$, so $\tan 2\theta = \frac{1}{2}$

Use the identity for tan to rearrange the equation.

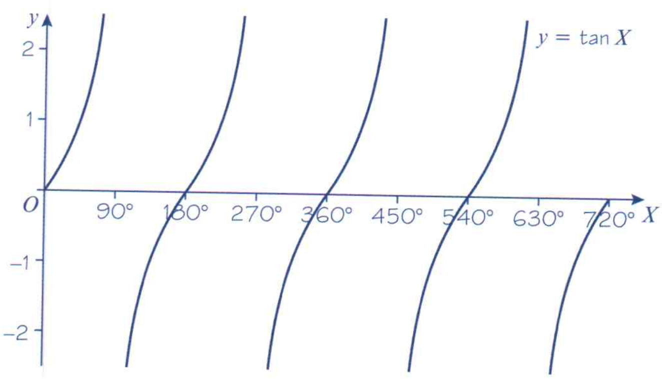
Let $X = 2\theta$

So $\tan X = \frac{1}{2}$

As $X = 2\theta$, then as $0 \leqslant \theta \leqslant 360°$

The interval for X is $0 \leqslant X \leqslant 720°$

Let $X = 2\theta$, and double both values to find the interval for X.



Draw a graph of $\tan X$ for this interval.
Alternatively, you could use a CAST diagram as in part **a**.

The principal solution for X is $26.565...°$

Add multiples of 180°:

$X = 26.565...°, 206.565...°, 386.565...°, 566.565...°$

$\theta = 13.3°, 103°, 193°, 283°$

Convert your values of X back into values of θ.
Round each answer to a sensible degree of accuracy at the end.

You need to be able to solve equations of the form $\sin(ax + b) = c$, $\cos(ax + b) = c$ and $\tan(ax + b) = c$.

Example 15

Solve the equation $\sin(2x + 60°) = 0.3$ in the interval $0 \leqslant x \leqslant 180°$.

Let $X = 2x + 60°$

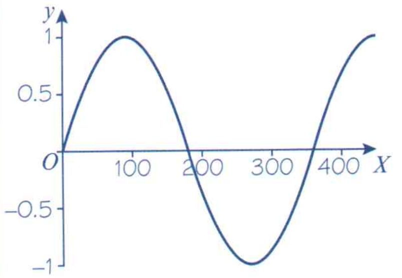
So $\sin X = 0.3$

The interval for X is

$2 \times 0° + 60° \leqslant X \leqslant 2 \times 180° + 60°$

So $60° \leqslant X \leqslant 420°$

Adjust the interval by multiplying by 2 then adding 60° to both values.



Draw a sketch of the sin graph for the given interval.

The principal value for X is 17.45...°

$X = 162.54...°, 377.45...°$

This is not in the given interval so it does not correspond to a solution of the equation. Use the symmetry of the sin graph to find other solutions.

For each value, subtract 60° then divide by 2:

$x = 51.27...°, 158.72...°$

Hence $x = 51.3°$ or $158.7°$

You could also use a CAST diagram to solve this problem.

Exercise 10E

1 Find the values of θ, in the interval $0 \leqslant \theta \leqslant 360°$, for which:

a $\sin 4\theta = 0$ **b** $\cos 3\theta = -1$ **c** $\tan 2\theta = 1$

d $\cos 2\theta = \frac{1}{2}$ **e** $\tan \frac{1}{2}\theta = -\frac{1}{\sqrt{3}}$ **f** $\sin(-\theta) = \frac{1}{\sqrt{2}}$

2 Solve the following equations in the interval given:

a $\tan(45° - \theta) = -1, 0 \leqslant \theta \leqslant 360°$ **b** $2\sin(\theta - 20°) = 1, 0 \leqslant \theta \leqslant 360°$

c $\tan(\theta + 75°) = \sqrt{3}, 0 \leqslant \theta \leqslant 360°$ **d** $\sin(\theta - 10°) = -\frac{\sqrt{3}}{2}, 0 \leqslant \theta \leqslant 360°$

e $\cos(50° + 2\theta) = -1, 0 \leqslant \theta \leqslant 360°$ **f** $\tan(3\theta + 25°) = -0.51, -90° < x \leqslant 180°$

3 Solve the following equations in the interval given:

a $3\sin 3\theta = 2\cos 3\theta, 0 \leqslant \theta \leqslant 180°$

b $4\sin(\theta + 45°) = 5\cos(\theta + 45°), 0 \leqslant \theta \leqslant 450°$

c $2\sin 2x - 7\cos 2x = 0, 0 \leqslant x \leqslant 180°$

d $\sqrt{3}\sin(x - 60°) + \cos(x - 60°) = 0, -180° \leqslant x \leqslant 180°$

E 4 Solve for $0 \leqslant x \leqslant 180°$ the equations:

a $\sin(x + 20°) = \frac{1}{2}$ **(4 marks)**

b $\cos 2x = -0.8$, giving your answers to 1 decimal place. **(4 marks)**

E 5 a Sketch for $0 \leqslant x \leqslant 360°$ the graph of $y = \sin(x + 60°)$ **(2 marks)**

b Write down the exact coordinates of the points where the graph meets the coordinate axes. **(3 marks)**

c Solve, for $0 \leqslant x \leqslant 360°$, the equation $\sin(x + 60°) = 0.55$, giving your answers to 1 decimal place. **(5 marks)**

E 6 a Given that $4\sin x = 3\cos x$, write down the value of $\tan x$. **(1 mark)**

b Solve, for $0 \leqslant \theta \leqslant 360°$, $4\sin 2\theta = 3\cos 2\theta$ giving your answers to 1 decimal place. **(5 marks)**

P 7 The equation $\tan kx = -\frac{1}{\sqrt{3}}$, where k is a constant and $k > 0$, has a solution at $x = 60°$

a Find a possible value of k. **(3 marks)**

b State, with justification, whether this is the only such possible value of k. **(1 mark)**

E 8 Solve the equation $\sin(3x - 45°) = \frac{1}{2}$ in the interval $0 \leqslant x \leqslant 180°$. **(4 marks)**

10.6 Equations and identities

You need to be able to solve quadratic equations in $\sin\theta$, $\cos\theta$ or $\tan\theta$. This may give rise to two sets of solutions.

$5\sin^2 x + 3\sin x - 2 = 0$ — This is a quadratic equation in the form $5A^2 + 3A - 2 = 0$ where $A = \sin x$.

$(5\sin x - 2)(\sin x + 1) = 0$ — Factorise

$5\sin x - 2 = 0$ $\quad$ $\sin x + 1 = 0$ — Setting each factor equal to zero produces two linear equations in $\sin x$.

Example 16

Solve for θ, in the interval $0 \leqslant x \leqslant 360°$, the equations

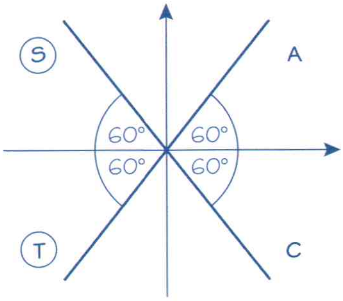
a $2\cos^2\theta - \cos\theta - 1 = 0$ $\quad$ b $\sin^2(\theta - 30°) = \frac{1}{2}$

a $2\cos^2\theta - \cos\theta - 1 = 0$

So $(2\cos\theta + 1)(\cos\theta - 1) = 0$ — Compare with $2x^2 - x - 1 = (2x + 1)(x - 1)$

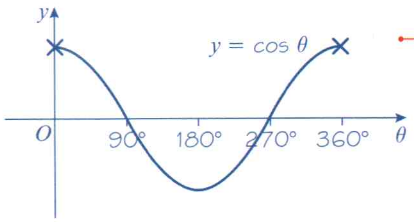
So $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$ — Set each factor equal to 0 to find two sets of solutions.

$\cos\theta = -\frac{1}{2}$ so $\theta = 120°$



120° makes an angle of 60° with the horizontal. But cosine is negative in the 2nd and 3rd quadrants so $\theta = 120°$ or $\theta = 240°$.

$\theta = 120°$ or $\theta = 240°$



Sketch the graph of $y = \cos\theta$.

Or $\cos\theta = 1$ so $\theta = 0$ or 360°

So the solutions are

$\theta = 0°, 120°, 240°, 360°$

There are four solutions within the given interval.

b $\sin^2(\theta - 30°) = \frac{1}{2}$

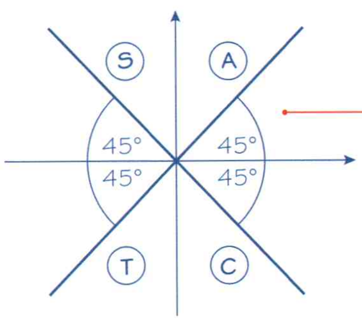
$\sin(\theta - 30°) = \frac{1}{\sqrt{2}}$

or $\sin(\theta - 30°) = -\frac{1}{\sqrt{2}}$

The solutions of $x^2 = k$ are $x = \pm\sqrt{k}$.

So $\theta - 30° = 45°$ or $\theta - 30° = -45°$

Use your calculator to find one solution for each equation.



Draw a diagram to find the quadrants where sine is positive and the quadrants where sine is negative.

So from $\sin(\theta - 30°) = \frac{1}{\sqrt{2}}$

$\theta - 30° = 45°, 135°$

and from $\sin(\theta - 30°) = -\frac{1}{\sqrt{2}}$

$\theta - 30° = 225°, 315°$

So the solutions are: $\theta = 75°, 165°, 255°, 345°$

In some equations you may need to use the identity $\sin^2\theta + \cos^2\theta \equiv 1$.

Example 17

Find the values of x, in the interval $-180° \leqslant x \leqslant 180°$, satisfying the equation $2\cos^2 x + 9\sin x = 3\sin^2 x$.

$2\cos^2 x + 9\sin x = 3\sin^2 x$

$2(1 - \sin^2 x) + 9\sin x = 3\sin^2 x$

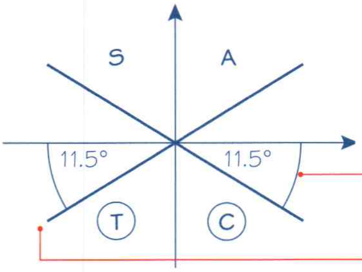
$5\sin^2 x - 9\sin x - 2 = 0$

So $(5\sin x + 1)(\sin x - 2) = 0$

$\sin x = -\frac{1}{5}$

As $\sin^2 x + \cos^2 x \equiv 1$, you are able to rewrite $\cos^2 x$ as $(1 - \sin^2 x)$, and so form a quadratic equation in $\sin x$.

Watch out The factor $(\sin x - 2)$ does not produce any solutions, because $\sin x = 2$ has no solutions.



Your calculator value of x is $x = -11.5°$ (1 d.p.). Insert into the CAST diagram.

The smallest angle in the interval, in the 3rd quadrant, is $(-180 + 11.5) = -168.5°$; there are no values between 0 and 180°.

The solutions are $-168.5°$ and $-11.5°$ (1 d.p.)

Exercise 10F

1 Solve for θ, in the interval $0 \leqslant \theta \leqslant 360°$, the following equations. Give your answers to 3 significant figures where they are not exact.

a $4\cos^2\theta = 1$ **b** $2\sin^2\theta - 1 = 0$ **c** $3\sin^2\theta + \sin\theta = 0$

d $\tan^2\theta - 2\tan\theta - 10 = 0$ **e** $2\cos^2\theta - 5\cos\theta + 2 = 0$ **f** $\sin^2\theta - 2\sin\theta - 1 = 0$

g $\tan^2 2\theta = 3$

Hint In part **e**, only one factor leads to valid solutions.

2 Solve for θ, in the interval $-180° \leqslant \theta \leqslant 180°$, the following equations. Give your answers to 3 significant figures where they are not exact.

a $\sin^2 2\theta = 1$ **b** $\tan^2\theta = 2\tan\theta$

c $\cos\theta(\cos\theta - 2) = 1$ **d** $4\sin\theta = \tan\theta$

3 Solve for θ, in the interval $0 \leqslant \theta \leqslant 180°$, the following equations. Give your answers to 3 significant figures where they are not exact.

a $4(\sin^2\theta - \cos\theta) = 3 - 2\cos\theta$ **b** $2\sin^2\theta = 3(1 - \cos\theta)$ **c** $4\cos^2\theta - 5\sin\theta - 5 = 0$

4 Solve for θ, in the interval $-180° \leqslant \theta \leqslant 180°$, the following equations. Give your answers to 3 significant figures where they are not exact.

a $5\sin^2\theta = 4\cos^2\theta$ **b** $\tan\theta = \cos\theta$

E 5 Find all the solutions, in the interval $0 \leqslant x \leqslant 360°$, to the equation $8\sin^2 x + 6\cos x - 9 = 0$ giving each solution to one decimal place. **(6 marks)**

E 6 Find, for $0 \leqslant x \leqslant 360°$, all the solutions of $\sin^2 x + 1 = \frac{7}{2}\cos^2 x$ giving each solution to one decimal place. **(6 marks)**

E/P 7 Show that the equation $2\cos^2 x + \cos x - 6 = 0$ has no solutions. **(3 marks)**

E/P 8 **a** Show that the equation $\cos^2 x = 2 - \sin x$ can be written as $\sin^2 x - \sin x + 1 = 0$. **(2 marks)**

b Hence show that the equation $\cos^2 x = 2 - \sin x$ has no solutions. **(3 marks)**

Problem-solving

If you have to answer a question involving the number of solutions to a quadratic equation, see if you can make use of the discriminant.

E/P 9 $\tan^2 x - 2\tan x - 4 = 0$

a Show that $\tan x = p \pm \sqrt{q}$ where p and q are numbers to be found. **(3 marks)**

b Hence solve the equation $\tan^2 x - 2\tan x - 4 = 0$ in the interval $0 \leqslant x \leqslant 540°$. **(5 marks)**

Challenge

1 Solve the equation $\cos^2 3\theta - \cos 3\theta = 2$ in the interval $-180° \leqslant \theta \leqslant 180°$.

2 Solve the equation $\tan^2(\theta - 45°) = 1$ in the interval $0 \leqslant \theta \leqslant 360°$.

Mixed exercise 10

1 Write each of the following as a trigonometric ratio of an acute angle:

a $\cos 237°$ **b** $\sin 312°$ **c** $\tan 190°$

2 Without using your calculator, work out the values of:

a $\cos 270°$ **b** $\sin 225°$ **c** $\cos 180°$ **d** $\tan 240°$ **e** $\tan 135°$

P 3 Given that angle A is obtuse and $\cos A = -\sqrt{\frac{7}{11}}$, show that $\tan A = \frac{-2\sqrt{7}}{7}$

P 4 Given that angle B is reflex and $\tan B = +\frac{\sqrt{21}}{2}$, find the exact value of: **a** $\sin B$ **b** $\cos B$

5 Simplify the following expressions:

a $\cos^4\theta - \sin^4\theta$ **b** $\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta$

c $\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta$

6 **a** Given that $2(\sin x + 2\cos x) = \sin x + 5\cos x$, find the exact value of $\tan x$.

b Given that $\sin x\cos y + 3\cos x\sin y = 2\sin x\sin y - 4\cos x\cos y$, express $\tan y$ in terms of $\tan x$.

P 7 Prove that, for all values of θ:

a $(1 + \sin\theta)^2 + \cos^2\theta \equiv 2(1 + \sin\theta)$ **b** $\cos^4\theta + \sin^2\theta \equiv \sin^4\theta + \cos^2\theta$

(P) **8** Without attempting to solve them, state how many solutions the following equations have in the interval $0 \leqslant \theta \leqslant 360°$. Give a brief reason for your answer.

a $2\sin\theta = 3$

b $\sin\theta = -\cos\theta$

c $2\sin\theta + 3\cos\theta + 6 = 0$

d $\tan\theta + \dfrac{1}{\tan\theta} = 0$

(E) **9** **a** Factorise $4xy - y^2 + 4x - y$. **(2 marks)**

b Solve the equation $4\sin\theta\cos\theta - \cos^2\theta + 4\sin\theta - \cos\theta = 0$, in the interval $0 \leqslant \theta \leqslant 360°$. **(5 marks)**

(E) **10** **a** Express $4\cos 3\theta - \sin(90° - 3\theta)$ as a single trigonometric function. **(1 mark)**

b Hence solve $4\cos 3\theta - \sin(90° - 3\theta) = 2$ in the interval $0 \leqslant \theta \leqslant 360°$. Give your answers to 3 significant figures. **(3 marks)**

(E/P) **11** Given that $2\sin 2\theta = \cos 2\theta$:

a Show that $\tan 2\theta = 0.5$. **(1 mark)**

b Hence find the values of θ, to one decimal place, in the interval $0 \leqslant \theta \leqslant 360°$ for which $2\sin 2\theta = \cos 2\theta$. **(4 marks)**

12 Find all the values of θ in the interval $0 \leqslant \theta \leqslant 360°$ for which:

a $\cos(\theta + 75°) = 0.5$,

b $\sin 2\theta = 0.7$, giving your answers to one decimal place.

(E) **13** Find the values of x in the interval $0 < x < 270°$ which satisfy the equation

$$\frac{\cos 2x + 0.5}{1 - \cos 2x} = 2$$

(6 marks)

(E) **14** Find, in degrees, the values of θ in the interval $0 < \theta < 360°$ for which
$2\cos^2\theta - \cos\theta - 1 = \sin^2\theta$
Give your answers to 1 decimal place, where appropriate. **(6 marks)**

(E/P) **15** A teacher asks one of his students to solve the equation $2\sin 3x = 1$ for $-360° \leqslant x \leqslant 360°$. The attempt is shown below:

$\sin 3x = \frac{1}{2}$
$3x = 30°$
$x = 10°$
Additional solution at $180° - 10° = 170°$

a Identify two mistakes made by the student. **(2 marks)**

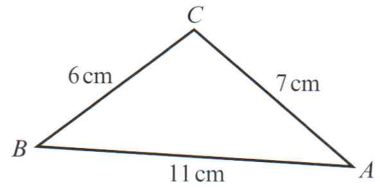
b Solve the equation. **(2 marks)**

16 **a** Sketch the graphs of $y = 3\sin x$ and $y = 2\cos x$ on the same set of axes $(0 \leqslant x \leqslant 360°)$.

b Write down how many solutions there are in the given range for the equation $3\sin x = 2\cos x$.

c Solve the equation $3\sin x = 2\cos x$ algebraically, giving your answers to one decimal place.

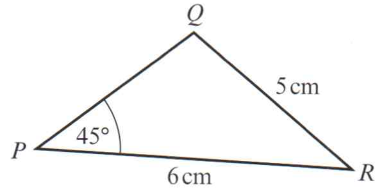
(E) **17** The diagram shows the triangle ABC with $AB = 11$ cm, $BC = 6$ cm and $AC = 7$ cm.



a Find the exact value of $\cos B$, giving your answer in simplest form. **(3 marks)**

b Hence find the exact value of $\sin B$. **(2 marks)**

(E/P) **18** The diagram shows triangle PQR with $PR = 6$ cm, $QR = 5$ cm and angle $QPR = 45°$.



a Show that $\sin Q = \dfrac{3\sqrt{2}}{5}$ **(3 marks)**

b Given that Q is obtuse, find the exact value of $\cos Q$. **(2 marks)**

(E/P) **19** **a** Show that the equation $3\sin^2 x - \cos^2 x = 2$ can be written as $4\sin^2 x = 3$. **(2 marks)**

b Hence solve the equation $3\sin^2 x - \cos^2 x = 2$ in the interval $-180° \leqslant x \leqslant 180°$. **(7 marks)**

(E) **20** Find all the solutions to the equation $3\cos^2 x + 1 = 4\sin x$ in the interval $-360° \leqslant x \leqslant 360°$, giving your answers to 1 decimal place. **(6 marks)**

(E) **21** Consider the function f(x) defined by

$$f(x) \equiv 3 + 2\sin(2x + k),\ 0 < x < 360°$$

where k is a constant and $0 < k < 360°$. The curve with equation $y = f(x)$ passes through the point with coordinates $(15, 3 + \sqrt{3})$.

a Show that $k = 30°$ is a possible value for k and find the other possible value of k. **(3 marks)**

b Given that $k = 30°$, solve the equation $f(x) = 1$. **(5 marks)**

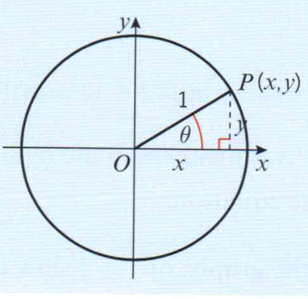
Challenge

Solve the equation $\tan^4 x - 3\tan^2 x + 2 = 0$ in the interval $0 \leqslant x \leqslant 360°$.

Summary of key points

1 For a point $P(x, y)$ on a unit circle such that OP makes an angle θ with the positive x-axis:

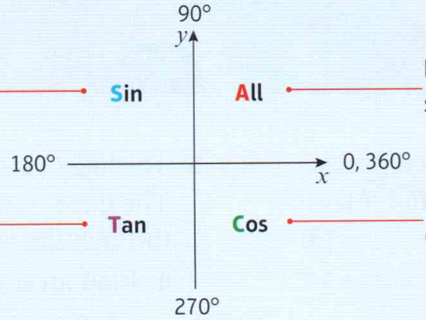
- $\cos\theta = x = x$-coordinate of P
- $\sin\theta = y = y$-coordinate of P
- $\tan\theta = \dfrac{y}{x}$ = gradient of OP



2 You can use the quadrants to determine whether each of the trigonometric ratios is positive or negative.

For an angle θ in the second quadrant, only $\sin\theta$ is positive.

For an angle θ in the third quadrant, only $\tan\theta$ is positive.



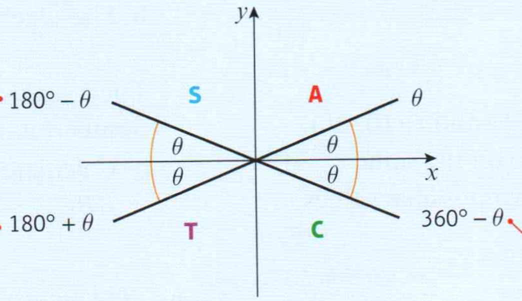
For an angle θ in the first quadrant, $\sin\theta$, $\cos\theta$ and $\tan\theta$ are all positive.

For an angle θ in the fourth quadrant, only $\cos\theta$ is positive.

3 You can use these rules to find sin, cos or tan of any positive or negative angle using the corresponding **acute** angle made with the x-axis, θ.

$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$



$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$

4 The trigonometric ratios of 30°, 45° and 60° have exact forms, given below:

$\sin 30° = \frac{1}{2}$ $\quad$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ $\tan 45° = 1$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \frac{1}{2}$ $\quad$ $\tan 60° = \sqrt{3}$

5 For all values of θ, $\sin^2\theta + \cos^2\theta \equiv 1$

6 For all values of θ such that $\cos\theta \neq 0$, $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$

7
- Solutions to $\sin\theta = k$ and $\cos\theta = k$ only exist when $-1 \leqslant k \leqslant 1$
- Solutions to $\tan\theta = p$ exist for all values of p.

8 When you use the inverse trigonometric functions on your calculator, the angle you get is called the **principal value**.

9 Your calculator will give principal values in the following ranges:
- $\cos^{-1}$ in the range $0 \leqslant \theta \leqslant 180°$
- $\sin^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$
- $\tan^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$