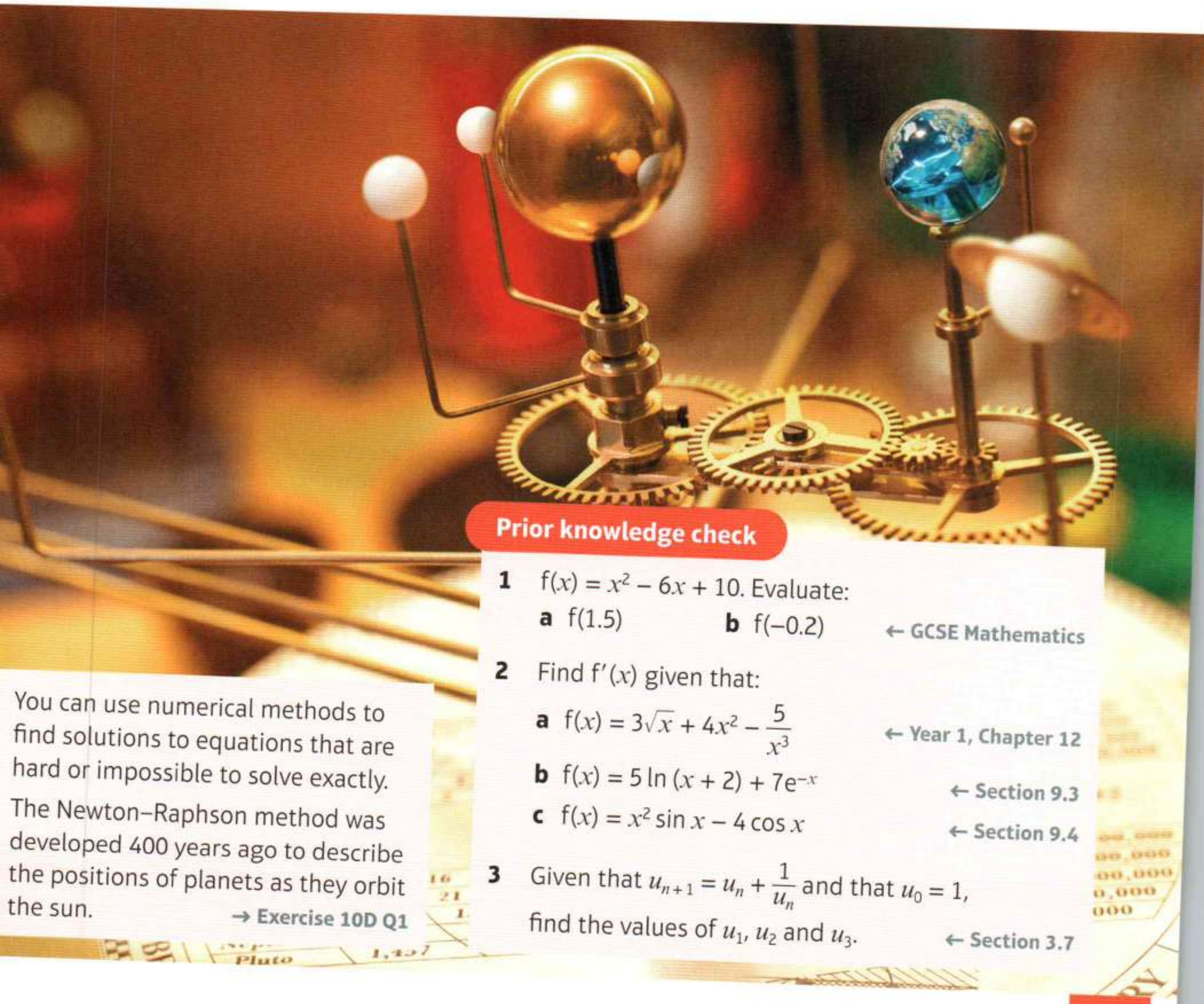
Numerical methods

10

Objectives

After completing this chapter you should be able to:

- Locate roots of $f(x) = 0$ by considering changes of sign → pages 274–277
- Use iteration to find an approximation to the root of the equation $f(x) = 0$ → pages 278–282
- Use the Newton–Raphson procedure to find approximations to the solutions of equations of the form $f(x) = 0$ → pages 282–285
- Use numerical methods to solve problems in context → pages 286–289



You can use numerical methods to find solutions to equations that are hard or impossible to solve exactly.

The Newton–Raphson method was developed 400 years ago to describe the positions of planets as they orbit the sun. → Exercise 10D Q1

Prior knowledge check

1. $f(x) = x^2 - 6x + 10$. Evaluate:
   a $f(1.5)$ **b** $f(-0.2)$ ← GCSE Mathematics
2. Find $f'(x)$ given that:
   a $f(x) = 3\sqrt{x} + 4x^2 - \frac{5}{x^3}$ ← Year 1, Chapter 12
   b $f(x) = 5\ln(x + 2) + 7e^{-x}$ ← Section 9.3
   c $f(x) = x^2 \sin x - 4\cos x$ ← Section 9.4
3. Given that $u_{n+1} = u_n + \frac{1}{u_n}$ and that $u_0 = 1$, find the values of u_1, u_2 and u_3. ← Section 3.7

10.1 Locating roots

A root of a function is a value of x for which $f(x) = 0$. The graph of $y = f(x)$ will cross the x-axis at points corresponding to the roots of the function.

Links The following two things are identical:
- the roots of the function $f(x)$
- the roots of the equation $f(x) = 0$ ← Year 1, Section 2.3

You can sometimes show that a root exists within a given interval by showing that the function changes sign (from positive to negative, or vice versa) within the interval.

- **If the function $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has at least one root, x, which satisfies $a < x < b$.**

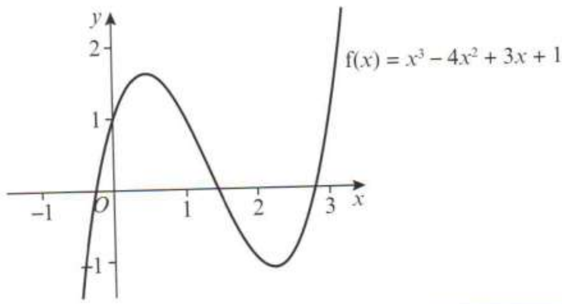
Notation **Continuous** means that the function does not 'jump' from one value to another. If the graph of the function has a vertical asymptote between a and b then the function is not continuous on $[a, b]$.

Example 1

The diagram shows a sketch of the curve $y = f(x)$, where $f(x) = x^3 - 4x^2 + 3x + 1$.

a Explain how the graph shows that $f(x)$ has a root between $x = 2$ and $x = 3$.

b Show that $f(x)$ has a root between $x = 1.4$ and $x = 1.5$.



a The graph crosses the x-axis between $x = 2$ and $x = 3$. This means that a root of $f(x)$ lies between $x = 2$ and $x = 3$.

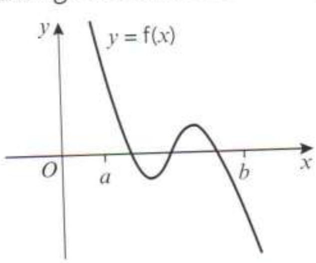
The graph of $y = f(x)$ crosses the x-axis whenever $f(x) = 0$.

b $f(1.4) = (1.4)^3 - 4(1.4)^2 + 3(1.4) + 1 = 0.104$
$f(1.5) = (1.5)^3 - 4(1.5)^2 + 3(1.5) + 1 = -0.125$
There is a change of sign between 1.4 and 1.5, so there is at least one root between $x = 1.4$ and $x = 1.5$.

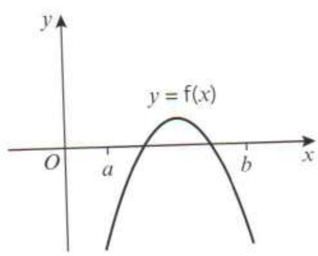
$f(1.4) > 0$ and $f(1.5) < 0$, so there is a change of sign.

$f(x)$ changes sign in the interval $[1.4, 1.5]$, so $f(x)$ must equal zero within this interval.

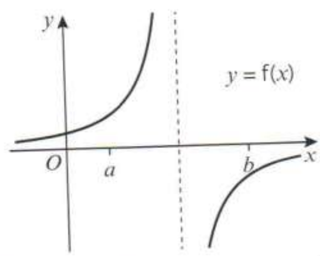
There are three situations you need to watch out for when using the change of sign rule to locate roots. A change of sign does not necessarily mean there is exactly one root, and the absence of a sign change does not necessarily mean that a root does not exist in the interval.



There are multiple roots within the interval $[a, b]$. In this case there is an **odd number** of roots



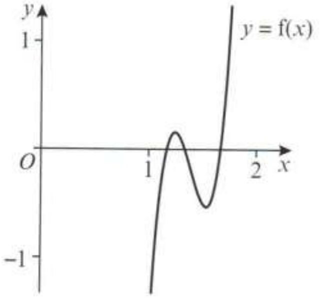
There are multiple roots within the interval $[a, b]$, but a sign change does not occur. In this case there is an **even number** of roots.



There is a vertical asymptote within interval $[a, b]$. A sign change does occur, but there is no root.

Example 2

The graph of the function $f(x) = 54x^3 - 225x^2 + 309x - 140$ is shown in the diagram.



A student observes that f(1.1) and f(1.6) are both negative and states that f(x) has no roots in the interval (1.1, 1.6).

a Explain by reference to the diagram why the student is incorrect.

b Calculate f(1.3) and f(1.5) and use your answer to explain why there are at least 3 roots in the interval $1.1 < x < 1.7$.

a The diagram shows that there could be two roots in the interval (1.1, 1.6).

b $f(1.1) = -0.476 < 0$
$f(1.3) = 0.088 > 0$
$f(1.5) = -0.5 < 0$
$f(1.7) = 0.352 > 0$
There is a change of sign between 1.1 and 1.3, between 1.3 and 1.5 and between 1.5 and 1.7, so there are at least three roots in the interval $1.1 < x < 1.7$.

Notation The interval (1.1, 1.6) is the set of all real numbers, x, that satisfy $1.1 < x < 1.6$.

Calculate the values of f(1.1), f(1.3), f(1.5) and f(1.7). Comment on the sign of each answer.

f(x) changes sign at least three times in the interval $1.1 < x < 1.7$ so f(x) must equal zero at least three times within this interval.

Example 3

a Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Explain how your diagram shows that the function $f(x) = \ln x - \frac{1}{x}$ has only one root.

b Show that this root lies in the interval $1.7 < x < 1.8$.

c Given that the root of f(x) is α, show that $\alpha = 1.763$ correct to 3 decimal places.

a

$\ln x - \frac{1}{x} = 0 \Rightarrow \ln x = \frac{1}{x}$

The equation $\ln x = \frac{1}{x}$ has only one solution, so f(x) has only one root.

Sketch $y = \ln x$ and $y = \frac{1}{x}$ on the same axes.
Notice that the curves do intersect.

f(x) has a root where f(x) = 0.

The curves meet at only one point, so there is only one value of x that satisfies the equation $\ln x = \frac{1}{x}$

b $f(x) = \ln x - \frac{1}{x}$

$f(1.7) = \ln 1.7 - \frac{1}{1.7} = -0.0576...$

$f(1.8) = \ln 1.8 - \frac{1}{1.8} = 0.0322...$

There is a change of sign between 1.7 and 1.8, so there is at least one root in the interval $1.7 < x < 1.8$.

c $f(1.7625) = -0.00064... < 0$
$f(1.7635) = 0.00024... > 0$
There is a change of sign in the interval (1.7625, 1.7635) so $1.7625 < \alpha < 1.7635$, so $\alpha = 1.763$ correct to 3 d.p.

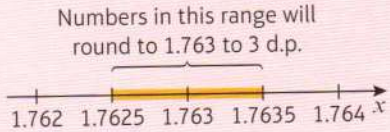
Online Locate the root of $f(x) = \ln x - \frac{1}{x}$ using technology.

$f(1.7) < 0$ and $f(1.8) > 0$, so there is a change of sign.

You need to state that there is a change of sign in your conclusion.

Problem-solving

To determine a root to a given degree of accuracy you need to show that it lies within a range of values that will all round to the given value.



Exercise 10A

1 Show that each of these functions has at least one root in the given interval.

a $f(x) = x^3 - x + 5, -2 < x < -1$

b $f(x) = x^2 - \sqrt{x} - 10, 3 < x < 4$

c $f(x) = x^3 - \frac{1}{x} - 2, -0.5 < x < -0.2$

d $f(x) = e^x - \ln x - 5, 1.65 < x < 1.75$

(E) 2 $f(x) = 3 + x^2 - x^3$

a Show that the equation $f(x) = 0$ has a root, α, in the interval [1.8, 1.9]. **(2 marks)**

b By considering a change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.864$ correct to 3 decimal places. **(3 marks)**

(E) 3 $h(x) = \sqrt[3]{x} - \cos x - 1$, where x is in radians.

a Show that the equation $h(x) = 0$ has a root, α, between $x = 1.4$ and $x = 1.5$. **(2 marks)**

b By choosing a suitable interval, show that $\alpha = 1.441$ is correct to 3 decimal places. **(3 marks)**

(E) 4 $f(x) = \sin x - \ln x, x > 0$, where x is in radians.

a Show that $f(x) = 0$ has a root, α, in the interval [2.2, 2.3]. **(2 marks)**

b By considering a change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 2.219$ correct to 3 decimal places. **(3 marks)**

(P) 5 $f(x) = 2 + \tan x, 0 < x < \pi$, where x is in radians.

a Show that $f(x)$ changes sign in the interval [1.5, 1.6].

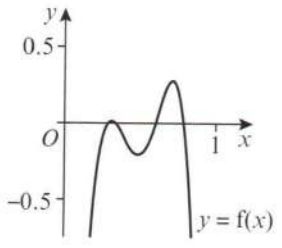
b State with a reason whether $f(x)$ has a root in the interval [1.5, 1.6].

6 A student observes that the function $f(x) = \frac{1}{x} + 2$, $x \neq 0$, has a change of sign on the interval $[-1, 1]$. The student writes:

> $y = f(x)$ has a vertical asymptote within this interval so even though there is a change of sign, $f(x)$ has no roots in this interval.

By means of a sketch, or otherwise, explain why the student is incorrect.

7 $f(x) = (105x^3 - 128x^2 + 49x - 6)\cos 2x$, where x is in radians.
The diagram shows a sketch of $y = f(x)$.



a Calculate $f(0.2)$ and $f(0.8)$.

b Use your answer to part **a** to make a conclusion about the number of roots of $f(x)$ in the interval $0.2 < x < 0.8$.

c Further calculate $f(0.3)$, $f(0.4)$, $f(0.5)$, $f(0.6)$ and $f(0.7)$.

d Use your answers to parts **a** and **c** to make an improved conclusion about the number of roots of $f(x)$ in the interval $0.2 < x < 0.8$.

8 **a** Using the same axes, sketch the graphs of $y = e^{-x}$ and $y = x^2$.

b Explain why the function $f(x) = e^{-x} - x^2$ has only one root.

c Show that the function $f(x) = e^{-x} - x^2$ has a root between $x = 0.70$ and $x = 0.71$.

9 **a** On the same axes, sketch the graphs of $y = \ln x$ and $y = e^x - 4$.

b Write down the number of roots of the equation $\ln x = e^x - 4$.

c Show that the equation $\ln x = e^x - 4$ has a root in the interval $(1.4, 1.5)$.

10 $h(x) = \sin 2x + e^{4x}$

a Show that there is a stationary point, α, of $y = h(x)$ in the interval $-0.9 < x < -0.8$. **(4 marks)**

b By considering the change of sign of $h'(x)$ in a suitable interval, verify that $\alpha = -0.823$ correct to 3 decimal places. **(2 marks)**

11 **a** On the same axes, sketch the graphs of $y = \sqrt{x}$ and $y = \frac{2}{x}$ **(2 marks)**

b With reference to your sketch, explain why the equation $\sqrt{x} = \frac{2}{x}$ has exactly one real root. **(1 mark)**

c Given that $f(x) = \sqrt{x} - \frac{2}{x}$, show that the equation $f(x) = 0$ has a root r, where $1 < r < 2$. **(2 marks)**

d Show that the equation $\sqrt{x} = \frac{2}{x}$ may be written in the form $x^p = q$, where p and q are integers to be found. **(2 marks)**

e Hence write down the exact value of the root of the equation $\sqrt{x} - \frac{2}{x} = 0$. **(1 mark)**

12 $f(x) = x^4 - 21x - 18$

a Show that there is a root of the equation $f(x) = 0$ in the interval $[-0.9, -0.8]$. **(3 marks)**

b Find the coordinates of any stationary points on the graph $y = f(x)$. **(3 marks)**

c Given that $f(x) = (x - 3)(x^3 + ax^2 + bx + c)$, find the values of the constants a, b and c. **(3 marks)**

d Sketch the graph of $y = f(x)$. **(3 marks)**

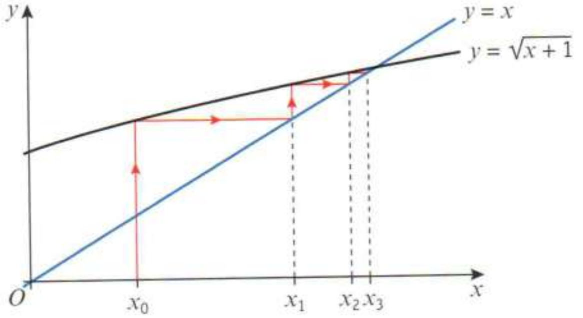
10.2 Iteration

An iterative method can be used to find a value of x for which $f(x) = 0$. To perform an iterative procedure, it is usually necessary to manipulate the algebraic function first.

- **To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.**

Some iterations will **converge** to a root. This can happen in two ways. One way is that successive iterations get closer and closer to the root from the same direction. Graphically these iterations create a series of steps. The resulting diagram is sometimes referred to as a **staircase diagram**.

$f(x) = x^2 - x - 1$ can produce the iterative formula $x_{n+1} = \sqrt{x_n + 1}$ when $f(x) = 0$. Let $x_0 = 0.5$. Successive iterations produce the following staircase diagram.



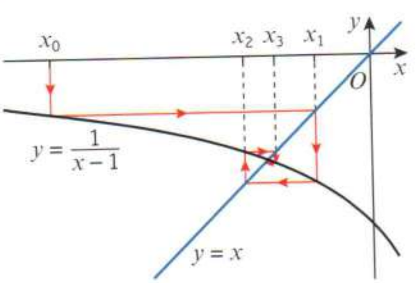
Read up from x_0 on the vertical axis to the curve $y = \sqrt{x+1}$ to find x_1. You can read across to the line $y = x$ to 'map' this value back onto the x-axis. Repeating the process shows the values of x_n converging to the root of the equation $x = \sqrt{x+1}$, which is also the root of $f(x)$.

The other way that an iteration converges is that successive iterations alternate being below the root and above the root. These iterations can still converge to the root and the resulting graph is sometimes called a **cobweb diagram**.

Watch out By rearranging the same function in different ways you can find different iterative formulae, which may converge differently.

$f(x) = x^2 - x - 1$ can produce the iterative formula $x_{n+1} = \dfrac{1}{x_n - 1}$ when $f(x) = 0$. Let $x_0 = -2$.

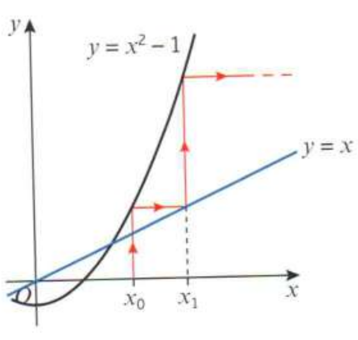
Successive iterations produce the cobweb diagram, shown on the right.



Not all iterations or starting values converge to a root. When an iteration moves away from a root, often increasingly quickly, you say that it **diverges**.

$f(x) = x^2 - x - 1$ can produce the iterative formula $x_{n+1} = x_n^2 - 1$ when $f(x) = 0$. Let $x_0 = 2$.

Successive iterations diverge from the root, as shown in the diagram.



Example 4

$f(x) = x^2 - 4x + 1$

a Show that the equation $f(x) = 0$ can be written as $x = 4 - \frac{1}{x}, x \neq 0$.

$f(x)$ has a root, α, in the interval $3 < x < 4$.

b Use the iterative formula $x_{n+1} = 4 - \frac{1}{x_n}$ with $x_0 = 3$ to find the value of x_1, x_2 and x_3.

a $f(x) = 0$

$x^2 - 4x + 1 = 0$

$x^2 = 4x - 1$

Add $4x$ to each side and subtract 1 from each side.

$x = 4 - \frac{1}{x}, x \neq 0$

Divide each term by x. This step is only valid if $x \neq 0$.

b $x_1 = 4 - \frac{1}{x_0} = 3.666666...$

$x_2 = 4 - \frac{1}{x_1} = 3.72727...$

$x_3 = 4 - \frac{1}{x_2} = 3.73170...$

Online Use the iterative formula to work out x_1, x_2 and x_3. You can use your calculator to find each value quickly.

Example 5

$f(x) = x^3 - 3x^2 - 2x + 5$

a Show that the equation $f(x) = 0$ has a root in the interval $3 < x < 4$.

b Use the iterative formula $x_{n+1} = \sqrt{\frac{x_n^3 - 2x_n + 5}{3}}$ to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places and taking:

i $x_0 = 1.5$ **ii** $x_0 = 4$

a $f(3) = (3)^3 - 3(3)^2 - 2(3) + 5 = -1$

$f(4) = (4)^3 - 3(4)^2 - 2(4) + 5 = 13$

There is a change of sign in the interval $3 < x < 4$, and f is continuous, so there is a root of $f(x)$ in this interval.

The graph crosses the x-axis between $x = 3$ and $x = 4$.

b i $x_1 = \sqrt{\frac{x_0^3 - 2x_0 + 5}{3}} = 1.3385...$

$x_2 = \sqrt{\frac{x_1^3 - 2x_1 + 5}{3}} = 1.2544...$

$x_3 = \sqrt{\frac{x_2^3 - 2x_2 + 5}{3}} = 1.2200...$

Each iteration gets closer to a root, so the sequence $x_0, x_1, x_2, x_3, ...$ is **convergent**.

ii $x_1 = \sqrt{\frac{x_0^3 - 2x_0 + 5}{3}} = 4.5092...$

$x_2 = \sqrt{\frac{x_1^3 - 2x_1 + 5}{3}} = 5.4058...$

$x_3 = \sqrt{\frac{x_2^3 - 2x_2 + 5}{3}} = 7.1219...$

Each iteration gets further from a root, so the sequence $x_0, x_1, x_2, x_3,...$ is **divergent**.

Online Explore the iterations graphically using technology.

Exercise 10B

(P) **1** $f(x) = x^2 - 6x + 2$

a Show that $f(x) = 0$ can be written as:

i $x = \frac{x^2 + 2}{6}$ **ii** $x = \sqrt{6x - 2}$ **iii** $x = 6 - \frac{2}{x}$

b Starting with $x_0 = 4$, use each iterative formula to find a root of the equation $f(x) = 0$. Round your answers to 3 decimal places.

c Use the quadratic formula to find the roots to the equation $f(x) = 0$, leaving your answer in the form $a \pm \sqrt{b}$, where a and b are constants to be found.

(P) **2** $f(x) = x^2 - 5x - 3$

a Show that $f(x) = 0$ can be written as:

i $x = \sqrt{5x + 3}$ **ii** $x = \frac{x^2 - 3}{5}$

b Let $x_0 = 5$. Show that each of the following iterative formulae gives different roots of $f(x) = 0$.

i $x_{n+1} = \sqrt{5x_n + 3}$ **ii** $x_{n+1} = \frac{x_n^2 - 3}{5}$

(E/P) **3** $f(x) = x^2 - 6x + 1$

a Show that the equation $f(x) = 0$ can be written as $x = \sqrt{6x - 1}$. **(1 mark)**

b Sketch on the same axes the graphs of $y = x$ and $y = \sqrt{6x - 1}$. **(2 marks)**

c Write down the number of roots of $f(x)$. **(1 mark)**

d Use your diagram to explain why the iterative formula $x_{n+1} = \sqrt{6x_n - 1}$ converges to a root of $f(x)$ when $x_0 = 2$. **(1 mark)**

$f(x) = 0$ can also be rearranged to form the iterative formula $x_{n+1} = \frac{x_n^2 + 1}{6}$

e By sketching a diagram, explain why the iteration diverges when $x_0 = 10$. **(2 marks)**

(P) **4** $f(x) = xe^{-x} - x + 2$

a Show that the equation $f(x) = 0$ can be written as $x = \ln\left|\frac{x}{x - 2}\right|, x \neq 2$.

$f(x)$ has a root, α, in the interval $-2 < x < -1$.

b Use the iterative formula $x_{n+1} = \ln\left|\frac{x_n}{x_n - 2}\right|, x \neq 2$ with $x_0 = -1$ to find, to 2 decimal places, the values of x_1, x_2 and x_3.

5 $f(x) = x^3 + 5x^2 - 2$

a Show that $f(x) = 0$ can be written as:

i $x = \sqrt[3]{2 - 5x^2}$ **ii** $x = \frac{2}{x^2} - 5$ **iii** $x = \sqrt{\frac{2 - x^3}{5}}$

b Starting with $x_0 = 10$, use the iterative formula in part **a (ii)** to find a root of the equation $f(x) = 0$. Round your answer to 3 decimal places.

c Starting with $x_0 = 1$, use the iterative formula in part **a (iii)** to find a different root of the equation $f(x) = 0$. Round your answer to 3 decimal places.

d Explain why this iterative formulae cannot be used when $x_0 = 2$.

6 $f(x) = x^4 - 3x^3 - 6$

a Show that the equation $f(x) = 0$ can be written as $x = \sqrt[3]{px^4 + q}$, where p and q are constants to be found. **(2 marks)**

b Let $x_0 = 0$. Use the iterative formula $x_{n+1} = \sqrt[3]{px_n^4 + q}$, together with your values of p and q from part **a**, to find, to 3 decimal places, the values of x_1, x_2 and x_3. **(3 marks)**

The root of $f(x) = 0$ is α.

c By choosing a suitable interval, prove that $\alpha = -1.132$ to 3 decimal places. **(3 marks)**

7 $f(x) = 3\cos(x^2) + x - 2$

a Show that the equation $f(x) = 0$ can be written as $x = \left(\arccos\left(\frac{2 - x}{3}\right)\right)^{\frac{1}{2}}$ **(2 marks)**

b Use the iterative formula $x_{n+1} = \left(\arccos\left(\frac{2 - x_n}{3}\right)\right)^{\frac{1}{2}}$, $x_0 = 1$ to find, to 3 decimal places, the values of x_1, x_2 and x_3. **(3 marks)**

c Given that $f(x) = 0$ has only one root, α, show that $\alpha = 1.1298$ correct to 4 decimal places. **(3 marks)**

8 $f(x) = 4\cot x - 8x + 3$, $0 < x < \pi$, where x is in radians.

a Show that there is a root α of $f(x) = 0$ in the interval [0.8, 0.9]. **(2 marks)**

b Show that the equation $f(x) = 0$ can be written in the form $x = \frac{\cos x}{2\sin x} + \frac{3}{8}$ **(3 marks)**

c Use the iterative formula $x_{n+1} = \frac{\cos x_n}{2\sin x_n} + \frac{3}{8}$, $x_0 = 0.85$ to calculate the values of x_1, x_2 and x_3 giving your answers to 4 decimal places. **(3 marks)**

d By considering the change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 0.831$ correct to 3 decimal places. **(2 marks)**

9 $g(x) = e^{x-1} + 2x - 15$

a Show that the equation $g(x) = 0$ can be written as $x = \ln(15 - 2x) + 1$, $x < \frac{15}{2}$ **(2 marks)**

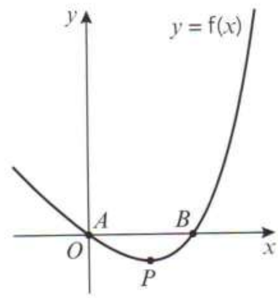
The root of $g(x) = 0$ is α.

The iterative formula $x_{n+1} = \ln(15 - 2x_n) + 1$, $x_0 = 3$, is used to find a value for α.

b Calculate the values of x_1, x_2 and x_3 to 4 decimal places. **(3 marks)**

c By choosing a suitable interval, show that $\alpha = 3.16$ correct to 2 decimal places. **(3 marks)**

E/P **10** The diagram shows a sketch of part of the curve with equation $y = f(x)$, where $f(x) = xe^x - 4x$. The curve cuts the x-axis at the points A and B and has a minimum turning point at P, as shown in the diagram.



a Work out the coordinates of A and the coordinates of B. **(3 marks)**

b Find $f'(x)$. **(3 marks)**

c Show that the x-coordinate of P lies between 0.7 and 0.8. **(2 marks)**

d Show that the x-coordinate of P is the solution to the equation $x = \ln\left(\frac{4}{x+1}\right)$. **(3 marks)**

To find an approximation for the x-coordinate of P, the iterative formula $x_{n+1} = \ln\left(\frac{4}{x_n + 1}\right)$ is used.

e Let $x_0 = 0$. Find the values of x_1, x_2, x_3 and x_4. Give your answers to 3 decimal places. **(3 marks)**

10.3 The Newton–Raphson method

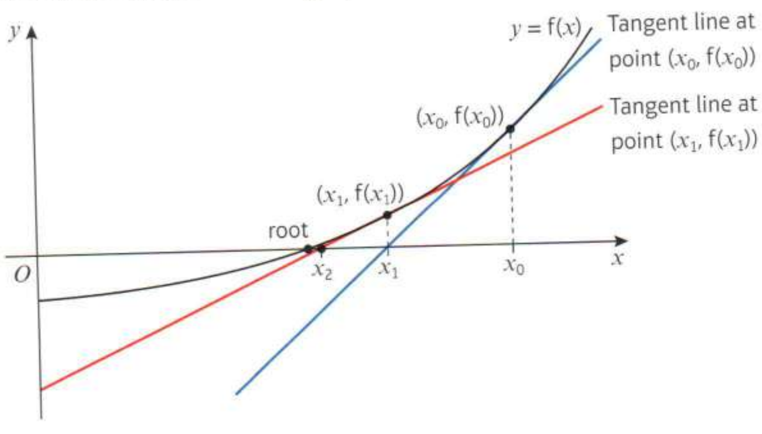
The Newton–Raphson method can be used to find numerical solutions to equations of the form $f(x) = 0$. You need to be able to differentiate $f(x)$ to use this method.

- **The Newton–Raphson formula is**

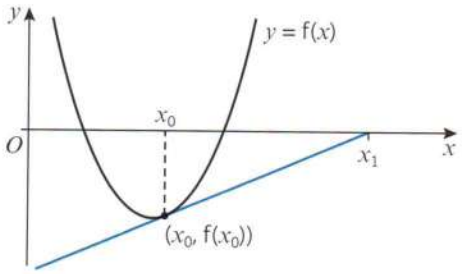
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Notation The Newton–Raphson method is sometimes called the Newton–Raphson **process** or the Newton–Raphson **procedure**.

The method uses tangent lines to find increasingly accurate approximations of a root. The value of x_{n+1} is the point at which the tangent to the graph at $(x_n, f(x_n))$ intersects the x-axis.

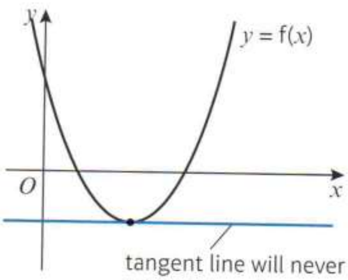


If the starting value is not chosen carefully, the Newton–Raphson method can converge on a root very slowly, or can fail completely. If the initial value, x_0, is near a turning point or the derivative at this point, $f'(x_0)$, is close to zero, then the tangent at $(x_0, f(x_0))$ will intercept the x-axis a long way from x_0.



Because x_0 is close to a turning point the gradient of the tangent at $(x_0, f(x_0))$ is small, so it intercepts the x-axis a long way from x_0.

If any value, x_i, in the Newton–Raphson method is **at** a turning point, the method will fail because $f'(x_i) = 0$ and the formula would result in division by zero, which is not valid. Graphically, the tangent line will run parallel to the x-axis, therefore never intersecting.

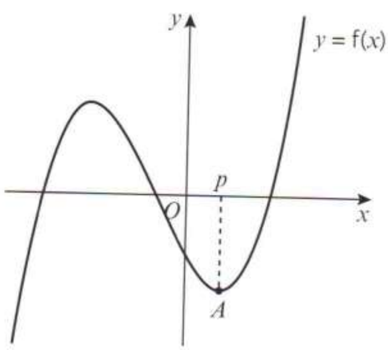


Example 6

The diagram shows part of the curve with equation $y = f(x)$, where $f(x) = x^3 + 2x^2 - 5x - 4$.

The point A, with x-coordinate p, is a stationary point on the curve.

The equation $f(x) = 0$ has a root, α, in the interval $1.8 < \alpha < 1.9$.



a Explain why $x_0 = p$ is not suitable to use as a first approximation to α when applying the Newton–Raphson method to $f(x)$.

b Using $x_0 = 2$ as a first approximation to α, apply the Newton–Raphson procedure twice to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

c By considering the change of sign in $f(x)$ over an appropriate interval, show that your answer to part **b** is accurate to 3 decimal places.

a It's a turning point, so $f'(p) = 0$, and you cannot divide by zero in the Newton-Raphson formula.

b $f'(x) = 3x^2 + 4x - 5$

Using $x_0 = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{2}{15}$$

$$x_1 = 1.8\dot{6}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 1.8\dot{6} - \frac{0.139\,8517}{12.919\,992}$$

$$x_2 = 1.8558$$

$x_2 = 1.856$ to three decimal places

c $f(1.8555) = -0.00348 < 0$,
$f(1.8565) = 0.00928 > 0$.
Sign change in interval [1.8555, 1.8565] therefore $x = 1.856$ is accurate to 3 decimal places.

Use $\frac{d}{dx}(ax^n) = anx^{n-1}$

Use the Newton–Raphson process twice.

Substitute $x_1 = 1.8\dot{6}$ into the Newton–Raphson formula.

Use a spreadsheet package to find successive Newton–Raphson approximations.

Online Explore how the Newton–Raphson method works graphically and algebraically using technology.

Exercise 10C

1 $f(x) = x^3 - 2x - 1$

a Show that the equation $f(x) = 0$ has a root, α, in the interval $1 < \alpha < 2$.

b Using $x_0 = 1.5$ as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

(E) **2** $f(x) = x^2 - \frac{4}{x} + 6x - 10,\ x \neq 0$.

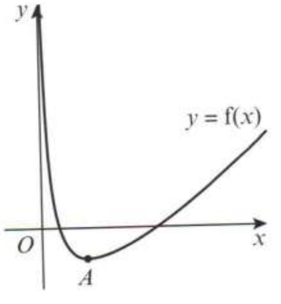
a Use differentiation to find $f'(x)$. **(2 marks)**

The root, α, of the equation $f(x) = 0$ lies in the interval $[-0.4, -0.3]$.

b Taking -0.4 as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4 marks)**

(E/P) **3** The diagram shows part of the curve with equation

$y = f(x)$, where $f(x) = x^{\frac{3}{2}} - e^{-x} + \frac{1}{\sqrt{x}} - 2,\ x > 0$.



The point A, with x-coordinate q, is a stationary point on the curve.
The equation $f(x) = 0$ has a root α in the interval [1.2, 1.3].

a Explain why $x_0 = q$ is not suitable to use as a first approximation when applying the Newton–Raphson method. **(1 mark)**

b Taking $x_0 = 1.2$ as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4 marks)**

E 4 $f(x) = 1 - x - \cos(x^2)$

a Show that the equation $f(x) = 0$ has a root α in the interval $1.4 < \alpha < 1.5$. **(1 mark)**

b Using $x_0 = 1.4$ as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. **(4 marks)**

c By considering the change of sign of $f(x)$ over an appropriate interval, show that your answer to part **b** is correct to 3 decimal places. **(2 marks)**

E 5 $f(x) = x^2 - \frac{3}{x^2}, x \geqslant 0$

a Show that a root α of the equation $f(x) = 0$ lies in the interval [1.3, 1.4]. **(1 mark)**

b Differentiate $f(x)$ to find $f'(x)$. **(2 marks)**

c By taking 1.3 as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(3 marks)**

P 6 $y = f(x)$, where $f(x) = x^2 \sin x - 2x + 1$. The points P, Q, and R are roots of the equation. The points A and B are stationary points, with x-coordinates a and b respectively.

a Show that the curve has a root in each of the following intervals:

i [0.6, 0.7] **(1 mark)**

ii [1.2, 1.3] **(1 mark)**

iii [2.4, 2.5] **(1 mark)**

b Explain why $x_0 = a$ is not suitable to use as a first approximation to α when applying the Newton–Raphson method to $f(x)$. **(1 mark)**

c Using $x_0 = 2.4$ as a first approximation, apply the Newton–Raphson method to $f(x)$ to obtain a second approximation. Give your answer to 3 decimal places. **(4 marks)**

E 7 $f(x) = \ln(3x - 4) - x^2 + 10, x > \frac{4}{3}$

a Show that $f(x) = 0$ has a root α in the interval [3.4, 3.5]. **(2 marks)**

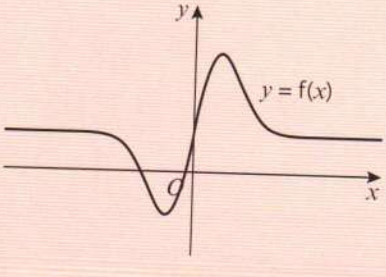
b Find $f'(x)$. **(2 marks)**

c Taking 3.4 as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to obtain a second approximation for α, giving your answer to 3 decimal places. **(3 marks)**

Challenge

$f(x) = \frac{1}{5} + xe^{-x^2}$

The diagram shows a sketch of the curve $y = f(x)$. The curve has a horizontal asymptote at $y = \frac{1}{5}$.



a Prove that the Newton–Raphson method will fail to converge on a root of $f(x) = 0$ for all values of $x_0 > \frac{1}{\sqrt{2}}$

b Taking −0.5 as a first approximation, use the Newton–Raphson method to find the root of $f(x) = 0$ that lies in the interval [−1, 0], giving your answer to 3 d.p.

10.4 Applications to modelling

You can use the techniques from this chapter to find solutions to models of real-life situations.

Example 7

The price of a car in £s, x years after purchase, is modelled by the function

$$f(x) = 15\,000\,(0.85)^x - 1000 \sin x,\ x > 0$$

a Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.

b Show that $f(x)$ has a root between 19 and 20.

c Find $f'(x)$.

d Taking 19.5 as a first approximation, apply the Newton–Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

e Criticise this model with respect to the value of the car as it gets older.

a $f(10) = 15\,000\,(0.85)^{10} - 1000 \sin 10$
$= 3497.13\ldots$

After 10 years the value of the car is £3500 to the nearest £100.

Substitute $x = 10$ into the $f(x)$.
Unless otherwise stated, assume that angles are measured in radians.

b $f(19) = 15\,000\,(0.85)^{19} - 1000 \sin 19$
$= 534.11\ldots > 0$

$f(20) = 15\,000\,(0.85)^{20} - 1000 \sin 20$
$= -331.55\ldots < 0$

Substitute $x = 19$ and $x = 20$ into $f(x)$.

There is a change of sign between 19 and 20, so there is at least one root in the interval $19 < x < 20$.

$f(x)$ changes sign in the interval [19, 20], and $f(x)$ is continuous, so $f(x)$ must equal zero within this interval.

c $f'(x) = (15\,000)(0.85)^x(\ln 0.85) - 1000 \cos x$

Use the fact that $\frac{d}{dx}(a^x) = a^x \ln a$.

d $f(19.5) = 15\,000\,(0.85)^{19.5} - 1000 \sin 19.5$
$= 25.0693\ldots$

$f'(19.5) = (15\,000)(0.85)^{19.5}(\ln 0.85) - 1000 \cos 19.5 = -898.3009\ldots$

Substitute $x = 19.5$ into $f(x)$ and $f'(x)$.

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

$$= 19.5 - \frac{25.0693\ldots}{-898.3009\ldots}$$

$$= 19.528$$

Apply the Newton–Raphson method once to obtain an improved second estimate.

e In reality, the car can never have a negative value so this model is not reasonable for cars that are approximately 20 or more years old.

Exercise 10D

P 1 An astronomer is studying the motion of a planet moving along an elliptical orbit. She formulates the following model relating the angle moved at a given time, E radians, to the angle the planet would have moved if it had been travelling on a circular path, M radians:

$$M = E - 0.1\sin E,\ E \geqslant 0$$

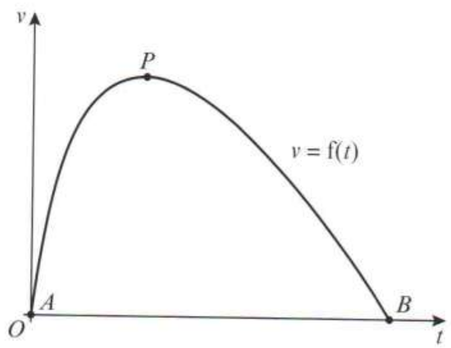
In order to predict the position of the planet at a particular time, the astronomer needs to find the value of E when $M = \frac{\pi}{6}$

a Show that this value of E is a root of the function $f(x) = x - 0.1\sin x - k$ where k is a constant to be determined.

b Taking 0.6 as a first approximation, apply the Newton–Raphson procedure once to $f(x)$ to obtain a second approximation for the value of E when $M = \frac{\pi}{6}$

c By considering a change of sign on a suitable interval of $f(x)$, show that your answer to part **b** is correct to 3 decimal places.

P 2 The diagram shows a sketch of part of the curve with equation $v = f(t)$, where $f(t) = \left(10 - \frac{1}{2}(t + 1)\right)\ln(t + 1)$. The function models the velocity in m/s of a skier travelling in a straight line.



a Find the coordinates of A and B.

b Find $f'(t)$.

c Given that P is a stationary point on the curve, show that the t-coordinate of P lies between 5.8 and 5.9.

d Show that the t-coordinate of P is the solution to

$$t = \frac{20}{1 + \ln(t + 1)} - 1$$

An approximation for the t-coordinate of P is found using the iterative formula

$$t_{n+1} = \frac{20}{1 + \ln(t_n + 1)} - 1$$

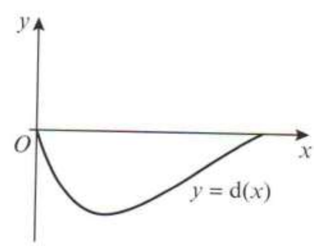
e Let $t_0 = 5$. Find the values of t_1, t_2 and t_3. Give your answers to 3 decimal places.

3 The depth of a stream is modelled by the function

$$d(x) = e^{-0.6x}(x^2 - 3x),\ 0 \leqslant x \leqslant 3$$

where x is the distance in metres from the left bank of the stream and $d(x)$ is the depth of the stream in metres.

The diagram shows a sketch of $y = d(x)$.



a Explain the condition $0 \leqslant x \leqslant 3$.

b Show that $d'(x) = -\frac{1}{5}e^{-0.6x}(ax^2 + bx + c)$, where a, b and c are constants to be found.

c Show that $d'(x) = 0$ can be written in the following ways:

i $x = \sqrt{\frac{19x - 15}{3}}$ **ii** $x = \frac{3x^2 + 15}{19}$ **iii** $x = \frac{19x - 15}{3x}$

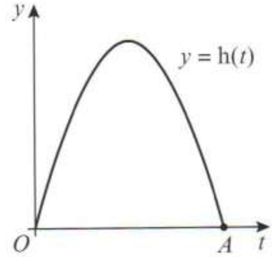
d Let $x_0 = 1$. Show that only one of the three iterations converges to a stationary point of $y = d(x)$, and find the x-coordinate at this point correct to 3 decimal places.

e Find the maximum depth of the river in metres to 2 decimal places.

(E/P) **4** Ed throws a ball for his dog. The vertical height of the ball is modelled by the function

$$h(t) = 40\sin\left(\frac{t}{10}\right) - 9\cos\left(\frac{t}{10}\right) - 0.5t^2 + 9,\ t \geqslant 0$$

$y = h(t)$ is shown in the diagram.



a Show that the t-coordinate of A is the solution to

$$t = \sqrt{18 + 80\sin\left(\frac{t}{10}\right) - 18\cos\left(\frac{t}{10}\right)}$$ **(3 marks)**

To find an approximation for the t-coordinate of A, the iterative formula

$t_{n+1} = \sqrt{18 + 80\sin\left(\frac{t_n}{10}\right) - 18\cos\left(\frac{t_n}{10}\right)}$ is used.

b Let $t_0 = 8$. Find the values of t_1, t_2, t_3 and t_4. Give your answers to 3 decimal places. **(3 marks)**

c Find $h'(t)$. **(2 marks)**

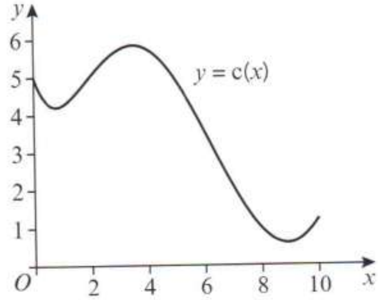
d Taking 8 as a first approximation, apply the Newton–Raphson method once to $h(t)$ to obtain a second approximation for the time when the height of the ball is zero. Give your answer to 3 decimal places. **(3 marks)**

e Hence suggest an improvement to the range of validity of the model. **(2 marks)**

(E/P) **5** The annual number of non-violent crimes, in thousands, in a large town x years after the year 2000 is modelled by the function

$$c(x) = 5e^{-x} + 4\sin\left(\frac{x}{2}\right) + \frac{x}{2},\ 0 \leqslant x \leqslant 10$$

The diagram shows the graph of $y = c(x)$.



a Find $c'(x)$. **(2 marks)**

b Show that the roots of the following equations correspond to the turning points on the graph of $y = c(x)$.

i $x = 2\arccos\left(\frac{5}{2}e^{-x} - \frac{1}{4}\right)$ **(2 marks)**

ii $x = \ln\left(\frac{10}{4\cos\left(\frac{x}{2}\right) + 1}\right)$ **(2 marks)**

c Let $x_0 = 3$ and $x_{n+1} = 2\arccos\left(\frac{5}{2}e^{-x_n} - \frac{1}{4}\right)$. Find the values of x_1, x_2, x_3 and x_4. Give your answers to 3 decimal places. **(3 marks)**

d Let $x_0 = 1$ and $x_{n+1} = \ln\left(\frac{10}{4\cos\left(\frac{x_n}{2}\right) + 1}\right)$. Find the values of x_1, x_2, x_3 and x_4. Give your answers to 3 decimal places. **(3 marks)**

A councillor states that the number of non-violent crimes in the town was increasing between October 2000 and June 2003.

e State, with reasons whether the model supports this claim. **(2 marks)**

Mixed exercise 10

/P **1** $f(x) = x^3 - 6x - 2$

a Show that the equation $f(x) = 0$ can be written in the form $x = \pm\sqrt{a + \frac{b}{x}}$, and state the values of the integers a and b. **(2 marks)**

$f(x) = 0$ has one positive root, α.

The iterative formula $x_{n+1} = \sqrt{a + \frac{b}{x_n}}$, $x_0 = 2$ is used to find an approximate value for α.

b Calculate the values of x_1, x_2, x_3 and x_4 to 4 decimal places. **(3 marks)**

c By choosing a suitable interval, show that $\alpha = 2.602$ is correct to 3 decimal places. **(3 marks)**

/P **2** $f(x) = \frac{1}{4 - x} + 3$

a Calculate f(3.9) and f(4.1). **(2 marks)**

b Explain why the equation $f(x) = 0$ does not have a root in the interval $3.9 < x < 4.1$. **(2 marks)**

The equation $f(x) = 0$ has a single root, α.

c Use algebra to find the exact value of α. **(2 marks)**

/P **3** $p(x) = 4 - x^2$ and $q(x) = e^x$.

a On the same axes, sketch the curves of $y = p(x)$ and $y = q(x)$. **(2 marks)**

b State the number of positive roots and the number of negative roots of the equation $x^2 + e^x - 4 = 0$. **(1 mark)**

c Show that the equation $x^2 + e^x - 4 = 0$ can be written in the form $x = \pm(4 - e^x)^{\frac{1}{2}}$ **(2 marks)**

The iterative formula $x_{n+1} = -(4 - e^{x_n})^{\frac{1}{2}}$, $x_0 = -2$, is used to find an approximate value for the negative root.

d Calculate the values of x_1, x_2, x_3 and x_4 to 4 decimal places. **(3 marks)**

e Explain why the starting value $x_0 = 1.4$ will not produce a valid result with this formula. **(2 marks)**

E/P **4** $g(x) = x^5 - 5x - 6$

a Show that $g(x) = 0$ has a root, α, between $x = 1$ and $x = 2$. **(2 marks)**

b Show that the equation $g(x) = 0$ can be written as $x = (px + q)^{\frac{1}{r}}$, where p, q and r are integers to be found. **(2 marks)**

The iterative formula $x_{n+1} = (px + q)^{\frac{1}{r}}$, $x_0 = 1$ is used to find an approximate value for α.

c Calculate the values of x_1, x_2 and x_3 to 4 decimal places. **(3 marks)**

d By choosing a suitable interval, show that $\alpha = 1.708$ is correct to 3 decimal places. **(3 marks)**

E/P **5** $g(x) = x^2 - 3x - 5$

a Show that the equation $g(x) = 0$ can be written as $x = \sqrt{3x + 5}$. **(1 mark)**

b Sketch on the same axes the graphs of $y = x$ and $y = \sqrt{3x + 5}$. **(2 marks)**

c Use your diagram to explain why the iterative formula $x_{n+1} = \sqrt{3x_n + 5}$ converges to a root of $g(x)$ when $x_0 = 1$. **(1 mark)**

$g(x) = 0$ can also be rearranged to form the iterative formula $x_{n+1} = \dfrac{x_n^2 - 5}{3}$

d With reference to a diagram, explain why this iterative formula diverges when $x_0 = 7$. **(3 marks)**

E/P **6** $f(x) = 5x - 4\sin x - 2$, where x is in radians.

a Show that $f(x) = 0$ has a root, α, between $x = 1.1$ and $x = 1.15$. **(2 marks)**

b Show that $f(x) = 0$ can be written as $x = p\sin x + q$, where p and q are rational numbers to be found. **(2 marks)**

c Starting with $x_0 = 1.1$, use the iterative formula $x_{n+1} = p\sin x_n + q$ with your values of p and q to calculate the values of x_1, x_2, x_3 and x_4 to 3 decimal places. **(3 marks)**

E/P **7** **a** On the same axes, sketch the graphs of $y = \dfrac{1}{x}$ and $y = x + 3$. **(2 marks)**

b Write down the number of roots of the equation $\dfrac{1}{x} = x + 3$. **(1 mark)**

c Show that the positive root of the equation $\dfrac{1}{x} = x + 3$ lies in the interval $(0.30, 0.31)$. **(2 marks)**

d Show that the equation $\dfrac{1}{x} = x + 3$ may be written in the form $x^2 + 3x - 1 = 0$. **(2 marks)**

e Use the quadratic formula to find the positive root of the equation $x^2 + 3x - 1 = 0$ to 3 decimal places. **(2 marks)**

E/P **8** $g(x) = x^3 - 7x^2 + 2x + 4$

a Find $g'(x)$. **(2 marks)**

A root α of the equation $g(x) = 0$ lies in the interval $[6.5, 6.7]$.

b Taking 6.6 as a first approximation to α, apply the Newton–Raphson process once to $g(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4 marks)**

c Given that $g(1) = 0$, find the exact value of the other two roots of $g(x)$. **(3 marks)**

d Calculate the percentage error of your answer in part **b**. **(2 marks)**

E/P **9** $f(x) = 2\sec x + 2x - 3, -\frac{\pi}{2} < x < \frac{\pi}{2}$ where x is in radians.

a Show that $f(x) = 0$ has a solution, α, in the interval $0.4 < x < 0.5$. **(2 marks)**

b Taking 0.4 as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4 marks)**

c Show that $x = -1.190$ is a different solution, β, of $f(x) = 0$ correct to 3 decimal places. **(2 marks)**

E/P **10** $f(x) = e^{0.8x} - \frac{1}{3 - 2x}, x \neq \frac{3}{2}$

a Show that the equation $f(x) = 0$ can be written as $x = 1.5 - 0.5e^{-0.8x}$. **(3 marks)**

b Use the iterative formula $x_{n+1} = 1.5 - 0.5e^{-0.8x_n}$ with $x_0 = 1.3$ to obtain x_1, x_2 and x_3. Hence write down one root of $f(x) = 0$ correct to 3 decimal places. **(2 marks)**

c Show that the equation $f(x) = 0$ can be written in the form $x = p\ln(3 - 2x)$, stating the value of p. **(3 marks)**

d Use the iterative formula $x_{n+1} = p\ln(3 - 2x_n)$ with $x_0 = -2.6$ and the value of p found in part **c** to obtain x_1, x_2 and x_3. Hence write down a second root of $f(x) = 0$ correct to 2 decimal places. **(2 marks)**

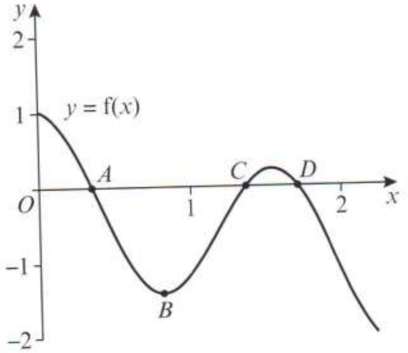
E/P **11** **a** By writing $y = x^x$ in the form $\ln y = x\ln x$, show that $\frac{dy}{dx} = x^x(\ln x + 1)$. **(4 marks)**

b Show that the function $f(x) = x^x - 2$ has a root, α, in the interval [1.4, 1.6]. **(2 marks)**

c Taking $x_0 = 1.5$ as a first approximation to α, apply the Newton–Raphson procedure once to obtain a second approximation to α, giving your answer to 4 decimal places. **(4 marks)**

d By considering a change of sign of $f(x)$ over a suitable interval, show that $\alpha = 1.5596$, correct to 4 decimal places. **(3 marks)**

E/P **12** The diagram shows part of the curve with equation $y = f(x)$, where $f(x) = \cos(4x) - \frac{1}{2}x$.



a Show that the curve has a root in the interval [1.3, 1.4]. **(2 marks)**

b Use differentiation to find the coordinates of point B. Write each coordinate correct to 3 decimal places. **(3 marks)**

c Using the iterative formula $x_{n+1} = \frac{1}{4}\arccos\left(\frac{1}{2}x_n\right)$, with $x_0 = 0.4$, find the values of x_1, x_2, x_3 and x_4. Give your answers to 4 decimal places. **(3 marks)**

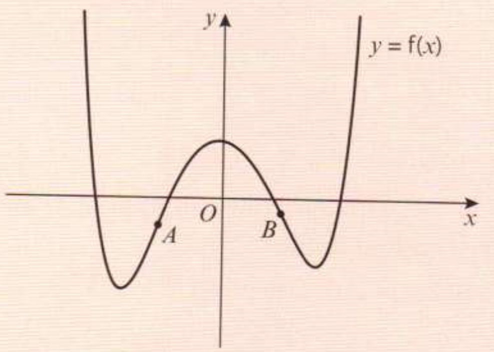
d Using $x_0 = 1.7$ as a first approximation to the root at D, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to the root, giving your answer to 3 decimal places. **(4 marks)**

e By considering the change of sign of $f(x)$ over an appropriate interval, show that the answer to part **d** is accurate to 3 decimal places. **(2 marks)**

Challenge

$f(x) = x^6 + x^3 - 7x^2 - x + 3$

The diagram shows a sketch of $y = f(x)$. Points A and B are the points of inflection on the curve.



a Show that equation $f''(x) = 0$ can be written as:

i $x = \dfrac{7 - 15x^4}{3}$ **ii** $x = \dfrac{7}{15x^3 + 3}$ **iii** $x = \sqrt[4]{\dfrac{7 - 3x}{15}}$

b By choosing a suitable iterative formula and starting value, find an approximation for the x-coordinate of B, correct to 3 decimal places.

c Explain why you cannot use the same iterative formula to find an approximation for the x-coordinate of A.

d Use the Newton–Raphson method to find an estimate for the x-coordinate of A, correct to 3 decimal places.

Summary of key points

1 If the function $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has at least one root, x, which satisfies $a < x < b$.

2 To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.

3 The Newton–Raphson formula for approximating the roots of a function $f(x)$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$