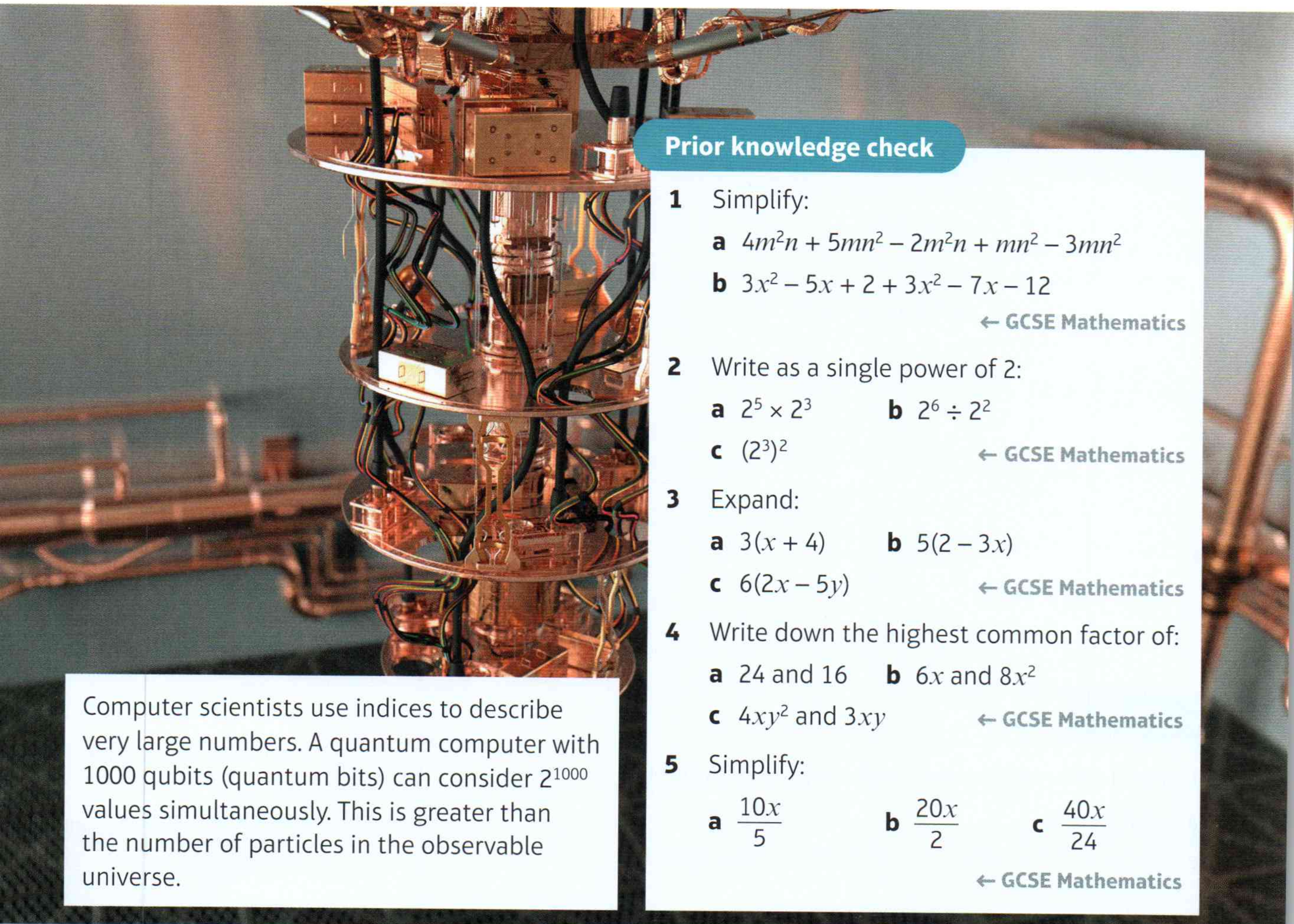
Algebraic expressions 1

Objectives

After completing this chapter you should be able to:

- Multiply and divide integer powers → pages 2–3
- Expand a single term over brackets and collect like terms → pages 3–4
- Expand the product of two or three expressions → pages 4–6
- Factorise linear, quadratic and simple cubic expressions → pages 6–9
- Know and use the laws of indices → pages 9–11
- Simplify and use the rules of surds → pages 12–13
- Rationalise denominators → pages 13–16



Computer scientists use indices to describe very large numbers. A quantum computer with 1000 qubits (quantum bits) can consider 2^{1000} values simultaneously. This is greater than the number of particles in the observable universe.

Prior knowledge check

1 Simplify:

a $4m^2n + 5mn^2 - 2m^2n + mn^2 - 3mn^2$

b $3x^2 - 5x + 2 + 3x^2 - 7x - 12$

← GCSE Mathematics

2 Write as a single power of 2:

a $2^5 \times 2^3$ **b** $2^6 \div 2^2$

c $(2^3)^2$

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3 Expand:

a $3(x + 4)$ **b** $5(2 - 3x)$

c $6(2x - 5y)$

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4 Write down the highest common factor of:

a 24 and 16 **b** $6x$ and $8x^2$

c $4xy^2$ and $3xy$

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5 Simplify:

a $\frac{10x}{5}$ **b** $\frac{20x}{2}$ **c** $\frac{40x}{24}$

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1.1 Index laws

- **You can use the laws of indices to simplify powers of the same base.**
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $(ab)^n = a^n b^n$

Notation

x^5 — x: This is the **base**.

5: This is the **index**, **power** or **exponent**.

Example 1

Simplify these expressions:

a $x^2 \times x^5$ **b** $2r^2 \times 3r^3$ **c** $\frac{b^7}{b^4}$ **d** $6x^5 \div 3x^3$ **e** $(a^3)^2 \times 2a^2$ **f** $(3x^2)^3 \div x^4$

a $x^2 \times x^5 = x^{2+5} = x^7$

Use the rule $a^m \times a^n = a^{m+n}$ to simplify the index.

b $2r^2 \times 3r^3 = 2 \times 3 \times r^2 \times r^3$

$= 6 \times r^{2+3} = 6r^5$

Rewrite the expression with the numbers together and the r terms together.

$2 \times 3 = 6$

$r^2 \times r^3 = r^{2+3}$

c $\frac{b^7}{b^4} = b^{7-4} = b^3$

Use the rule $a^m \div a^n = a^{m-n}$ to simplify the index.

d $6x^5 \div 3x^3 = \frac{6}{3} \times \frac{x^5}{x^3}$

$x^5 \div x^3 = x^{5-3} = x^2$

$= 2 \times x^2 = 2x^2$

e $(a^3)^2 \times 2a^2 = a^6 \times 2a^2$

Use the rule $(a^m)^n = a^{mn}$ to simplify the index.

$= 2 \times a^6 \times a^2 = 2a^8$

$a^6 \times a^2 = a^{6+2} = a^8$

f $\frac{(3x^2)^3}{x^4} = 3^3 \times \frac{(x^2)^3}{x^4}$

Use the rule $(ab)^n = a^n b^n$ to simplify the numerator.

$(x^2)^3 = x^{2 \times 3} = x^6$

$= 27 \times \frac{x^6}{x^4} = 27x^2$

$\frac{x^6}{x^4} = x^{6-4} = x^2$

Example 2

Expand these expressions and simplify if possible:

a $-3x(7x - 4)$ **b** $y^2(3 - 2y^3)$

c $4x(3x - 2x^2 + 5x^3)$ **d** $2x(5x + 3) - 5(2x + 3)$

Watch out A minus sign outside brackets changes the sign of every term inside the brackets.

a $-3x(7x - 4) = -21x^2 + 12x$

b $y^2(3 - 2y^3) = 3y^2 - 2y^5$

c $4x(3x - 2x^2 + 5x^3)$
$= 12x^2 - 8x^3 + 20x^4$

d $2x(5x + 3) - 5(2x + 3)$
$= 10x^2 + 6x - 10x - 15$
$= 10x^2 - 4x - 15$

$-3x \times 7x = -21x^{1+1} = -21x^2$
$-3x \times (-4) = +12x$

$y^2 \times (-2y^3) = -2y^{2+3} = -2y^5$

Remember a minus sign outside the brackets changes the signs within the brackets.

Simplify $6x - 10x$ to give $-4x$.

Example 3

Simplify these expressions:

a $\dfrac{x^7 + x^4}{x^3}$ **b** $\dfrac{3x^2 - 6x^5}{2x}$ **c** $\dfrac{20x^7 + 15x^3}{5x^2}$

a $\dfrac{x^7 + x^4}{x^3} = \dfrac{x^7}{x^3} + \dfrac{x^4}{x^3}$
$= x^{7-3} + x^{4-3} = x^4 + x$

b $\dfrac{3x^2 - 6x^5}{2x} = \dfrac{3x^2}{2x} - \dfrac{6x^5}{2x}$
$= \dfrac{3}{2}x^{2-1} - 3x^{5-1} = \dfrac{3x}{2} - 3x^4$

c $\dfrac{20x^7 + 15x^3}{5x^2} = \dfrac{20x^7}{5x^2} + \dfrac{15x^3}{5x^2}$
$= 4x^{7-2} + 3x^{3-2} = 4x^5 + 3x$

Divide each term of the numerator by x^3.

x^1 is the same as x.

Divide each term of the numerator by $2x$.

Simplify each fraction:
$\dfrac{3x^2}{2x} = \dfrac{3}{2} \times \dfrac{x^2}{x} = \dfrac{3}{2} \times x^{2-1}$
$-\dfrac{6x^5}{2x} = -\dfrac{6}{2} \times \dfrac{x^5}{x} = -3 \times x^{5-1}$

Divide each term of the numerator by $5x^2$.

Exercise 1A

1 Simplify these expressions:

a $x^3 \times x^4$ **b** $2x^3 \times 3x^2$ **c** $\dfrac{k^3}{k^2}$

d $\dfrac{4p^3}{2p}$ **e** $\dfrac{3x^3}{3x^2}$ **f** $(y^2)^5$

g $10x^5 \div 2x^3$ **h** $(p^3)^2 \div p^4$ **i** $(2a^3)^2 \div 2a^3$

j $8p^4 \div 4p^3$ **k** $2a^4 \times 3a^5$ **l** $\dfrac{21a^3b^7}{7ab^4}$

m $9x^2 \times 3(x^2)^3$ **n** $3x^3 \times 2x^2 \times 4x^6$ **o** $7a^4 \times (3a^4)^2$

p $(4y^3)^3 \div 2y^3$ **q** $2a^3 \div 3a^2 \times 6a^5$ **r** $3a^4 \times 2a^5 \times a^3$

2 Expand and simplify if possible:

a $9(x-2)$ **b** $x(x+9)$ **c** $-3y(4-3y)$

d $x(y+5)$ **e** $-x(3x+5)$ **f** $-5x(4x+1)$

g $(4x+5)x$ **h** $-3y(5-2y^2)$ **i** $-2x(5x-4)$

j $(3x-5)x^2$ **k** $3(x+2)+(x-7)$ **l** $5x-6-(3x-2)$

m $4(c+3d^2)-3(2c+d^2)$ **n** $(r^2+3t^2+9)-(2r^2+3t^2-4)$

o $x(3x^2-2x+5)$ **p** $7y^2(2-5y+3y^2)$ **q** $-2y^2(5-7y+3y^2)$

r $7(x-2)+3(x+4)-6(x-2)$ **s** $5x-3(4-2x)+6$

t $3x^2-x(3-4x)+7$ **u** $4x(x+3)-2x(3x-7)$ **v** $3x^2(2x+1)-5x^2(3x-4)$

3 Simplify these fractions:

a $\dfrac{6x^4+10x^6}{2x}$ **b** $\dfrac{3x^5-x^7}{x}$ **c** $\dfrac{2x^4-4x^2}{4x}$

d $\dfrac{8x^3+5x}{2x}$ **e** $\dfrac{7x^7+5x^2}{5x}$ **f** $\dfrac{9x^5-5x^3}{3x}$

1.2 Expanding brackets

To find the **product** of two expressions you **multiply** each term in one expression by each term in the other expression.

Multiplying each of the 2 terms in the first expression by each of the 3 terms in the second expression gives 2 × 3 = 6 terms.

$x\times$

$$(x+5)(4x-2y+3) = x(4x-2y+3)+5(4x-2y+3)$$
$$= 4x^2-2xy+3x+20x-10y+15$$
$$= 4x^2-2xy+23x-10y+15$$

$5\times$

Simplify your answer by collecting like terms.

Example 4

Expand these expressions and simplify if possible:

a $(x+5)(x+2)$ **b** $(x-2y)(x^2+1)$ **c** $(x-y)^2$ **d** $(x+y)(3x-2y-4)$

a $(x+5)(x+2)$

$= x^2+2x+5x+10$

$= x^2+7x+10$

b $(x-2y)(x^2+1)$

$= x^3+x-2x^2y-2y$

Multiply x by $(x+2)$ and then multiply 5 by $(x+2)$.

Simplify your answer by collecting like terms.

$-2y\times x^2=-2x^2y$

There are no like terms to collect.

c $(x - y)^2$
$= (x - y)(x - y)$
$= x^2 - xy - xy + y^2$
$= x^2 - 2xy + y^2$

$(x - y)^2$ means $(x - y)$ multiplied by itself.

$-xy - xy = -2xy$

d $(x + y)(3x - 2y - 4)$
$= x(3x - 2y - 4) + y(3x - 2y - 4)$
$= 3x^2 - 2xy - 4x + 3xy - 2y^2 - 4y$
$= 3x^2 + xy - 4x - 2y^2 - 4y$

Multiply x by $(3x - 2y - 4)$ and then multiply y by $(3x - 2y - 4)$.

Example 5

Expand these expressions and simplify if possible:

a $x(2x + 3)(x - 7)$ **b** $x(5x - 3y)(2x - y + 4)$ **c** $(x - 4)(x + 3)(x + 1)$

a $x(2x + 3)(x - 7)$
$= (2x^2 + 3x)(x - 7)$
$= 2x^3 - 14x^2 + 3x^2 - 21x$
$= 2x^3 - 11x^2 - 21x$

Start by expanding one pair of brackets: $x(2x + 3) = 2x^2 + 3x$

You could also have expanded the second pair of brackets first: $(2x + 3)(x - 7) = 2x^2 - 11x - 21$
Then multiply by x.

b $x(5x - 3y)(2x - y + 4)$
$= (5x^2 - 3xy)(2x - y + 4)$
$= 5x^2(2x - y + 4) - 3xy(2x - y + 4)$
$= 10x^3 - 5x^2y + 20x^2 - 6x^2y + 3xy^2 - 12xy$
$= 10x^3 - 11x^2y + 20x^2 + 3xy^2 - 12xy$

Be careful with minus signs. You need to change every sign in the second pair of brackets when you multiply it out.

c $(x - 4)(x + 3)(x + 1)$
$= (x^2 - x - 12)(x + 1)$
$= x^2(x + 1) - x(x + 1) - 12(x + 1)$
$= x^3 + x^2 - x^2 - x - 12x - 12$
$= x^3 - 13x - 12$

Choose one pair of brackets to expand first, for example:
$(x - 4)(x + 3) = x^2 + 3x - 4x - 12$
$= x^2 - x - 12$

You multiplied together three linear terms, so the final answer contains an x^3 term.

Exercise 1B

1 Expand and simplify if possible:

a $(x + 4)(x + 7)$ **b** $(x - 3)(x + 2)$ **c** $(x - 2)^2$
d $(x - y)(2x + 3)$ **e** $(x + 3y)(4x - y)$ **f** $(2x - 4y)(3x + y)$
g $(2x - 3)(x - 4)$ **h** $(3x + 2y)^2$ **i** $(2x + 8y)(2x + 3)$
j $(x + 5)(2x + 3y - 5)$ **k** $(x - 1)(3x - 4y - 5)$ **l** $(x - 4y)(2x + y + 5)$
m $(x + 2y - 1)(x + 3)$ **n** $(2x + 2y + 3)(x + 6)$ **o** $(4 - y)(4y - x + 3)$
p $(4y + 5)(3x - y + 2)$ **q** $(5y - 2x + 3)(x - 4)$ **r** $(4y - x - 2)(5 - y)$

2 Expand and simplify if possible:

a $5(x+1)(x-4)$ **b** $7(x-2)(2x+5)$ **c** $3(x-3)(x-3)$

d $x(x-y)(x+y)$ **e** $x(2x+y)(3x+4)$ **f** $y(x-5)(x+1)$

g $y(3x-2y)(4x+2)$ **h** $y(7-x)(2x-5)$ **i** $x(2x+y)(5x-2)$

j $x(x+2)(x+3y-4)$ **k** $y(2x+y-1)(x+5)$ **l** $y(3x+2y-3)(2x+1)$

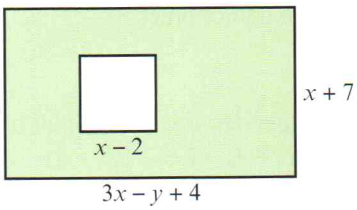
m $x(2x+3)(x+y-5)$ **n** $2x(3x-1)(4x-y-3)$ **o** $3x(x-2y)(2x+3y+5)$

p $(x+3)(x+2)(x+1)$ **q** $(x+2)(x-4)(x+3)$ **r** $(x+3)(x-1)(x-5)$

s $(x-5)(x-4)(x-3)$ **t** $(2x+1)(x-2)(x+1)$ **u** $(2x+3)(3x-1)(x+2)$

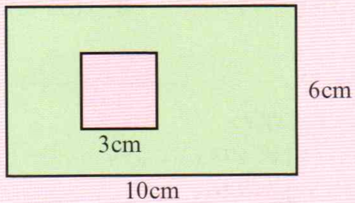
v $(3x-2)(2x+1)(3x-2)$ **w** $(x+y)(x-y)(x-1)$ **x** $(2x-3y)^3$

(P) **3** The diagram shows a rectangle with a square cut out.
The rectangle has length $3x - y + 4$ and width $x + 7$.
The square has length $x - 2$.
Find an expanded and simplified expression for the shaded area.



Problem-solving

Use the same strategy as you would use if the lengths were given as numbers:



(P) **4** A cuboid has dimensions $x + 2$ cm, $2x - 1$ cm and $2x + 3$ cm.
Show that the volume of the cuboid is $4x^3 + 12x^2 + 5x - 6\text{ cm}^3$.

(E/P) **5** Given that $(2x + 5y)(3x - y)(2x + y) = ax^3 + bx^2y + cxy^2 + dy^3$, where a, b, c and d are constants, find the values of a, b, c and d. **(2 marks)**

Challenge

Expand and simplify $(x + y)^4$.

Links You can use the binomial expansion to expand expressions like $(x + y)^4$ quickly. → Section 8.3

1.3 Factorising

You can write expressions as a **product of their factors**.

- **Factorising is the opposite of expanding brackets.**

Expanding brackets →

$4x(2x + y) = 8x^2 + 4xy$

$(x + 5)^3 = x^3 + 15x^2 + 75x + 125$

$(x + 2y)(x - 5y) = x^2 - 3xy - 10y^2$

← Factorising

Example 6

Factorise these expressions completely:

a $3x + 9$ **b** $x^2 - 5x$ **c** $8x^2 + 20x$ **d** $9x^2y + 15xy^2$ **e** $3x^2 - 9xy$

a $3x + 9 = 3(x + 3)$ — 3 is a common factor of $3x$ and 9.

b $x^2 - 5x = x(x - 5)$ — x is a common factor of x^2 and $-5x$.

c $8x^2 + 20x = 4x(2x + 5)$ — 4 and x are common factors of $8x^2$ and $20x$. So take $4x$ outside the brackets.

d $9x^2y + 15xy^2 = 3xy(3x + 5y)$ — 3, x and y are common factors of $9x^2y$ and $15xy^2$. So take $3xy$ outside the brackets.

e $3x^2 - 9xy = 3x(x - 3y)$ — x and $-3y$ have no common factors so this expression is completely factorised.

■ **A quadratic expression has the form $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.**

Notation Real numbers are all the positive and negative numbers, or zero, including fractions and surds.

To factorise a quadratic expression:

- Find two factors of ac that add up to b — For the expression $2x^2 + 5x - 3$, $ac = -6 = -1 \times 6$ and $-1 + 6 = 5 = b$.
- Rewrite the b term as a sum of these two factors — $2x^2 - x + 6x - 3$
- Factorise each pair of terms — $= x(2x - 1) + 3(2x - 1)$
- Take out the common factor — $= (x + 3)(2x - 1)$

■ $x^2 - y^2 = (x + y)(x - y)$

Notation An expression in the form $x^2 - y^2$ is called the **difference** of two squares.

Example 7

Factorise:

a $x^2 - 5x - 6$ **b** $x^2 + 6x + 8$ **c** $6x^2 - 11x - 10$ **d** $x^2 - 25$ **e** $4x^2 - 9y^2$

a $x^2 - 5x - 6$ — Here $a = 1$, $b = -5$ and $c = -6$.

$ac = -6$ and $b = -5$

So $x^2 - 5x - 6 = x^2 + x - 6x - 6$ — (1) Work out the two factors of $ac = -6$ which add to give you $b = -5$. $-6 + 1 = -5$ (2) Rewrite the b term using these two factors.

$= x(x + 1) - 6(x + 1)$ — (3) Factorise first two terms and last two terms.

$= (x + 1)(x - 6)$ — (4) $x + 1$ is a factor of both terms, so take that outside the brackets. This is now completely factorised.

b $x^2 + 6x + 8$
$= x^2 + 2x + 4x + 8$ — $ac = 8$ and $2 + 4 = 6 = b$.
$= x(x + 2) + 4(x + 2)$ — Factorise.
$= (x + 2)(x + 4)$

c $6x^2 - 11x - 10$
$= 6x^2 - 15x + 4x - 10$ — $ac = -60$ and $4 - 15 = -11 = b$.
$= 3x(2x - 5) + 2(2x - 5)$ — Factorise.
$= (2x - 5)(3x + 2)$

d $x^2 - 25$ — This is the difference of two squares as the two terms are x^2 and 5^2.
$= x^2 - 5^2$
$= (x + 5)(x - 5)$ — The two x terms, $5x$ and $-5x$, cancel each other out.

e $4x^2 - 9y^2$ — This is the same as $(2x)^2 - (3y)^2$.
$= 2^2x^2 - 3^2y^2$
$= (2x + 3y)(2x - 3y)$

Example 8

Factorise completely:

a $x^3 - 2x^2$ **b** $x^3 - 25x$ **c** $x^3 + 3x^2 - 10x$

a $x^3 - 2x^2 = x^2(x - 2)$ — You can't factorise this any further.

b $x^3 - 25x = x(x^2 - 25)$ — x is a common factor of x^3 and $-25x$. So take x outside the brackets. $x^2 - 25$ is the difference of two squares.
$= x(x^2 - 5^2)$
$= x(x + 5)(x - 5)$

c $x^3 + 3x^2 - 10x = x(x^2 + 3x - 10)$ — Write the expression as a product of x and a quadratic factor.
$= x(x + 5)(x - 2)$ — Factorise the quadratic to get three linear factors.

Exercise 1C

1 Factorise these expressions completely:

a $4x + 8$	**b** $6x - 24$	**c** $20x + 15$
d $2x^2 + 4$	**e** $4x^2 + 20$	**f** $6x^2 - 18x$
g $x^2 - 7x$	**h** $2x^2 + 4x$	**i** $3x^2 - x$
j $6x^2 - 2x$	**k** $10y^2 - 5y$	**l** $35x^2 - 28x$
m $x^2 + 2x$	**n** $3y^2 + 2y$	**o** $4x^2 + 12x$
p $5y^2 - 20y$	**q** $9xy^2 + 12x^2y$	**r** $6ab - 2ab^2$
s $5x^2 - 25xy$	**t** $12x^2y + 8xy^2$	**u** $15y - 20yz^2$
v $12x^2 - 30$	**w** $xy^2 - x^2y$	**x** $12y^2 - 4yx$

2 Factorise:

a $x^2 + 4x$ **b** $2x^2 + 6x$ **c** $x^2 + 11x + 24$
d $x^2 + 8x + 12$ **e** $x^2 + 3x - 40$ **f** $x^2 - 8x + 12$
g $x^2 + 5x + 6$ **h** $x^2 - 2x - 24$ **i** $x^2 - 3x - 10$
j $x^2 + x - 20$ **k** $2x^2 + 5x + 2$ **l** $3x^2 + 10x - 8$
m $5x^2 - 16x + 3$ **n** $6x^2 - 8x - 8$
o $2x^2 + 7x - 15$ **p** $2x^4 + 14x^2 + 24$
q $x^2 - 4$ **r** $x^2 - 49$
s $4x^2 - 25$ **t** $9x^2 - 25y^2$ **u** $36x^2 - 4$
v $2x^2 - 50$ **w** $6x^2 - 10x + 4$ **x** $15x^2 + 42x - 9$

Hint For part **n**, take 2 out as a common factor first. For part **p**, let $y = x^2$.

3 Factorise completely:

a $x^3 + 2x$ **b** $x^3 - x^2 + x$ **c** $x^3 - 5x$
d $x^3 - 9x$ **e** $x^3 - x^2 - 12x$ **f** $x^3 + 11x^2 + 30x$
g $x^3 - 7x^2 + 6x$ **h** $x^3 - 64x$ **i** $2x^3 - 5x^2 - 3x$
j $2x^3 + 13x^2 + 15x$ **k** $x^3 - 4x$ **l** $3x^3 + 27x^2 + 60x$

(P) **4** Factorise completely $x^4 - y^4$. **(2 marks)**

Problem-solving

Watch out for terms that can be written as a function of a function: $x^4 = (x^2)^2$

(E) **5** Factorise completely $6x^3 + 7x^2 - 5x$. **(2 marks)**

Challenge

Write $4x^4 - 13x^2 + 9$ as the product of four linear factors.

1.4 Negative and fractional indices

Indices can be negative numbers or fractions.

$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x,$

similarly $\underbrace{x^{\frac{1}{n}} \times x^{\frac{1}{n}} \times \ldots \times x^{\frac{1}{n}}}_{n \text{ terms}} = x^{\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}} = x^1 = x$

Notation **Rational** numbers are those that can be written as $\frac{a}{b}$ where a and b are integers.

- **You can use the laws of indices with any rational power.**
  - $a^{\frac{1}{m}} = \sqrt[m]{a}$
  - $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
  - $a^{-m} = \frac{1}{a^m}$
  - $a^0 = 1$

Notation $a^{\frac{1}{2}} = \sqrt{a}$ is the positive square root of a.
For example $9^{\frac{1}{2}} = \sqrt{9} = 3$ but $9^{\frac{1}{2}} \neq -3$.

Example 9

Simplify:

a $\dfrac{x^3}{x^{-3}}$ **b** $x^{\frac{1}{2}} \times x^{\frac{3}{2}}$ **c** $(x^3)^{\frac{2}{3}}$ **d** $2x^{1.5} \div 4x^{-0.25}$ **e** $\sqrt[3]{125x^6}$ **f** $\dfrac{2x^2 - x}{x^5}$

a $\dfrac{x^3}{x^{-3}} = x^{3-(-3)} = x^6$

Use the rule $a^m \div a^n = a^{m-n}$.

b $x^{\frac{1}{2}} \times x^{\frac{3}{2}} = x^{\frac{1}{2}+\frac{3}{2}} = x^2$

This could also be written as $\sqrt{x}$.
Use the rule $a^m \times a^n = a^{m+n}$.

c $(x^3)^{\frac{2}{3}} = x^{3 \times \frac{2}{3}} = x^2$

Use the rule $(a^m)^n = a^{mn}$.

d $2x^{1.5} \div 4x^{-0.25} = \frac{1}{2}x^{1.5-(-0.25)} = \frac{1}{2}x^{1.75}$

Use the rule $a^m \div a^n = a^{m-n}$.
$1.5 - (-0.25) = 1.75$

e $\sqrt[3]{125x^6} = (125x^6)^{\frac{1}{3}}$

$= (125)^{\frac{1}{3}}(x^6)^{\frac{1}{3}} = \sqrt[3]{125}(x^{6 \times \frac{1}{3}}) = 5x^2$

Using $a^{\frac{1}{m}} = \sqrt[m]{a}$.

f $\dfrac{2x^2 - x}{x^5} = \dfrac{2x^2}{x^5} - \dfrac{x}{x^5}$

Divide each term of the numerator by x^5.

$= 2 \times x^{2-5} - x^{1-5} = 2x^{-3} - x^{-4}$

$= \dfrac{2}{x^3} - \dfrac{1}{x^4}$

Using $a^{-m} = \dfrac{1}{a^m}$

Example 10

Evaluate:

a $9^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $49^{\frac{3}{2}}$ **d** $25^{-\frac{3}{2}}$

a $9^{\frac{1}{2}} = \sqrt{9} = 3$

Using $a^{\frac{1}{m}} = \sqrt[m]{a}$. $9^{\frac{1}{2}} = \sqrt{9}$

b $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

This means the cube root of 64.

c $49^{\frac{3}{2}} = (\sqrt{49})^3$

$= 7^3 = 343$

Using $a^{\frac{n}{m}} = \sqrt[m]{a^n}$.
This means the square root of 49, cubed.

d $25^{-\frac{3}{2}} = \dfrac{1}{25^{\frac{3}{2}}} = \dfrac{1}{(\sqrt{25})^3}$

$= \dfrac{1}{5^3} = \dfrac{1}{125}$

Using $a^{-m} = \dfrac{1}{a^m}$

Online Use your calculator to enter negative and fractional powers.

Example 11

Given that $y = \frac{1}{16}x^2$ express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{2}}$ **b** $4y^{-1}$

a $y^{\frac{1}{2}} = \left(\frac{1}{16}x^2\right)^{\frac{1}{2}}$

$= \frac{1}{\sqrt{16}}x^{2 \times \frac{1}{2}} = \frac{x}{4}$

b $4y^{-1} = 4\left(\frac{1}{16}x^2\right)^{-1}$

$= 4\left(\frac{1}{16}\right)^{-1}x^{2 \times (-1)} = 4 \times 16x^{-2}$

$= 64x^{-2}$

Substitute $y = \frac{1}{16}x^2$ into $y^{\frac{1}{2}}$.

$\left(\frac{1}{16}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{16}}$ and $(x^2)^{\frac{1}{2}} = x^{2 \times \frac{1}{2}}$

$\left(\frac{1}{16}\right)^{-1} = 16$ and $x^{2 \times -1} = x^{-2}$

Problem-solving

Check that your answers are in the correct form. If k and n are constants they could be positive or negative, and they could be integers, fractions or surds.

Exercise 1D

1 Simplify:

a $x^3 \div x^{-2}$ **b** $x^5 \div x^7$ **c** $x^{\frac{3}{2}} \times x^{\frac{5}{2}}$

d $(x^2)^{\frac{3}{2}}$ **e** $(x^3)^{\frac{5}{3}}$ **f** $3x^{0.5} \times 4x^{-0.5}$

g $9x^{\frac{2}{3}} \div 3x^{\frac{1}{6}}$ **h** $5x^{\frac{7}{5}} \div x^{\frac{2}{5}}$ **i** $3x^4 \times 2x^{-5}$

j $\sqrt{x} \times \sqrt[3]{x}$ **k** $(\sqrt{x})^3 \times (\sqrt[3]{x})^4$ **l** $\frac{(\sqrt[3]{x})^2}{\sqrt{x}}$

2 Evaluate:

a $25^{\frac{1}{2}}$ **b** $81^{\frac{3}{2}}$ **c** $27^{\frac{1}{3}}$

d 4^{-2} **e** $9^{-\frac{1}{2}}$ **f** $(-5)^{-3}$

g $\left(\frac{3}{4}\right)^0$ **h** $1296^{\frac{3}{4}}$ **i** $\left(\frac{25}{16}\right)^{\frac{3}{2}}$

j $\left(\frac{27}{8}\right)^{\frac{2}{3}}$ **k** $\left(\frac{6}{5}\right)^{-1}$ **l** $\left(\frac{343}{512}\right)^{-\frac{2}{3}}$

3 Simplify:

a $(64x^{10})^{\frac{1}{2}}$ **b** $\frac{5x^3 - 2x^2}{x^5}$ **c** $(125x^{12})^{\frac{1}{3}}$ **d** $\frac{x + 4x^3}{x^3}$

e $\frac{2x + x^2}{x^4}$ **f** $\left(\frac{4}{9}x^4\right)^{\frac{3}{2}}$ **g** $\frac{9x^2 - 15x^5}{3x^3}$ **h** $\frac{5x + 3x^2}{15x^3}$

(E) **4** **a** Find the value of $81^{\frac{1}{4}}$. **(1 mark)**

b Simplify $x(2x^{-\frac{1}{3}})^4$. **(2 marks)**

(E) **5** Given that $y = \frac{1}{8}x^3$ express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{3}}$ **(2 marks)**

b $\frac{1}{2}y^{-2}$ **(2 marks)**

6 Solve

a $x^{\frac{1}{2}} = 7$ b $y^{\frac{4}{3}} = 81$

c $x^{-\frac{3}{2}} = 8$ d $z^{-\frac{3}{4}} = 1000$

E/P 7 Solve $27\sqrt{x} = \frac{1}{x}$ **(2 marks)**

Hint You can undo a fractional index by raising it to the power of its reciprocal: $\left(y^{\frac{4}{3}}\right)^{\frac{3}{4}} = y$

Problem-solving Multiply both sides by x then simplify the indices.

1.5 Surds

If n is an integer that is **not** a square number, then any multiple of $\sqrt{n}$ is called a surd. Examples of surds are $\sqrt{2}$, $\sqrt{19}$ and $5\sqrt{2}$.

Surds are examples of **irrational numbers**. The decimal expansion of a surd is never-ending and never repeats, for example $\sqrt{2} = 1.414213562\ldots$

Notation Irrational numbers cannot be written in the form $\frac{a}{b}$ where a and b are integers. Surds are examples of **irrational numbers**.

You can use surds to write exact answers to calculations.

- **You can manipulate surds using these rules:**
  - $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
  - $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Example 12

Simplify:

a $\sqrt{12}$ b $\frac{\sqrt{20}}{2}$ c $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

a $\sqrt{12} = \sqrt{(4 \times 3)}$

$= \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

Look for a factor of 12 that is a square number. Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$. $\sqrt{4} = 2$

b $\frac{\sqrt{20}}{2} = \frac{\sqrt{4} \times \sqrt{5}}{2}$

$\sqrt{20} = \sqrt{4} \times \sqrt{5}$

$\sqrt{4} = 2$

$= \frac{2 \times \sqrt{5}}{2} = \sqrt{5}$

Cancel by 2.

c $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

$= 5\sqrt{6} - 2\sqrt{6}\sqrt{4} + \sqrt{6} \times \sqrt{49}$

$\sqrt{6}$ is a common factor.

$= \sqrt{6}(5 - 2\sqrt{4} + \sqrt{49})$

Work out the square roots $\sqrt{4}$ and $\sqrt{49}$.

$= \sqrt{6}(5 - 2 \times 2 + 7)$

$5 - 4 + 7 = 8$

$= \sqrt{6}(8)$

$= 8\sqrt{6}$

Example 13

Expand and simplify if possible:

a $\sqrt{2}(5 - \sqrt{3})$ **b** $(2 - \sqrt{3})(5 + \sqrt{3})$

a $\sqrt{2}(5 - \sqrt{3})$
$= 5\sqrt{2} - \sqrt{2}\sqrt{3}$
$= 5\sqrt{2} - \sqrt{6}$

$\sqrt{2} \times 5 - \sqrt{2} \times \sqrt{3}$

Using $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

b $(2 - \sqrt{3})(5 + \sqrt{3})$
$= 2(5 + \sqrt{3}) - \sqrt{3}(5 + \sqrt{3})$
$= 10 + 2\sqrt{3} - 5\sqrt{3} - \sqrt{9}$
$= 7 - 3\sqrt{3}$

Expand the brackets completely before you simplify.

Collect like terms: $2\sqrt{3} - 5\sqrt{3} = -3\sqrt{3}$

Simplify any roots if possible: $\sqrt{9} = 3$

Exercise 1E

1 Do not use your calculator for this exercise. Simplify:

a $\sqrt{28}$ **b** $\sqrt{72}$ **c** $\sqrt{50}$

d $\sqrt{32}$ **e** $\sqrt{90}$ **f** $\frac{\sqrt{12}}{2}$

g $\frac{\sqrt{27}}{3}$ **h** $\sqrt{20} + \sqrt{80}$ **i** $\sqrt{200} + \sqrt{18} - \sqrt{72}$

j $\sqrt{175} + \sqrt{63} + 2\sqrt{28}$ **k** $\sqrt{28} - 2\sqrt{63} + \sqrt{7}$ **l** $\sqrt{80} - 2\sqrt{20} + 3\sqrt{45}$

m $3\sqrt{80} - 2\sqrt{20} + 5\sqrt{45}$ **n** $\frac{\sqrt{44}}{\sqrt{11}}$ **o** $\sqrt{12} + 3\sqrt{48} + \sqrt{75}$

2 Expand and simplify if possible:

a $\sqrt{3}(2 + \sqrt{3})$ **b** $\sqrt{5}(3 - \sqrt{3})$ **c** $\sqrt{2}(4 - \sqrt{5})$

d $(2 - \sqrt{2})(3 + \sqrt{5})$ **e** $(2 - \sqrt{3})(3 - \sqrt{7})$ **f** $(4 + \sqrt{5})(2 + \sqrt{5})$

g $(5 - \sqrt{3})(1 - \sqrt{3})$ **h** $(4 + \sqrt{3})(2 - \sqrt{3})$ **i** $(7 - \sqrt{11})(2 + \sqrt{11})$

E **3** Simplify $\sqrt{75} - \sqrt{12}$ giving your answer in the form $a\sqrt{3}$, where a is an integer. **(2 marks)**

1.6 Rationalising denominators

If a fraction has a surd in the denominator, it is sometimes useful to **rearrange** it so that the denominator is a **rational** number. This is called rationalising the denominator.

- **The rules to rationalise denominators are:**
  - **For fractions in the form $\frac{1}{\sqrt{a}}$, multiply the numerator and denominator by $\sqrt{a}$.**
  - **For fractions in the form $\frac{1}{a + \sqrt{b}}$, multiply the numerator and denominator by $a - \sqrt{b}$.**
  - **For fractions in the form $\frac{1}{a - \sqrt{b}}$, multiply the numerator and denominator by $a + \sqrt{b}$.**

Example 14

Rationalise the denominator of:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{3+\sqrt{2}}$ **c** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ **d** $\frac{1}{(1-\sqrt{3})^2}$

a $\frac{1}{\sqrt{3}} = \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$ — Multiply the numerator and denominator by $\sqrt{3}$.

$= \frac{\sqrt{3}}{3}$ — $\sqrt{3} \times \sqrt{3} = (\sqrt{3})^2 = 3$

b $\frac{1}{3+\sqrt{2}} = \frac{1 \times (3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$ — Multiply numerator and denominator by $(3-\sqrt{2})$.

$= \frac{3-\sqrt{2}}{9-3\sqrt{2}+3\sqrt{2}-2}$ — $\sqrt{2} \times \sqrt{2} = 2$

$= \frac{3-\sqrt{2}}{7}$ — $9-2=7$, $-3\sqrt{2}+3\sqrt{2}=0$

c $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{(\sqrt{5}+\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$ — Multiply numerator and denominator by $\sqrt{5}+\sqrt{2}$.

$= \frac{5+\sqrt{5}\sqrt{2}+\sqrt{2}\sqrt{5}+2}{5-2}$ — $-\sqrt{2}\sqrt{5}$ and $\sqrt{5}\sqrt{2}$ cancel each other out.

$= \frac{7+2\sqrt{10}}{3}$ — $\sqrt{5}\sqrt{2} = \sqrt{10}$

d $\frac{1}{(1-\sqrt{3})^2} = \frac{1}{(1-\sqrt{3})(1-\sqrt{3})}$ — Expand the brackets.

$= \frac{1}{1-\sqrt{3}-\sqrt{3}+\sqrt{9}}$ — Simplify and collect like terms. $\sqrt{9} = 3$

$= \frac{1}{4-2\sqrt{3}}$

$= \frac{1 \times (4+2\sqrt{3})}{(4-2\sqrt{3})(4+2\sqrt{3})}$ — Multiply the numerator and denominator by $4+2\sqrt{3}$.

$= \frac{4+2\sqrt{3}}{16+8\sqrt{3}-8\sqrt{3}-12}$ — $\sqrt{3} \times \sqrt{3} = 3$; $16-12=4$, $8\sqrt{3}-8\sqrt{3}=0$

$= \frac{4+2\sqrt{3}}{4} = \frac{2+\sqrt{3}}{2}$

Exercise 1F

1 Simplify:

a $\frac{1}{\sqrt{5}}$ **b** $\frac{1}{\sqrt{11}}$ **c** $\frac{1}{\sqrt{2}}$ **d** $\frac{\sqrt{3}}{\sqrt{15}}$

e $\frac{\sqrt{12}}{\sqrt{48}}$ **f** $\frac{\sqrt{5}}{\sqrt{80}}$ **g** $\frac{\sqrt{12}}{\sqrt{156}}$ **h** $\frac{\sqrt{7}}{\sqrt{63}}$

2 Rationalise the denominators and simplify:

a $\frac{1}{1+\sqrt{3}}$ **b** $\frac{1}{2+\sqrt{5}}$ **c** $\frac{1}{3-\sqrt{7}}$ **d** $\frac{4}{3-\sqrt{5}}$ **e** $\frac{1}{\sqrt{5}-\sqrt{3}}$

f $\frac{3-\sqrt{2}}{4-\sqrt{5}}$ **g** $\frac{5}{2+\sqrt{5}}$ **h** $\frac{5\sqrt{2}}{\sqrt{8}-\sqrt{7}}$ **i** $\frac{11}{3+\sqrt{11}}$ **j** $\frac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

k $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$ **l** $\frac{\sqrt{41}+\sqrt{29}}{\sqrt{41}-\sqrt{29}}$ **m** $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

3 Rationalise the denominators and simplify:

a $\frac{1}{(3-\sqrt{2})^2}$ **b** $\frac{1}{(2+\sqrt{5})^2}$ **c** $\frac{4}{(3-\sqrt{2})^2}$

d $\frac{3}{(5+\sqrt{2})^2}$ **e** $\frac{1}{(5+\sqrt{2})(3-\sqrt{2})}$ **f** $\frac{2}{(5-\sqrt{3})(2+\sqrt{3})}$

E/P **4** Simplify $\frac{3-2\sqrt{5}}{\sqrt{5}-1}$ giving your answer in the form $p + q\sqrt{5}$, where p and q are rational numbers. **(4 marks)**

Problem-solving

You can check that your answer is in the correct form by writing down the values of p and q and checking that they are rational numbers.

Mixed exercise 1

1 Simplify:

a $y^3 \times y^5$ **b** $3x^2 \times 2x^5$ **c** $(4x^2)^3 \div 2x^5$ **d** $4b^2 \times 3b^3 \times b^4$

2 Expand and simplify if possible:

a $(x + 3)(x - 5)$ **b** $(2x - 7)(3x + 1)$ **c** $(2x + 5)(3x - y + 2)$

3 Expand and simplify if possible:

a $x(x + 4)(x - 1)$ **b** $(x + 2)(x - 3)(x + 7)$ **c** $(2x + 3)(x - 2)(3x - 1)$

4 Expand the brackets:

a $3(5y + 4)$ **b** $5x^2(3 - 5x + 2x^2)$ **c** $5x(2x + 3) - 2x(1 - 3x)$ **d** $3x^2(1 + 3x) - 2x(3x - 2)$

5 Factorise these expressions completely:

a $3x^2 + 4x$ **b** $4y^2 + 10y$ **c** $x^2 + xy + xy^2$ **d** $8xy^2 + 10x^2y$

6 Factorise:

a $x^2 + 3x + 2$ **b** $3x^2 + 6x$ **c** $x^2 - 2x - 35$ **d** $2x^2 - x - 3$
e $5x^2 - 13x - 6$ **f** $6 - 5x - x^2$

7 Factorise:

a $2x^3 + 6x$ **b** $x^3 - 36x$ **c** $2x^3 + 7x^2 - 15x$

8 Simplify:

a $9x^3 \div 3x^{-3}$ **b** $(4^{\frac{3}{2}})^{\frac{1}{3}}$ **c** $3x^{-2} \times 2x^4$ **d** $3x^{\frac{1}{3}} \div 6x^{\frac{2}{3}}$

9 Evaluate:

a $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ **b** $\left(\frac{225}{289}\right)^{\frac{3}{2}}$

10 Simplify:

a $\frac{3}{\sqrt{63}}$ **b** $\sqrt{20} + 2\sqrt{45} - \sqrt{80}$

11 **a** Find the value of $35x^2 + 2x - 48$ when $x = 25$.

b By factorising the expression, show that your answer to part **a** can be written as the product of two prime factors.

12 Expand and simplify if possible:

a $\sqrt{2}(3 + \sqrt{5})$ **b** $(2 - \sqrt{5})(5 + \sqrt{3})$ **c** $(6 - \sqrt{2})(4 - \sqrt{7})$

13 Rationalise the denominator and simplify:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2} - 1}$ **c** $\frac{3}{\sqrt{3} - 2}$ **d** $\frac{\sqrt{23} - \sqrt{37}}{\sqrt{23} + \sqrt{37}}$ **e** $\frac{1}{(2 + \sqrt{3})^2}$ **f** $\frac{1}{(4 - \sqrt{7})^2}$

14 **a** Given that $x^3 - x^2 - 17x - 15 = (x + 3)(x^2 + bx + c)$, where b and c are constants, work out the values of b and c.

b Hence, fully factorise $x^3 - x^2 - 17x - 15$.

(E) **15** Given that $y = \frac{1}{64}x^3$ express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{3}}$ **(1 mark)**

b $4y^{-1}$ **(1 mark)**

(E/P) **16** Show that $\frac{5}{\sqrt{75} - \sqrt{50}}$ can be written in the form $\sqrt{a} + \sqrt{b}$, where a and b are integers. **(5 marks)**

(E) **17** Expand and simplify $(\sqrt{11} - 5)(5 - \sqrt{11})$. **(2 marks)**

(E) **18** Factorise completely $x - 64x^3$. **(3 marks)**

(E/P) **19** Express $27^{2x + 1}$ in the form 3^y, stating y in terms of x. **(2 marks)**

E/P **20** Solve the equation $8 + x\sqrt{12} = \frac{8x}{\sqrt{3}}$

Give your answer in the form $a\sqrt{b}$ where a and b are integers. **(4 marks)**

P **21** A rectangle has a length of $(1 + \sqrt{3})$ cm and area of $\sqrt{12}$ cm^2.
Calculate the width of the rectangle in cm.
Express your answer in the form $a + b\sqrt{3}$, where a and b are integers to be found.

E **22** Show that $\frac{(2 - \sqrt{x})^2}{\sqrt{x}}$ can be written as $4x^{-\frac{1}{2}} - 4 + x^{\frac{1}{2}}$. **(2 marks)**

E/P **23** **a** Given that $243\sqrt{3} = 3^a$, find the value of a. **(2 marks)**

b Given further that $3^x \times 27^y = 243\sqrt{3}$, express y as a function of x. **(2 marks)**

E/P **24** Given that $\frac{4x^3 + x^{\frac{5}{2}}}{\sqrt{x}}$ can be written in the form $4x^a + x^b$, write down the value of a and the value of b. **(2 marks)**

Challenge

a Simplify $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

b Hence show that $\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \ldots + \frac{1}{\sqrt{24} + \sqrt{25}} = 4$

Summary of key points

1 You can use the laws of indices to simplify powers of the **same base**.
- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$

2 Factorising is the opposite of expanding brackets.

3 A quadratic expression has the form $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.

4 $x^2 - y^2 = (x + y)(x - y)$

5 You can use the laws of indices with any rational power.
- $a^{\frac{1}{m}} = \sqrt[m]{a}$
- $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
- $a^{-m} = \frac{1}{a^m}$
- $a^0 = 1$

6 You can manipulate surds using these rules:
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

7 The rules to rationalise denominators are:
- Fractions in the form $\frac{1}{\sqrt{a}}$, multiply the numerator and denominator by $\sqrt{a}$.
- Fractions in the form $\frac{1}{a + \sqrt{b}}$, multiply the numerator and denominator by $a - \sqrt{b}$.
- Fractions in the form $\frac{1}{a - \sqrt{b}}$, multiply the numerator and denominator by $a + \sqrt{b}$.